A Stochastic Integral by a Near-Martingale

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A STOCHASTIC INTEGRAL BY A NEAR-MARTINGALE

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Abstract. In this paper we discuss the new stochastic integral in [1] in terms of the Itô isometry. We prove the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL). Moreover, we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem.

1. Introduction

A new stochastic integral was introduced in [1]. The Itô isometry based on the new integral for special processes was discussed in [10]. The Doob-Meyer decomposition theorem for continuous near-submartingales was also discussed in [3]. This stochastic integral has been studied from different points of view [2, 4, 7, 8, 9] and references cited therein.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a \leq t \leq b}$ be a basic probability space with a filtration $\mathcal{F}_t$, and $B = \{B(t); a \leq t \leq b\}$ a $\{\mathcal{F}_t\}$-Brownian motion on $(\Omega, \mathcal{F}, P)$. A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be instantly independent of $\mathcal{F}_t$ if $g(t)$ is independent of $\mathcal{F}_t$ for all $t \in [a, b]$. A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $L^2_{\text{ind}}([a, b] \times \Omega)$ if the process satisfies the following conditions:

1. $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\mathcal{F}_t$.
2. $\int_a^b E[|g(t)|^2]dt < \infty$.
3. $g$ is right-continuous in $t$.

A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $L^2_{\text{ind}}(\Omega, L^2([a, b]))$ if the process satisfies the following conditions:

1. $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\mathcal{F}_t$.
2. $\int_a^b |g(t)|^2dt < \infty$, a.e.

In this article we discuss the new stochastic integral through the Itô isometry. In Section 2 we discuss the stochastic integral by the Brownian motion $B$ for processes in $L^2_{\text{ind}}([a, b] \times \Omega)$ through the Itô isometry with its properties. In Section 3 we extend the stochastic integral to that on a class $L^2_{\text{ind}}(\Omega, L^2([a, b]))$ which is larger than the space in Section 2. In Section 4 we give the proof of the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL). This theorem is important to discuss the new integral in [1] for its extension. In the last

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section we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem. This is a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

2. Stochastic Integrals on $L_{\text{ind}}^2([a, b] \times \Omega)$

Let $g$ be in $L_{\text{ind}}^2([a, b] \times \Omega)$. Then $g$ is called to be an instantly independent step process if there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ and instantly independent random variables $\eta_i$, $i = 1, 2, \ldots, n$ with $E[\eta_i^2] < \infty$ such that

$$g(t, \omega) = \sum_{i=1}^{n} \eta_i(\omega)1_{(t_{i-1}, t_i)}(t), \quad \omega \in \Omega, t \in [a, b]. \quad (2.1)$$

We denote the set of all instantly independent step processes by $\text{Step}_{\text{ind}}([a, b] \times \Omega)$.

For any $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$ given as (2.1), we define $\mathcal{J}(g)$ by

$$\mathcal{J}(g) := \sum_{i=1}^{n} \eta_i(B(t_i) - B(t_{i-1})).$$

Then we have the following.

**Lemma 2.1.** For any $g, h \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$ and $a, b \in \mathbb{R}$, it holds that

$$\mathcal{J}(ag + bh) = a\mathcal{J}(g) + b\mathcal{J}(h).$$

**Lemma 2.2.** For any $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$, the following equalities hold.

(1) $E[\mathcal{J}(g)] = 0$.

(2) $E[|\mathcal{J}(g)|^2] = \int_{a}^{b} E[|g(t)|^2] dt$.

**Proof.** Let $g$ be a function in $\text{Step}_{\text{ind}}([a, b] \times \Omega)$ given as (2.1).

(1): Since, for any $1 \leq i \leq n$, $\eta_i$ is independent to $B(t_i) - B(t_{i-1})$, we have

$$E[\eta_i(B(t_i) - B(t_{i-1}))] = E[\eta_i]E[B(t_i) - B(t_{i-1})] = 0.$$

Therefore, $E[\mathcal{J}(g)] = 0$.

(2): If $i < j$, we have

$$E[\eta_i\eta_j(B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] = E[(B(t_i) - B(t_{i-1}))E[\eta_i\eta_j(B(t_j) - B(t_{j-1}))] = 0.$$

If $i = j$, we have

$$E[\eta_i^2(B(t_i) - B(t_{i-1}))^2] = E[(B(t_i) - B(t_{i-1}))^2]E[\eta_i^2] = (t_i - t_{i-1})E[\eta_i^2].$$

Therefore, we obtain

$$E[|\mathcal{J}(g)|^2] = \sum_{i,j=1}^{n} E[\eta_i\eta_j(B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] = \int_{a}^{b} E[|g(t)|^2] dt.$$
Lemma 2.3. For any \( g \in L^2_{\text{ind}}([a, b] \times \Omega) \), there exists \( \{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega) \) such that
\[
\lim_{n \to \infty} \int_a^b E[|g(t) - g_n(t)|^2]dt = 0
\]
holds.

Let \( g \in L^2_{\text{ind}}([a, b] \times \Omega) \). Then there exists \( \{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega) \) such that
\[
\lim_{n \to \infty} \int_a^b E[|g(t) - g_n(t)|^2]dt = 0.
\]
By Lemmas 2.1, 2.2 and 2.3, we have
\[
E[|J(g_n) - J(g_m)|^2] = \int_a^b E[|g_n(t) - g_m(t)|^2]dt \xrightarrow{n,m \to \infty} 0.
\]
Therefore, \( \{J(g_n)\}_{n=1}^{\infty} \) is a Cauchy sequence in \( L^2(\Omega) \). By the completeness of \( L^2(\Omega) \), there exists \( J(g) \in L^2(\Omega) \) such that
\[
J(g) = \lim_{n \to \infty} J(g_n), \quad \text{in } L^2(\Omega).
\]
Thus we can define the stochastic integral \( \int_a^b g(t)dB(t) \) by
\[
\int_a^b g(t)dB(t) := J(g)
\]
as an element of \( L^2(\Omega) \). This is well-defined. In fact, assume that there exist \( \{g_n(t)\}_{n=0}^{\infty}, \{h_n(t)\}_{n=0}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega) \) such that
\[
\lim_{n \to \infty} \int_a^b E[|g(t) - g_n(t)|^2]dt = 0, \quad \lim_{n \to \infty} \int_a^b E[|g(t) - h_n(t)|^2]dt = 0.
\]
Then we can see that
\[
E[|J(g_n) - J(h_n)|^2] = \int_a^b E[|g_n(t) - h_n(t)|^2]dt
\]
\[
= \int_a^b E[|g(t) - g_n(t)|^2]dt + \int_a^b E[|g(t) - h_n(t)|^2]dt \xrightarrow{n \to \infty} 0.
\]
Therefore, \( \lim_{n \to \infty} J(g_n) = \lim_{n \to \infty} J(h_n) \) in \( L^2(\Omega) \).

Theorem 2.4. For any \( g \in L^2_{\text{ind}}([a, b] \times \Omega) \), \( J(g) \) has the following properties:

1. \( E[J(g)] = 0 \).
2. \( E[|J(g)|^2] = \int_a^b E[|g(t)|^2]dt \).

Proof. (1) follows from \( E[J(g)] = \lim_{n \to \infty} E[J(g_n)] = 0 \). (2) follows from
\[
E[|J(g)|^2] = \lim_{n \to \infty} E[|J(g_n)|^2] = \lim_{n \to \infty} \int_a^b E[|g_n(t)|^2]dt = \int_a^b E[|g(t)|^2]dt.
\]
\( \square \)
Corollary 2.5. For any \( g, h \in L^2_{\text{ind}}([a, b] \times \Omega) \), the equality
\[
E \left[ \int_a^b g(t) dB(t) \int_a^b h(t) dB(t) \right] = \int_a^b E[g(t)h(t)] dt
\]
holds.

Proof. By Theorem 2.4, we have
\[
E \left[ \left| \int_a^b g(t) dB(t) + \int_a^b h(t) dB(t) \right|^2 \right] = \int_a^b E[|g(t) + h(t)|^2] dt.
\]
Then we can see that
\[
E \left[ \left| \int_a^b g(t) dB(t) + \int_a^b h(t) dB(t) \right|^2 \right] = E \left[ \left( \int_a^b g(t) dB(t) \right)^2 \right]
+ 2 \left( \int_a^b g(t) dB(t) \right) \left( \int_a^b h(t) dB(t) \right) + \left( \int_a^b h(t) dB(t) \right)^2
= \int_a^b E[|g(t)|^2] dB(t)
+ 2 \int_a^b g(t) dB(t) \int_a^b h(t) dB(t) + \int_a^b E[|h(t)|^2] dB(t).
\]
On the other hand, we get
\[
\int_a^b E[|g(t) + h(t)|^2] dt
= \int_a^b E[|g(t)|^2] dB(t) + 2 \int_a^b E[g(t)h(t)] dt + \int_a^b E[|h(t)|^2] dB(t).
\]
Consequently, we obtain
\[
E \left[ \int_a^b g(t) dB(t) \int_a^b h(t) dB(t) \right] = \int_a^b E[f(t)g(t)] dt.
\]

\[\square\]

Example 2.6. For any \( g \in L^2_{\text{ind}}([a, b] \times \Omega) \), the stochastic process
\[
\left\{ \int_t^b g(s) dB(s); \; a \leq t \leq b \right\}
\]
is an instantly independent process of \( \mathcal{F}_t \).
3. Stochastic Integrals on $L_{\text{ind}}(\Omega, L^2[a,b])$

Lemma 3.1. For any $g \in L_{\text{ind}}(\Omega, L^2[a,b])$, there exists a sequence $\{g_n\}_{n=0}^\infty \subset L_{\text{ind}}^2([a,b] \times \Omega)$ such that

$$\lim_{n \to \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \text{ a.e.}$$

Proof. For any $n \in \mathbb{N}$, we set

$$g_n(t, \omega) = \begin{cases} g(t, \omega), & \int_t^b |g(s, \omega)|^2 ds \leq n, \\ 0, & \int_t^b |g(s, \omega)|^2 ds > n. \end{cases}$$

Then $\{g_n(t); \ a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$ and

$$\int_a^b |g_n(t, \omega)|^2 dt = \int_{\tau_n(\omega)}^b |g(t, \omega)|^2 dt, \text{ a.e. } \omega$$

holds, where $\tau_n(\omega) = \inf \left\{ t; \int_t^b |g(s, \omega)|^2 ds \leq n \right\}$. Therefore, we have

$$\int_a^b |g_n(t)|^2 dt \leq n, \text{ a.e. } \omega.$$ 

Since $a \leq t \leq b$ and $g \in L_{\text{ind}}(\Omega, L^2[a,b])$, it holds that

$$\int_a^b |g(t, \omega)|^2 dt \leq n, \text{ a.e. } \omega \in \Omega$$

for a large $n$. Then we have $g(t, \omega) = g_n(t, \omega)$ for all $t \in [a,b]$. Consequently, we obtain

$$\lim_{n \to \infty} \int_a^b |g_n(t, \omega) - g(t, \omega)|^2 dt = 0, \text{ a.e. } \omega.$$ 

\Box

Lemma 3.2. Let $g \in \text{Step}_{\text{ind}}([a,b] \times \Omega)$. Then for any $\epsilon > 0$, there exists $c > 0$ such that

$$P \left( \left| \int_a^b g(t)dB(t) \right| > \epsilon \right) \leq \frac{c}{\epsilon^2} + P \left( \int_a^b |g(t)|^2 dt > c \right).$$

Proof. For any $c > 0$, we define $g_c(t, \omega)$ by

$$g_c(t, \omega) = \begin{cases} g(t, \omega), & \int_t^b |g(s, \omega)|^2 ds \leq c, \\ 0, & \int_t^b |g(s, \omega)|^2 ds > c. \end{cases}$$

Since

$$\left\{ \left| \int_a^b g(t)dB(t) \right| > \epsilon \right\} \subset \left\{ \left| \int_a^b g_c(t)dB(t) \right| > \epsilon \right\} \cup \left\{ \int_a^b g(t)dB(t) \neq \int_a^b g_c(t)dB(t) \right\},$$

we get

$$P \left( \left| \int_a^b g(t)dB(t) \right| > \epsilon \right) \leq P \left( \left| \int_a^b g_c(t)dB(t) \right| > \epsilon \right) + P \left( \int_a^b |g(t)|^2 dt > c \right).$$

Since

$$P \left( \left| \int_a^b g_c(t)dB(t) \right| > \epsilon \right) \leq \frac{c}{\epsilon^2},$$

we obtain

$$P \left( \left| \int_a^b g(t)dB(t) \right| > \epsilon \right) \leq \frac{c}{\epsilon^2} + P \left( \int_a^b |g(t)|^2 dt > c \right).$$
for any $\epsilon > 0$ and $c > 0$, we have
\[
P \left( \left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \\
\leq P \left( \left| \int_a^b g_c(t) dB(t) \right| > \epsilon \right) + P \left( \int_a^b g(t) dB(t) \neq \int_a^b g_c(t) dB(t) \right).
\]
Then since $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$, we have
\[
\left\{ \int_a^b g(t) dB(t) \neq \int_a^b g_c(t) dB(t) \right\} \subset \left\{ \int_a^b |g(t)|^2 dt > c \right\}
\]
Therefore,
\[
P \left( \left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \leq P \left( \left| \int_a^b g_c(t) dB(t) \right| > \epsilon \right) + P \left( \int_a^b |g(t)|^2 dt > c \right).
\]
By the Chebyshev inequality, we obtain
\[
P \left( \left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \leq \frac{1}{\epsilon^2} E \left[ \left| \int_a^b g_c(t) dB(t) \right|^2 \right] + P \left( \int_a^b |g(t)|^2 dt > c \right)
\]
\[
= \frac{1}{\epsilon^2} \int_a^b E[|g_c(t)|^2] dt + P \left( \int_a^b |g(t)|^2 dt > c \right)
\]
\[
\leq \frac{c}{\epsilon^2} + P \left( \int_a^b |g(t)|^2 dt > c \right).
\]
\[
\frac{c}{\epsilon^2} + P \left( \int_a^b |g(t)|^2 dt > c \right).
\]

Lemma 3.3. For any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, there exists a sequence $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that
\[
\lim_{n \to \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}
\]

Proof. By Lemma 3.1, for any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, we can take $\{h_n\}_{n=1}^{\infty} \subset L^2_{\text{ind}}([a, b] \times \Omega)$ such that
\[
\lim_{n \to \infty} \int_a^b |h_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}
\]
For any $n$, applying Lemma 2.3 to $h_n$, there exists $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that
\[
E \left[ \int_a^b |g_n(t) - h_n(t)|^2 dt \right] < \frac{1}{n}.
\]
Then we have
\[
\left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right\}
\subset \left\{ \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon}{4} \right\} \cup \left\{ \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right\}
\]
for all \(\varepsilon > 0\). Hence, for all \(\varepsilon > 0\),
\[
P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right)
\leq P \left( \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon}{4} \right) + P \left( \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right).
\]
Therefore, by the Chebyshev inequality,
\[
P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right)
\leq \frac{4}{\varepsilon} E \left[ \int_a^b |g_n(t) - h_n(t)|^2 dt \right] + P \left( \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right)
\leq \frac{4}{n\varepsilon} + P \left( \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right)
\]
for all \(\varepsilon > 0\). Consequently, we obtain
\[
\lim_{n \to \infty} P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right) = 0
\]
for all \(\varepsilon > 0\). This means the assertion:
\[
\lim_{n \to \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}
\]

By Lemma 3.3, for any \(g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])\), there exists \(\{g_n\}_{n=1}^\infty \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)\) such that
\[
\lim_{n \to \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}
\]
Then by Lemma 3.2, for any \(\varepsilon > 0\),
\[
P(|\mathcal{J}(g_n) - \mathcal{J}(g_m)| > \varepsilon) \leq \frac{\varepsilon^2}{2} + P \left( \int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right).
\]
Since
\[
\left\{ \int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right\} \\
\subset \left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\} \cup \left\{ \int_a^b |g_m(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\},
\]
we have
\[
P \left( \int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right) \\
\leq P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) + P \left( \int_a^b |g_m(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right).
\]
Hence, since there exists \( N \in \mathbb{N} \) such that
\[
P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) < \frac{\varepsilon}{4}
\]
for all \( n \geq N \) by Lemma 3.3, it holds that
\[
P \left( \int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right) < \frac{\varepsilon}{2}
\]
for all \( n, m \geq N \). Consequently, for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that
\[
P (|\mathcal{J}(g_n) - \mathcal{J}(g_m)| > \varepsilon) < \varepsilon
\]
for all \( n, m \geq N \). This implies that \( \{\mathcal{J}(g_n)\} \) converges in probability. Thus we define the stochastic integral \( \int_a^b g(t)dB(t) \) by
\[
\int_a^b g(t)dB(t) = \lim_{n \to \infty} \mathcal{J}(g_n), \quad \text{in probability}.
\]
This is well-defined. In fact, suppose that there exist sequences \( \{g_n(t)\}_{n=0}^{\infty} \) and \( \{h_n(t)\}_{n=0}^{\infty} \in \text{Step}_{\text{ind}}([a, b] \times \Omega) \) such that
\[
\lim_{n \to \infty} \int_a^b |g(t) - g_n(t)|^2 dt = 0, \lim_{n \to \infty} \int_a^b |g(t) - h_n(t)|^2 dt = 0 \quad \text{in probability}.
\]
Then by Lemma 3.2, for any \( \varepsilon > 0 \), we have
\[
P (|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon) \leq \frac{\varepsilon}{2} + P \left( \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right).
\]
Since
\[
\left\{ \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right\} \\
\subset \left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\} \cup \left\{ \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\},
\]
we have
\[
P \left( \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right)
\leq P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) + P \left( \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right).
\]
By Lemma 3.3, for any \( \varepsilon > 0 \), there exist \( N_1 \in \mathbb{N} \) and \( N_2 \in \mathbb{N} \) such that
\[
P \left( \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) < \frac{\varepsilon}{4}, \text{ for all } n \geq N_1,
\]
and
\[
P \left( \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) < \frac{\varepsilon}{4}, \text{ for all } n \geq N_2.
\]
Therefore, putting \( N = \max\{N_1, N_2\} \), we have
\[
P \left( \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right) < \frac{\varepsilon}{2}
\]
for all \( n, m \geq N \). Consequently,
\[
P (|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon) < \varepsilon
\]
holds for all \( n \geq N \). Thus, we obtain \( \lim_{n \to \infty} \mathcal{J}(g_n) = \lim_{n \to \infty} \mathcal{J}(h_n) \) in probability.

4. The Doob-Meyer Decomposition by the Near-martingale

Let \( (\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a \leq t \leq b} \) be a basic probability space with a filtration \( \{\mathcal{F}_t\}_{a \leq t \leq b} \). A stochastic process \( \{X(t); a \leq t \leq b\} \) is called to be the near-martingale with filtration \( \{\mathcal{F}_t\}_{a \leq t \leq b} \) if it satisfies the following conditions:

1. \( E[|X(t)|] < \infty \) for all \( a \leq t \leq b \),
2. \( E[X(s)|\mathcal{F}_a] = E[X(s)|\mathcal{F}_a] \) for all \( s < t \).

If the condition
3. \( E[X(t)|\mathcal{F}_s] \geq E[X(s)|\mathcal{F}_s] \) for all \( s < t \)
holds instead of the condition (2), the stochastic process \( \{X(t); a \leq t \leq b\} \) is called to be the near-submartingale with the filtration \( \{\mathcal{F}_t\}_{a \leq t \leq b} \).

**Theorem 4.1.** ([5]) Let \( X = \{X(t); n \in \mathbb{N}\} \) be a near-submartingale. Then, there exist a near-martingale \( N = \{N(n); n \in \mathbb{N}\} \) and an increasing process \( A = \{A(n); n \in \mathbb{N}\} \) such that
\[
X(n) = N(n) + A(n), n \in \mathbb{N},
\]
where \( A \) is called to be the increasing process if it satisfies the following conditions:

1. \( A(1) = 0 \),
2. for each \( n \geq 2 \), \( A(n) \) is \( \mathcal{F}_{n-1} \)-measurable,
3. for any \( m \leq n \), \( A(m) \leq A(n) \), a.e.

**Theorem 4.2.** Let \( X(t) = \int_a^b g(s)dB(s) \) for any \( a \leq t \leq b \) and \( g \in L^2_{\text{cadlag}}([a, b] \times \Omega) \). Then the stochastic process \( \{X(t); a \leq t \leq b\} \) is a near-martingale with \( \{\mathcal{F}_t\}_{a \leq t \leq b} \).
Proof. Let \( g \in \text{Step}_\text{ind}([a, b] \times \Omega) \). Then \( g \) has the form

\[
g(u, \omega) = \sum_{i=1}^{n} \eta_i(\omega)1_{[t_{i-1}, t_i]}(u), \quad s = t_0 < t_1 < \cdots < t_j = t < \cdots < t_n = b,
\]

where \( \eta_i, i = 0, 1, 2, \cdots, n, \) are random variables which independent to \( \mathcal{F}_t \) satisfying \( E[\eta_i^2] < \infty \). Then we obtain

\[
E \left[ \int_{t}^{b} g(u)dB(u) \bigg| \mathcal{F}_s \right] = E \left[ \sum_{i=j+1}^{n} \eta_i(B(t_i) - B(t_{i-1})) \bigg| \mathcal{F}_s \right] + \sum_{i=1}^{j} E \left[ \eta_i \right] E \left[ (B(t_i) - B(t_{i-1})) \bigg| \mathcal{F}_s \right]
\]

Next we prove the theorem in the case of \( g \in L^2_{\text{ind}}([a, b] \times \Omega) \). By Lemma 2.3, there exists \( \{g_n\}_{n=1}^{\infty} \subset \text{Step}_\text{ind}([a, b] \times \Omega) \) such that \( \lim_{n \to \infty} \int_{a}^{b} E[|g(t) - g_n(t)|^2]dt = 0 \). Let

\[
X^{(n)}(t) = \int_{t}^{b} g_n(u)dB(u), \quad n \in \mathbb{N}.
\]

Then \( \{X^{(n)}(t); \ a \leq t \leq b\} \) is a near-martingale for each \( n \in \mathbb{N} \) from above argument. For any \( s < t \), we have

\[
E[X(t) - X(s)|\mathcal{F}_s] = E[X(t) - X^{(n)}(t)|\mathcal{F}_s] + E[X^{(n)}(s) - X(s)|\mathcal{F}_s].
\]

Since

\[
E[|E[X(t) - X^{(n)}(t)|\mathcal{F}_s]|^2] \leq E[E[|X(t) - X^{(n)}(t)|^2|\mathcal{F}_s]]
\]

\[
= E[|X(t) - X^{(n)}(t)|^2]
\]

\[
= \int_{t}^{b} E[|g(u) - g_n(u)|^2]du
\]

\[
\leq \int_{a}^{b} E[|g(u) - g_n(u)|^2]du \xrightarrow{n \to \infty} 0,
\]

and by taking subsequence of \( \{X^{(n)}(t)\} \), we get

\[
E[X(t) - X^{(n)}(t)|\mathcal{F}_s] \xrightarrow{n \to \infty} 0, \quad \text{a. e.}
\]
Similarly, we have
\[ E[X(s) - X^{(n)}(s)|\mathcal{F}_s] \xrightarrow{n \to \infty} 0, \quad \text{a. e.} \]
Consequently, we obtain
\[ E[X(t) - X(s)|\mathcal{F}_s] = 0, \quad \text{a. e.} \]
This implies
\[ E[X(t)|\mathcal{F}_s] = E[X(s)|\mathcal{F}_s], \quad \text{a. e.} \]
\[ \square \]
From now on, we assume that the submartingale and the near-submartingale are right-continuous. Let \( \{\mathcal{F}_t; t \geq 0\} \) be a right-continuous filtration and set
\[ \mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t. \]

**Definition 4.3.** Let \( X = \{X(t), t \in \mathbb{R}_+\} \) be a near-submartingale (respectively, near-martingale). Suppose there exists an \( \mathcal{F}_\infty \)-measurable and integrable random variable \( X(\infty) \) such that
\[ E[X(t)|\mathcal{F}_t] \leq E[X(\infty)|\mathcal{F}_t], \quad \text{(respectively, } E[X(t)|\mathcal{F}_t] = E[X(\infty)|\mathcal{F}_t]) \]
for all \( t \in \mathbb{R}_+ (\equiv [0, \infty)) \). Then we call \( \{X(t), t \in \mathbb{R}_+ (\equiv [0, \infty])\} \) a closed near-submartingale (respectively, closed near-martingale).

**Definition 4.4.** An \( (\mathcal{F}_t) \)-adapted right-continuous process \( A = \{A(t); t \in \mathbb{R}_+\} \) is called an increasing process if \( A(t) \) is an increasing function in \( t \) and \( A(0) = 0 \) almost surely.

**Definition 4.5.** An integrable increasing process \( A \) is called a natural increasing process if it satisfies the equality
\[ E \left[ \int_0^t X(s)dA(s) \right] = E \left[ \int_0^t X(s-)dA(s) \right], \quad \forall t \in \mathbb{R}_+ \]
for all bounded martingales \( X \).

Let \( X = \{X(\lambda); \lambda \in \Lambda\} \) be a system of integrable random variables on a probability space \( (\Omega, \mathcal{F}, P) \). If \( X \) satisfies
\[ \sup_{\lambda \in \Lambda} \int_{\{|X(\lambda)| > c\}} |X(\lambda)|dP \xrightarrow{c \to \infty} 0, \]
then \( X \) is called to be uniformly integrable. A near-submartingale \( X = \{X(t), t \in \mathbb{R}_+\} \) is called to have the Doob-Meyer decomposition if \( X \) is expressed in the form
\[ X(t) = N(t) + A(t), \quad \forall t \in \mathbb{R}_+ \]
for some near-martingale \( N \) and natural increasing process \( A \).
Lemma 4.6. Let $A, B$ be natural increasing processes. Then, if $A - B$ is a near-martingale, for any bounded $(\mathcal{F}_t)$-adapted process $f = \{f(t) : t \geq 0\}$, the equality

$$E \left[ \int_0^t f(s) dA(s) \right] = E \left[ \int_0^t f(s) dB(s) \right]$$

holds.

Proof. Let $N(t) = A(t) - B(t)$ for all $t \in \mathbb{R}_+$. Take a partition of $[0, t]$:

$$\delta := \{0 = t_0 < \cdots < t_n = t\}.$$

Then since $N$ is a near-martingale, we get

$$E \left[ \sum_{k=1}^n f(t_{k-1})(N(t_k) - N(t_{k-1})) \right]$$

$$= E \left[ \sum_{k=1}^n E[f(t_{k-1})(N(t_k) - N(t_{k-1}))|\mathcal{F}_{t_{k-1}}] \right]$$

$$= E \left[ \sum_{k=1}^n f(t_{k-1})(E[N(t_k)|\mathcal{F}_{t_{k-1}}] - E[N(t_{k-1})|\mathcal{F}_{t_{k-1}}]) \right] = 0.$$

Therefore,

$$E \left[ \sum_{k=1}^n f(t_{k-1})(A(t_k) - A(t_{k-1})) \right] = E \left[ \sum_{k=1}^n f(t_{k-1})(B(t_k) - B(t_{k-1})) \right]$$

holds. Here, setting $f^k(s) = f(t_k), t_k < s \leq t_{k+1}$, $k = 0, 1, \ldots, n - 1$, we have

$$E \left[ \int_0^t f^k(s) dA(s) \right] = E \left[ \int_0^t f^k(s) dB(s) \right].$$

Consequently, by $|\delta| \to 0$ and the left-continuity, we obtain

$$E \left[ \int_0^t f(s) dA(s) \right] = E \left[ \int_0^t f(s) dB(s) \right].$$

\[\square\]

Lemma 4.7. (cf. [5]) Let $A$ be an integrable increasing process. Then $A$ is natural if and only if

$$E[X(t)A(t)] = E \left[ \int_0^t X(s-) dA(s) \right]$$

holds for any bounded martingale $X$.

Theorem 4.8. The Doob-Meyer decomposition of a near-submartingale is uniquely determined if it exists.

Proof. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale. Suppose that both of $X = M + A$ and $X = N + B$ are the Doob-Meyer decompositions. Then since $A - B$ is a near-martingale and by Lemma 4.6, for any bounded martingale $\{Y(t); t \in \mathbb{R}_+\}$, we have

$$E \left[ \int_0^t Y(s-) dA(s) \right] = E \left[ \int_0^t Y(s-) dB(s) \right].$$
Since $A, B$ is natural increasing and by Lemma 4.7, we have
\[ E[Y(t)A(t)] = E[Y(t)B(t)]. \]

For any bounded random variable $Y$, we define $Y = \{Y(t); t \in \mathbb{R}_+\}$ by $Y(t) := E[Y|F_t]$ for all $t \in \mathbb{R}_+$. Then, $Y$ is a $(F_t)_{t \in \mathbb{R}_+}$-martingale, and therefore, we have
\begin{align*}
E[Y(A(t)) &= E[E[Y(A(t))|F_t]] = E[Y(A(t))] \\
&= E[Y(t)B(t)] = E[E[YB(t)|F_t]] = E[YB(t)].
\end{align*}

Consequently, putting $Y = 1_A$ for all $A \in \mathcal{F}$, we obtain $P(A(t) = B(t)) = 1$ for each $t \in \mathbb{R}_+$. This implies
\[ P(\forall t \in \mathbb{R}_+; A(t) = B(t)) = 1 \]
by the right-continuity of $A(t)$ and $B(t)$.

Let $\mathcal{T}$ be the set of stopping times and set $\mathcal{T}_a := \{\tau \in \mathcal{T}; \tau(\omega) \leq a, \quad \forall \omega \in \Omega\}.$
A closed near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to be in the class $(D)$ if $X(\tau)$ is uniformly integrable for any $\tau \in \mathcal{T}$. A near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to be in the class $(DL)$ if $X(\tau)$ is uniformly integrable for any $a > 0$ and $\tau \in \mathcal{T}_a$.

**Lemma 4.9.** (cf. [5]) $\{A^n_n; n \in \mathbb{N}\}$ is uniformly integrable.

**Theorem 4.10.** Let $X$ be a near-submartingale in the class $(DL)$. If $X(t) \rightarrow X(\infty)$ a. e. and there exists an integrable random variable $Y$ such that $|X_t| \leq Y$ for all $t \geq 0$, then $X$ has the Doob-Meyer decomposition $X = N + A$. Moreover, if $X$ is in the class $(D)$, then $N$ and $A$ in the decomposition of $X$ are uniformly integrable.

**Proof.** It is enough to prove the theorem in the case of a near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ in the class $(D)$. Let $Y(t)$ be $Y(t) = X(t) - E[X(\infty)|F_t]$ for all $t \in \mathbb{R}_+$. Then, $\{Y(t), t \in \mathbb{R}_+\}$ is a near-submartingale, and hence $\lim_{t \rightarrow \infty} Y(t) = 0$, a. e. Let $\{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale satisfying $\lim_{t \rightarrow \infty} X(t) = 0$, a. e. Take a sequence $\delta_n = \{t_j^{(n)} = \frac{j}{n}, j \in \mathbb{N}\}$, $n = 1, 2, 3, \ldots$ of partitions of $[0, \infty)$.

For an arbitrarily fixed $\delta_n$, we denote $t_j^{(n)}$ by $t_j$ simply. For each $n$, we define an increasing process $A^n(t), t \in \delta_n$ by
\[ A^n(t_k) = \sum_{i=1}^{k-1} \{E[X(t_{j+1})|F_{t_j}] - E[X(t_{j})|F_{t_j}]\}, \quad t_j \in \delta_n. \]

Then by Lemma 4.9, $A^n(\infty)$ is uniformly integrable. Therefore, there exist some subsequence $A^{n_l}(\infty), l = 1, 2, \ldots$ and an integrable random variable $A(\infty)$ such that $A^{n_l}(\infty) \rightarrow A(\infty)$ in $L^1$. For any $t \in \mathbb{R}_+$, we define $A(t)$ by
\[ A(t) = E[X(t)|F_t] + E[A(\infty)|F_t]. \] (4.1)
Then $A$ is a $(\mathcal{F}_t)$-adapted process. Since

$$E[A^{n}\ell(\infty)|\mathcal{F}_0] = \lim_{k \to \infty} E \left[ \sum_{j=0}^{k-1} \left( E[X(t_{j+1})|\mathcal{F}_{t_j}] - E[X(t_j)|\mathcal{F}_{t_j}] \right) \right] \bigg| \mathcal{F}_0$$

$$= \lim_{k \to \infty} \left\{ E[X(t_k)|\mathcal{F}_0] - E[X(0)|\mathcal{F}_0] \right\}$$

$$= -E[X(0)|\mathcal{F}_0], \quad t_k \in \delta_n,$$

for any $\ell = 1, 2, \cdots$, we have

$$A(0) = E[X(0)|\mathcal{F}_0] + \lim_{\ell \to \infty} E[A^{n\ell}(\infty)|\mathcal{F}_0] = 0.$$ 

We next prove that $A$ is a natural increasing process. Take $s$ and $t$ with $s < t$ in $\bigcup_n \delta_n$. Then since $s, t \in \delta_{n_\ell}$ for a large $n_\ell \in \mathbb{N}$, by Theorem 4.1, we have

$$E[X(s)|\mathcal{F}_s] + E[A^{n_\ell}(\infty)|\mathcal{F}_s] \leq E[X(t)|\mathcal{F}_t] + E[A^{n_\ell}(\infty)|\mathcal{F}_t].$$

Taking $n_\ell \to \infty$, we get

$$E[X(s)|\mathcal{F}_s] + E[A(\infty)|\mathcal{F}_s] \leq E[X(t)|\mathcal{F}_t] + E[A(\infty)|\mathcal{F}_t], \quad \text{a. e.}$$

Hence, $A(s) \leq A(t)$. Since $\bigcup_n \delta_n$ is dense in $\mathbb{R}_+$, we obtain $A(s) \leq A(t)$ for all $s < t$. This implies that $A$ is an increasing process. For any bounded closed martingale $Z$, we can see that

$$E[Z(\infty)A^n(\infty)] = \sum_k E[Z(\infty)(A^n(t_{k+1}) - A^n(t_k))]$$

$$= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(\infty)|\mathcal{F}_{t_k}]]$$

$$= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(t_k)|\mathcal{F}_{t_k}]]$$

$$= \sum_k E[Z(t_k)(A^n(t_{k+1}) - A^n(t_k))], \quad t_k \in \delta_n.$$

On the other hand, since

$$E[A(t) - A(s)|\mathcal{F}_s] = E[X(t) - X(s)|\mathcal{F}_s]$$

by taking conditional expectations under $\mathcal{F}_s$ in (4.1), we have

$$E[A(t_{k+1}) - A(t_k)|\mathcal{F}_{t_k}]$$

$$= E[X(t_{k+1})|\mathcal{F}_{t_k}] - E[X(t_k)|\mathcal{F}_{t_k}]$$

$$= A^n(t_{k+1}) - A^n(t_k).$$

Therefore, it holds that

$$E[Z(\infty)A^n(\infty)] = \sum_k E[Z(t_k)(A(t_{k+1}) - A(t_k))].$$

Taking $n \to \infty$, we obtain

$$E[Z(\infty)A(\infty)] = E \left[ \int_0^\infty Z(s-)dA(s) \right].$$
This implies that $A$ is natural. Since
\[
E[X(t) - A(t) | \mathcal{F}_s] = E[E[X(t) - A(t) | \mathcal{F}_t] | \mathcal{F}_s] = E[-E[A(\infty) | \mathcal{F}_t] | \mathcal{F}_s] = -E[A(\infty) | \mathcal{F}_s] = E[X(s) - A(s) | \mathcal{F}_s],
\]
the near-martingale part of $X$ is given by $X - A$. \qed

5. A Stochastic Integral by a Near-martingale

Let $0 \leq a < b$. Let $\mathcal{F}_t := \sigma(B(b) - B(s); t < s \leq b) \vee \mathcal{N}$ for any $t \in [a, b]$, and $C([a, b])$ the Banach space of all continuous functions on $[a, b]$ with norm $\| \cdot \|_\infty$ given by $\|f\|_\infty := \sup_{t \in [a, b]} |f(t)|$, $f \in C([a, b])$. Define $B(C([a, b]))$ by the smallest $\sigma$-field including the family of open sets in $C(([a, b]))$, which is called the topological Borel field. Denote by $P_W$ the Wiener measure on $B(C([a, b]))$. For any $\mathcal{F}_t$-adapted process $g = \{g(t); a \leq t \leq b\}$ we consider
\[
N(t) := \int_t^b g(u)dB(u), \ t \in [a, b]. \tag{5.1}
\]
Then, $g$ is an instantly independent process of $(\mathcal{F}_t)$ and $N = \{N(t); a \leq t \leq b\}$ is a near-martingale and also an instantly independent process of $(\mathcal{F}_t)$. Since $g(t)$ is $\mathcal{F}_t$-measurable for any $t \in [a, b]$, then $g(t)$ can be expressed in the form
\[
g(t) = G(B(b) - B(s); t < s \leq b)
\]
for some $B(C([a, b]))$-measurable function $G$ for any $t \in [a, b]$.

By Theorem 4.10, there exists a unique natural increasing process $A = \{A(t); a \leq t \leq b\}$ such that $-N^2 - A$ is a near-martingale. We denote $A$ by $\langle N \rangle = \{(N)(t); a \leq t \leq b\}$. Here, we have
\[
E[(N(t) - N(s))^2 | \mathcal{F}_s] = E[(N)(t) - (N)(s) | \mathcal{F}_s]
\]
for any $s < t$. Let
\[
L^2(\langle N \rangle) := \left\{ X; X \text{ is predictable and satisfies } E \left[ \int_a^t |X(t)|^2d\langle N \rangle(t) \right] < \infty \forall t \right\}.
\]
For any $X$ in $L^2(\langle N \rangle)$, we define semi-norms $\|X\|_{L^2}(\langle N \rangle)$, $a \leq t \leq b$, by
\[
\|X\|_{L^2}(\langle N \rangle) := E \left[ \int_a^t |X|^2d\langle N \rangle(t) \right]^{1/2}.
\]
Then $L^2(\langle N \rangle)$ is the complete metric space with semi-norms $\|X\|_{L^2}(\langle N \rangle)$, $a \leq t \leq b$.

For any $f \in C([a, b])$ and partition $\Delta : a = t_0 < t_1 < \cdots < t_n = b$, we put
\[
f_\Delta = \sum_{k=1}^n f(B(t_{k-1}))1_{[t_{k-1}, t_k)}
\]
and define the stochastic integral $\int_a^b f_\Delta(B(t))dN(t)$ by
\[
\int_a^b f_\Delta(B(t))dN(t) := \sum_{k=1}^n f(B(t_{k-1}))(N(t_k) - N(t_{k-1})), \ \text{in } L^2(\Omega).
\]
Then we have the following:

**Proposition 5.1.** For any \( f \in C([a, b]) \) and partition
\[ \Delta : a = t_0 < t_1 < \cdots < t_n = b, \]
the process \( \int_a f_\Delta dN \) is an \( L^2 \) near-martingale and satisfies
\[
\left\langle \int_a f_\Delta(B(\cdot))dN(\cdot) \right\rangle(t) = \int_a f_\Delta(B(t))d(\langle N \rangle(t)),
\]
(5.2)
\[
E \left[ \left( \int_a f_\Delta(B(t))dN(t) \right)^2 \right] = \|f_\Delta(B(\cdot))\|^2_\langle \langle N \rangle \rangle^2
\]
(5.3)
for all \( a \leq t \leq b. \)

**Proof.** Let \( t > s > a \) and \( f \in C([a, b]). \) Then for any partition
\[ \Delta : s = t_0 < t_1 < \cdots < t_n = b, \]
we can see that
\[
E \left[ \left( \int_s^t f_\Delta(B(t))dN(t) \right)^2 \right]_{\mathcal{F}_s}
\]
\[
= \sum_{k=1}^n E[f_{\Delta_k}(\Delta_k N(t))^2]|\mathcal{F}_{t_{k-1}}]\mathcal{F}_s)
+ 2 \sum_{k>\ell} E[f_{\Delta_k}f_{\Delta_\ell} \Delta_k N(t)\Delta_\ell N(t)|\mathcal{F}_{t_{k-1}}]\mathcal{F}_{t_{\ell-1}}]\mathcal{F}_s),
\]
where \( f_{\Delta_k} := f(B(t_{k-1})) \), and \( \Delta_k N(t) := N(t_k) - N(t_{k-1}) \) for \( k = 1, 2, \ldots, n. \)
By Corollary 2.5 and Theorem 2.6, we have
\[
E[\Delta_k N(t)\Delta_\ell N(t)|\mathcal{F}_{t_{k-1}}] = 0.
\]
Therefore, we get
\[
E \left[ \left( \int_s^t f_\Delta(B(u))dN(u) \right)^2 \right]_{\mathcal{F}_s}
\]
\[
= \sum_{k=1}^n E[f(B(t_{k-1}))^2 E[\langle N \rangle(t_k) - \langle N \rangle(t_{k-1})]|\mathcal{F}_{t_{k-1}}]\mathcal{F}_s]
\]
\[
= E \left[ \sum_{k=1}^n f(B(t_{k-1}))^2(\langle N \rangle(t_k) - \langle N \rangle(t_{k-1})) \right]_{\mathcal{F}_s}
\]
\[
= E \left[ \int_s^t f_\Delta(B(u))^2d(\langle N \rangle(u)) \right]_{\mathcal{F}_s}.
\]
This implies (5.2), and taking the expectation of the both sides of (5.2), we obtain (5.3).

For any $f \in C([a, b])$, we have $f_\Delta(B(t)) \rightarrow f(B(t))$ in $L^2((N))$ as $|\Delta| := \max\{t_k - t_{k-1}; \; k = 1, 2, \ldots, n\} \rightarrow 0$. Therefore by Proposition 5.1, we can define $\int_a^b f(B(t))dN(t)$ by

$$\int_a^b f(B(t))dN(t) := \lim_{|\Delta| \rightarrow 0} \int_a^b f_\Delta(B(t))dN(t) \text{ in } L^2(\Omega).$$

The stochastic integral $\int_a^b f(B(t))g(t)dB(t)$ with $g(t)$ from (5.1) can be regarded as $-\int_a^b f(B(t))dN(t)$. This is a generalization of [10] and a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

References