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RANDOM MATRICES, CONTINUOUS CIRCULAR SYSTEMS AND THE TRIANGULAR OPERATOR

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Abstract. Using suitably defined continuous analogs of the matricial circular systems and the direct integral of Hilbert spaces $\mathcal{H}=\int_{\Gamma}^{\oplus}\mathcal{H}(\gamma)d\gamma$, we study the operators living in $\mathcal{H}$ which give the asymptotic joint *-distributions of complex independent Gaussian random matrices with not necessarily equal variances of the entries. These operators are decomposed in terms of continuous circular systems $\{\zeta(x,y;u): x,y\in[0,1], u\in\mathcal{U}\}$ acting between the fibers of $\mathcal{H}$, the continuous analogs of matricial circular systems obtained when the Gaussian entries are block-identically distributed. In the case of square matrices with i.i.d. entries, we obtain the circular operators of Voiculescu, whereas in the case of upper-triangular matrices with i.i.d. entries, we obtain the triangular operators of Dykema and Haagerup. We apply this approach to give a bijective proof of the formula for the moments of T^*T, where T is a triangular operator, using the enumeration formula of Chauve, Dulucq and Rechnizter for alternating ordered rooted trees.

1. Introduction

Independent Gaussian random matrices with suitably normalized complex i.i.d. entries are asymptotically free with respect to the normalized trace composed with classical expectation. The limit mixed (*-) moments can be expressed in terms of mixed (*-) moments of free circular operators. This fundamental result was shown by Voiculescu [16], who also found a relation to free group factors [17].

Asymptotic freeness of Voiculescu was later generalized in many directions, in particular, to the non-Gaussian random matrices by Dykema [3] and to Gaussian band matrices by Shlyakhtenko [13]. In the latter case, where the Gaussian variables are not assumed to be identically distributed, scalar valued freeness [18] is not sufficient to describe the asymptotics of matrices and one has to use freeness with amalgamation over some subalgebra, a generalization of freeness, in which a state is replaced by an operator-valued conditional expectation with values in this subalgebra. This approach was later used by Benaych-Georges [1] in his study of rectangular block random matrices, who introduced a rectangular analog of Voiculescu's R-transform [18]. Analytic methods like operator-valued transforms

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were also applied to the study of blocks of random matrices (most of these results are mentioned in the recently published monograph of Mingo and Speicher [11]).

The main point of our approach is that our realizations of the limit joint *-distributions of Gaussian random matrices are built from operators living in Hilbert spaces rather than in Hilbert modules. Our combinatorics also has new features since it is based on coloring noncrossing pair partitions rather than on using nested evaluations which are needed when applying the conditional expectation and operator-valued free probability. Although one can translate our approach to that of the operator-valued free probability, the new language provides a new realization of limit distributions as well as connections to some nice results in combinatorics. For instance, the combinatorics of colored partitions allows us to find a connection between certain results on moments of the triangular operator defined by Dykema and Haagerup [4] with known enumeration results for alternating ordered rooted trees. The triangular operator got some attention in the context of the famous invariant subspace problem in the papers of Dykema and Haagerup on the so-called DT-operators and their decompositions [4,5], including the circular operator of Voiculescu and the triangular operator T. The moments of T^*T were also found by Dykema and Haagerup in [4] and the more general case of the moments of ${T^*}^k T^k$ was treated by Śniady [14], who used the so-called generalized circular elements and the combinatorics of nested evaluations.

As far as matricial circular systems are concerned, they describe the asymptotics of blocks of independent Gaussian random matrices under partial traces. If we are given an ensemble of independent non-Hermitian $n \times n$ Gaussian random matrices $\{Y(u,n) : u \in \mathcal{U}\}$ with suitably normalized independent block-identically distributed (*i.b.i.d.*) complex entries for each natural n, then the mixed *-moments of their (in general, rectangular) blocks $S_{p,q}(u,n)$ converge under normalized partial traces to the mixed *-moments of certain bounded operators, which we write informally

$$\lim_{n\to\infty} S_{p,q}(u,n) = \zeta_{p,q}(u),$$

where $u \in \mathcal{U}$ and $1 \leq p,q \leq r$ and the operators $\zeta_{p,q}(u)$ are called *matricial circular operators* [10]. Their counterparts in the operator-valued free probability are operators of the form $F_p c(u) F_q$, where $\{c(u) : u \in \mathcal{U}\}$ is a family of circular elements living in the Fock space over Hilbert A-module, where A is the algebra of $r \times r$ diagonal matrices with canonical generators $F_1, \ldots, F_r$.

In this paper, we use continuous analogs of $\zeta_{p,q}(u)$, namely $\zeta(x,y;u)$, where, roughly speaking, $p/r \to x$, $q/r \to y$ as $r \to \infty$. The families of these operators are called continuous circular systems. Using direct integrals involving these systems, we obtain realizations of the asymptotic *-joint distributions of matrices $Y(u,n)$ in terms of operators of the form

$$\zeta(g,u) = \int_{\Gamma_1}^{\oplus} g(x,y)\zeta(x,y;u)dxdy,$$

where $\Gamma_1 = [0,1] \times [0,1]$ and $g \in L^\infty(\Gamma_1)$.

In particular, if $g = \chi_\Delta$, where $\Delta = \{(x,y) \in \Gamma_1 : x < y\}$, then we obtain the family of free triangular operators of Dykema and Haagerup [4]. We apply this approach to study the mixed *-moments of the triangular operators. In particular, we provide a new and natural bijective proof of the formula for the moments

$$\varphi((T^*T)^n) = \frac{n^n}{(n+1)!},$$

shown by Dykema and Haagerup [4] by a different method, where T is a triangular operator, using the nice enumeration result of Chauve, Dulucq and Rechnitzer [2] for alternating ordered rooted tress. In this context, let us remark that there is a more general bijective proof of Śniady [15], based on a certain algorithm of counting total orders on directed trees.

The paper is organized as follows. In Section 2 we recall the notions related to matricial circular systems. In Section 3, we introduce the direct integral of Hilbert spaces, on which we define a family of creation and annihilation operators. In Section 4, we introduce continuous circular systems as isometries between suitably defined fiber Hilbert spaces. Mixed *-moments of matricial circular operators and their convergence to the mixed *-moments of the operators decomposed in terms of the continuous circular systems are discussed in Section 5. In Section 6, we apply our approach to the triangular operators and we provide a bijective proof of the formula for the moments of T^*T.

We adopt the convention that the stars which indicate adjoints are written closely to the main symbol, for instance $\wp^*(g,u)$ and $\wp^*(x,y;u)$ are the adjoints of $\wp(g,u)$ and $\wp(x,y;u)$, respectively.

2. Matricial Circular Systems

Let $[r] := \{1,2,\ldots,r\}$ and let $\mathcal{U}$ be a countable set. To each $(p,q) \in \mathcal{J} \subset [r]\times[r]$ and $u \in \mathcal{U}$ we associate a Hilbert space $\mathcal{H}_{p,q}(u)$. Using this family of Hilbert spaces, we can construct the matricially free Fock space of tracial type (see [7,9,10]).

Definition 2.1. By the *matricially free Fock space of tracial type* we understand the direct sum of Hilbert spaces

$$\mathcal{M} = \bigoplus_{q=1}^{r} \mathcal{M}_q,$$

where each summand is of the form

$$\mathcal{M}_q = \mathbb{C}\Omega_q \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{p_1,\ldots,p_m \\ u_1,\ldots,u_n}} \mathcal{H}_{p_1,p_2}(u_1) \otimes \mathcal{H}_{p_2,p_3}(u_2) \otimes \ldots \otimes \mathcal{H}_{p_m,q}(u_m),$$

where Ω_q is a unit vector for any $q \in [r]$, endowed with the canonical inner product. We denote by Ψ_q the state associated with Ω_q.

Let us recall a number of basic facts and notions from [8,9,10].

(1) In the special case when each $\mathcal{H}_{p,q}(u) = \mathbb{C}e_{p,q}(u)$ for any p,q,u, where $e_{p,q}(u)$ is a unit vector, the *matricially free creation operators* associated with matrices $B(u) = (b_{p,q}(u))$ of non-negative real numbers (covariance

matrices) are bounded operators whose non-trivial action onto the canonical orthonormal basis is

$$\begin{aligned} \wp_{p,q}(u)\Omega_q &= \sqrt{b_{p,q}(u)}e_{p,q}(u), \\ \wp_{p,q}(u)(e_{q,t}(s)) &= \sqrt{b_{p,q}(u)}(e_{p,q}(u)\otimes e_{q,t}(s)), \\ \wp_{p,q}(u)(e_{q,t}(s)\otimes w) &= \sqrt{b_{p,q}(u)}(e_{p,q}(u)\otimes e_{q,t}(s)\otimes w), \end{aligned}$$

for any $p, q, t \in [r]$ and $u, s \in \mathcal{U}$, where $e_{q,t}(s)\otimes w$ is a basis vector. Action onto the remaining basis vectors gives zero. The corresponding *matricially free annihilation operators* are their adjoints denoted $\wp^*_{p,q}(u)$.

(2) Matricially free creation operators can be realized as operator-valued matrices [10]. If we are given a C^*-probability space $(\mathcal{A}, \phi)$ and a family of free creation operators $\{\ell(p,q,u) : p,q \in [r],\ u \in \mathcal{U}\}$ with covariances $b_{p,q}(u)$, respectively, which are *-free with respect to ϕ, and $(e(p,q))$ is the array of matrix units in $M_r(\mathbb{C})$, then

$$\wp_{p,q}(u) \quad = \quad \ell(p,q,u)\otimes e(p,q)$$

for any $p, q \in [r]$ and $u \in \mathcal{U}$. This equality holds in the sense that the mixed *-moments of the operators $\wp_{p,q}(u)$ under the states Ψ_j agree with the corresponding mixed *-moments of the above matrices under the states $\Phi_j = \phi\otimes\psi_j$, where ψ_j is the state associated with the canonical basis vector $e(j)$ of $\mathbb{C}^r$, where $j \in [r]$.

(3) The *matricial circular operators* are of the form

$$\zeta_{p,q}(u) = \wp_{p,q}(u') + \wp^*_{q,p}(u''),$$

where $u \in \mathcal{U} = [t]$, $p, q \in [r]$ and u', u'' are different copies of u. As we showed in [9], they can be realized as operator-valued matrices

$$\zeta_{p,q}(u) = c(p,q,u)\otimes e(p,q),$$

where $p, q \in [r]$, $u \in \mathcal{U}$, $\{c(p,q,u) : p,q \in [r],\ u \in \mathcal{U}\}$ is a family of free generalized circular operators, i.e. $c(p,q,u) = \ell(p,q,u') + \ell^*(q,p,u'')$. We use the term 'generalized' since, in general, the covariances of the creation operators are arbitrary. If all creation operators are standard, this is a family of free circular operators. The corresponding family of arrays of operators is called the *matricial circular system*.

3. Direct Integrals

We would like to construct a continuous analog of the matricially free Fock space of tracial type. For that purpose, we will use the formalism of direct integrals (for more on direct integrals, see, for instance, [6]). However, two different direct integral decompositions of the considered Fock space and of the canonical operator fields acting on this Fock space will be helpful.

We begin with a decomposition which is a straightforward generalization of the discrete matricially free Fock space of tracial type. For $I = [0,1]$, let

$$\Gamma = \bigoplus_{n=0}^{\infty} \Gamma_n$$

be the direct sum of measure spaces, where $\Gamma_n = I^{n+1}$ is equipped with the Lebesgue measure denoted $d\gamma_n$, and let us denote by $d\gamma$ the corresponding direct sum of measures on the set Γ.

Definition 3.1. By the *continuous matricially free Fock space* we understand the direct integral of Hilbert spaces of the form

$$\mathcal{H} = \int_{\Gamma}^{\oplus} \mathcal{H}(\gamma) d\gamma,$$

where Hilbert spaces are associated to $\gamma \in \Gamma$ as follows:

(1) if $\gamma = x \in \Gamma_0 = I$, then

$$\mathcal{H}(\gamma) = \mathbb{C}\Omega(x),$$

where $\Omega(x)$ is a unit vector,

(2) if $\gamma = (x_1, x_2, \ldots, x_{n+1}) \in \Gamma_n$ and $n \in \mathbb{N}$, then

$$\mathcal{H}(\gamma) = \mathcal{H}(x_1, x_2) \otimes \mathcal{H}(x_2, x_3) \otimes \ldots \otimes \mathcal{H}(x_n, x_{n+1}),$$

where each $\mathcal{H}(x, y)$ is a separable Hilbert space, and each $\mathcal{H}(\gamma)$ is equipped with the canonical inner product,

(3) the canonical inner product in $\mathcal{H}$ is then given by

$$\langle F, G \rangle = \int_{\Gamma} \langle F(\gamma), G(\gamma) \rangle \, d\gamma,$$

where $F = \int_{\Gamma}^{\oplus} F(\gamma) d\gamma, G = \int_{\Gamma}^{\oplus} G(\gamma) d\gamma \in \mathcal{H}$ are measurable square integrable fields with the natural assumption that $F(\gamma), G(\gamma) \in \mathcal{H}(\gamma)$.

Remark 3.2. Let us collect certain basic facts about the Hilbert spaces defined above.

(1) The continuous family of unit vectors $\{\Omega(x) : x \in I\}$ replaces the finite set of vacuum vectors $\{\Omega_1, \ldots, \Omega_r\}$ used in the discrete case. The corresponding direct integral

$$\mathcal{H}_0 := \int_{I}^{\oplus} \mathcal{H}(x) dx \cong L^2(I)$$

will be called the *vacuum space*. In this paper, we will be mainly concerned with the function on I which is constantly equal to one since it corresponds to the canonical trace on the algebra of random matrices. However, weighted traces will, in general, lead to different elements of $L^2(I)$.

(2) In the case when $\mathcal{H}(x,y) \cong \mathcal{G}$ for any $(x,y) \in \Gamma_1$, where $\mathcal{G}$ is a separable Hilbert space (with an orthonormal basis indexed by $\mathcal{U}$), we also have isomorphisms for higher order integrals

$$\mathcal{H}_n := \int_{\Gamma_n}^{\oplus} \mathcal{H}(\gamma) d\gamma_n \cong L^2(\Gamma_n, \mathcal{G}^{\otimes n}),$$

where $n \geq 1$ and $L^2(\Gamma_n, \mathcal{G}^{\otimes n})$ denotes the Hilbert space of square integrable $\mathcal{G}^{\otimes n}$-valued functions over the set Γ_n with respect to $d\gamma_n$ ($d\gamma$ restricted to Γ_n). Thus, in this particular case, we have the isomorphism

$$\mathcal{H} \cong L^2(I) \oplus \bigoplus_{n=1}^{\infty} L^2(\Gamma_n, \mathcal{G}^{\otimes n}).$$

(3) Fields $F = \int_{\Gamma}^{\oplus} F(\gamma) d\gamma, G = \int_{\Gamma}^{\oplus} G(\gamma) d\gamma \in \mathcal{H}$ have direct sum decompositions

$$F = \sum_{n=0}^{\infty} F_n \text{ and } G = \sum_{n=0}^{\infty} G_n,$$

where $F_n, G_n \in \int_{\Gamma_n} \mathcal{H}(\gamma) d\gamma$ in the natural sense. Under the above isomorphism assumptions, $F_0, G_0 \in L^2(I)$ and $F_n, G_n \in L^2(\Gamma_n, \mathcal{G}^{\otimes n})$ for $n \geq 1$. In most computations, it is enough to consider these to be of the form

$$\begin{aligned} F_n(\gamma) &= f_1(x_1, x_2) \otimes \ldots \otimes f_n(x_n, x_{n+1}), \\ G_n(\gamma) &= g_1(x_1, x_2) \otimes \ldots \otimes g_n(x_n, x_{n+1}), \end{aligned}$$

for $\gamma = (x_1, \ldots, x_{n+1})$ and $n \geq 1$, with $f_i(x_i, x_{i+1}), g_i(x_i, x_{i+1}) \in \mathcal{G}$ for any i.

(4) The canonical inner product in $\mathcal{H}$ decomposes as

$$\langle F, G \rangle = \sum_{n=0}^{\infty} \int_{\Gamma_n} \langle F_n(\gamma), G_n(\gamma) \rangle \, d\gamma_n$$

for any $F, G \in \mathcal{H}$, and an analogous equation holds for squared norms.

This setting is suitable for introducing continuous analogs of sums of matricially free creation operators

$$\wp(u) = \sum_{p,q=1}^{r} \wp_{p,q}(u),$$

where the covariance of each $\wp_{p,q}(u)$ is assumed to be $b_{p,q}(u) \geq 0$, $u \in \mathcal{U}$. In particular, if $b_{p,q}(u) = d_p$ for any p, q, u, where $d_1 + \cdots + d_r = 1$, then $\{\wp(u) : u \in \mathcal{U}\}$ is a family of standard free creation operators. We would like to find a continuous analog of these decompositions, using direct integrals.

The continuous analogs of the matricially free creation operators will be denoted $\wp(f)$, where f is an essentially bounded $\mathcal{G}$-valued function on Γ_1, namely $f \in L^{\infty}(\Gamma_1, \mathcal{G})$, where the square Γ_1 is equipped with the two-dimensional Lebesgue measure.

Definition 3.3. For given $f \in L^\infty(\Gamma_1, \mathcal{G})$, let us define bounded linear operators $\wp(f)$ on $\mathcal{H}$ by

$$\wp(f)\left(\int_I^\oplus F_0(x_1)dx_1\right) = \int_{\Gamma_1}^\oplus f(x,x_1)F_0(x_1)dxdx_1$$

for any $F_0 \in L^2(I)$, and

$$\wp(f)\left(\int_{\Gamma_n}^\oplus F_n(x_1,\ldots,x_{n+1})dx_1\ldots dx_{n+1}\right)$$
$$= \int_{\Gamma_{n+1}}^\oplus f(x,x_1)\otimes F_n(x_1,\ldots,x_{n+1})dxdx_1\ldots dx_{n+1}$$

for any $F_n \in L^2(\Gamma_n, \mathcal{G}^{\otimes n})$, where $n \in \mathbb{N}$. In the special case when $f = g \otimes e(u)$, where $e(u)$ is some basis unit vector of $\mathcal{G}$, under the identification $L^\infty(\Gamma_1,\mathcal{G}) \cong L^\infty(\Gamma_1)\otimes\mathcal{G}$, we will write $\wp(g,u)$ instead of $\wp(f)$.

In order to give formulas for the adjoints of $\wp(f)$, we first need to define bounded operators which multiply each $F(\gamma)$ by an essentially bounded function k of the first coordinate of γ. The explicit definition is given below.

Definition 3.4. For $k \in L^\infty(I)$, define bounded linear operators

$$M(k,\gamma) : \mathcal{H}(\gamma) \to \mathcal{H}(\gamma)$$

for any $\gamma = (x_1,\ldots,x_{n+1}) \in \Gamma_n$ and $n \geq 0$ by

$$M(k,\gamma)F_n(\gamma) = k(x_1)F_n(\gamma),$$

and the associated decomposable operator by

$$M(k) := \int_\Gamma^\oplus M(k,\gamma)d\gamma,$$

which is bounded on $\mathcal{H}$.

The operator $M(k)$ reminds the gauge operator on the free Fock space associated with the multiplication operator by k, but one important difference is that $M(k)$ is non trivial on the vacuum space unless k vanishes outside of the set of measure zero. Moreover, we will use the shorthand notations

$$\begin{aligned} F_n(\gamma) &= f_1(x_1,x_2)\otimes\ldots\otimes f_n(x_n,x_{n+1}), \\ F_{n-1}(\gamma') &= f_2(x_2,x_3)\otimes\ldots\otimes f_n(x_n,x_{n+1}), \end{aligned}$$

where $\gamma = (x_1,\ldots,x_{n+1}) \in \Gamma_n$, $\gamma' = (x_2,\ldots,x_{n+1}) \in \Gamma_{n-1}$ and each $f(x_i,x_{i+1})$ is an element of the Hilbert space $\mathcal{G}$.

Proposition 3.5. *The adjoints of the operators $\wp(f)$ are given by*

$$\begin{aligned} \wp^*(f)\int_I^\oplus F_0(\gamma)d\gamma_0 &= 0 \\ \wp^*(f)\int_{\Gamma_n}^\oplus F_n(\gamma)d\gamma_n &= \int_{\Gamma_{n-1}}^\oplus M(k,\gamma')F_{n-1}(\gamma')d\gamma'_{n-1} \end{aligned}$$

where

$$k(x_2) = \int_0^1 \langle f_1(x_1, x_2), f(x_1, x_2)\rangle dx_1,$$

and $\langle .,. \rangle$ *is the canonical inner product in* $\mathcal{G}$*.*

Proof. The first formula is obvious since the range of $\wp(f)$ is contained in the orthogonal complement of $L^2(I)$. To prove the second formula, we can take $F_n(\gamma)$ and $G_n(\gamma)$ to be simple tensors of the form

$$\begin{aligned} F_n(\gamma) &= f_1(x_1, x_2) \otimes \ldots \otimes f_n(x_n, x_{n+1}), \\ G_n(\gamma) &= g_1(x_1, x_2) \otimes \ldots \otimes g_n(x_n, x_{n+1}), \end{aligned}$$

where $\gamma = (x_1, \ldots, x_{n+1})$. Then

$$\begin{aligned} &\langle \wp(f) \int_{\Gamma_{n-1}}^{\oplus} G_{n-1}(\gamma') d\gamma'_{n-1}, \int_{\Gamma_n}^{\oplus} F_n(\gamma) d\gamma_n \rangle \\ &= \langle \int_{\Gamma_n}^{\oplus} f(x_1, x_2) \otimes G_{n-1}(\gamma') d\gamma_{n-1}, \int_{\Gamma_n}^{\oplus} F_n(\gamma) d\gamma_n \rangle \\ &= \int_{\Gamma_n} \langle f(x_1, x_2), f_1(x_1, x_2)\rangle \langle G_{n-1}(\gamma'), F_{n-1}(\gamma')\rangle dx_1 \ldots dx_{n+1} \\ &= \int_{\Gamma_{n-1}} \left(\int_I \langle f(x_1, x_2), f_1(x_1, x_2)\rangle dx_1 \right) \langle G_{n-1}(\gamma'), F_{n-1}(\gamma')\rangle dx_2 \ldots dx_{n+1} \\ &= \langle \int_{\Gamma_{n-1}}^{\oplus} G_{n-1}(\gamma') d\gamma'_{n-1}, \wp^*(f) \int_{\Gamma_n}^{\oplus} F_n(\gamma) d\gamma_n(\gamma) \rangle, \end{aligned}$$

where $\gamma = (x_1, \ldots, x_{n+1})$ and $\gamma' = (x_2, \ldots, x_{n+1})$. The proof is completed. □

Corollary 3.6. *For any* $f, f_1 \in L^\infty(\Gamma_1, \mathcal{G})$*, it holds that*

$$\wp^*(f)\wp(f_1) = M(k),$$

where k is of the same form as in Proposition 3.5.

Remark 3.7. Let us consider some special cases and one property of the operators studied above.

(1) It is easy to see that if the functions f, f_1 do not depend on the second coordinate, i.e. $f(x_1, x_2) = \widetilde{f}(x_1)$ and $f_1(x_1, x_2) = \widetilde{f}_1(x_1)$, then

$$k(x_2) = \int_0^1 \langle \widetilde{f}_1(x_1), \widetilde{f}(x_1)\rangle dx_1,$$

for any x_2 and thus $M(k)$ reduces to the multiplication by a constant and thus we can write the relation

$$\wp^*(f)\wp(f_1) = \langle f_1, f\rangle = \langle \widetilde{f}_1, \widetilde{f}\rangle,$$

and thus the operators $\wp(f), \wp^*(f)$ reduce to free creation and annihilation operators, respectively, with the natural inner product for square integrable $\mathcal{G}$-valued functions.

(2) In the above case, if we take two functions of the form: $f_1 = \chi_{\Gamma_1} \otimes e(u')$ and $f_2 = \chi_{\Gamma_1} \otimes e(u'')$, where χ_{Γ_1} is the characteristic function of the square, and denote the associated creation operators by $\wp(u'), \wp(u'')$, respectively, then it is easy to see that $\{\zeta(u) : u \in \mathcal{U}\}$, where

$$\zeta(u) = \wp(u') + \wp^*(u'')$$

and $u' \neq u''$, viewed as two 'copies' of u, is a family of free circular operators (in other words, instead of the set $\mathcal{U}$ we have to consider a twice bigger set of indices).

(3) If we use characteristic functions of the triangle and take $f = \chi_\Delta \otimes e(u)$ and $f_1 = \chi_\Delta \otimes e(u_1)$, where $\Delta = \{(x, y) : 0 \leq x < y \leq 1\}$ and $e(u), e(u_1)$ are orthonormal basis vectors in $\mathcal{G}$, then

$$k(x_2) = \delta_{u,u_1} \int_0^{x_2} dx_1 = \delta_{u,u_1} x_2,$$

and thus $M(k)$ reduces to the multiplication by x_2 times the Kronecker delta related to basis vectors, and thus the relation between the creation and annihilation operators becomes

$$\wp^*(f)\wp(f_1) = \delta_{u,u_1} M(\mathrm{id}),$$

which corresponds to the case when we deal with strictly upper triangular Gaussian random matrices and the operatorial limit is the triangular operator.

(4) For simplicity, we will assume from now on that $f = g \otimes e(u)$ and that g does not depend on u. Let us observe that if (g_n) is a sequence of functions from $L^\infty(\Gamma_1)$ which converges in norm to $g \in L^\infty(\Gamma_1)$, then the corresponding sequences of operators considered above converge strongly on $\mathcal{H}$, namely $s - \lim_{n\to\infty} \wp(g_n, u) = \wp(g, u)$, $s - \lim_{n\to\infty} \wp^*(g_n, u) = \wp^*(g, u)$ and $s - \lim_{n\to\infty} M(g_n) = M(g)$.

4. Continuous Circular Systems

Other decompositions of $\mathcal{H}$ are also relevant since they give useful decompositions of the operators of interest. We will introduce decompositions in which the sets of fibers are relatively small (indexed by I), but the fibers themselves are 'long'. These decompositions allow us to introduce continuous analogs of matricial circular systems and interpret the operators of interest as integrals of two-dimensional 'densities'.

Definition 4.1. For each $x \in [0, 1]$, let us define the associated fiber Hilbert space that begins with x:

$$\mathcal{N}(x) := \int_{\Gamma(y)}^{\oplus} \mathcal{N}(\gamma) d\widetilde{\gamma}$$

$$\cong \mathbb{C}\Omega(x) \oplus \int_I^{\oplus} \mathcal{H}(x, y) dy \oplus \int_{\Gamma_1}^{\oplus} \mathcal{H}(x, y) \otimes \mathcal{H}(y, z) dy dz \oplus \dots,$$

where $\Gamma(x)=\{(x,\gamma):\gamma\in\Gamma\}$ for any fixed $x\in I$, with $\widetilde{\gamma}(\{x\})=1$ and $\widetilde{\gamma}(\{x\}\times A)=\lambda(A)$ (the Lebesgue measure of A) for any $A\subset\Gamma$, and let

$$\mathcal{H}=\int_I^{\oplus}\mathcal{N}(x)dx,$$

be the associated direct integral decomposition. All Hilbert spaces involved are equipped with canonical inner products.

Definition 4.2. In a similar fashion, for all $(x,y)\in\Gamma_1$, define Hilbert spaces

$$\mathcal{N}(x,y):=\mathcal{H}(x,y)\oplus\int_{\Gamma_0}^{\oplus}\mathcal{H}(x,y)\otimes\mathcal{H}(y,z)dz\oplus\ldots,$$

equipped with the canonical inner products and let

$$\mathcal{H}\ominus\mathcal{H}_0=\int_{\Gamma_1}^{\oplus}\mathcal{N}(x,y)dxdy,$$

be the associated direct integral decomposition, where Hilbert spaces involved are equipped with canonical inner products.

Definition 4.3. Let us suppose that $\{e(y,z;u):u\in\mathcal{U}\}$ is a countable orthonormal basis of $\mathcal{H}(y,z)$ for each $(y,z)\in\Gamma_1$. For any given $x,y\in I$ and $u\in\mathcal{U}$, define isometries $\wp(x,y;u):\mathcal{N}(y)\to\mathcal{N}(x,y)$ by the direct integral extension of

$$\begin{aligned}\wp(x,y;u)\Omega(y) &= e(x,y;u),\\ \wp(x,y;u)e(y,z;s) &= e(x,y;u)\otimes e(y,z;s),\\ \wp(x,y;u)(e(y,z;s)\otimes w) &= e(x,y;u)\otimes e(y,z;s)\otimes w,\end{aligned}$$

for any $x,y,z\in I$ and $u,s\in\mathcal{U}$, where $e(y,z;s)\otimes w$ is a basis vector of some tensor product $\mathcal{H}(y,z)\otimes\mathcal{H}(z,z_1)\otimes\ldots\otimes\mathcal{H}(z_{n-1},z_n)$.

Remark 4.4. Equivalently, we could act with $\wp(x,y;u)$ onto direct integrals in the last two equations. For instance, the second equation would then take the form

$$\wp(x,y;u)\int_I^{\oplus}g(y,z)e(y,z;s)dz=\int_I^{\oplus}g(y,z)(e(x,y;u)\otimes e(y,z;s))dz,$$

but it is more convenient to completely decompose the considered fibers since we get simpler formulas which are in correspondence with the discrete case. As far as this correspondence is concerned, in contrast to the discrete case, we do not include scalars in the definition of $\wp(x,y;u)$ in order to avoid lengthy formulas. These scalars, playing the role of covariances, are included in the function g when we deal with $\wp(g,u)$ to the effect that $|g(x,y)|^2$ is the continuous analog of $b_{p,q}(u)$ (as we mentioned earlier, we shall assume for simplicity that these covariances do not depend on u).

Proposition 4.5. *For any $x,y\in I$ and $u\in\mathcal{U}$, let $\wp^*(x,y;u):\mathcal{N}(x,y)\to\mathcal{N}(y)$ be the bounded operator defined by the direct integral extension of the formal formulas*

$$\begin{aligned}\wp^*(x,y;u)e(x,y;u) &= \Omega(y),\\ \wp^*(x,y;u)(e(x,y;u)\otimes w) &= w,\end{aligned}$$

for any $x, y \in I$ and $u \in \mathcal{U}$ and w as above, and setting them to be zero on the remaining basis vectors. Then the operator $\wp^*(x, y; u)$ is the adjoint of $\wp(x, y; u)$ for any x, y, u.

Proof. These formulas are obtained by straightforward computations. □

Definition 4.6. Using the continuous family $\{\wp(x, y; u) : x, y \in I, u \in \mathcal{U}\}$ and the family of their adjoints, one then defines the continuous analogs of the matricial circular operators as

$$\zeta(x, y; u) = \wp(x, y; u') + \wp^*(y, x, u''),$$

for $(x, y) \in \Gamma_1$ and $u \in \mathcal{U}$ and u', u'' are copies of u, as in Remark 3.7. This definition is in agreement with the definition of matricial circular systems and therefore the family

$$\{\zeta(x, y; u) : x, y \in I,\ u \in \mathcal{U}\}$$

will be called the *continuous circular system.*

Proposition 4.7. *If $f(x, y) = g(x, y) \otimes e(u)$, the matrix elements of operators $\wp(f) = \wp(g, u)$ and their adjoints of the form*

$$\langle \wp(g, u)h_1, h_2\rangle = \int_{\Gamma_1} g(x, y)\langle \wp(x, y; u)h_1(y), h_2(x, y)\rangle dxdy,$$

$$\langle h_1, \wp^*(g, u)h_2\rangle = \int_{\Gamma_1} \overline{g(x, y)}\langle h_1(y), \wp^*(x, y; u)h_2(x, y)\rangle dxdy,$$

where $g \in L^\infty(\Gamma_1)$ and $u \in \mathcal{U}$, are well defined for any $h_1 = \int_I^\oplus h_1(y)dy$ and $h_2 = \int_{\Gamma_1}^\oplus h_2(x, y)dxdy$ according to the decompositions of $\mathcal{H}$ and $\mathcal{H} \ominus \mathcal{H}_0$, in Definitions 4.1 and 4.2, respectively.

Proof. Observe that the integrals on the RHS are well defined since h_1 and h_2 have square integrable norms by assumption, each $\wp(x, y; u)$ is an isometry from $\mathcal{N}(y)$ to $\mathcal{N}(x, y)$ and g is essentially bounded on Γ_1. By Definition 3.3 and Proposition 3.5, the integrals on the RHS give the desired matrix elements. This completes the proof. □

In the above situation, we can write a decomposition of the creation operators $\wp(g, u)$ in the direct integral form

$$\wp(g, u) = \int_{\Gamma_1}^\oplus g(x, y)\wp(x, y; u)dxdy,$$

and an analogous formula for the annihilation operators $\wp^*(g, u)$, namely

$$\wp^*(g, u) = \int_{\Gamma_1}^\oplus \overline{g(x, y)}\wp^*(x, y; u)dxdy.$$

We use the symbol $\oplus$ with a slight abuse of notation since the considered families of integrands are 'almost decomposable' with respect to the direct integral decomposition of Definition 4.2. The operators $\wp(x, y; u)$ ($\wp^*(x, y; u)$) can be interpreted as operators creating (annihilating) vector $e(u)$ of color x 'under condition y'. The 'condition' y refers to the color of the vector onto which the operator $\wp(x, y; u)$ acts. If the given pairing is a block in the mixed *-moment of creation and annihilation operators, integration over x of the associated $|g(x,y)|^2$ gives the contribution of the given pairing to the mixed *-moment.

It is also natural to define the corresponding 'circular operator'

$$\zeta(g,u) = \int_{\Gamma_1}^{\oplus} (g(x,y)\wp(x,y;u') + \overline{g(y,x)}\wp^*(y,x;u''))dxdy$$

which becomes a circular operator $\zeta(u)$ if $g = \chi_{\Gamma_1}$. Similarly, if we set $g = \chi_{\Gamma_1}$, we obtain canonical creation and annihilation operators associated with the basis vector $e(u)$, denoted $\wp(u)$ and $\wp^*(u)$.

5. Mixed *-moments

We would like to discuss the mixed *-moments of the creation operators and of certain operators obtained from them. A very interesting example is that of the triangular operator obtained as the limit realization of sctrictly upper triangular Gaussian random matrices.

In our previous works, we have studied the combinatorics of *-moments of various operators (creation, semicircular, circular, etc.) in the matricial (discrete) case [7,8,9,10]. It was based on the class of *colored labeled noncrossing partitions*. Labels are associated to independent random matrices and colors are related to their blocks. The idea of block coloring is very straightforward. In the discrete case, we colored the blocks with natural numbers from the finite set $[r]$ if the matrix is decomposed into r^2 blocks. The color of each block depended on the color of its *nearest outer block* (see [7] for more details). If we dealt with more than one matrix, we also assigned labels to blocks, but labels are rather easy to deal with since they just have to match within a block. (if only one label is used, we just use colored noncrossing partitions).

The basic definitions and notations concerning partitions are given below (some of them extend the classical terminology [12]). If π is a non-crossing pair-partition of the set $[m]$, where m is an even positive integer, which is denoted $\pi \in \mathcal{NC}_m^2$, the set

$$B(\pi) = \{V_1, \ldots, V_s\}$$

is the set of its blocks, where $m = 2s$. If $V_i = \{l(i), r(i)\}$ and $V_j = \{l(j), r(j)\}$ are two blocks of π with left legs $l(i)$ and $l(j)$ and right legs $r(i)$ and $r(j)$, respectively, then V_i is *inner* with respect to V_j if $l(j) < l(i) < r(i) < r(j)$. In that case V_j is *outer* with respect to V_i. It is the *nearest outer block* of V_i if there is no block $V_k = \{l(k), r(k)\}$ such that $l(j) < l(k) < l(i) < r(i) < r(k) < r(j)$. It is easy to see that the nearest outer block, if it exists, is unique, and we write in this case $V_j = o(V_i)$. If V_i does not have an outer block, we set $o(V_i) = V_0$, where $V_0 = \{0, m+1\}$ is the additional block called *imaginary* (marked with a dashed line in Fig. 1). The partition of the set $\{0, 1, \ldots, m+1\}$ consisting of the blocks of π and of the imaginary block will be denoted by $\widehat{\pi}$.

Now, the new idea in this paper is that in the limit $r \to \infty$ the combinatorics is still described by colored labeled noncrossing pair partitions, but the discrete set of colors $[r]$ is replaced by the interval $[0,1]$. As we shall see in Section 6, the set of colors will be $x_1, \ldots, x_{s+1}$ lying in the interval $[0,1]$ and the weights will

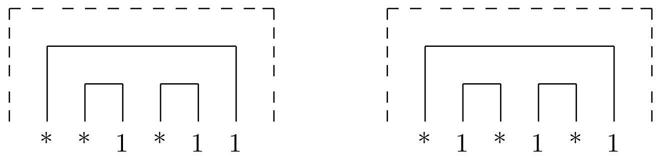


FIGURE 1. The same noncrossing pair partition $\pi \in \mathcal{NC}_6^2$ in two different situations. On the left: π is adapted to $(*, *, 1, *, 1, 1)$. On the right: π is not adapted to $(*, 1, *, 1, *, 1)$. For simplicity, labels are omitted. The imaginary block is marked with a dashed line.

be products of $g(x_i, x_{o(i)})$ replacing products of $b_{p,q}$ used in [7], whenever V_i is a block and $o(V_i)$ is its nearest outer block.

In order to go from the discrete case to the continuous one, take a closer look at pairs of creation and annihilation operators. If a tuple $((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$, where $\epsilon_j \in \{1, *\}$ and $u_j \in \mathcal{U}$ for any j, is given, where m is even, we will say that $\pi \in \mathcal{NC}_m^2$ is *adapted* to it if $u_i = u_j$ whenever $\{i, j\}$ is a block and $(\epsilon_i, \epsilon_j) = (*, 1)$ whenever $\{i, j\}$ is a block and $i < j$. This notion of adaptedness (illustrated in Fig. 1) is convenient when speaking of *-moments of creation operators. Clearly, if a tuple is given, it may have at most one noncrossing pair partition adapted to it in the above sense. If such a partition exists, then it means that the considered mixed *-moment of creation operators does not vanish. Let us remark that in the case of *-moments of 'circular operators' we need another (weaker) notion of adaptedness since in that case $(\epsilon_i, \epsilon_j) \in \{(*, 1), (1, *)\}$. Therefore, one should be careful not to confuse the combinatorics of the *-moments of creation operators with that of *-moments of 'circular operators' (that one will be discussed in Section 6).

We need to define a continuous analog of the state Ψ considered in the discrete case. For simplicity, we can take $d_q = 1/r$ for any q, which corresponds to the decomposition of the random matrices into blocks which are asymptotically square and of equal sizes. We will use the state $\varphi : B(\mathcal{H}) \to \mathbb{C}$ of the form

$$\varphi = \int_I^{\oplus} \varphi(\gamma) d\gamma,$$

where $\varphi(\gamma) = \varphi(x)$ is the vacuum state associated with $\Omega(x)$, namely

$$\varphi(F) = \int_I \langle F(x)\Omega(x), \Omega(x)\rangle dx,$$

where $F = \int_I^{\oplus} F(x)dx \in B(\mathcal{H})$ according to the decomposition of $\mathcal{H}$ into fibers that end with $x \in I$, which is a natural continuous analog of the state Ψ when $d_q = 1/r$ for all q obtained by taking the limit $r \to \infty$.

Computations of mixed *-moments of interest always reduce to the mixed *-moments of the *creation* operators. Therefore, let us first establish a connection on this level with the use of the operators $\wp(g, u)$ introduced in Section 3. We consider the case when $b_{p,q}(u)=d_p$ for any p,q,u. For further simplicity, one can assume that $d_p=1/r$ for all p, but the result given below holds for any asymptotic dimensions.

Lemma 5.1. *For any $p_1,q_1,\ldots,p_m,q_m\in[r]$, $u_1,\ldots,u_m\in\mathcal{U}$, $\epsilon_1,\ldots,\epsilon_m\in\{1,*\}$ and any $m\in\mathbb{N}$, it holds that*

$$\Psi(\wp_{p_1,q_1}^{\epsilon_1}(u_1)\ldots\wp_{p_m,q_m}^{\epsilon_m}(u_m))=\varphi(\wp^{\epsilon_1}(f_1)\ldots\wp^{\epsilon_m}(f_m)),$$

where $f_j=g_j\otimes e(u_j)$ for $j\in[m]$ and g_j is the characteristic function of the rectangle $I_{p_j}\times I_{q_j}\subset\Gamma_1$ for any $1\leq j\leq m$, where $I=I_1\cup\ldots\cup I_r$ is the partition of I into disjoint non-empty intervals with natural ordering.

Proof. Let us observe that for any fixed $r\in\mathbb{N}$ an isometric embedding $\theta:\mathcal{M}\to\mathcal{H}$ is given by

$$\begin{aligned}\theta(\Omega_q)&=\frac{1}{\sqrt{d_q}}\int_{I_q}^{\oplus}\Omega(x)dx,\\ \theta(e_{p_1,p_2}(u_1)\otimes\cdots\otimes e_{p_m,p_{n+1}}(u_n))&=\frac{1}{\sqrt{d_{p_1}\cdots d_{p_{n+1}}}}\\ &\times\int_{I_{p_1}\times\cdots\times I_{p_{n+1}}}^{\oplus}e(x_1,x_2;u_1)\otimes\cdots\otimes e(x_n,x_{n+1};u_n)dx_1\ldots dx_{n+1},\end{aligned}$$

for any $q,p_1,\ldots,p_{n+1}\in[r]$ and $u_1,\ldots,u_n\in\mathcal{U}$. It is then easy to check directly that the mixed *-moments of the operators $\wp_{p,q}(u)$ in the state Ψ agree with the corresponding mixed *-moments of the operators $\wp(g,u)$, where g is the characteristic function of $I_{p,q}$, respectively. This completes the proof. □

The combinatorics of mixed *-moments of matricially free creation operators can be expressed in terms of noncrossing pair partitions adapted to stars and labels. It is not hard to see that they also describe the combinatorics of the mixed *-moments of the much more general family of operators $\wp(f)$, where $f=g\otimes e(u)$, in which matricially free creation operators are included if one takes characteristic functions of rectangles as above. The main reason is that they encode two main facts: blocks must corresponds to pairings of creation and annihilation operators which have the same label, but the contribution of each partition depends on the inner products.

Proposition 5.2. *Let $f_k=g_k\otimes e(u_k)$ and $\epsilon_k\in\{1,*\}$, where $k\in[m]$ and $m=2s$, be such that there exists a unique non-crossing pair partition $\pi\in\mathcal{NC}_m^2$ adapted to $((\epsilon_1,u_1)\ldots,(\epsilon_m,u_m))$. Then*

$$\varphi(\wp^{\epsilon_1}(f_1)\ldots\wp^{\epsilon_m}(f_m))=\int_{\Gamma_s}\prod_{j=1}^{s}\langle f_{r(j)}(x_j,x_{o(j)}),f_{l(j)}(x_j,x_{o(j)})\rangle dx_0dx_1\ldots dx_s,$$

where $V_j=\{l(j),r(j)\}$, $j=1,\ldots,s$, are the blocks of π with $l(j)<r(j)$, with x_0 assigned to the imaginary block of π and $\langle .,.\rangle$ is the canonical inner product in $\mathcal{G}$.

Proof. Each pairing of a creation and annihilation operator produces a function k of one argument in the operator $M(k)$. Here, we just compute the inner products in $\mathcal{G}$ which appear in the definition of such k for all pairings, which gives

$$\langle f_{r(j)}(x_j, x_{o(j)}), f_{l(j)}(x_j, x_{o(j)})\rangle = \overline{g_{r(j)}(x_j, x_{o(j)})}g_{l(j)}(x_j, x_{o(j)})$$

for each pairing, and then integrate the product over all the variables $x_0, \ldots, x_s$, which gives the desired formula. □

Proposition 5.3. *Under the above assumptions, if $g_{l(j)} = g_{r(j)} = \chi_j$ for all $j \in [s]$, where $\chi_1, \ldots, \chi_s$ are characteristic functions of some measurable subsets of Γ_1, then*

$$\varphi(\wp^{\epsilon_1}(f_1) \ldots \wp^{\epsilon_m}(f_m)) = Vol(\pi),$$

where $Vol(\pi)$ is the volume of the region $V(\pi) \subseteq \Gamma_s$ defined by $x_0, x_1, \ldots, x_s$ for which $\chi_j(x_j, x_{o(j)}) = 1$, for all $j = 1, \ldots, s$, with x_0 assigned to the imaginary block.

Proof. If we set the functions associated with the left and right legs of the block π_j to be equal to χ_k, then in the proof of Proposition 5.2 we have

$$\overline{g_{r(j)}(x_j, x_{o(j)})}g_{l(j)}(x_j, x_{o(j)}) = \chi_j(x_j, x_{o(j)}),$$

which gives a condition on two variables, x_j and $x_{o(j)}$, from among $s+1$ variables describing the $s+1$-dimensional cube Γ_s. Each inner product in the formula of Proposition 5.2 leads to a similar condition, which completes the proof. □

Example 5.4. Consider the mixed *-moment associated with the pair partition $\pi = \{\{1,6\}, \{2,3\}, \{4,5\}\}$, where it is natural to assume that $f_1 = f_6$, $f_2 = f_3$ and $f_4 = f_5$. Then

$$\begin{aligned}&\varphi(\wp^*(f_1)\wp^*(f_2)\wp(f_2)\wp^*(f_4)\wp(f_4)\wp(f_1))\\ &= \int_{\Gamma_3} \|f_1(y,x)\|^2 \|f_2(z,y)\|^2 \|f_4(w,y)\|^2 \; dwdzdydx.\end{aligned}$$

In the case when $f_j = g_j \otimes e(u_j)$ for $j = 1, 2, 4$ and the numerical valued functions are the characteristic functions of the triangle $\Delta = \{(x,y) : x < y\}$, this integral is equal to

$$Vol(\pi) = \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^y dw = \frac{1}{12}.$$

Finally, let us return to the asymptotic *-distributions of Gaussian random matrices with i.b.i.d. entries. We assume that we have r^2 blocks for each $r \in \mathbb{N}$. Later we will go with r to infinity. Therefore, at this point it seems appropriate to include r in the symbols denoting random matrices as well as limit operators.

Proposition 5.5. *For any $r \in \mathbb{N}$, let $\{Y(u,n,r) : u \in \mathcal{U}\}$ be a family of square $n \times n$ independent complex Gaussian random matrices with i.b.i.d. entries for any natural n and any $u \in \mathcal{U}$. Then,*

$$\lim_{n\to\infty} \tau(n)(Y^{\epsilon_1}(u_1,n,r) \ldots Y^{\epsilon_m}(u_m,n,r)) = \Psi(\eta^{\epsilon_1}(u_1,r) \ldots \eta^{\epsilon_m}(u_m,r))$$

for any $u_1, \ldots, u_m \in \mathcal{U}$ and $\epsilon(1), \ldots, \epsilon(m) \in \{1, *\}$, where

$$\eta(u,r) = \sum_{p,q=1}^{r} \zeta_{p,q}(u,r)$$

for any $u \in \mathcal{U}$ and the operators $\zeta_{p,q}(u,r)$ are discrete (generalized) matricial circular operators corresponding to given r.

Proof. The proof of this result was given in [10]. □

The next step consists in taking the limit of the *-moments on the RHS as $r \to \infty$. We assume that $d_p = 1/r$ for any p and any r and, for simplicity, we assume that the block covariances $b_{p,q}(u,r)$, defined for all $p,q \in [r]$ and all u,r, do not depend on u. From these block covariances we built a sequence of simple functions

$$b_r(x,y) = \sum_{p,q=1}^{r} b_{p,q}(u,r)\chi_{I_p,I_q}(x,y)$$

and assume that it converges to some $g \in L^{\infty}(\Gamma_1)$ as $r \to \infty$. Then we compute the limit of the mixed *-moments expressed as liner combinations of *-moments of the type given by Lemma 5.1.

Theorem 5.6. *Let $\{Y(u,n,r) : u \in \mathcal{U}, r \in \mathbb{N}\}$ be a family of independent $n \times n$ random matrices for any $n \in \mathbb{N}$, such that*

(1) *each $Y(u,n,r)$ consists of r^2 blocks of equal size with i.b.i.d. complex Gaussian entries,*
(2) *the sequence of simple functions (b_r) converges to g in $L^{\infty}(\Gamma_1)$ as $r \to \infty$.*

Then

$$\lim_{r\to\infty}\lim_{n\to\infty} \tau(n)(Y^{\epsilon_1}(u_1,n,r)\ldots Y^{\epsilon_m}(u_m,n,r)) = \varphi(\zeta^{\epsilon_1}(g,u_1)\ldots\zeta^{\epsilon_m}(g,u_m)),$$

where $\zeta(g,u_j) = \wp(g,u_j') + \wp^*(g^t,u_j'')$, with $g^t(x,y) = g(y,x)$ and all labels u_j', u_j'', $j \in [m]$, different, and where $\varphi = \int_I^{\oplus}\varphi(\gamma)d\gamma$.

Proof. The second limit ($n \to \infty$) was computed in Proposition 5.5. It is easy to see that the moments obtained there, namely $\Psi(\eta^{\epsilon_1}(u_1,r)\ldots\eta^{\epsilon_m}(u_m,r))$, can be written as linear combinations of mixed *-moments of creation and annihilation operators of continuous type in the state φ, namely such as those given in Lemma 5.1, since

$$\eta(u,r) = \sum_{p,q=1}^{r}\zeta_{p,q}(u,r) = \sum_{p,q}^{r}(\wp_{p,q}(u',r) + \wp_{q,p}^*(u'',r)),$$

where the matricial creation and annihilation operators are assumed to have covariances independent of u, but otherwise arbitrary nonnegative numbers, i.e. $b_{p,q}(u) = b_{p,q}$ for any u. When we express the RHS in terms of operators of the form $\wp(g_{p,q},u)$ and their adjoints, where $g_{p,q} = b_{p,q}\chi_{I_p\times I_q}$ for any p,q, we can write the above sum as

$$\zeta(g_r,u) = \wp(g_r,u') + \wp^*(g_r^t,u''),$$

where $g_r(x,y) = \sum_{p,q=1}^{r} g_{p,q}$ and g_r^t stands for the transpose of g_r. Now, if (g_r) converges to g in $L^{\infty}(\Gamma_1)$, then, by Definition 3.3 and Proposition 5.2, the mixed *-moments of $\wp(g_r,u)$ in the state φ converge to the corresponding mixed *-moments of $\wp(g,u)$, which entails convergence of the mixed *-moments of $\zeta(g_r,u)$ in the state φ to the mixed *-moments of $\zeta(g,u)$. This completes the proof. □

6. Triangular Operator and Labeled Ordered Rooted Trees

Let us apply Theorem 5.6 to independent strictly upper triangular Gaussian random matrices, whose limits are free triangular operators [4]. We express the limit *-moments of such matrices in terms of operators $\zeta(u)=\zeta(\chi_\Delta,u)$ where χ_Δ is the characteristic function of the triangle Δ. Our result gives a new Hilbert space realization of the limit *-moments, equivalent to the von Neumann algebra approach of Dykema and Haagerup in [4], where the triangular operator T was introduced. Note that our approach to the combinatorics of its *-moments is also different than the algorithm in [4, Lemma 2.4].

By a family of independent strictly upper triangular Gaussian random matrices we will understand a family of complex $n\times n$ matrices $Y(u,n)$, where $u\in\mathcal{U}$, whose entries above the main diagonal form a family of complex Gaussian random variables whose real and imaginary parts form a family of $n(n-1)$ i.i.d. Gaussian random variables for each u (also independent for different $u\in\mathcal{U}$), each having mean zero and variance $1/2n$.

Theorem 6.1. *Let $\{Y(u,n):u\in\mathcal{U}\}$ be a family of independent strictly upper triangular Gaussian random matrices for any $n\in\mathbb{N}$. Then*

$$\lim_{n\to\infty}\tau(n)(Y^{\epsilon_1}(u_1,n)\dots Y^{\epsilon_m}(u_m,n))=\varphi(\zeta^{\epsilon_1}(u_1)\dots\zeta^{\epsilon_m}(u_m))$$

for any $\epsilon_1,\dots,\epsilon_m\in\{1,*\}$ and $u_1,\dots,u_m\in\mathcal{U}\}$, where $\zeta(u_j)=\zeta(\chi_\Delta,u_j)$, $j\in[m]$, with $\Delta=\{(x,y)\in\Gamma_1: x<y\}$ and $\varphi=\int_I^\oplus\varphi(\gamma)d\gamma$.

Proof. We know from [4] that the limits of the mixed *-moments of independent strictly upper triangular Gaussian random matrices under $\tau(n)$ as $n\to\infty$ exist and are, by definition, equal to the mixed *-moments of free triangular operators $T(u)$, namely

$$\lim_{n\to\infty}\tau(n)(Y^{\epsilon_1}(u_1,n)\cdots Y^{\epsilon_m}(u_m,n))=\varphi(T^{\epsilon_1}(u_1)\cdots T^{\epsilon_m}(u_m))$$

for any $\epsilon_1,\dots,\epsilon_m\in\{1,*\}$ and any $u_1,\dots,u_m\in\mathcal{U}\}$. At the same time, it can be justified that the above limits are equal to the limit moments of Theorem 5.6, where matrices $\{Y(u,n,r):u\in\mathcal{U}\}$ are $n\times n$ independent block strictly upper triangular matrices with r^2 blocks for all natural n and r, the non-vanishing blocks being $S_{p,q}(u,n,r)$ for $p<q$. For instance, an estimate in terms of Schatten p-norms $\|A\|_p=\sqrt[p]{\mathrm{tr}(n)(|A|^p)}$ for $p\geq 1$, where $\mathrm{tr}(n)$ is the normalized trace, can be used. It holds that

$$|\mathrm{tr}(n)(A)|\leq\|A\|_1\leq\|A\|_p\leq\|A\|$$

for any $p\geq 1$. Therefore, let $Y_j=Y^{\epsilon_j}(u_j,n)$ and $Y_j'=Y^{\epsilon_j}(u_j,n,r)$ for $j\in[m]$ and any n,r. Then, applying the above inequalities to the trace

$$\mathrm{tr}(n)(Y_1\dots Y_m-Y_1'\dots Y_m')=\sum_{j=1}^m\mathrm{tr}(n)(Y_1'\dots Y_{j-1}'(Y_j-Y_j')Y_{j+1}\dots Y_m),$$

and using repeatedly the Hölder inequality $\|AB\|_s\leq\|A\|_p\|B\|_q$, where $s^{-1}=p^{-1}+q^{-1}$, we obtain an upper bound for the absolute value of this trace of the form

$$(m+1)\cdot\max_{1\leq j\leq m}\|Y_j-Y_j'\|_{m+1}\cdot(\max_{1\leq k\leq m}\{\|Y_k\|_m,\|Y_k'\|_m\})^{m-1}.$$

Therefore, for large n there exists R such that if $r > R$, then the difference between the mixed *-moments of primed and unprimed matrices can be made arbitrarily small since the norms $\|Y_j - Y_j'\|_{m+1}$ can be made arbitrarily small for large n and large $r > R$. Therefore, we can use Theorem 5.6 to express the limit mixed *-moments of the strictly upper triangular matrices in terms of the operators $\zeta(\xi_\Delta, u)$, respectively, where ξ_Δ is the characteristic function of the triangle $\Delta = \{(x,y) \in \Gamma_1 : x < y\}$. In other words, we can identify the triangular operators $T(u)$ with $\zeta(\xi_\Delta, u)$, $u \in \mathcal{U}$. This completes the proof. □

In order to give these *-moments in a more explicit form, let us assign continuous colors to blocks of $\pi \in \mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$, where $m = 2s$, and analyze relation between these colors. By $\mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$ we denote the set of noncrossing pair partitions of $[m]$, such that $u_i = u_j$ and $\epsilon_i \neq \epsilon_j$ whenever $\{i, j\}$ is a block. We should remember that stars refer here to operators of the form

$$\zeta(f, u) = \wp(f, u') + \wp^*(f^t, u''),$$

where $f = \chi_\Delta$. For simplicity, let us write $\zeta(f, u) = \zeta(u)$, $\wp(f, u') = \wp(u')$ and $\wp(f^t, u'') = \wp(u'')$. Each pairing that gives a nonzero contribution must be of the form $(\wp^*(u'), \wp(u'))$ or $(\wp^*(u''), \wp(u''))$. The first one is obtained when $\zeta^*(u)$ is associated with the left leg of a block and $\zeta(u)$ is associated with the right leg, whereas in the second one the stars are interchanged. In any case, only one leg of a block can be marked with a star. We do not star the legs of the imaginary block.

Definition 6.2. Let V_j be a block of $\pi \in \mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$ and let $V_{o(j)}$ be its nearest outer block. We distinguish four types of blocks (see Fig. 2):

(1) *type 1*: the right leg of V_j and the left leg of $V_{o(j)}$ are starred,
(2) *type 2*: the left leg of V_j and the right leg of $V_{o(j)}$ are starred,
(3) *type 3*: the left legs of both V_j and $V_{o(j)}$ are starred,
(4) *type 4*: the right legs of both V_j and $V_{o(j)}$ are starred.

Remark 6.3. Our combinatorics is now based on coloring the blocks of noncrossing pair partitions with numbers from the interval $[0, 1]$ and finding relations betwen them.

(1) Let us color the blocks of $\pi \in \mathcal{NC}^2((\epsilon_1, u_1), \ldots, (\epsilon_m, u_m))$, where $m = 2s$, and the imaginary block with $s + 1$ continuous colors form $[0, 1]$: $x_1, \ldots, x_{s+1}$, assigned to $V_1, \ldots, V_{s+1}$, respectively. It is convenient to number those blocks and colors starting from the right, as shown in Fig. 4.
(2) Now, using these inequalities, we can associate a region $V(\pi) \subset \Gamma_2$ to each π. Namely, let

$$V(\pi) = \{x \in \Gamma_s : \ x_j < x_{o(j)} \text{ if } V_j \in B'(\pi) \ \wedge \ x_j > x_{o(j)} \text{ if } V_j \in B''(\pi)\},$$

where $B'(\pi)$ and $B''(\pi)$ stand for blocks of π whose left legs are starred and unstarred, respectively.

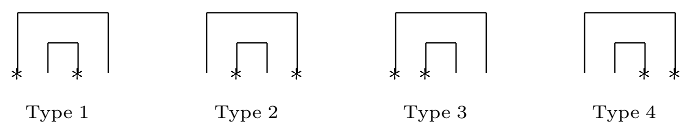


FIGURE 2. Four types of pairs $(V, o(V))$, where V is a block and $o(V)$ is the nearest outer block of V, which depend on which legs are starred.

Corollary 6.4. *The non-vanishing mixed *-moments of the free triangular operators in the state* φ *take the form*

$$\varphi(T^{\epsilon_1}(u_1)\dots T^{\epsilon_m}(u_m)) = \sum_{\pi\in\mathcal{NC}^2((\epsilon_1,u_1),\dots,(\epsilon_m,u_m))} Vol(\pi),$$

where $m=2s$ *and* $Vol(\pi)$ *is the volume of the region* $V(\pi)$.

Proof. We know that we can replace $T(u_j)$ by $\zeta(u_j)$. Now, without loss of generality, we can assume that $u_j = u$ for all j since the general case just gives the additional condition on π that $u_i = u_j$ whenever $\{i,j\}$ is a block. We will omit u and write $\wp_1 = \wp(u')$ and $\wp_2 = \wp(u'')$ and replace $\mathcal{NC}^2((\epsilon_1,u),\dots,(\epsilon_m,u))$ by $\mathcal{NC}^2(\epsilon_1,\dots,\epsilon_m)$. We have

$$\varphi(T^{\epsilon_1}\dots T^{\epsilon_m}) = \sum_{\pi\in\mathcal{NC}^2(\epsilon_1,\dots,\epsilon_m)} \varphi\left(\wp_{j_1(\pi)}^{\epsilon_1(\pi)}\dots\wp_{j_m(\pi)}^{\epsilon_m(\pi)}\right),$$

where π 'chooses' whether to take the pairing $(\wp_1^*,\wp_1)$ or $(\wp_2^*,\wp_2)$, namely

$$\wp_{j_i(\pi)}^{\epsilon_i(\pi)} = \begin{cases} \wp_1 & \text{if } \epsilon_i = 1 \text{ and } i\in\mathcal{R}(\pi) \\ \wp_2 & \text{if } \epsilon_i = * \text{ and } i\in\mathcal{R}(\pi) \\ \wp_2^* & \text{if } \epsilon_i = 1 \text{ and } i\in\mathcal{L}(\pi) \\ \wp_1^* & \text{if } \epsilon_i = * \text{ and } i\in\mathcal{L}(\pi) \end{cases},$$

where $\mathcal{R}(\pi)$ and $\mathcal{L}(\pi)$ stand for the right and left legs of π, respectively. If $j>1$, then we choose the color x_j assigned to block V_j to be the first coordinate of χ_Δ associated with the pairing of type $(\wp_1^*,\wp_1)$, or the first coordinate of χ_Δ^t associated with the pairing of type $(\wp_2^*,\wp_2)$, depending on whether the left leg of V_j is starred or unstarred, respectively. Since we have $f=\chi_\Delta$ in each operator ζ that appears in the moment $\varphi(\zeta^{\epsilon_1}\dots\zeta^{\epsilon_m})$, let us observe that if the left leg of V_j is starred, then $x_j < x_{o(j)}$ is obtained from Remark 3.7 on the form of $M(k)$, with $k(x_{o(j)}) = \int_{x_j<x_{o(j)}} dx_j$. In turn, if the left leg of V_j is unstarred, then instead of χ_Δ, we take its transpose in the corresponding pairing of type $(\wp_2^*,\wp_2)$ which amounts to taking $M(k)$ with $k(x_{o(j)}) = \int_{x_j>x_{o(j)}} dx_j$ (since $u_j = u_{o(j)}$, by the adaptedness assumption on π). This completes the proof. □

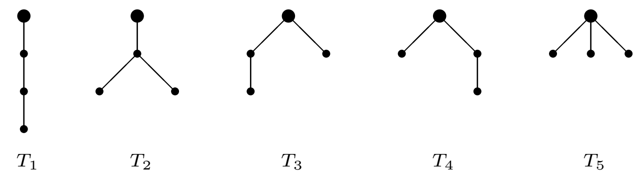


FIGURE 3. Ordered rooted trees on 4 vertices

Remark 6.5. Let us make some remarks which will be useful in establishing a bijective proof of the formula for the moments $\varphi((T^*T)^n)$ for the triangular operator T.

(1) There is an obvious bijection between $\mathcal{NC}_{2n}^2$ and the set of associated extended pair-partitions of $[0, 2n+1]$, for which we will use the same notation and we will identify $\pi \in \mathcal{NC}_{2n}^2$ with its extension $\pi \cup \{0, 2n+1\}$.

(2) For any integer n, let $\mathcal{O}_n$ be the set of ordered rooted trees on the set of $n+1$ vertices (see Example 6.6). There is a natural bijection

$$\gamma : \mathcal{O}_n \to \mathcal{NC}_{2n}^2$$

given by the rule: vertex v is a child of vertex w if and only if $\gamma(w)$ is the nearest outer block of $\gamma(v)$, with the root of T corresponding to the imaginary block of $\pi = \gamma(T)$.

(3) Suppose now that to each vertex of $\mathcal{O}_n$ we assign a *label* from the set $[n+1]$. The bijection γ leads to a natural bijection

$$\gamma' : \mathcal{L}_n \to \mathcal{CNC}_{2n}^2,$$

between the set of labeled ordered rooted trees on $n+1$ vertices, $\mathcal{L}_n$, where the vertices are labeled by different numbers from $[n+1]$, and colored non-crossing pair-partitions of the set $[0, 2n+1]$, denoted $\mathcal{CNC}_{2n}^2$, extended by the (colored) imaginary block (we identify π with $\pi \cup \{0, 2n+1\}$), where blocks (including the imaginary block) are colored by different numbers from the set $[n+1]$.

(4) It is well known that the number of labeled ordered rooted trees on $n+1$ vertices is given by

$$|\mathcal{L}_n| = (n+1)!C_n = \frac{(2n)!}{n!},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

Example 6.6. The set $\mathcal{O}_4$ of ordered rooted trees on 4 vertices consists of the trees given in Fig. 3, where the root is distinguished by a larger circle. In ordered trees, the children of any vertex are ordered and that is why T_3 and T_4 are inequivalent since different children of the roots have off-springs. In turn, $\mathcal{NC}_6^2$ consists of the non-crossing pair partitions shown in Fig. 4, where we also draw the imaginary

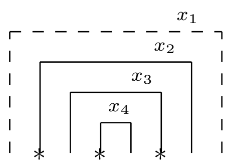
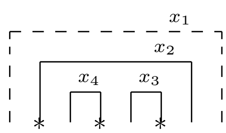
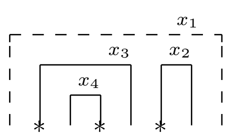
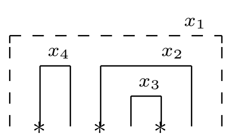
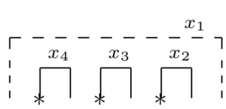
π	$R(\pi)$	$S(\pi)$	$Vol(\pi)$
π_1 : 	$x_2 < x_1$ $x_2 < x_3$ $x_4 < x_3$	$x_4 < x_2 < x_3 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_2 < x_4 < x_1 < x_3$ $x_4 < x_2 < x_1 < x_3$ $x_2 < x_1 < x_4 < x_3$	$\frac{5}{24}$
π_2 : 	$x_2 < x_1$ $x_2 < x_4$ $x_2 < x_3$	$x_2 < x_3 < x_4 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_2 < x_4 < x_1 < x_3$ $x_2 < x_3 < x_1 < x_4$ $x_2 < x_1 < x_3 < x_4$ $x_2 < x_1 < x_4 < x_3$	$\frac{6}{24}$
π_3 : 	$x_2 < x_1$ $x_3 < x_4$ $x_3 < x_1$	$x_2 < x_3 < x_4 < x_1$ $x_3 < x_2 < x_4 < x_1$ $x_3 < x_4 < x_2 < x_1$ $x_3 < x_2 < x_1 < x_4$ $x_2 < x_3 < x_1 < x_4$	$\frac{5}{24}$
π_4 : 	$x_2 < x_1$ $x_4 < x_1$ $x_2 < x_3$	$x_4 < x_2 < x_3 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_2 < x_3 < x_4 < x_1$ $x_4 < x_2 < x_1 < x_3$ $x_2 < x_4 < x_1 < x_3$	$\frac{5}{24}$
π_5 : 	$x_2 < x_1$ $x_3 < x_1$ $x_4 < x_1$	$x_4 < x_3 < x_2 < x_1$ $x_3 < x_4 < x_2 < x_1$ $x_4 < x_2 < x_3 < x_1$ $x_2 < x_4 < x_3 < x_1$ $x_3 < x_2 < x_4 < x_1$ $x_2 < x_3 < x_4 < x_1$	$\frac{6}{24}$

FIGURE 4. Noncrossing pair partitions of [6], each extended by an imaginary block, corresponding to $\varphi((T^*T)^3)$. Starred legs correspond to T^*. We assign continuous colors x_j to blocks, where $j \in [4]$. The corresponding regions $R(\pi)$ inside the cube $[0,1]^4$ are given by three inequalities for colors of the blocks of π and have volumes $Vol(\pi)$. Each region consists of simplices $S(\pi)$ defined by linearly ordered colors. Altogether we get $27 = 3^3$ simplices.

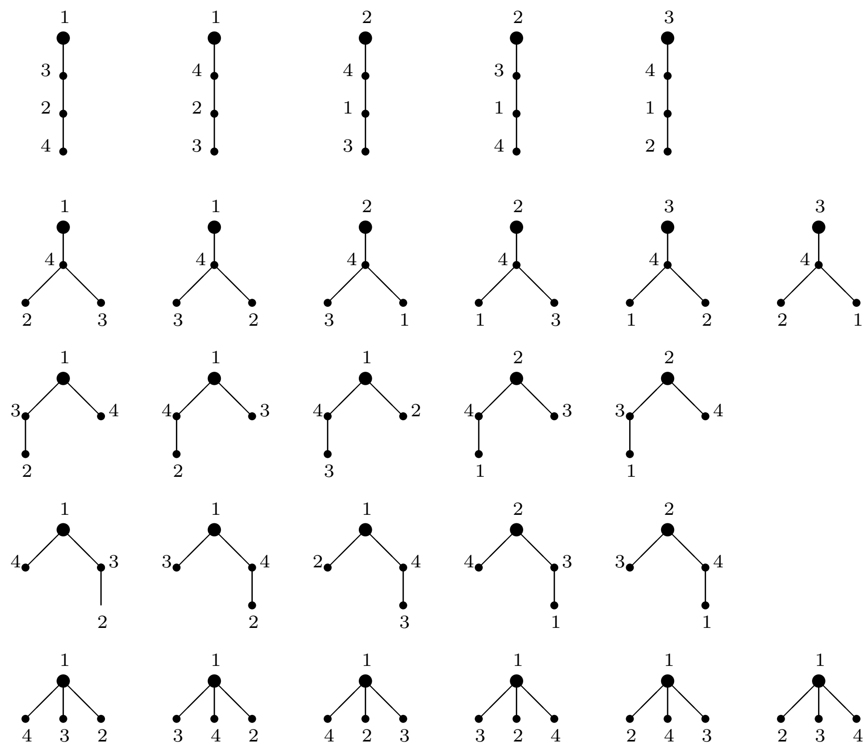


FIGURE 5. The set $\mathcal{A}_4$ of alternating ordered rooted trees of type I. This set is in bijection with the set of noncrossing colored pair partitions with totally ordered colors given in Fig. 4, or with the corresponding simplices. The alternating ordered rooted trees are listed in the same order as the corresponding simplices in Fig. 4.

blocks which can be identified with the roots of the corresponding trees. The natural bijection $\gamma : \mathcal{O}_4 \rightarrow \mathcal{NC}_6^2$ is given by $\gamma(T_k) = \pi_k$. In fact, that is why there are 5 trees of this type since $C_3 = 5$ and Catalan numbers count ordered rooted trees due to the bijection γ. In turn, there are $6!/3! = 120$ different labeled ordered rooted trees on 4 vertices if we label them by the 4-element set in an arbitrary way.

Definition 6.7. Let $(v_1, v_2, v_3, v_4, \ldots)$ be a path in an labeled ordered rooted tree T on $n+1$ vertices, which means that v_j is a son of v_{j-1}, and let $(x_1, x_2, x_3, x_4, \ldots)$ be the corresponding sequence of labels. Then T is called *alternating* if this sequence satisfies one of the inequalities,

$$x_1 > x_2 < x_3 > x_4 \ldots, \quad \text{or} \quad x_1 < x_2 > x_3 < x_4 \ldots,$$

i.e. the differences of labels corresponding to the neighboring vertices alternate in sign. These two types of conditions split the set of alternating ordered rooted trees on $n+1$ vertices into two subsets of the same cardinalities, which we will call *of type I* and *of type II*, respectively. Denote by $\mathcal{A}_n$ the set of alternating ordered rooted trees on $n+1$ vertices of type I.

Remark 6.8. There is a nice enumeration result of Chauve, Dulucq and Rechnitzer [2] which says that

$$|\mathcal{A}_n| = n^n$$

for any natural n. Recall that $\mathcal{A}_n$ stands for the set of alternating ordered rooted trees of type I on $n+1$ vertices. Thus, the number of all alternating ordered rooted trees is $2n^n$. Note also that a typical formula refers to trees on n vertices.

Example 6.9. Among the labeled ordered rooted trees on 4 vertices, there are $2 \times 3^3 = 54$ alternating ones. In this example, each type contains $3^3 = 27$ alternating ordered rooted trees. The complete set $\mathcal{A}_4$ of all alternating ordered rooted trees of type I is given in Figure 5. In view of the above bijection results, the cardinality of all non-crossing pair partitions of the 6-element set with alternating colorings is also 54 and there are 27 partitions in which each block of odd depth has a smaller color than its nearest outer block (equivalently, the color of the imaginary block, which is assumed to have zero depth, is greater than the colors of all blocks for which the imaginary block is the nearest outer block).

Corollary 6.10. *The moments of T^*T, where T is the triangular operator, are*

$$M_n = \varphi((T^*T)^n) = \frac{n^n}{(n+1)!}$$

for any $n \in \mathbb{N}$.

Proof. Let $\epsilon_j = *$ if j is odd and $\epsilon_j = 1$ if j is even. In this special case, it is easy to see that $\mathcal{NC}^2(\epsilon_1, \ldots, \epsilon_{2n}) \cong \mathcal{NC}^2_{2n}$. Observe that in the case of alternating starred and unstarred legs there can be no pairs $(V, o(V))$ of type 3 and 4 since otherwise there would be unequal numbers of starred and unstarred legs between the right leg of V and the left leg of $o(V)$, which would mean that there must be a block between V and $o(V)$, which is a contradiction. Therefore, blocks with starred left and right legs must alternate as we take a sequence of blocks $(V_{i_1}, \ldots, V_{i_p})$, where each block is the nearest outer block of its successor. By Corollary 6.4, we need to compute $Vol(\pi)$ for each $\pi \in \mathcal{NC}^2_{2n}$. Each of the corresponding regions $R(\pi)$ is defined by a set of n inequalities for colors $x_1, \ldots, x_{n+1}$. Irrespective of what symbols represent these colors, in order to satisfy these inequalities, we have to find the number of total orderings of the form

$$x_{j_1} < x_{j_2} < \ldots < x_{j_{n+1}}$$

which satisfy the given inequalities. The number of these total orderings is equal to the number of $n+1$-simplices, each of volume $1/(n+1)!$, into which $R(\pi)$ decomposes. The key observation is that in order to compute the number of these total orderings corresponding to π (under conditions given by n inequalities which express orders between the colors of each V and $o(V)$) it suffices to count in how many ways we can label blocks of π with natural numbers from $[n+1]$ in such a way that orders between these numbers alternate as we go down each sequence of type $(V_{i_1}, \ldots, V_{i_p})$. More specifically, to colors $x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}$ in the total ordering (defining a simplex) given above we assign numbers $n+1, n, \ldots, 1$, respectively, thus we assign number 1 to the biggest color and the number $n+1$ - to the smallest one. Now, if we use the bijection between labeled ordered rooted trees and colored noncrossing pair partitions, it suffices to enumerate all alternating ordered rooted trees on $n+1$ vertices. The enumeration result of [2] mentioned above completes the proof. □

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