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## AN ASYMPTOTIC COMPARISON OF TWO TIME-HOMOGENEOUS PAM MODELS

HYUN-JUNG KIM\* AND SERGEY VLADIMIR LOTOTSKY

ABSTRACT. Both Wick-Itô-Skorokhod and Stratonovich interpretations of the Parabolic Anderson model (PAM) lead to solutions that are real analytic as functions of the noise intensity  $\varepsilon$ , and, in the limit  $\varepsilon \rightarrow 0$ , the difference between the two solutions is of order  $\varepsilon^2$  and is non-random.

### 1. Introduction

Let  $W = W(x)$ ,  $x \in [0, \pi]$  be a standard Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . With no loss of generality, we assume that all realizations of  $W$  are in  $\mathcal{C}^{1/2-}((0, \pi))$ , that is, Hölder continuous of every order less than  $1/2$ .

Consider the equations

$$\begin{aligned} \frac{\partial u_\diamond(t, x; \varepsilon)}{\partial t} &= \frac{\partial^2 u_\diamond(t, x; \varepsilon)}{\partial x^2} + \varepsilon u_\diamond(t, x; \varepsilon) \diamond \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ u_\diamond(t, 0; \varepsilon) &= u_\diamond(t, \pi; \varepsilon) = 0, \quad u_\diamond(0, x; \varepsilon) = \varphi(x), \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} \frac{\partial u_\circ(t, x; \varepsilon)}{\partial t} &= \frac{\partial^2 u_\circ(t, x; \varepsilon)}{\partial x^2} + \varepsilon u_\circ(t, x; \varepsilon) \circ \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ u_\circ(t, 0; \varepsilon) &= u_\circ(t, \pi; \varepsilon) = 0, \quad u_\circ(0, x; \varepsilon) = \varphi(x). \end{aligned} \quad (1.2)$$

Equation (1.1) is the Wick-Itô-Skorokhod formulation of the parabolic Anderson model with potential  $\varepsilon \dot{W}$ ; equation (1.2) is the corresponding Stratonovich (or geometric rough path) formulation. These equations, with  $\varepsilon = 1$ , are studied in [1] and [2], respectively.

The objective of the paper is to show that

- The solutions of (1.1) and (1.2) are real-analytic functions of  $\varepsilon$ : with suitable functions  $u_\diamond^{(n)}$ , and  $u_\circ^{(n)}$ , the equalities

$$u_\diamond(t, x; \varepsilon) = u_\diamond(t, x; 0) + \sum_{n=1}^{\infty} \varepsilon^n u_\diamond^{(n)}(t, x) \quad (1.3)$$

$$u_\circ(t, x; \varepsilon) = u_\circ(t, x; 0) + \sum_{n=1}^{\infty} \varepsilon^n u_\circ^{(n)}(t, x) \quad (1.4)$$

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- hold for all  $t > 0$ ,  $x \in [0, \pi]$ ,  $\varepsilon > 0$ , and every realization of  $W$ ;
- The first two terms in (1.3) and (1.4) are the same so that

$$|u_\circ(t, x; \varepsilon) - u_\circ(t, x; \varepsilon)| = O(\varepsilon^2), \quad \varepsilon \rightarrow 0, \quad (1.5)$$

for all  $t > 0$  and  $x \in [0, \pi]$ , and every realization of  $W$ .

Equalities (1.3) and (1.4) are in the spirit of [5]. Equality (1.5) is similar to [9, Proposition 4.1]; see also [8].

The precise statement of the main result is in Section 2, and the proof is in Sections 3, 4, and 5.

## 2. The Main Result

Denote by  $p = p(t, x, y)$  the heat semigroup on  $[0, \pi]$  with zero boundary conditions:

$$p(t, x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \sin(kx) \sin(ky), \quad t > 0, \quad x, y \in [0, \pi]. \quad (2.1)$$

Let  $\varphi = \varphi(x)$  be a continuous function on  $[0, \pi]$ , and let  $u = u(t, x)$  be the solution of the heat equation

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= \frac{\partial^2 u(t, x)}{\partial x^2}, \quad t > 0, \quad 0 < x < \pi, \\ u(t, 0) &= u(t, \pi) = 0, \quad u(0, x) = \varphi(x), \end{aligned} \quad (2.2)$$

that is,

$$u(t, x) = \int_0^\pi p(t, x, y) \varphi(y) dy. \quad (2.3)$$

Next, define the function  $\mathbf{u} = \mathbf{u}(t, x)$  by

$$\mathbf{u}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) u(s, y) dW(y) ds. \quad (2.4)$$

That is,  $\mathbf{u}$  is the mild solution of

$$\begin{aligned} \frac{\partial \mathbf{u}(t, x)}{\partial t} &= \frac{\partial^2 \mathbf{u}(t, x)}{\partial x^2} + u(t, x) \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ \mathbf{u}(t, 0) &= \mathbf{u}(t, \pi) = \mathbf{u}(0, x) = 0. \end{aligned} \quad (2.5)$$

Because  $u$  is non-random, no stochastic integral is required to define  $\mathbf{u}$ .

**Proposition 2.1.** *If  $\varphi \in \mathcal{C}((0, \pi))$ , then  $\mathbf{u}$  is a continuous function of  $t$  and  $x$  for all  $t > 0$  and  $x \in [0, \pi]$ .*

*Proof.* This follows by the Kolmogorov continuity criterion:  $\mathbf{u}$  is a Gaussian random field and direct computations show

$$\mathbb{E}(\mathbf{u}(t + \tau, x + h) - \mathbf{u}(t, x))^2 \leq C(t)(\tau^2 + h^2)^{1/4} \max_{x \in [0, \pi]} |\varphi(x)|;$$

cf. [1, Sections 6 and 7]. □

Next, define the functions  $u_\diamond^{(n)} = u_\diamond^{(n)}(t, x)$ ,  $n = 0, 1, 2 \dots$ ,  $t \geq 0$ ,  $x \in [0, \pi]$ , by  $u_\diamond^{(0)}(t, x) = u(t, x)$ , and, for  $n \geq 1$ ,  $u_\diamond^{(n)}$  is the mild solution of

$$\begin{aligned} \frac{\partial u_\diamond^{(n)}(t, x)}{\partial t} &= \frac{\partial^2 u_\diamond^{(n)}(t, x)}{\partial x^2} + u_\diamond^{(n-1)}(t, x) \diamond \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ u_\diamond^{(n)}(t, 0) &= u_\diamond^{(n)}(t, \pi) = u_\diamond^{(n)}(0, x) = 0. \end{aligned} \quad (2.6)$$

In other words,

$$u_\diamond^{(n)}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) u_\diamond^{(n-1)}(s, y) \diamond dW(y) ds, \quad n \geq 1, \quad (2.7)$$

and, in particular,  $u_\diamond^{(1)} = u$ .

Similarly, define the functions  $u_\circ^{(n)} = u_\circ^{(n)}(t, x)$ ,  $n = 0, 1, 2 \dots$ ,  $t \geq 0$ ,  $x \in [0, \pi]$ , by  $u_\circ^{(0)}(t, x) = u(t, x)$ , and, for  $n \geq 1$ ,  $u_\circ^{(n)}$  is the mild solution of

$$\begin{aligned} \frac{\partial u_\circ^{(n)}(t, x)}{\partial t} &= \frac{\partial^2 u_\circ^{(n)}(t, x)}{\partial x^2} + u_\circ^{(n-1)}(t, x) \circ \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ u_\circ^{(n)}(t, 0) &= u_\circ^{(n)}(t, \pi) = u_\circ^{(n)}(0, x) = 0. \end{aligned} \quad (2.8)$$

In other words,

$$u_\circ^{(n)}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) u_\circ^{(n-1)}(s, y) \circ dW(y) ds, \quad n \geq 1, \quad (2.9)$$

and, in particular,  $u_\circ^{(1)} = u$ .

The main result of the paper can now be stated as follows.

**Theorem 2.2.** *Let  $\varphi \in \mathcal{C}((0, \pi))$ . Then*

- (1) *Equality (1.3) holds with  $u_\diamond^{(n)}$  from (2.7).*
- (2) *Equality (1.4) holds with  $u_\circ^{(n)}$  from (2.9).*
- (3) *Equality (1.5) holds and*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \left( u_\circ(t, x; \varepsilon) - u_\diamond(t, x; \varepsilon) \right) = \int_0^\pi p^{(3)}(t, x, z) \varphi(z) dz, \quad (2.10)$$

where

$$p^{(3)}(t, x, z) = \int_0^\pi \int_0^t \int_0^s p(t-s, x, y) p(s-r, y, y) p(r, y, z) dr ds dy.$$

The proof is carried out in the following three sections.

### 3. The Wick-Itô-Skorokhod Case

The objective of this section is the proof of (1.3).

The solution of (1.1) is defined as a chaos solution (cf. [6, Theorems 3.10]). It is a continuous in  $(t, x)$  function (cf. [1, Sections 6 and 7]) and has a representation as a series

$$u_\circ(t, x; \varepsilon) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x; \varepsilon) \xi_\alpha, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{J} &= \left\{ \boldsymbol{\alpha} = (\alpha_k, k \geq 1) : \alpha_k \in \{0, 1, 2, \dots\}, \sum_k \alpha_k < \infty \right\}, \\ \xi_{\boldsymbol{\alpha}} &= \prod_k \left( \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right), \quad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \\ \xi_k &= \int_0^\pi \mathbf{m}_k(x) dW(x), \quad \mathbf{m}_k(x) = \sqrt{2/\pi} \sin(kx), \end{aligned}$$

and, with  $|\boldsymbol{\alpha}| = \sum_k \alpha_k$ ,  $|(\mathbf{0})| = 0$ ,  $|\boldsymbol{\epsilon}(k)| = 1$ ,

$$\begin{aligned} u_{(\mathbf{0})}(t, x; \varepsilon) &= u(t, x), \\ u_{\boldsymbol{\alpha}}(t, x; \varepsilon) &= \varepsilon \sum_k \sqrt{\alpha_k} \int_0^t \int_0^\pi p(t-s, x, y) u_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}(s, y; \varepsilon) \mathbf{m}_k(y) dy ds; \end{aligned}$$

see [1, Section 3] for details. In particular,

$$\sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t, x; 1) \xi_{\boldsymbol{\alpha}} = \sum_{|\boldsymbol{\alpha}|=n-1} \int_0^t \int_0^\pi p(t-s, x, y) u_{\boldsymbol{\alpha}}(s, y; 1) \xi_{\boldsymbol{\alpha}} \diamond dW(y) ds. \quad (3.2)$$

Comparing (2.7) with (3.2) shows that

$$u_{\diamond}^{(n)}(t, x) = \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t, x; 1) \xi_{\boldsymbol{\alpha}}. \quad (3.3)$$

In other words, (1.3) is equivalent to (3.1).

Next,

$$\begin{aligned} \mathbb{E}|u_{\diamond}^{(n)}(t, x)| &= \mathbb{E} \left| \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t, x; 1) \xi_{\boldsymbol{\alpha}} \right| \leq \left( \mathbb{E} \left( \sum_{|\boldsymbol{\alpha}|=n} u_{\boldsymbol{\alpha}}(t, x; 1) \xi_{\boldsymbol{\alpha}} \right)^2 \right)^{1/2} \\ &= \left( \sum_{|\boldsymbol{\alpha}|=n} |u_{\boldsymbol{\alpha}}(t, x; 1)|^2 \right)^{1/2} \leq C^n(t) n^{-n/4} \sup_{x \in (0, \pi)} |\varphi(x)|^{1/2}, \end{aligned}$$

where the last inequality follows by [1, Theorem 4.1]. As a result,

$$\sum_{n \geq 0} \varepsilon^n \mathbb{E}|u_{\diamond}^{(n)}(t, x)| < \infty,$$

that is, the series converges absolutely with probability one for all  $t > 0$ ,  $x \in [0, \pi]$ , and  $\varepsilon \in \mathbb{R}$ .

*This concludes the proof of (1.3).*

#### 4. The Stratonovich Case

The objective of this section is to prove (1.4). To simplify the presentation, we use the following notations:

$$\Lambda = (-\Delta)^{1/2}, \quad H^\theta = \Lambda^{-\theta}(L_2((0, \pi))), \quad \|\cdot\|_\theta = \|\Lambda^\theta \cdot\|_{L_2((0, \pi))}, \quad \theta \in \mathbb{R},$$

$$p * g(t, s, x) = \int_0^\pi p(t-s, x, y)g(s, y) dy,$$

where  $\Delta$  is the Laplace operator on  $(0, \pi)$  with zero boundary conditions and  $p$  is the heat kernel (2.1).

By direct computation,

$$\|p * g\|_\gamma(t, s) \leq C_{T, \theta} (t-s)^{-\theta/2} \|g\|_{\gamma-\theta}(s), \quad \theta > 0, \quad \gamma \in \mathbb{R}, \quad t \in (s, T]; \quad (4.1)$$

cf. [3, Lemma 7.3].

Consider the equation

$$\frac{\partial v(t, x)}{\partial t} = \frac{\partial^2 v(t, x)}{\partial x^2} + v(t, x) \circ \dot{W}(x) + f(t, x) \circ \dot{W}(x), \quad t > 0, \quad (4.2)$$

which includes (1.2) as a particular case. By definition, a solution (classical, mild, generalized, etc.) of (4.2) is a suitable limit, as  $\epsilon \rightarrow 0$ , of the corresponding solutions of

$$\frac{\partial v_\epsilon(t, x)}{\partial t} = \frac{\partial^2 v_\epsilon(t, x)}{\partial x^2} + v_\epsilon(t, x)V_\epsilon(x) + f(t, x)V_\epsilon(x), \quad t > 0, \quad (4.3)$$

where  $V_\epsilon$ ,  $\epsilon > 0$  are smooth functions on  $[0, \pi]$  such that

$$\sup_\epsilon \left\| \int V_\epsilon \right\|_{C^{1/2}} < \infty, \quad \lim_{\epsilon \rightarrow 0} \sup_{x \in [0, \pi]} \left| \int_0^x V_\epsilon(y) dy - W(x) \right| = 0.$$

By [2, Theorem 3.5],

- The generalized solution of (4.2) is the same as the generalized solution of the equation

$$v_t = \left( v_x + W(x)v + W(x)f \right)_x - W(x)v_x - W(x)f_x; \quad (4.4)$$

the subscripts  $t$  and  $x$ , as in  $f_x$ , denote the corresponding partial derivatives;

- The mild solution of (4.2) is the solution of the integral equation

$$v(t, x) = \int_0^t p * ((f + v)W)_x(t, s, x) ds - \int_0^t p * ((f + v)_x W)(t, s, x) ds + \int_0^\pi p(t, x, y)\varphi(y) dy. \quad (4.5)$$

On the one hand, mild and generalized solutions of (4.2) are the same: just use  $\mathbf{m}_k$  as the test functions. On the other hand, different definitions of the solution lead to different regularity results.

By standard parabolic regularity, if  $\varphi \in H^0$  and  $f \in L_2((0, T); H^\gamma)$ ,  $\gamma \in (1/2, 1]$ , then there is a unique generalized solution of (4.4) in the normal triple  $(H^1, H^0, H^{-1})$  and

$$v \in L_2((0, T); H^1) \cap \mathcal{C}((0, T); H^0) \quad (4.6)$$

for every realization of  $W$ ; cf. [4, Theorem 3.4.1]. Note that we cannot claim  $v \in \mathcal{C}((0, T); H^\gamma)$  even if  $\varphi \in H^\gamma$ . In fact, because  $W \in \mathcal{C}^{1/2-}$  is a point-wise multiplier in  $H^\gamma$  for  $\gamma \in (-1/2, 1/2)$  [3, Lemma 5.2], an attempt to find a traditional regularity result for equation (4.4) in a normal triple  $(H^{r+1}, H^r, H^{r-1})$  leads to an irreconcilable pair of restrictions on  $r$ : to have  $Wf \in L_2((0, T); H^r)$  we need  $r < 1/2$ , whereas to have  $Wf_x \in L_2((0, T); H^{r-1})$  we need  $r - 1 > -1/2$  or  $r > 1/2$ .

Accordingly, to derive a bound on  $\|v\|_\gamma(t)$  for  $t > 0$ , we use the mild formulation (4.5).

**Proposition 4.1.** *Let  $\gamma \in (1/2, 1)$ ,  $f \in L_2((0, T); H^\gamma)$ ,  $\varphi \in H^0$ , and let  $v$  be the mild solution of (4.2) with  $v|_{t=0} = \varphi$ . Then, for every  $T > 0$  and every realization of  $W$ , there exists a number  $C_\circ$  such that*

$$\|v\|_\gamma(t) \leq C_\circ \left( t^{-\gamma/2} \|\varphi\|_0 + \int_0^t (t-s)^{-\gamma} \|f\|_\gamma(s) ds \right). \quad (4.7)$$

*Proof.* Throughout the proof,  $C$  denotes a number depending on  $\gamma$ ,  $T$ , and the norm of  $W$  in the space  $\mathcal{C}^{1-\gamma}$ . The value of  $C$  can change from one instance to another. With no loss of generality, we assume that  $\varphi$  and  $f$  are smooth functions with compact support.

To begin, let us show that if  $V$  is the mild solution of

$$\frac{\partial V(t, x)}{\partial t} = \frac{\partial^2 V(t, x)}{\partial x^2} + f(t, x) \circ \dot{W}(x), \quad t > 0,$$

$V(t, 0) = V(t, \pi) = 0$ ,  $V|_{t=0} = \varphi$ , then

$$\|V\|_\gamma(t) \leq Ct^{-\gamma/2} \|\varphi\|_0 + C \int_0^t (t-s)^{-\gamma} \|f\|_\gamma(s) ds. \quad (4.8)$$

Indeed, by (4.5),

$$V(t, x) = \int_0^t p * (fW)_x(t, s, x) ds - \int_0^t p * (f_x W)(t, s, x) ds + \int_0^\pi p(t, x, y) \varphi(y) dy.$$

Using (4.1) with  $\theta = \gamma$ ,

$$\left\| \int_0^\pi p(t, \cdot, y) \varphi(y) dy \right\|_\gamma \leq Ct^{-\gamma/2} \|\varphi\|_0.$$

Then

$$\|V\|_\gamma(t) \leq \int_0^t \|p * (fW)_x\|_\gamma(t, s) ds + \int_0^t \|p * (f_x W)\|_\gamma(t, s) ds + Ct^{-\gamma/2} \|\varphi\|_0. \quad (4.9)$$

To estimate the first term on the right hand side of (4.9), we use (4.1) with  $\theta = 2\gamma$ . Then

$$\|p * (fW)_x\|_\gamma(t, s) \leq C(t-s)^{-\gamma} \|(fW)_x\|_{-\gamma}(s) \leq C(t-s)^{-\gamma} \|fW\|_{1-\gamma}(s),$$

and, because  $W \in \mathcal{C}^{1/2-}((0, \pi))$  is a (point-wise) multiplier in  $H^{1-\gamma}$ ,

$$\|fW\|_{1-\gamma}(s) \leq C_W \|f\|_{1-\gamma}(s);$$

recall that  $0 < 1 - \gamma < 1/2$ . Finally, as  $1 - \gamma < \gamma$ ,

$$\|p * (fW)_x\|_{\gamma}(t, s) \leq C(t-s)^{-\gamma} \|f\|_{\gamma}(s). \quad (4.10)$$

To estimate the second term on the right hand side of (4.9), we use (4.1) with  $\theta = 1$ . Then

$$\|p * (f_x W)\|_{\gamma}(t, s) \leq \frac{C}{\sqrt{t-s}} \|f_x W\|_{\gamma-1}(s) \leq \frac{C}{\sqrt{t-s}} \|f_x\|_{\gamma-1}(s),$$

that is,

$$\|p * (f_x W)\|_{\gamma}(t, s) \leq C(t-s)^{-1/2} \|f\|_{\gamma}(s). \quad (4.11)$$

To establish (4.8), we now combine (4.9), (4.10), and (4.11), keeping in mind that  $(t-s)^{-1/2} \leq C(t-s)^{-\gamma}$  because  $\gamma > 1/2$ .

Next, (4.8) applied to (4.2) implies

$$\|v\|_{\gamma}(t) \leq C \int_0^t (t-s)^{-\gamma} \|v\|_{\gamma}(s) ds + C \int_0^t (t-s)^{-\gamma} \|f\|_{\gamma}(s) ds + Ct^{-\gamma/2} \|\varphi\|_0,$$

and then a generalization of the Gronwall inequality (e.g. [10, Corollary 2]) leads to (4.7).  $\square$

**Corollary 4.2.** *If  $\varphi \in H^0$  and  $\gamma \in (1/2, 1)$ , then, for every  $T > 0$ ,  $a > 0$ , and every realization of  $W$ , there exists a number  $\tilde{C}_\circ$  such that the mild solution of (1.2) satisfies*

$$\sup_{|\varepsilon| < a} \|u_\circ(t, \cdot, \varepsilon)\|_{\gamma} \leq \tilde{C}_\circ t^{-\gamma/2} \|\varphi\|_0, \quad t \in (0, T]. \quad (4.12)$$

Next, define the functions  $u_\circ^{(n), \varepsilon} = u_\circ^{(n), \varepsilon}(t, x)$ ,  $n = 0, 1, 2, \dots$ ,  $t \geq 0$ ,  $x \in [0, 1]$ ,  $\varepsilon \in \mathbb{R}$ , by  $u_\circ^{(0), \varepsilon}(t, x) = u_\circ(t, x; \varepsilon)$  and, for  $n \geq 1$ ,

$$\begin{aligned} \frac{\partial u_\circ^{(n), \varepsilon}(t, x)}{\partial t} &= \frac{\partial^2 u_\circ^{(n), \varepsilon}(t, x)}{\partial x^2} + \varepsilon u_\circ^{(n), \varepsilon}(t, x) \circ \dot{W}(x) + u_\circ^{(n-1), \varepsilon}(t, x) \circ \dot{W}(x), \\ u_\circ^{(n), \varepsilon}(t, 0) &= u_\circ^{(n), \varepsilon}(t, \pi) = 0, \quad u_\circ^{(n), \varepsilon}(0, x) = 0. \end{aligned} \quad (4.13)$$

In particular,

$$u_\circ^{(n), 0}(t, x) = u_\circ^{(n)}(t, x).$$

Note that all equations in (4.13) are of the form (4.2).

**Proposition 4.3.** *If  $\varphi \in H^0$ , then, for every  $\gamma \in (1/2, 2/3)$  and every realization of  $W$ ,*

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \frac{1}{(\varepsilon - \varepsilon_0)^n} \left\| u_\circ(t, \cdot; \varepsilon) - \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k u_\circ^{(k), \varepsilon_0}(t, \cdot) \right\|_{\gamma} = 0, \quad (4.14)$$

$n \geq 0$ ,  $\varepsilon_0 \in \mathbb{R}$ ,  $t \geq 0$ .



*Proof.* Throughout the proof,  $C$  denotes a number depending on  $\gamma$ ,  $T$ ,  $\varepsilon_0$ , and the norm of  $W$  in the space  $\mathcal{C}^{1-\gamma}$ . Define

$$\mathbf{v}_\varepsilon^{(n)}(t, x) = \frac{1}{(\varepsilon - \varepsilon_0)^n} \left( u_\circ(t, x; \varepsilon) - \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k u_\circ^{(k), \varepsilon_0}(t, x) \right). \quad (4.15)$$

By (1.2),

$$\begin{aligned} \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k \frac{\partial u_\circ^{(k), \varepsilon_0}(t, x)}{\partial t} &= \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k \frac{\partial^2 u_\circ^{(k), \varepsilon_0}(t, x)}{\partial x^2} \\ &\quad + \varepsilon_0 \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k u_\circ^{(k), \varepsilon_0}(t, x) \circ \dot{W}(x) \\ &\quad + (\varepsilon - \varepsilon_0) \sum_{k=1}^n (\varepsilon - \varepsilon_0)^{k-1} u_\circ^{(k-1), \varepsilon_0}(t, x) \circ \dot{W}(x), \end{aligned} \quad (4.16)$$

so that

$$\begin{aligned} \frac{\partial \mathbf{v}_\varepsilon^{(0)}(t, x)}{\partial t} &= \frac{\partial^2 \mathbf{v}_\varepsilon^{(0)}(t, x)}{\partial x^2} + \varepsilon_0 \mathbf{v}_\varepsilon^{(0)}(t, x) \circ \dot{W}(x) + (\varepsilon - \varepsilon_0) u_\circ(t, x; \varepsilon) \circ \dot{W}(x), \\ \frac{\partial \mathbf{v}_\varepsilon^{(n)}(t, x)}{\partial t} &= \frac{\partial^2 \mathbf{v}_\varepsilon^{(n)}(t, x)}{\partial x^2} + \varepsilon_0 \mathbf{v}_\varepsilon^{(n)}(t, x) \circ \dot{W}(x) + \mathbf{v}_\varepsilon^{(n-1)}(t, x) \circ \dot{W}(x), \quad n \geq 1, \end{aligned} \quad (4.17)$$

$\mathbf{v}_\varepsilon^{(n)}(0, x) = 0$ ,  $n \geq 0$ , and (4.14) becomes

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \|\mathbf{v}_\varepsilon^{(n)}\|_\gamma(t) = 0, \quad n \geq 0, \quad t \geq 0, \quad \varepsilon_0 \in \mathbb{R}. \quad (4.18)$$

Note that all equations in (4.17) are of the form (4.2), and (4.18) trivially holds for  $t = 0$ . Accordingly, combining the second equation in (4.17) with (4.7),

$$\|\mathbf{v}_\varepsilon^{(n)}\|_\gamma(t) \leq C \int_0^t (t-s)^{-\gamma} \|\mathbf{v}_\varepsilon^{(n-1)}\|_\gamma(s) ds,$$

$n \geq 1$ , and then, for  $t > 0$ , (4.18) follows by induction: for  $n = 0$ , (4.12) yields

$$\begin{aligned} \|\mathbf{v}_\varepsilon^{(0)}\|_\gamma(t) &\leq |\varepsilon - \varepsilon_0| C \int_0^t (t-s)^{-\gamma} \|u_\circ(s, \cdot, \varepsilon)\|_\gamma ds \\ &\leq C |\varepsilon - \varepsilon_0| \|\varphi\|_0 \int_0^t (t-s)^{-\gamma} s^{-\gamma/2} ds \leq C |\varepsilon - \varepsilon_0| \|\varphi\|_0 t^{1-(3/2)\gamma} \rightarrow 0, \quad \varepsilon \rightarrow \varepsilon_0; \end{aligned}$$

similarly, for  $n \geq 1$ ,

$$\|\mathbf{v}_\varepsilon^{(n)}\|_\gamma(t) \leq C^{(n)} |\varepsilon - \varepsilon_0| \|\varphi\|_0,$$

because  $1 - (3/2)\gamma > 0$ . □

**Proposition 4.4.** *If  $\varphi \in H^0$  and  $\gamma \in (1/2, 1)$ , then*

$$\lim_{n \rightarrow \infty} c^n \sup_{|\varepsilon| < a} \|u_\circ^{(n), \varepsilon}\|_\gamma(t) = 0, \quad t \geq 0, \quad (4.19)$$

for all  $c > 0$ ,  $a > 0$ , and every realization of  $W$ .

*Proof.* Throughout this proof,  $C$  denotes a number depending on  $\gamma$ ,  $T$ ,  $a$ , and the norm of  $W$  in the space  $\mathcal{C}^{1-\gamma}$ .

Combining (4.13) and (4.7),

$$\|u_{\circ}^{(n),\varepsilon}\|_{\gamma}(t) \leq C \int_0^t (t-s)^{r-1} \|u_{\circ}^{(n-1),\varepsilon}\|_{\gamma}(s) ds.$$

By iteration and (4.12), with  $r = 1 - \gamma > 0$ ,

$$\begin{aligned} \sup_{|\varepsilon| < a} \|u_{\circ}^{(n),\varepsilon}\|_{\gamma}(t) &\leq C^n \|\varphi\|_0 \\ &\times \int_0^t \int_0^{s_{n-1}} \dots \int_0^{s_2} (t-s_n)^{r-1} (s_n-s_{n-1})^{r-1} \dots (s_2-s_1)^{r-1} s_1^{-\gamma/2} ds_1 \dots ds_n \\ &= C^n \|\varphi\|_0 \frac{(\Gamma(r))^n \Gamma(1-(\gamma/2))}{\Gamma(nr+1)} t^{nr-(\gamma/2)}, \end{aligned}$$

where  $\Gamma$  is the Gamma function

$$\Gamma(y) = \int_0^{\infty} t^{y-1} e^{-t} dt.$$

Then (4.19) follows by the Stirling formula.  $\square$

Equality (1.4) now follows:

- By the Sobolev embedding theorem, every element, or equivalence, class from  $H^{\gamma}$ ,  $\gamma > 1/2$ , has a representative that is a continuous function on  $[0, \pi]$ ;
- By Proposition 4.3 and the Taylor formula,

$$u_{\circ}(t, x) = u(t, x) + \sum_{k=1}^n u_{\circ}^{(k),0}(t, x) \varepsilon^k + R_n(t, x);$$

- By Proposition 4.4,

$$\lim_{n \rightarrow \infty} R_n(t, x) = 0.$$

This concludes the proof of (1.4).

### 5. The Correction Term

The objective of this section is the proof of (2.10).

Using (1.3) and (1.4), and remembering that  $u_{\circ}^{(1)} = u_{\diamond}^{(1)} = \mathbf{u}$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \left( u_{\circ}(t, x; \varepsilon) - u_{\circ}(t, x; \varepsilon) \right) &= u_{\circ}^{(2)}(t, x) - u_{\diamond}^{(2)}(t, x) \\ &= \int_0^t \int_0^{\pi} p(t-s, x, y) (\mathbf{u}(s, y) \circ dW(y) - \mathbf{u}(s, y) \diamond dW(y)) ds. \end{aligned} \tag{5.1}$$

By definition,

$$\xi_k \diamond \xi_n = \begin{cases} \xi_k \xi_n, & k \neq n, \\ \xi_n^2 - 1, & k = n, \end{cases}$$

and therefore

$$\xi_k \xi_n - \xi_k \diamond \xi_n = \begin{cases} 0 & k \neq n, \\ 1, & k = n, \end{cases} \quad (5.2)$$

Then (5.2) and [7, Theorem 3.1.2] imply that, for a function  $f = f(x)$  of the form

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \xi_k,$$

with  $f_k$  non-random and satisfying

$$\sum_k \int_0^\pi |f_k(x)| dx < \infty, \quad (5.3)$$

the following equality holds:

$$\int_0^\pi f(x) \circ dW(x) - \int_0^\pi f(x) \diamond dW(x) = \sum_{k=1}^{\infty} \left( \int_0^\pi f_k(x) \mathbf{m}_k(x) dx \right). \quad (5.4)$$

Condition (5.3) ensures that the sum on the right-hand side of (5.4) converges absolutely.

Next, recall that, by (2.4),

$$\mathbf{u}(s, y) = \sum_{k=1}^{\infty} \left( \int_0^\pi \int_0^s p(s-r, y, z) u(r, z) \mathbf{m}_k(z) dr dz \right) \xi_k.$$

For fixed  $s \in [0, T]$  and  $y \in [0, \pi]$ , define

$$g(z) = \int_0^s p(s-r, y, z) u(r, z) dr, \quad g_k = \int_0^\pi g(z) \mathbf{m}_k(z) dz.$$

Then

$$\mathbf{u} = \sum_{k=1}^{\infty} g_k \xi_k,$$

and (5.3) in this case will follow from uniform, in  $(s, y)$  convergence of

$$\sum_{k=1}^{\infty} |g_k|,$$

which, by Bernstein's theorem [11, Theorem VI.3-1], will, in turn, follow from

$$|g(z+h) - g(z)| \leq Ch^\delta \quad (5.5)$$

with  $\delta \in (1/2, 1)$  and  $C$  independent of  $s, y, z$ .

Recall that

$$u(r, z) = \sum_{k=1}^{\infty} \varphi_k e^{-k^2 r} \mathbf{m}_k(z), \quad \varphi_k = \int_0^\pi \varphi(x) \mathbf{m}_k(x) dx,$$

and, by integral comparison,

$$\sum_{k=1}^{\infty} k^p e^{-k^2 t} \leq \frac{C(p)}{t^{(1+p)/2}}, \quad p \geq 0.$$

Also,

$$|\sin(k(z+h)) - \sin(kz)| \leq k^\delta h^\delta, \delta \in (0, 1),$$

and the maximum principle implies  $|u(r, z)| \leq C$ . Then

$$p(s-r, y, z) \leq \frac{C}{\sqrt{s-r}}, \quad |p(s-r, y, z+h) - p(s-r, y, z)| \leq \frac{Ch^\delta}{(s-r)^{(1+\delta)/2}},$$

$$|u(r, z+h) - u(r, z)| \leq \frac{Ch^\delta}{r^{(1+\delta)/2}},$$

and (5.5) follows because

$$\int_0^s \frac{dr}{(r(s-r))^{(1+\delta)/2}} < \infty$$

for  $\delta \in (1/2, 1)$ .

We now apply (5.4) to (5.1):

$$\begin{aligned} & \int_0^t \int_0^\pi p(t-s, x, y) (\mathbf{u}(s, y) \circ dW(y) - \mathbf{u}(s, y) \diamond dW(y)) ds \\ &= \int_0^t \int_0^\pi \sum_{n=1}^\infty \left( \int_0^\pi \left( \int_0^s p(s-r, y, z) u(r, z) dr \right) \mathbf{m}_n(z) dz \right) \mathbf{m}_n(y) p(t-s, x, y) dy ds \\ &= \int_0^\pi \int_0^t \int_0^s p(t-s, x, y) p(s-r, y, y) u(r, y) dr ds dy, \end{aligned}$$

which, in view of (2.3) and the Fubini theorem, is the same as (2.10).

*This concludes the proof of (2.10).*

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