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Bounds on Smooth Matrix Coefficients on L^2 -spaces

Hongyu He

Abstract. Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let K be a maximal compact subgroup. In this paper, we give an upper bound for K -finite and \mathfrak{k} -smooth matrix coefficients of the regular representation $L^2(X)$ where X is a differentiable G -space equipped with a G -invariant measure, under an assumption about $\text{supp}(L^2(X)) \cap \hat{G}_K$. Furthermore, we show that this bound holds for unitary representations that are weakly contained in $L^2(X)$. Our result generalizes a result of Cowling-Haagerup-Howe [2]. As an example, we discuss the matrix coefficients of the $O(p, q)$ representation $L^2(\mathbb{R}^{p+q})$.

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1. Introduction

Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let (π, \mathcal{H}_π) be a unitary representation of G . One important problem in harmonic analysis is to decompose (π, \mathcal{H}_π) into a direct integral of irreducible unitary representations with multiplicities. More precisely, there exists a Borel measure $d\sigma$ on the unitary dual \hat{G} such that

$$\mathcal{H}_\pi = \int_{(\sigma, \mathcal{H}_\sigma) \in \hat{G}} \mathcal{H}_\sigma \hat{\otimes} M_\sigma d\sigma.$$

Here \hat{G} is equipped with the Fell topology and M_σ records the multiplicity of σ ([4] [14]). Very often, to determine the direct integral decomposition, one has to first determine the support of π , namely, the closed subset of \hat{G} consisting of all

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representations that are weakly contained in (π, \mathcal{H}_π) ([4], [14]). Then one can define a certain transform for each $\sigma \in \text{supp}(\pi)$ to decompose \mathcal{H}_π .

Let X be a (differentiable) G -space that carries a G -invariant measure. Then $L^2(X)$ becomes a unitary representation of G . For many X , determining $\text{supp}(L^2(X))$ remains an open problem, especially the discrete part. In this paper, we want to point a way that may lead to some new development. For a set of vectors S , let $\langle S \rangle$ be the complex linear space spanned by S . Let u be a vector in \mathcal{H}_π . If u is cyclic, that is, $\langle \pi(G)u \rangle$ is dense in \mathcal{H}_π , then the matrix coefficient $(\pi(g)u, u)$ determines $\text{supp}(\pi)$ uniquely. The purpose of this paper is to give some basic estimate of the smooth matrix coefficients of $L^2(X)$. Smooth matrix coefficients here mean the matrix coefficients for smooth vectors. We show that all K -finite and \mathfrak{k} -smooth matrix coefficients are bounded above by some function related to Harish-Chandra's Ξ function. Our estimate equally applies to representations that are weakly contained in $L^2(X)$, in particular those in $\text{supp}(L^2(X))$. We follow the approach taking by Cowling, Haagerup and Howe in treating the tempered representations ([2]).

Before we state our result, we fix some notations. Fix an Iwasawa decomposition KAN . Let Σ^+ be the set of positive restricted roots from N . Let ρ be the half sum of positive restricted roots. Let $\mathfrak{a}_\mathbb{C}^* = \text{Hom}_\mathbb{R}(\mathfrak{a}, \mathbb{C})$ and $\mathfrak{a}^* = \text{Hom}_\mathbb{R}(\mathfrak{a}, \mathbb{R})$. Let \mathfrak{a}^+ be a closed Weyl chamber defined by Σ^+ and by $W(G, \mathfrak{a})$ (See Page 124 [9]). Let $\lambda, \lambda' \in \mathfrak{a}_\mathbb{C}^*$. We say that λ is dominated by λ' if

$$\Re(\lambda')(H) \geq \Re(\lambda)(H) \quad (\forall H \in \mathfrak{a}^+).$$

We write $\lambda \preceq \lambda'$. \preceq defines a partial ordering on $\mathfrak{a}_\mathbb{C}^*$.

Let \hat{G}_K be the spherical unitary dual. Then \hat{G}_K can be identified with a closed subset of

$$\mathfrak{a}_\mathbb{C}^*/W(G : \mathfrak{a}).$$

Fix a dominant Weyl chamber in \mathfrak{a}^* corresponding to \mathfrak{a}^+ . We say that $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is dominant if $\Re(\lambda)$ is in the dominant Weyl chamber. **Identify \hat{G}_K with a closed subset of dominant $\mathfrak{a}_\mathbb{C}^*$.** Let Ξ be Harish-Chandra's basic spherical function.

Fix a maximal torus T in K and a positive root system. Let r_K be the rank of K and l_K be the number of positive roots of K . Let ρ_K be the half sum of the positive roots. Let V_λ be an irreducible unitary representation of K with highest weight λ . Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$ if \mathfrak{k} is semisimple. If \mathfrak{k} contains a nontrivial center, define $C(\mathfrak{k})$ to be the element in the center of the universal enveloping algebra $U(\mathfrak{k})$ satisfying

$$C(\mathfrak{k})|_{V_\lambda} = [-(\rho_K + \lambda, \rho_K + \lambda) + (\rho_K, \rho_K)]I$$

for every $V_\lambda \in \hat{K}$. Let d_λ be the dimension of V_λ .

Theorem 1.1 (Main Theorem). *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let X be a G -space endowed with a G -invariant measure dx . Suppose that $\text{supp}(L^2(X)) \cap \hat{G}_K$, as a subset of dominant $\mathfrak{a}_\mathbb{C}^*$, is dominated by a real λ_0 . Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(X)$ (see [4], [14]).*

1. *Let u, v be two K -finite vectors in \mathcal{H}_π . Let S_1 be the K -types appearing in $\langle \pi(K)u \rangle$. Let S_2 be the K -types appearing in $\langle \pi(K)v \rangle$. Then for any $H \in \mathfrak{a}^+$ and $k_1, k_2 \in K$, we have*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H). \quad (1.1)$$

2. *Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$. Let u, v be two \mathfrak{k} smooth vectors (See Definition 5.1). Then there exists a positive constant C , independent of u, v , such that for any $k_1, k_2 \in K, H \in \mathfrak{a}^+$,*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq C \exp \lambda_0(H) \Xi(\exp H) \|(C(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_{\kappa+r\kappa}} u\| \|(C(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_{\kappa+r\kappa}} v\|. \quad (1.2)$$

In particular, these estimates hold for irreducible unitary representations in $\text{supp}(\pi)$.

Note that in many cases, the spherical support $\text{supp}(L^2(X)) \cap \hat{G}_K$ is easier to determine than $\text{supp}(L^2(X))$.

The proof of the main theorem contains three ingredients. The first ingredient comes from a uniform bound for the spherical functions in [9]. The second ingredient comes from a paper by Cowling-Haagerup-Howe that bounds the K -finite matrix coefficients of tempered representations by $\Xi(g)$. The third ingredient comes from a bound on the dimension of an irreducible unitary representations of K . In Theorem 6.1, we give a result only assuming that X has a K -invariant measure.

There are bounds for smooth matrix coefficients for unitary representations in [1], [7]. The tempered case, that is $\lambda_0 = 0$, was treated in [2]. The bound for the smooth matrix coefficients of tempered representations was treated recently by Sun [13]. The ideas in this paper are quite standard, not new. Nevertheless, we believe that our estimates can shed lights on the structure of $\text{supp}(L^2(X))$, as well as some other applications. Let us take the example of $L^2(\mathbb{R}^{p+q})$ as a unitary representation of $O(p, q)$. The spectral decomposition of $L^2(\mathbb{R}^{p+q})$ was established by Strichartz in general and others in some special cases. See [12] and the references therein. Applying our main theorem, we have

Theorem 1.2. *Suppose that $q \geq p$ and $pq > 1$. Let $G = O(p, q)$ and $K = O(p) \times O(q)$. Let $C(\mathfrak{k})$ be the Casimir operator. Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(\mathbb{R}^{p+q})$. Let u, v be two \mathfrak{k} -smooth vectors in \mathcal{H}_π . Let*

$\lambda_t = (\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{q-p}{2}, t)$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, if $q - p > 2$, we have

$$|(\pi(k_1 \exp H k_2)u, v)| \leq C \exp \lambda_{\frac{q-p}{2}-1}(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\|; \quad (1.3)$$

if $q - p = 0, 1, 2$, we have

$$|(\pi(g)u, v)| \leq C \phi_{\lambda_0}(g) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\|. \quad (1.4)$$

Here $\phi_{\lambda_0}(g)$ is the spherical function corresponding to $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*/W(G, \mathfrak{a})$.

Let ϵ be a small positive number. Our theorem implies that $(\pi(k_1 \exp H k_2)u, v)$ decays faster than $C \exp(-1 + \epsilon)(|H_1| + |H_2| + \dots + |H_p|)$ if $q - p \geq 2$. If $q = p + 1$ then $(\pi(k_1 \exp H k_2)u, v)$ decays faster than $C \exp(-1 + \epsilon)(|H_1| + |H_2| + \dots + \frac{1}{2}|H_p|)$. If $q = p$, then $(\pi(k_1 \exp H k_2)u, v)$ decays faster than $C \exp(-1 + \epsilon)(|H_1| + |H_2| + \dots + |H_{p-1}|)$. These results are slightly different from what one would expect.

A more intriguing problem is to find a bound for the \mathfrak{k} -smooth matrix coefficients from below. Clearly, the \mathfrak{k} -smooth matrix coefficients of $L^2(\mathbb{R}^{p+q})$ cannot decay arbitrarily fast unless $\min(p, q) = 1$. Having an upper bound, if one can find a bound from below, one can potentially narrow down the possible τ in $\text{supp}(L^2(\mathbb{R}^{p+q}))$, which is already known. For those X that $\text{supp}(L^2(X))$ is not known, we hope that this approach will yield some new results.

2. Bounds for K -invariant Matrix Coefficients

Let G be a semisimple Lie group with a finite number of connected components and a finite center. G may be disconnected. Let K be a maximal compact subgroup. Fix an Iwasawa decomposition KAN . Let Σ^+ be the positive restricted roots corresponding to N . For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let $\phi_\lambda(g)$ be the corresponding spherical function. $\phi_\lambda(g)$ is both left and right K -invariant. We have the following (see Ch. 7.8 [9])

1. For λ real, $\phi_\lambda(g) > 0$ for all g ;
2. For $\phi_\lambda(g) = \phi_{w\lambda}(g)$ for any $w \in W(G : \mathfrak{a})$;
3. $|\phi_\lambda(g)| \leq \phi_{\Re\lambda}(g)$.

Let \mathfrak{a}^+ be a closed positive Weyl chamber satisfying the property that

$$\alpha(H) \geq 0, \quad (\forall H \in \mathfrak{a}^+, \alpha \in \Sigma^+).$$

\mathfrak{a}^+ determines a dominant Weyl chamber in \mathfrak{a}^* by identifying \mathfrak{a} with \mathfrak{a}^* . $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is said to be **dominant** if $\Re\lambda$ is in the dominant Weyl chamber. If λ is dominant and real, we have

$$\phi_\lambda(\exp H) \leq \exp(\lambda(H))\phi_0(\exp H)$$

for any $H \in \mathfrak{a}^+$ (see Ch. 7.8 [9]). Here $\phi_0(g)$ is Harish-Chandra's Ξ function. Essentially, the formulae above give bounds for K -invariant functions for each irreducible representation.

Let \hat{G} be the unitary dual of G . Let (π, \mathcal{H}) be a unitary representation of G . Let $\text{supp}(\pi)$ or sometimes $\text{supp}(\mathcal{H})$ be the support of π , namely the closed subset of \hat{G} consisting of those that are weakly contained in π (See Ch 18.1 [4] or Ch 14.10 [14]). If $\text{supp}(\pi)$ is a subset of $\text{supp}(\pi')$, we say that π is weakly contained in π' .

An irreducible admissible representation is said to be spherical if it has a K -fixed vector. Infinitesimal equivalence classes of spherical admissible representations are in one-to-one correspondence with

$$\mathfrak{a}_{\mathbb{C}}^*/W(G : \mathfrak{a}).$$

See Ch .IV [6] for example. The unitary spherical dual is often denoted by \hat{G}_K . We parametrize \hat{G}_K by a closed subset of **dominant** λ . We write the corresponding spherical unitary representation as $(\pi_\lambda, \mathcal{H}_\lambda)$.

Theorem 2.1. *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let (π, \mathcal{H}) be a unitary representation of G . Suppose that $\text{supp}(\pi) \cap \hat{G}_K$ is dominated by a real λ_0 . Then*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq \exp \lambda_0(H) \Xi(\exp H) \|u\| \|v\|$$

for any $k_1, k_2 \in K$, $H \in \mathfrak{a}^+$ and K -fixed vectors $u, v \in \mathcal{H}$.

The proof will be based on local results about $\phi_\lambda(g)$ we mentioned earlier and the direct integral theory (see for example Ch 14. [14]).

Proof. Decompose the unitary representation (π, \mathcal{H}) into a direct integral

$$\int_{\hat{G}} \mathcal{H}_s \hat{\otimes} M_s d\mu_s$$

where M_s records the multiplicity. Write

$$u = \int_{\hat{G}_K \cap \text{supp}(\pi)} u_s d\mu_s, \quad v = \int_{\hat{G}_K \cap \text{supp}(\pi)} v_s d\mu_s.$$

Then we have

$$\begin{aligned} \|u\|^2 &= \int_{\hat{G}_K \cap \text{supp}(\pi)} \|u_s\|^2 d\mu_s, & \|v\|^2 &= \int_{\hat{G}_K \cap \text{supp}(\pi)} \|v_s\|^2 d\mu_s \\ (\pi(g)u, v) &= \int_{\hat{G}_K \cap \text{supp}(\pi)} (\pi(g)u_s, v_s) d\mu_s. \end{aligned}$$

Notice that here s are all dominant in $\mathfrak{a}_{\mathbb{C}}^*$ and u_s, v_s are K -invariant. Now by our assumption, for every $H \in \mathfrak{a}^+$,

$$\begin{aligned}
& |(\pi(k_1 \exp H k_2)u, v)| \\
& \leq \int_{\hat{G}_K \cap \text{supp}(\pi)} |(\pi(\exp H)u_s, v_s)| d\mu_s \\
& = \int_{\hat{G}_K \cap \text{supp}(\pi)} |\phi_s(\exp H)| |(u_s, v_s)| d\mu_s \\
& \leq \int_{\hat{G}_K \cap \text{supp}(\pi)} \exp s(H) \Xi(\exp H) \|u_s\| \|v_s\| d\mu_s \tag{2.1} \\
& \leq \int_{\hat{G}_K \cap \text{supp}(\pi)} \exp \lambda_0(H) \Xi(\exp H) \|u_s\| \|v_s\| d\mu_s \\
& \leq \exp \lambda_0(H) \Xi(\exp H) \left(\int_{\hat{G}_K \cap \text{supp}(\pi)} \|u_s\|^2 d\mu_s \int_{\hat{G}_K \cap \text{supp}(\pi)} \|v_s\|^2 d\mu_s \right)^{\frac{1}{2}} \\
& = \exp \lambda_0(H) \Xi(\exp H) \|u\| \|v\|
\end{aligned}$$

□

In the case that π is supported on the tempered dual of G , $\lambda_0 = 0$. So we have

$$|(\pi(g)u, v)| \leq \Xi(g) \|u\| \|v\|.$$

This is proved in [2].

For u, v in other K -types of \mathcal{H} , it is not easy to bound $(\pi(g)u, v)$ by $\|u\|$ and $\|v\|$. Even if π is spherical, it is still not clear whether the type of bound in Theorem 2.1 is true. However, if π is supported on $\text{supp}(L^2(X))$ with X a G -space equipped with a G -invariant measure, we can find such a bound.

3. Bounds for K -finite Matrix Coefficients of $L^2(X)$

Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(G)$. Cowling, Haagerup and Howe obtain a sharp bound on the K -finite matrix coefficients of π .

Theorem 3.1 (Cowling-Haagerup-Howe [2]). *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(G)$. Let ξ and η be two K -finite vectors. Decompose the K invariant subspaces $\langle \pi(K)\xi \rangle$ and $\langle \pi(K)\eta \rangle$:*

$$\langle \pi(K)\xi \rangle = \bigoplus_{\tau \in \hat{K}} \bigoplus^{m(\tau)} \mathcal{H}_\tau, \quad \langle \pi(K)\eta \rangle = \bigoplus_{\tau \in \hat{K}} \bigoplus^{n(\tau)} \mathcal{H}_\tau.$$

Let $\dim(\mathcal{H}_\tau) = d_\tau$. Then $m(\tau) \leq d_\tau, n(\tau) \leq d_\tau$ and

$$|(\pi(g)\xi, \eta)| \leq (\dim \langle \pi(K)\xi \rangle)^{\frac{1}{2}} (\dim \langle \pi(K)\eta \rangle)^{\frac{1}{2}} \Xi(g).$$

In particular, if $\langle \pi(K)u \rangle \cong \oplus^{m(\tau)} \mathcal{H}_\tau$ and $\langle \pi(K)v \rangle \cong \oplus^{n(\sigma)} \mathcal{H}_\sigma$, then

$$|(\pi(g)u, v)| \leq d_\tau d_\sigma \|u\| \|v\| \Xi(g).$$

Let X be a differentiable G -space equipped with a G -invariant measure dx . Let G act on $L^2(X)$ by

$$L(g)f(x) = f(g^{-1}x) \quad (g \in G, x \in X).$$

We call $(L, L^2(X))$ a regular representation. One of the most important problems in harmonic analysis is to find the $\text{supp}(L^2(X))$. In many cases, the set $\text{supp}(L^2(X)) \cap \hat{G}_K$ is relatively easy to find, since \hat{G}_K is better understood than \hat{G} . In Theorem 2.1, we find a bound for the K -invariant matrix coefficients, assuming that $\text{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by a $\lambda_0 \in \mathfrak{a}^*$. Borrowing an idea from [2], we can show that similar bounds apply to all K -finite matrix coefficients of $L^2(X)$. Now this does not tell you much if X has finite volume because the trivial representation will appear in $L^2(X)$. But if X has infinite volume, bounds on K -finite matrix coefficients can shed lights on the structure of $\text{supp}(L^2(X))$. At the end of this paper, we will use the hyperboloid as an example to illustrate our point.

Theorem 3.2. *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let X be a G -space equipped with a G -invariant measure dx . Suppose that $\text{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let ϕ, ψ be continuous K -finite functions. Then for any $H \in \mathfrak{a}^+$,*

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \dim(\langle L(K)\phi \rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi \rangle)^{\frac{1}{2}} \|\phi\|_2 \|\psi\|_2 \exp \lambda_0(H) \Xi(\exp H).$$

Before we give the proof, let us recall the following lemma (See [2], for example).

Lemma 3.3. *Let ϕ be a continuous function on a K -homogeneous space X . Suppose that $\langle L(K)\phi \rangle$ is finite dimensional. Then*

$$\|\phi\|_\infty \leq \dim(\langle L(K)\phi \rangle)^{\frac{1}{2}} \|\phi\|_2.$$

Here L^2 -norm $\|*\|_2$ is taken over the K -invariant probability measure on X . In addition, if $\langle L(K)\phi \rangle$ consists of K -types from the set $S \subset \hat{K}$,

$$\dim(\langle L(K)\phi \rangle) \leq \sum_{\sigma_i \in S} d_{\sigma_i}^2.$$

Proof of Theorem 3.2: Use the notation from [2]. Let

$$\tilde{\phi}(x) = \sup_{k \in K} |\phi(kx)|, \quad \tilde{\psi}(x) = \sup_{k \in K} |\psi(kx)| \quad (x \in X).$$

Consider any K -orbit Kx_0 equipped with the K -invariant probability measure. We have

$$\int_{Kx_0} |\tilde{\phi}(kx_0)|^2 d[k] = \left(\sup_{k \in K} |\phi(kx_0)| \right)^2 \leq \dim(\langle L(K)\phi \rangle) \int_{Kx_0} |\phi(kx_0)|^2 d[k].$$

It follows that

$$\|\tilde{\phi}\|_2^2 \leq \dim(\langle L(K)\phi \rangle) \|\phi\|_2^2.$$

Hence $\|\tilde{\phi}\|_2 \leq \dim(\langle L(K)\phi \rangle)^{\frac{1}{2}} \|\phi\|_2$. For any $H \in \mathfrak{a}^+$, we have

$$\begin{aligned}
& |(L(k_1 \exp H k_2)\phi, \psi)| \\
& \leq |(L(\exp H)\tilde{\phi}, \tilde{\psi})| \\
& \leq \exp \lambda_0(H) \Xi(\exp H) \|\tilde{\phi}\|_2 \|\tilde{\psi}\|_2 \\
& \leq \dim(\langle L(K)\phi \rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi \rangle)^{\frac{1}{2}} \|\phi\|_2 \|\psi\|_2 \exp \lambda_0(H) \Xi(\exp H)
\end{aligned} \tag{3.1}$$

□

Now one can drop the requirement that ϕ, ψ are continuous.

Corollary 3.4. *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let X be a G -space equipped with a G -invariant measure dx . Suppose that $\text{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let ϕ, ψ be two L^2 K -finite functions on X . Let S_1 be the K -types appearing in $\langle L(K)\phi \rangle$. Let S_2 be the K -types appearing in $\langle L(K)\psi \rangle$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have*

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \exp \lambda_0(H) \Xi(\exp H) \|\phi\|_2 \|\psi\|_2 \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}}.$$

Proof. Choose two sequences of continuous functions

$$\phi_i \rightarrow \phi \quad \psi_i \rightarrow \psi$$

in L^2 -norm. Without loss of generality, suppose that

$$\text{supp}(\langle L(K)\phi_i \rangle) = S_1, \quad \text{supp}(\langle L(K)\psi_i \rangle) = S_2.$$

Otherwise, we can always project ϕ_i and ψ_i to respective K -types. By Theorem 3.2 and Lemma 3.3, for $H \in \mathfrak{a}^+$, we have

$$\begin{aligned}
& |(L(k_1 \exp H k_2)\phi_i, \psi_i)| \\
& \leq \dim(\langle L(K)\phi_i \rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi_i \rangle)^{\frac{1}{2}} \|\phi_i\|_2 \|\psi_i\|_2 \exp \lambda_0(H) \Xi(\exp H) \\
& \leq \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|\phi_i\|_2 \|\psi_i\|_2 \exp \lambda_0(H) \Xi(\exp H).
\end{aligned} \tag{3.2}$$

Taking pointwise limits, we obtain

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \exp \lambda_0(H) \Xi(\exp H) \|\phi\|_2 \|\psi\|_2 \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}}.$$

□

4. Bounds for K -finite Matrix Coefficients

Theorem 4.1. *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let X be a G -space equipped with a G -invariant measure dx . Suppose that $\text{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(X)$. Let u, v be two*

K -finite vectors in \mathcal{H}_π . Let S_1 be the K -types appearing in $\langle \pi(K)u \rangle$. Let S_2 be the K -types appearing in $\langle \pi(K)v \rangle$. Then for any $H \in \mathfrak{a}^+$,

$$|(\pi(k_1 \exp H k_2)u, v)| \leq \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H).$$

Proof. The ideas in this proof are essentially from [2]. For a unitary representation \mathcal{H} of K , let $\mathcal{H}(S_j)$ be the direct sum of its σ -isotypic subspaces with $\sigma \in S_j$.

Since (π, \mathcal{H}_π) is weakly contained in $L^2(X)$, $(\pi(g)u, v)$ can be approximated by finite sums

$$\sum_i (L(g)\phi_i, \psi_i) \quad (\phi_i, \psi_i \in L^2(X))$$

uniformly on compacta, subject to the condition that

$$\sum_i \|\phi_i\|_2 \|\psi_i\|_2 \leq \|u\| \|v\|.$$

Let P_{S_j} be the projector of $L^2(X)$ to $L^2(X)(S_j)$ ($j = 1, 2$). we can project ϕ_i and ψ_i to $L^2(X)(S_1)$ and $L^2(X)(S_2)$ respectively. Without loss of generality, assume that $\phi_i \in L^2(X)(S_1)$ and $\psi_i \in L^2(X)(S_2)$. By Cor. 3.4, we have the bound

$$|(L(k_1 \exp H k_2)\phi_i, \psi_i)| \leq \exp \lambda_0(H) \Xi(\exp H) \|\phi_i\|_2 \|\psi_i\|_2 \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} & \left| \sum (L(k_1 \exp H k_2)\phi_i, \psi_i) \right| \\ & \leq \sum |(L(k_1 \exp H k_2)\phi_i, \psi_i)| \\ & \leq \exp \lambda_0(H) \Xi(\exp H) \left\| \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \sum \|\phi_i\|_2 \|\psi_i\|_2 \right\|. \quad (4.1) \\ & \leq \exp \lambda_0(H) \Xi(\exp H) \left\| \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|u\| \|v\| \right\| \end{aligned}$$

It follows that

$$|(\pi(k_1 \exp H k_2)u, v)| \leq \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H).$$

□

I shall point out that our estimate clearly holds if π is in the support of $L^2(X)$. So our estimate can be used to exclude those π that are not in the support of $L^2(X)$.

5. Bounds for Smooth Matrix Coefficients

Let (π, \mathcal{H}_π) be a unitary representation weakly contained in $L^2(X)$. Now we can move forward to give a bound for \mathfrak{k} -smooth matrix coefficients of π . Very recently, B. Sun found a bound for the tempered representations for a bigger class of group G ([13]). Our idea is essentially the same.

Definition 5.1. Let (π, \mathcal{H}_π) be a unitary representation of a Lie group H . We say that a vector v is \mathfrak{h} smooth if $\pi(D)v$ is well-defined in \mathcal{H}_π for any $D \in U(\mathfrak{h})$.

Fix a maximal torus \mathfrak{t} and positive roots Σ^+ for the Lie algebra \mathfrak{k} . Let r_K be the dimension of \mathfrak{t} , l_K be the cardinality of Σ^+ , and ρ_K be the half sum of positive roots. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator in $U(\mathfrak{k})$. Parametrize \hat{K} by the highest weight λ . Then

$$\mathcal{C}(\mathfrak{k})|_{V_\lambda} = -\|\rho_K + \lambda\|^2 + \|\rho_K\|^2.$$

Clearly, for each positive root α ,

$$(\lambda + \rho_K, \alpha) \leq (\lambda + \rho_K, 2\rho_K) \leq (\lambda + \rho_K, \lambda + \rho_K) + (\rho_K, \rho_K).$$

If K is not Abelian, by Weyl's character formula,

$$\dim(V_\lambda) \leq (\|\lambda + \rho_K\|^2 + \|\rho_K\|^2)^{l_K}.$$

For all compact K ,

$$\dim(V_\lambda) \leq (\|\lambda + \rho_K\|^2 + \|\rho_K\|^2 + 1)^{l_K}.$$

So for $u \in V_\lambda$, we have

$$\dim(V_\lambda)\|u\| \leq \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K} u\|.$$

Theorem 5.2. Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let K be a maximal compact subgroup of G . Let r_K be the rank of K and l_K be the number of positive roots for \mathfrak{k} . Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator. Let X be a G -space equipped with a G -invariant measure. Suppose that $\text{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(X)$. Let u, v be two \mathfrak{k} -smooth vectors in \mathcal{H}_π . Then there exists a positive constant C , independent of u, v , such that for any $k_1, k_2 \in K$, $H \in \mathfrak{a}^+$

$$\begin{aligned} & |(\pi(k_1 \exp H k_2)u, v)| \leq \\ & C \exp \lambda_0(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} v\|. \end{aligned} \tag{5.1}$$

Here in the place of r_K one can use any integer greater than $\frac{r_K}{2}$.

Proof. Suppose that u, v are \mathfrak{k} -smooth. Decompose u, v according to the K -types:

$$u = \sum_{\lambda \in \hat{K}} u_\lambda, \quad v = \sum_{\lambda \in \hat{K}} v_\lambda.$$

Let $g = k_1 \exp H k_2$ and $H \in \mathfrak{a}^+$. Put $\rho_0 = \rho_K$. Then

$$\begin{aligned} & |(\pi(g)u, v)| \\ & \leq \sum_{\lambda, \mu} |(\pi(g)u_\lambda, v_\mu)| \\ & \leq \sum d_\mu d_\lambda \|u_\lambda\| \|v_\mu\| \Xi(g) \exp \lambda_0(H) \\ & = \Xi(g) \exp \lambda_0(H) \left[\sum d_\lambda \|u_\lambda\| \right] \left[\sum d_\mu \|v_\mu\| \right] \\ & \leq \Xi(g) \exp \lambda_0(H) \left[\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K} \|u_\lambda\| \right] \left[\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K} \|v_\mu\| \right] \\ & = \Xi(g) \exp \lambda_0(H) \left[\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K + r_K} \|u_\lambda\| (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-r_K} \right] \\ & \quad \left[\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K + r_K} \|v_\mu\| (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-r_K} \right] \\ & \leq \Xi(g) \exp \lambda_0(H) \left[\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{2l_K + 2r_K} \|u_\lambda\|^2 \right]^{\frac{1}{2}} \left[\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-2r_K} \right]^{\frac{1}{2}} \\ & \quad \left[\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{2l_K + 2r_K} \|v_\mu\|^2 \right]^{\frac{1}{2}} \left[\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-2r_K} \right]^{\frac{1}{2}} \\ & = C \Xi(g) \exp \lambda_0(H) \left[\sum \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} u_\lambda\|^2 \right]^{\frac{1}{2}} \left[\sum \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} v_\mu\|^2 \right]^{\frac{1}{2}} \\ & = C \Xi(g) \exp \lambda_0(H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} v\| \end{aligned} \tag{5.2}$$

Here $C = \sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2)^{-2r_K}$ which converges absolutely. \square

Of course, the estimate we obtain here can be improved substantially. For the purpose of this paper, it is sufficient. I shall also point out that for \mathfrak{k} -smooth vectors in $L^2(X)$, our bound can be established directly by bounding sup norm by the L^2 norm of some derivative. But this bound can not be passed from $L^2(X)$ to (π, \mathcal{H}_π) . Therefore, for (π, \mathcal{H}_π) , we must bound the K -finite matrix coefficients first and then pass this bound to all \mathfrak{k} smooth vectors.

6. X with K -invariant Measure

Sometimes, G -invariant measure does not exist for a G -space X . For example, when X is a flag variety, there is no G -invariant measure. Nevertheless, K -invariant measure always exists. Now suppose that X is equipped with only a K -invariant measure. Then $L^2(X)$ may no longer be a unitary representation of G . We can still define K -finite and \mathfrak{k} -smooth matrix coefficients. Suppose that there is a positive function $B(g)$ such that

$$|(L(g)\phi, \psi)| \leq B(g) \|\phi\|_2 \|\psi\|_2,$$

for any K -invariant function ϕ and ψ in $L^2(X)$. Then by similar arguments as in the proofs of Theorem 3.2 and Cor. 3.4 and Theorem 5.2, we obtain

Theorem 6.1. *Let G be a semisimple Lie group with a finite number of connected components and a finite center. Let X be a G -space endowed with a K -invariant measure dx . Suppose that there is positive function $B(g)$ such that*

$$|(L(g)\phi, \psi)| \leq B(g)\|\phi\|_2\|\psi\|_2,$$

for any K -invariant function ϕ and ψ in $L^2(X)$.

1. Let u, v be two K -finite functions in $L^2(X)$. Let S_1 be the K -types appearing in $\langle L(K)u \rangle$. Let S_2 be the K -types appearing in $\langle L(K)v \rangle$. Then we have

$$|(L(g)u, v)| \leq \left(\sum_{\sigma \in S_1} (d_\sigma)^2 \right)^{\frac{1}{2}} \left(\sum_{\tau \in S_2} (d_\tau)^2 \right)^{\frac{1}{2}} \|u\|_2 \|v\|_2 B(g).$$

2. Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$. Let u, v be two \mathfrak{k} smooth functions in $L^2(X)$. Then there exists a $C > 0$ such that for any $g \in G$

$$|(\pi(g)u, v)| \leq CB(g)\|(C(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K+r_K}u\|_2\|(C(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K+r_K}v\|_2.$$

7. Bounds for Smooth Matrix Coefficients of $L^2(\mathbb{R}^{p+q})$

Now we shall give an example here. Let $O(p, q)$ be the orthogonal group preserving the standard symmetric form

$$(x, y) = \sum_{i=1}^p x_i y_i - \sum_{j=p+1}^{p+q} x_j y_j \quad (x, y \in \mathbb{R}^{p+q}).$$

Consider $L^2(\mathbb{R}^{p+q})$, a regular representation of $O(p, q)$. R. Strichartz computed the spectrum of the pseudo Laplacian \square on $L^2(\mathbb{R}^{p+q})$ in full generality. Special cases were treated earlier. See [12] and the references therein. If $pq > 1$, besides the continuous spectrum, there are also discrete spectrum. Essentially, this determines the support of $L^2(\mathbb{R}^{p+q})$. The continuous spectrum comes from degenerate principal series and the discrete spectrum comes from some quotients of degenerate principal series. $L^2(\mathbb{R}^{p+q})$ was later studied by Rallis-Schiffman ([11]) and Howe ([3]) under the framework of dual reductive pair $(O(p, q), SL_2(\mathbb{R}))$. Howe proved that

$$L^2(\mathbb{R}^{p+q}) \cong \int_{s \in \widetilde{SL}_2(\mathbb{R})} \mathcal{H}_{\theta(s)} \otimes \mathcal{H}_s ds$$

Here $\widetilde{SL}_2(\mathbb{R})$ is the double cover of $SL_2(\mathbb{R})$, ds is a Borel measure on the unitary dual of $SL_2(\mathbb{R})$, and $\mathcal{H}_{\theta(s)}$ is an irreducible unitary representation of $O(p, q)$. The structure of the representation $\mathcal{H}_{\theta(s)}$ was studied by Molcanov ([10]) and later by Howe and Tan in greater details ([8]).

Let $G = O(p, q)$ and $K = O(p) \times O(q)$. Suppose that $pq > 1$. Then the real rank $r = \min(p, q)$. The half sum of positive restricted root

$$\rho = \left(\frac{p+q}{2} - 1, \frac{p+q}{2} - 2, \dots, \left| \frac{p-q}{2} \right| \right).$$

For the purpose of giving a bound for \mathfrak{k} -smooth matrix coefficients, we will need to know $\text{supp}(L^2(\mathbb{R}^{p+q})) \cap \hat{G}_K$. We will assume that $p \leq q$. This assumption won't effect our estimation. \hat{G}_K is parametrized by certain dominant λ , i.e.,

$$\Re(\lambda_1) \geq \Re(\lambda_2) \geq \dots \geq \Re(\lambda_p) \geq 0$$

up to a permutation and sign change. $\text{supp}(L^2(\mathbb{R}^{p+q})) \cap \hat{G}_K$ can be described as follows.

1. the continuous spectrum consists of $\lambda_{it} = \left(\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{q-p}{2}, it \right)$ with $t \geq 0$;
2. If $\frac{q-p}{2} > 1$, then the discrete spectrum consists of

$$\lambda_{\frac{q-p}{2}-2j-1} = \left(\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{q-p}{2}, \frac{q-p}{2} - 2j - 1 \right)$$

for all integer $j \in [0, \frac{q-p-2}{4})$.

If $p = q$, then there is no discrete spherical spectrum. $\mathcal{H}_{\lambda_{it}}$ will decompose into $\mathcal{H}_{\lambda_{it}}$ and $\mathcal{H}_{\lambda_{-it}}$ with respect to the group $SO_0(p, q)$. If $q - p = 1, 2$, $L^2(\mathbb{R}^{p+q})$ does not have any discrete spherical spectrum. By Theorem 2.1, we have

Theorem 7.1. *Suppose that $q \geq p$ and $pq > 1$. Let $\lambda_t = \left(\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{q-p}{2}, t \right)$. Let u, v be two K -invariant vectors in $L^2(\mathcal{R}^{p+q})$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have*

$$|(L(k_1 \exp H k_2)u, v)| \leq \exp \lambda_{\frac{q-p}{2}-1}(H) \Xi(\exp H) \|u\| \|v\| \quad (q - p > 2)$$

$$|(L(g)u, v)| \leq \phi_{\lambda_0}(g) \|u\| \|v\| \quad (q - p = 0, 1, 2).$$

By Theorem 5.2, we have the following

Theorem 7.2. *Suppose that $q \geq p$ and $pq > 1$. Let $G = O(p, q)$ and $K = O(p) \times O(q)$. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator. Let (π, \mathcal{H}_π) be a unitary representation that is weakly contained in $L^2(\mathbb{R}^{p+q})$. Let u, v be two \mathfrak{k} -smooth vectors in \mathcal{H}_π . Let $\lambda_t = \left(\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{q-p}{2}, t \right)$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have for $q - p > 2$*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq$$

$$C \exp \lambda_{\frac{q-p}{2}-1}(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\| \quad (7.1)$$

for $q - p = 0, 1, 2$

$$|(\pi(g)u, v)| \leq C \phi_{\lambda_0}(g) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\|.$$

Proof. It is clear that $r_K = \left[\frac{p}{2} \right] + \left[\frac{q}{2} \right] < p + q$ and $l_K < p^2 - p + q^2 - q$. Our assertion follows from Theorem 5.2. \square

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