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THE TWO-SUM OF RIBBON GRAPHS AND THE BOLLOBA´S-RIORDAN-TUTTE POLYNOMIAL

Brittan Farmer

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Abstract. For oriented ribbon graphs, Bollobás and Riordan have generalized the Tutte polynomial to the Bollobás-Riordan-Tutte (BRT) polynomial. We extend the operation of the two-sum of graphs to the two-sum of ribbon graphs pointed by an oriented edge. Following Las Vergnas, we define a five-variable pointed BRT polynomial and study its properties. As an application, we compute the BRT polynomial of two-sums.

1. Introduction

A graph consists of vertices and edges; each edge meets either one or two vertices. There are two important numbers attached to a graph: rank and nullity. These numbers are used to calculate the Tutte polynomial of a graph. Not all graphs can be drawn in the plane so that edges meet only at vertices. Sometimes a sphere with one or more handles is necessary. A ribbon graph is a graph drawn on a surface so that the complementary regions are disks. Counting the handles of this surface gives the genus of the ribbon graph. There is a polynomial on ribbon graphs called the Bollobás-Riordan-Tutte (BRT) polynomial, which in addition to rank and nullity incorporates the ribbon graph's genus.

Two graphs can be combined using the operation of the two-sum. It is well known how the Tutte polynomial of the result is related to the pointed Tutte polynomials of the original graphs [3, 4]. This thesis describes how to extend this result to ribbon graphs and the BRT polynomial.

In Section 2, ribbon graphs and their associated statistics and operations are properly defined. The BRT polynomial is given explicitly. In Section 3, the pointed BRT polynomial is introduced and its properties are presented. In Section 4, the pointed BRT polynomial is used to calculate the BRT polynomial of the two-sum of two ribbon graphs. An example is presented in Section 5, and a future research direction is discussed in Section 6.

2. Ribbon graphs

2.1. **Definition of ribbon graphs.** We will begin our discussion of ribbon graphs by defining graphs.

Definition 2.1. A *graph* is a vertex set V, an edge set E, and an incidence relation $R \subset V \times E$ such that each edge in E is in either one or two pairs in R. If $(v, e) \in R$, then we say edge e is incident to vertex v. If an edge $e \in E$ is incident to only one vertex $v \in V$, then we call it a *loop*.

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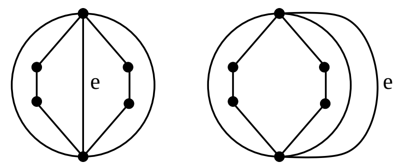

FIGURE 1. Two non-isomorphic embeddings of the same graph.

Definition 2.2. A *topological graph* is a one-dimensional cell complex, that is, a topological space X with a distinguished set of isolated points V and a set of disjoint intervals $\{I_i\}$ each homeomorphic to $(0,1)$ such that any homeomorphism $\pi_i : (0,1) \to I_i$ can be extended to a continuous map $\tilde{\pi}_i : [0,1] \to X$ with $\partial I_i := \tilde{\pi}_i(\{0,1\}) \subseteq V$. A vertex v is then incident to an edge I_i if $v \in \partial I_i$.

We embed a topological graph in a surface by viewing its vertices and intervals, or edges, as subsets of the surface. There are many ways to embed a graph because the edges are not incident to vertices in any particular order. See Figure 1 for two embeddings of the same graph. The only difference between the embeddings is the position of the edge e. The embedding on the left has four 4-valent faces and one 2-valent face. (The graphs are actually embedded on the sphere, so the outer region is a face.) The embedding on the right has one 6-valent face, two 4-valent faces, and two 2-valent faces. Since the embeddings have different face structures, we say that they are non-isomorphic. Some embeddings, such as those in Figure 1, can be viewed as ribbon graphs.

Definition 2.3. A *topological ribbon graph* is a graph with a fixed embedding in an oriented surface such that the complement is a disjoint union of polygons.

In this paper, we allow for polygons that have only two, one, or even, as in the case of the punctured sphere, no sides.

If the ribbon graph is disconnected, each connected component is embedded in a connected oriented surface.

In addition to this topological definition, there is also an equivalent combinatorial definition of ribbon graphs.

Definition 2.4. Let B be a set. Let $\sigma_0, \sigma_1 \in Sym(B)$ such that $\sigma_1^2 = id.$ and $\sigma_1(b) \neq b$ for all $b \in B$, i.e. σ_1 is a fixed point free involution. Let $i \in \mathbb{N}$. Then $\mathbb{D} = (B; \sigma_0, \sigma_1; i)$ is a *ribbon graph.* (In order that the definition can also describe the graph with one vertex and no edges, we include i, which gives the number of isolated, or edgeless, vertices in the ribbon graph.) There is a third permuation associated to this ribbon graph, $\sigma_2 = \sigma_1^{-1}\sigma_0^{-1}$.

Remark. Since σ_1 is a fixed point free involution, the cardinality of B must be even.

The topological ribbon graph can be constructed from the combinatorial definition. Let m denote the number of cycles of σ_2 and k_j denote the length of the j-th cycle of σ_2. We will construct the ribbon graph from m polygons, each with k_j sides, and i punctured spheres. We label each polygon with the elements of the corresponding cycle in counterclockwise order. The counterclockwise orientation of the face induces an orientation on its sides. We place the labels so that they are inside the polygon, near to the initial vertex of the corresponding edge. Each side b of the polygons is identified with the side $\sigma_1(b)$ so

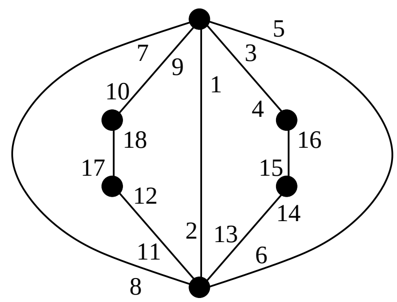

FIGURE 2. Constructing a topological ribbon graph from its permutations.

that the edges are oriented in opposite directions. The result is a ribbon graph. See Figure 2 for the topological ribbon graph defined by $B = \{1, ..., 18\}$, $\sigma_0 = (1\ 3\ 5\ 7\ 9)(2\ 11\ 8\ 6\ 13)(4\ 16)(10\ 18)(12\ 17)(14\ 15)$, $\sigma_1 = (1\ 2)...(2n-1\ 2n)...(17\ 18)$, $\sigma_2 = \sigma_1^{-1}\sigma_0^{-1} = (1\ 13\ 15\ 4)(2\ 9\ 18\ 12)(3\ 16\ 14\ 6)(5\ 8)(7\ 11\ 17\ 10)$, and $i = 0$. This ribbon graph is the same as the one on the left in Figure 1.

The orbits of σ_2 are clearly associated with the faces, or boundary components, of the ribbon graph. Notice that each edge is labelled by two elements of B, each label located at opposite endpoints of the edge. For this reason, we call the set B the set of half-edges of the ribbon graph. By reading off the labels as we travel counter-clockwise around each vertex, we recover the cycles of σ_0. Thus, σ_0 is called the vertex permuation of the ribbon graph.

To recover a regular graph from a ribbon graph, we define the vertex set V as the orbits of σ_0 and the edge set E as the orbits of σ_1. A vertex $v \in V$ is incident to an edge $e \in E$ if $v \cap e \neq \emptyset$ as a subset of B.

For a ribbon graph $\mathbb{D} = (B; \sigma_0, \sigma_1; i)$, we define a group $G_p(\sigma_0, \sigma_1)$ to be the subgroup of the permutation group of B generated by σ_0 and σ_1. We define a relation $\sim_k$ such that for $b_1, b_2 \in B$, $b_1 \sim_k b_2$ if there exists $\sigma \in G_p(\sigma_0, \sigma_1)$ such that $\sigma(b_1) = b_2$. We say that b_1 and b_2 are in the same component if and only if $b_1 \sim_k b_2$. It is easy to see that if b_1 and b_2 are in the same component, then there exists a path on the ribbon graph from b_1 to b_2, so b_1 and b_2 are on the same connected component. We can decompose B into disjoint component equivalence classes, and we denote the set of these classes by K.

We now define some statistics of the ribbon graph $\mathbb{D} = (B; \sigma_0, \sigma_1; i)$:

- $v(\mathbb{D})$ = number of vertices of $\mathbb{D}$ = number of orbits of $\sigma_0 + i = |V| + i$
- $e(\mathbb{D})$ = number of edges of $\mathbb{D}$ = number of orbits of $\sigma_1 = |E|$
- $bc(\mathbb{D})$ = number of boundary components of $\mathbb{D}$ = number of orbits of $\sigma_2 + i$
- $k(\mathbb{D})$ = number of connected components of $\mathbb{D} = |K| + i$

Next, we define some important operations on ribbon graphs.

Disjoint union. Suppose we have two dessins $\mathbb{D}_1 = (B_1; \sigma_0, \sigma_1; i)$ and $\mathbb{D}_2 = (B_2; \phi_0, \phi_1; j)$. We define their disjoint union by $\mathbb{D}_1 \amalg \mathbb{D}_2 := (B_1 \amalg B_2; \sigma_0 \amalg \phi_0, \sigma_1 \amalg \phi_1; i + j)$, where

$$\sigma_n \amalg \phi_n(b) = \begin{cases} \sigma_n(b) & \text{if } b \in B_1, \\ \phi_n(b) & \text{if } b \in B_2. \end{cases}$$

Now suppose $\mathbb{D} = (B; \sigma_0, \sigma_1; i)$ is a ribbon graph and $e \in E$, i.e. $e = \{b_1, b_2\}$, $b_1, b_2 \in B$, $\sigma_1(b_1) = b_2$.

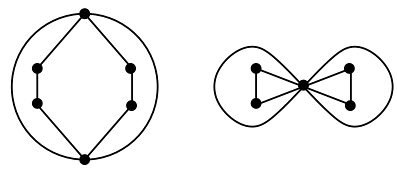

FIGURE 3. Edge deletion (left) and edge contraction (right).

Edge deletion. Deleting the edge e from the ribbon graph $\mathbb{D}$ results in $\mathbb{D} \setminus e = (B \setminus \{b_1, b_2\}, \sigma_0', \sigma_1', i')$. We define σ_0' by deleting the b_1 and b_2 elements from σ_0 and σ_1' by deleting the (b_1, b_2) orbit from σ_1. For each orbit of σ_0 that disappears in this process, we increase the number of isolated vertices by one, so $i' \in \{i, i+1, i+2\}$. Figure 3 shows the ribbon graph from Figure 2 with edge $e = \{1, 2\}$ deleted.

Edge contraction. Contracting edge e in the ribbon graph $\mathbb{D}$ results in $\mathbb{D}/e = (B \setminus \{b_1, b_2\}, \sigma_0'', \sigma_1'', i)$. (Note: In this paper, whenever we contract an edge e, we assume that e is not a loop, i.e. e has distinct initial and terminal vertices.) σ_1'' is simply σ_1 with the (b_1, b_2) orbit deleted. To form σ_0'', we order the orbits of σ_0 so that the orbits containing b_1 and b_2 are first, and b_1 and b_2 occur first in these orbits. We then concatenate these two orbits and delete b_1 and b_2. So if $\sigma_0 = (a_{11} = b_1, a_{12}, ..., a_{1m})(a_{21} = b_1, a_{22}, ..., a_{2n})...$, then $\sigma_0'' = (a_{12}, ..., a_{1m}, a_{22}, ..., a_{2n})...$. Figure 3 shows the ribbon graph from Figure 2 with edge $e = \{1, 2\}$ contracted.

Edge deletion and contraction are local operations, and thus create only small changes to a ribbon graph's statistics. Edge deletion does not change the number of vertices, and the number of edges decreases by one. The number of connected components can either remain unchanged or increase by one. If deleting an edge increases the number of connected components, this edge is called an *isthmus*. The number of boundary components can change in a few ways, depending on the type of edge deleted.

Notation. We will assign the following types to the edge e in the ribbon graph $\mathbb{D}$:

Type 1. The edge is an isthmus.

Type 2. The edge is not an isthmus and has different faces on each side.

Type 3. The edge is not an isthmus and has the same face on both sides.

These edge types are illustrated in Figure 2.1.

A Type 1 edge has the same face on both sides. When this edge is deleted, this face becomes two, one on each new connected component. The other faces are left unchanged. Thus, the total number of faces increases by one.

When a Type 2 edge is deleted, the two faces on each side of the edge become one face. Thus, the total number of faces decreases by one.

When a Type 3 edge is deleted, the number of faces does not change. However, the resulting ribbon graph can now be embedded onto a surface with a lower genus.

Contracting an edge changes the ribbon graph statistics in a predictable way, independent of edge type. The number of vertices and the number of edges both decrease by one. The number of boundary components and the number of connected components are left unchanged.

2.2. **The Bollobás-Riordan-Tutte polynomial.** To form a *spanning sub-ribbon graph* of the ribbon graph $\mathbb{D}$, we specify a subset of edges $S \subseteq E$. We then delete

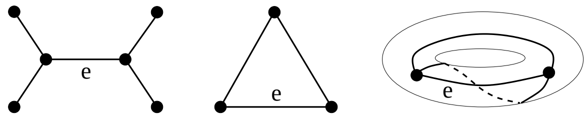

FIGURE 4. (left) e is a Type 1 edge; (center) e is a Type 2 edge; (right) e is a Type 3 edge.

all edges in $E \setminus S$. The spanning sub-ribbon graph is denoted by $\mathbb{D}(S)$, and will sometimes be called a *state*.

Definition 2.5. The *Bollobás-Riordan-Tutte (BRT) polynomial* [1, 2] of a ribbon graph $\mathbb{D}$ with edge set E is defined as a sum over all of the spanning sub-ribbon graphs of $\mathbb{D}$:

$$R(\mathbb{D}; X, Y, Z) = \sum_{S \subseteq E} X^{r(\mathbb{D})-r(\mathbb{D}(S))} Y^{n(\mathbb{D}(S))} Z^{g(\mathbb{D}(S))}$$

where

$$\begin{aligned} r(\mathbb{D}(S)) &= \text{rank of } \mathbb{D}(S) = v(\mathbb{D}(S)) - k(\mathbb{D}(S)) \\ n(\mathbb{D}(S)) &= \text{nullity of } \mathbb{D}(S) = e(\mathbb{D}(S)) - r(\mathbb{D}(S)) \\ g(\mathbb{D}(S)) &= \text{genus of } \mathbb{D}(S) = \frac{1}{2}[k(\mathbb{D}(S)) - bc(\mathbb{D}(S)) + n(\mathbb{D}(S))] \end{aligned} \tag{2.1}$$

Notation. For brevity, $r(\mathbb{D}(S))$, $n(\mathbb{D}(S))$, and $g(\mathbb{D}(S))$ will sometimes be denoted by $r_{\mathbb{D}}(S)$, $n_{\mathbb{D}}(S)$ and $g_{\mathbb{D}}(S)$.

Remark. We use the rank formulation for this polynomial. In [2], a mixed formulation of variables is used with X replaced by $X - 1$ in the above formula and Z replaced by Z^2. As with the Tutte polynomial, alternative definitions of the BRT polynomial can be given by contraction-deletion axioms [2] and quasitree expansions [5].

2.3. **Two-sum of pointed ribbon graphs.**

Definition 2.6. An orientation of an edge e, a cycle $\{e_1, e_2\}$ of σ_1, is a choice of ordering on the cycle. The *oriented edge* is denoted $\overrightarrow{e} := (e_1, e_2)$. If the edge is not a loop, this is equivalent to choosing initial and terminal vertices for the edge e.

Definition 2.7. A *pointed ribbon graph* is a pair of a ribbon graph $\mathbb{D}$ and an oriented edge $\overrightarrow{e}$, denoted $(\mathbb{D}, \overrightarrow{e})$.

If we have two pointed ribbon graphs, $(\mathbb{D}_1, \overrightarrow{e} = (e_1, e_2)) = (B_1; \sigma_{1,0}, \sigma_{1,1}; i)$ and $(\mathbb{D}_2, \overrightarrow{f} = (f_1, f_2)) = (B_2; \sigma_{2,0}, \sigma_{2,1}; j)$, and neither pointed edge is a loop, we would like to define their *parallel connection*, denoted $P((\mathbb{D}_1, \overrightarrow{e}), (\mathbb{D}_2, \overrightarrow{f}))$. This operation is well-defined for graphs, but for ribbon graphs, it presents difficulties. When we identify two vertices, we need to decide how to order the edges around this new vertex. This decision is motivated by a topological definition of parallel connection. We want to ensure that the parallel connection of the two ribbon graphs can be embedded into the connected sum of the surfaces into which the individual ribbon graphs are embedded. First, we make the convention that the

initial half-edge e_1 is written first in the vertex cycle to which it is incident and that the terminal half-edge e_2 is written last in its vertex cycle. For $\mathbb{D}_2$, write the vertex permutation so that the initial half-edge f_1 occurs last and the terminal half-edge f_2 occurs first:

$$\sigma_{1,0} = (\alpha_{1,1} = e_1, ..., \alpha_{1,k})(\alpha_{2,1}, ..., \alpha_{2,l} = e_2)... \text{ and}$$
$$\sigma_{2,0} = (\beta_{1,1}, ..., \beta_{1,m} = f_1)(\beta_{2,1} = f_2, ..., \beta_{2,n})....$$

Now, we double up each identified edge by introducing the edges $\overrightarrow{a} := (a_1, a_2)$ and $\overrightarrow{b} := (b_1, b_2)$. We have the new vertex permuations

$$\sigma'_{1,0} = (\alpha_{1,1} = e_1, ..., \alpha_{1,k}, a_1)(a_2, \alpha_{2,1}, ..., \alpha_{2,l} = e_2)... \text{ and}$$
$$\sigma'_{2,0} = (b_1, \beta_{1,1}, ..., \beta_{1,m} = f_1)(\beta_{2,1} = f_2, ..., \beta_{2,n}, b_2)....$$

This creates a new two-valent face in each of the ribbon graphs. We delete each of these faces and glue the two ribbon graphs together by identifying edges $\overrightarrow{e}$ and $\overrightarrow{f}$ as well as $\overrightarrow{a}$ and $\overrightarrow{b}$. This intermediate ribbon graph is denoted $(\mathbb{D}_1, \overrightarrow{e})\#(\mathbb{D}_2, \overrightarrow{f})$. Thus, $(\mathbb{D}_1, \overrightarrow{e})\#(\mathbb{D}_2, \overrightarrow{f}) = ((B_1 \amalg B_2 \amalg \{a_1, a_2\}) \setminus \{f_1, f_2\}; \sigma_0, \sigma_1; i + j)$. The new vertex permutation is

$$\sigma_0 = (\alpha_{1,1} = e_1, ..., \alpha_{1,k}, a_1, \beta_{1,1}, ..., \beta_{1,m-1})(a_2, \alpha_{2,1}, ..., \alpha_{2,l} = e_2, \beta_{2,2}, ..., \beta_{2,n})....$$

The new edge permutation σ_1 includes the cycle (a_1, a_2) as well as all of the cycles from $\sigma_{1,1}$ and $\sigma_{2,1}$ except (f_1, f_2). Now, the question is whether to delete edge $\overrightarrow{e}$ or edge $\overrightarrow{a}$. We choose, somewhat arbitrarily, to delete the edge $\overrightarrow{a}$. For an illustration of this process, see Figure 5.

If we had instead chosen to delete edge $\overrightarrow{e}$, we could have created a different ribbon graph. For an example of this possibility, see the last row of Figure 5. The parallel connection constructed by deleting $\overrightarrow{a}$ is pictured on the left. The parallel connection constructed by deleting $\overrightarrow{e}$ is pictured on the right. These are the same ribbon graphs from Figure 1, which we saw are non-isomorphic to one another.

We form the *two-sum*, denoted $(\mathbb{D}_1, \overrightarrow{e}) \oplus_2 (\mathbb{D}_2, \overrightarrow{f})$, by deleting both edges $\overrightarrow{a}$ and $\overrightarrow{e}$, which eliminates the ambiguity of the parallel connection definition.

3. The pointed Bollobás-Riordan-Tutte polynomial of pointed ribbon graphs

Brylawski [3] defined a pointed Tutte polynomial using a contraction-deletion relation. Since loops cannot be contracted in ribbon graphs, it is not possible to define a pointed BRT polynomial using this method. Las Vergnas [7] generalized Brylawski's polynomial to graphs (or, in the language of his article, matroids) pointed by sets. More importantly for this study, he gave a state sum definition for this polynomial, which we generalize in our definition of the pointed BRT polynomial. Most of the following properties are generalizations of those for Las Vergnas' pointed Tutte polynomial.

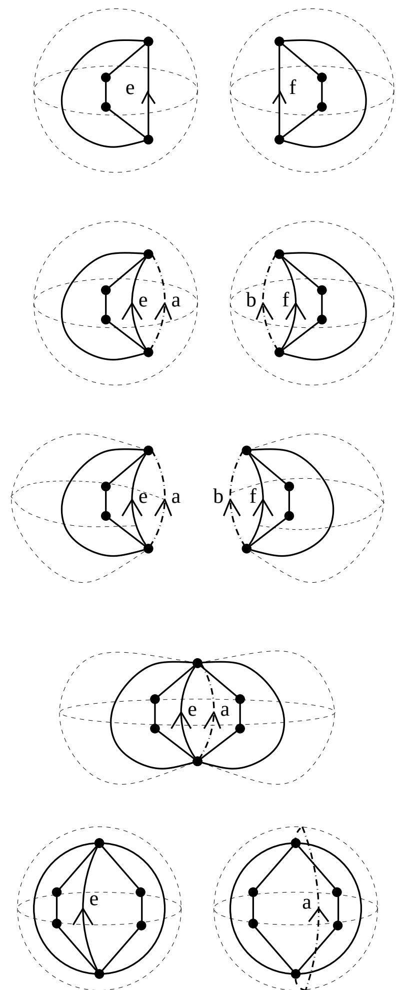

FIGURE 5. An illustration of the parallel connection of ribbon graphs and the connected sum of the surfaces of embedding.

Definition 3.1. The *pointed Bollobás-Riordan-Tutte polynomial* of a pointed ribbon graph $(\mathbb{D}, \overrightarrow{e})$ is a polynomial on five variables $\{X, Y, Z, \eta, \nu\}$ given by

$$\begin{aligned} &R(\mathbb{D}, \overrightarrow{e}; X, Y, Z, \eta, \nu) \\ &= \sum_{S \subseteq E \setminus e} X^{r(\mathbb{D}) - r_{\mathbb{D}}(S \cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S \cup e)} \eta^{r_{\mathbb{D}}(S \cup e) - r_{\mathbb{D}}(S)} \nu^{g_{\mathbb{D}}(S \cup e) - g_{\mathbb{D}}(S)} \end{aligned}$$

Note that this definition does not involve the orientation of the edge e. Henceforth we will write $R(\mathbb{D}, e; X, Y, Z, \eta, \nu)$.

The variable η indicates how much the rank changes when the pointed edge is deleted from a spanning sub-ribbon graph containing it. The variable ν does the same for the genus.

Proposition 3.1. *The ordinary (unpointed) BRT polynomial is related to the pointed BRT polynomial by the following equation:*

$$R(\mathbb{D}; X, Y, Z) = R(\mathbb{D}, e; X, Y, Z, X, Z^{-1}) + Y R(\mathbb{D}, e; X, Y, Z, Y^{-1}, 1)$$

Proof. We prove this by separating the state sum of the BRT polynomial into the sum over states which include e and those which do not.

$$\begin{aligned}
R(\mathbb{D}; X, Y, Z) &= \sum_{e\notin S\subseteq E} X^{r(\mathbb{D})-r_{\mathbb{D}}(S)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S)} + \sum_{e\in S\subseteq E} X^{r(\mathbb{D})-r_{\mathbb{D}}(S)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S)} \\
&= \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S)} + \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S\cup e)} Z^{g_{\mathbb{D}}(S\cup e)} \\
&= \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S\cup e)} X^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S)-g_{\mathbb{D}}(S\cup e)} \\
&\quad + \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S\cup e)-n_{\mathbb{D}}(S)} \\
&= \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S\cup e)} X^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)} (Z^{-1})^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} \\
&\quad + \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S\cup e)} Y^{|S\cup e|-r_{\mathbb{D}}(S\cup e)-[|S|-r_{\mathbb{D}}(S)]} \\
&= R(\mathbb{D}, e; X, Y, Z, X, Z^{-1}) + Y R(\mathbb{D}, e; X, Y, Z, Y^{-1}, 1),
\end{aligned}$$

because $|S \cup e| - |S| = 1$. □

3.1. **Ribbon Graph Operations and the Pointed Bollobás-Riordan-Tutte polynomial.** Suppose we have a ribbon graph $\mathbb{D}$ with edge set E. If S is a subset of $E \setminus e$, then $(\mathbb{D} \setminus e)(S) = \mathbb{D}(S)$ and, if e is not a loop, $(\mathbb{D}/e)(S) = \mathbb{D}(S \cup \{e\})/e$. By using earlier results about how ribbon graph statistics change under edge contraction, we have the following result:

$$\begin{aligned}
r_{\mathbb{D}\smallsetminus e}(S) &= r_{\mathbb{D}}(S) &\quad\text{and}\quad&& r_{\mathbb{D}/e}(S) &= r_{\mathbb{D}}(S\cup e) - r_{\mathbb{D}}(e) \\
n_{\mathbb{D}\smallsetminus e}(S) &= n_{\mathbb{D}}(S) &\quad\text{and}\quad&& n_{\mathbb{D}/e}(S) &= n_{\mathbb{D}}(S\cup e) - n_{\mathbb{D}}(e) \\
g_{\mathbb{D}\smallsetminus e}(S) &= g_{\mathbb{D}}(S) &\quad\text{and}\quad&& g_{\mathbb{D}/e}(S) &= g_{\mathbb{D}}(S\cup e)
\end{aligned} \tag{3.1}$$

Using (3.1), we find contraction and deletion relations for the pointed edge.

Proposition 3.2. (1) $R(\mathbb{D} \smallsetminus e) = X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})} R(\mathbb{D}, e; X, Y, Z, X, Z^{-1})$
(2) $R(\mathbb{D}/e) = Y^{r_{\mathbb{D}}(e)} R(\mathbb{D}, e; X, Y, Z, Y^{-1}, 1)$

Proof. (1) $R(\mathbb{D} \smallsetminus e) = \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D}\smallsetminus e)-r_{\mathbb{D}\smallsetminus e}(S)} Y^{n_{\mathbb{D}\smallsetminus e}(S)} Z^{g_{\mathbb{D}\smallsetminus e}(S)}$

$$\begin{aligned}
&= X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}\smallsetminus e}(S)} Y^{n_{\mathbb{D}\smallsetminus e}(S)} Z^{g_{\mathbb{D}\smallsetminus e}(S)} \\
&= X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S)} \\
&= X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S\cup e)} X^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)} (Z^{-1})^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} \\
&= X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})} R(\mathbb{D}, e; X, Y, Z, X, Z^{-1})
\end{aligned}$$

(2) $R(\mathbb{D}/e) = \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D}/e)-r_{\mathbb{D}/e}(S)}Y^{n_{\mathbb{D}/e}(S)}Z^{g_{\mathbb{D}/e}(S)}$

$$= \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(e)-[r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(e)]}Y^{n_{\mathbb{D}}(S\cup e)-n_{\mathbb{D}}(e)}Z^{g_{\mathbb{D}}(S\cup e)}$$

$$= Y^{-n_{\mathbb{D}}(e)} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)}Y^{n_{\mathbb{D}}(S)}Z^{g_{\mathbb{D}}(S\cup e)}Y^{n_{\mathbb{D}}(S\cup e)-n_{\mathbb{D}}(S)}$$

$$= Y^{-1+r_{\mathbb{D}}(e)} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)}Y^{n_{\mathbb{D}}(S)}Z^{g_{\mathbb{D}}(S\cup e)}Y^{1-r_{\mathbb{D}}(S\cup e)+r_{\mathbb{D}}(S)}$$

$$= Y^{r_{\mathbb{D}}(e)} \sum_{S\subseteq E\smallsetminus e} X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)}Y^{n_{\mathbb{D}}(S)}Z^{g_{\mathbb{D}}(S\cup e)}(Y^{-1})^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)}$$

$$= Y^{r_{\mathbb{D}}(e)}R(\mathbb{D}, e; X, Y, Z, Y^{-1}, 1)$$

□

We also have contraction and deletion relations for the unpointed edges in $(\mathbb{D}, e)$.

Lemma 3.1. *For any edge $f \in E \smallsetminus e$ which is not a loop,*

$$\begin{aligned} R(\mathbb{D}, e) =& X^{r(\mathbb{D})-r(\mathbb{D}\smallsetminus f)}R(\mathbb{D}\smallsetminus f, e; X, Y, Z, \eta, \nu) \\ &+ Y^{1-r_{\mathbb{D}}(f)}R(\mathbb{D}/f, e; X, Y, Z, \eta, \nu) \end{aligned}$$

Notation. We define the pointed BRT monomial corresponding to a state $S \subseteq E\smallsetminus e$ as:

$$\begin{aligned} &R_S(\mathbb{D}, e; X, Y, Z, \eta, \nu) \\ &= X^{r(\mathbb{D})-r_{\mathbb{D}}(S\cup e)}Y^{n_{\mathbb{D}}(S)}Z^{g_{\mathbb{D}}(S\cup e)}\eta^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)}\nu^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} \end{aligned}$$

Proof. We prove this by separating the pointed BRT polynomial into the sum over states which include f and those which do not.

$$R(\mathbb{D}, e) = \sum_{f\notin S\subseteq E\smallsetminus e} R_S(\mathbb{D}, e) + \sum_{f\in S\subseteq E\smallsetminus e} R_S(\mathbb{D}, e) \tag{3.2}$$

If $f \notin S \subseteq E \smallsetminus e$, then $S \cup e \subseteq E \smallsetminus f$, so using (3.1) we can write:

$$\begin{aligned} R_S(\mathbb{D}, e) &= X^{r(\mathbb{D})-r(\mathbb{D}\smallsetminus f)}X^{r(\mathbb{D}\smallsetminus f)-r_{\mathbb{D}\smallsetminus f}(S\cup e)}Y^{n_{\mathbb{D}\smallsetminus f}(S)}Z^{g_{\mathbb{D}\smallsetminus f}(S\cup e)} \\ &\qquad \times \eta^{r_{\mathbb{D}\smallsetminus f}(S\cup e)-r_{\mathbb{D}\smallsetminus f}(S)}\nu^{g_{\mathbb{D}\smallsetminus f}(S\cup e)-g_{\mathbb{D}\smallsetminus f}(S)} \\ &= X^{r(\mathbb{D})-r(\mathbb{D}\smallsetminus f)}R_S(\mathbb{D}\smallsetminus f, e; X, Y, Z, \eta, \nu) \end{aligned}$$

If $f \in S \subseteq E \smallsetminus e$, we can write S as $(S \smallsetminus f) \cup f$ and $S \cup e$ as $(S \smallsetminus f \cup e) \cup f$. Then, using Proposition 3.1 we can write:

$$\begin{aligned} &R_S(\mathbb{D}, e) \\ &= X^{r(\mathbb{D}/f)+r_{\mathbb{D}}(f)-(r_{\mathbb{D}/f}(S\smallsetminus f\cup e)+r_{\mathbb{D}}(f))}Y^{n_{\mathbb{D}/f}(S\smallsetminus f)+n_{\mathbb{D}}(f)}Z^{g_{\mathbb{D}/f}(S\smallsetminus f\cup e)} \\ &\qquad \times \eta^{r_{\mathbb{D}/f}(S\smallsetminus f\cup e)+r_{\mathbb{D}}(f)-(r_{\mathbb{D}}(S\smallsetminus f)+r_{\mathbb{D}}(f))}\nu^{g_{\mathbb{D}/f}(S\smallsetminus f\cup e)-g_{\mathbb{D}/f}(S\smallsetminus f)} \\ &= Y^{n_{\mathbb{D}}(f)}X^{r(\mathbb{D}/f)-r_{\mathbb{D}/f}(S\smallsetminus f\cup e)}Y^{n_{\mathbb{D}/f}(S\smallsetminus f)}Z^{g_{\mathbb{D}/f}(S\smallsetminus f\cup e)} \\ &\qquad \times \eta^{r_{\mathbb{D}/f}(S\smallsetminus f\cup e)-r_{\mathbb{D}}(S\smallsetminus f)}\nu^{g_{\mathbb{D}/f}(S\smallsetminus f\cup e)-g_{\mathbb{D}/f}(S\smallsetminus f)} \\ &= Y^{1-r_{\mathbb{D}}(f)}R_{S\smallsetminus f}(\mathbb{D}/f, e; X, Y, Z, \eta, \nu) \end{aligned}$$

Equation (3.2) becomes

$$\begin{aligned}
&R(\mathbb{D}, e)\\
&\quad = X^{r(\mathbb{D})-r(\mathbb{D}\setminus f)} \sum_{S\subseteq (E\setminus f)\setminus e} R_S(\mathbb{D}\setminus f, e)\\
&\qquad + Y^{1-r_{\mathbb{D}}(f)} \sum_{S\setminus f\subseteq (E\setminus f)\setminus e} R_{S\setminus f}(\mathbb{D}/f, e)\\
&\quad = X^{r(\mathbb{D})-r(\mathbb{D}\setminus f)} R(\mathbb{D}\setminus f, e; X, Y, Z, \eta, \nu) + Y^{1-r_{\mathbb{D}}(f)} R(\mathbb{D}/f, e; X, Y, Z, \eta, \nu)
\end{aligned}$$

□

Lemma 3.2. *If f is an isthmus of $\mathbb{D}$, then*

$$R(\mathbb{D}\setminus f, e) = R(\mathbb{D}/f, e)$$

Proof. Above, we have characterized how the number of vertices, edges, faces, and connected components are affected by the deletion and contraction of an isthmus. Now, using the definition of rank, nullity and genus, we see that for $S \subseteq E \setminus f$,

$$\begin{aligned}
r_{\mathbb{D}\setminus f}(S) &= r_{\mathbb{D}}(S\cup f) - 1 = r_{\mathbb{D}/f}(S)\\
n_{\mathbb{D}\setminus f}(S) &= n_{\mathbb{D}}(S\cup f) = n_{\mathbb{D}/f}(S)\\
g_{\mathbb{D}\setminus f}(S) &= g_{\mathbb{D}}(S\cup f) = g_{\mathbb{D}/f}(S)
\end{aligned}$$

Then,

$$\begin{aligned}
&R(\mathbb{D}\setminus f, e)\\
&= \sum_{S\subseteq E\setminus\{e,f\}} X^{r(\mathbb{D}\setminus f)-r_{\mathbb{D}\setminus f}(S\cup e)} Y^{n_{\mathbb{D}\setminus f}(S)} Z^{g_{\mathbb{D}\setminus f}(S\cup e)} \eta^{r_{\mathbb{D}\setminus f}(S\cup e)-r_{\mathbb{D}\setminus f}(S)} \nu^{g_{\mathbb{D}\setminus f}(S\cup e)-g_{\mathbb{D}\setminus f}(S)}\\
&= \sum_{S\subseteq E\setminus\{e,f\}} X^{r(\mathbb{D}/f)-r_{\mathbb{D}/f}(S\cup e)} Y^{n_{\mathbb{D}/f}(S)} Z^{g_{\mathbb{D}/f}(S\cup e)} \eta^{r_{\mathbb{D}/f}(S\cup e)-r_{\mathbb{D}/f}(S)} \nu^{g_{\mathbb{D}/f}(S\cup e)-g_{\mathbb{D}/f}(S)}\\
&= R(\mathbb{D}/f, e)
\end{aligned}$$

□

Proposition 3.3. *For any edge $f \in E \setminus e$,*

(1) *If f is an isthmus of $\mathbb{D}$,*

$$R(\mathbb{D}, e) = (X+1)R(\mathbb{D}\setminus f, e)$$

(2) *Otherwise, if f is neither an isthmus nor a loop of $\mathbb{D}$,*

$$R(\mathbb{D}, e) = R(\mathbb{D}\setminus f, e) + R(\mathbb{D}/f, e)$$

Proof.

(1) If f is an isthmus of $\mathbb{D}$, $r(\mathbb{D}) = r(\mathbb{D}\setminus f) + 1$ and $r_{\mathbb{D}}(f) = 1$. Thus by Lemmas 3.1 and 3.2,

$$\begin{aligned}
R(\mathbb{D}, e) &= XR(\mathbb{D}\setminus f, e) + R(\mathbb{D}/f, e)\\
&= XR(\mathbb{D}\setminus f, e) + R(\mathbb{D}\setminus f, e)\\
&= (X+1)R(\mathbb{D}\setminus f, e)
\end{aligned}$$

(2) If f is neither an isthmus nor a loop of $\mathbb{D}$, $r(\mathbb{D}) = r(\mathbb{D}\setminus f)$ and $r_{\mathbb{D}}(f) = 1$. Thus by Lemma 3.1,

$$R(\mathbb{D}, e) = R(\mathbb{D}\setminus f, e) + R(\mathbb{D}/f, e)$$

□

3.2. Decomposition of the pointed Bollobás-Riordan-Tutte polynomial.

Proposition 3.4. *For $S \subseteq E \smallsetminus e$,*

(1) *if edge e is Type 1 (an isthmus) in the state $S \cup e$, then $R_S(\mathbb{D}, e; 1, 1, 1, \eta, \nu) = \eta$.*
(2) *if edge e is Type 2 (not an isthmus, different faces on each side) in the state $S \cup e$, then $R_S(\mathbb{D}, e; 1, 1, 1, \eta, \nu) = 1$.*
(3) *if edge e is Type 3 (not an isthmus, same face on both sides) in the state $S \cup e$, then $R_S(\mathbb{D}, e; 1, 1, 1, \eta, \nu) = \nu$.*

Proof.

(1) Deleting edge e will decrease the rank by one and will not affect the genus. Thus,
$$\eta^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)}\nu^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} = \eta$$
(2) Deleting edge e will not affect the rank or the genus. Thus,
$$\eta^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)}\nu^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} = 1$$
(3) Deleting edge e will not affect the rank but will decrease the genus by one. Thus,
$$\eta^{r_{\mathbb{D}}(S\cup e)-r_{\mathbb{D}}(S)}\nu^{g_{\mathbb{D}}(S\cup e)-g_{\mathbb{D}}(S)} = \nu$$

□

Proposition 3.5. *The pointed BRT polynomial can be rewritten as a sum of three polynomials, where f, g, and h only involve X, Y, and Z:*

$$R(\mathbb{D}, e; X, Y, Z) = \eta f(X, Y, Z) + g(X, Y, Z) + \nu h(X, Y, Z)$$

Proof. We can associate a type to each state $S \subseteq E \setminus e$ based on the type of edge e in the state $S \cup e$. We can then separate the state sum into three pieces, depending on the type of state S:

$$\begin{aligned}
&R(\mathbb{D}, e; X, Y, Z, \eta, \nu) \\
&\quad = \sum_{\text{Type 1}} R_S(\mathbb{D}, e; X, Y, Z, \eta, \nu) + \sum_{\text{Type 2}} R_S(\mathbb{D}, e; X, Y, Z, \eta, \nu) \\
&\quad\quad + \sum_{\text{Type 3}} R_S(\mathbb{D}, e; X, Y, Z, \eta, \nu) \\
&\quad = \eta \sum_{\text{Type 1}} R_S(\mathbb{D}, e; X, Y, Z, 1, 1) + \sum_{\text{Type 2}} R_S(\mathbb{D}, e; X, Y, Z, 1, 1) \\
&\quad\quad + \nu \sum_{\text{Type 3}} R_S(\mathbb{D}, e; X, Y, Z, 1, 1) \\
&\quad = \eta f(X, Y, Z) + g(X, Y, Z) + \nu h(X, Y, Z)
\end{aligned}$$

□

$\mathbb{D}_2 \backslash \mathbb{D}_1$	Type 1	Type 2	Type 3
Type 1	Type 1	Type 2	Type 3
Type 2	Type 2	Type 2	Type 2
Type 3	Type 3	Type 2	Type 3

Table 1. Edge type after Parallel Connection

4. The Bollobás-Riordan-Tutte polynomial of the two-sum of pointed ribbon graphs

In a parallel connection, the total number of vertices is two less than the sum of the vertices in the two components; for edges, it is one less; for faces, it is equal; and for connected components it is one less. This determines how the relevant ribbon graph statistics change.

Lemma 4.1. *Let $(\mathbb{D}, e) = P((\mathbb{D}_1, e_1), (\mathbb{D}_2, e_2))$, the pointed parallel connection of $(\mathbb{D}_1, e_1)$ and $(\mathbb{D}_2, e_2)$. Let $S_1 \subseteq E_1 \setminus e_1$ and $S_2 \subseteq E_2 \setminus e_2$. If $S = S_1 \cup S_2$, then*

$$\begin{aligned} r_{\mathbb{D}}(S \cup e) &= r_{\mathbb{D}_1}(S_1 \cup e_1) + r_{\mathbb{D}_2}(S_2 \cup e_2) - 1 \\ n_{\mathbb{D}}(S \cup e) &= n_{\mathbb{D}_1}(S_1 \cup e_1) + n_{\mathbb{D}_2}(S_2 \cup e_2) \\ g_{\mathbb{D}}(S \cup e) &= g_{\mathbb{D}_1}(S_1 \cup e_1) + g_{\mathbb{D}_2}(S_2 \cup e_2) \end{aligned}$$

Proposition 4.1. *The types of edges e_1 in $\mathbb{D}_1(S_1 \cup e_1)$ and e_2 in $\mathbb{D}_2(S_2 \cup e_2)$ will determine the type of edge e in the parallel connection $\mathbb{D}$, according to Table 1.*

Main Theorem. *Let $(\mathbb{D}_1, e_1)$ and $(\mathbb{D}_2, e_2)$ be two pointed ribbon graphs with*

$$\begin{aligned} R(\mathbb{D}_1, e_1) &= \eta f_1(X, Y, Z) + g_1(X, Y, Z) + \nu h_1(X, Y, Z) \text{ and} \\ R(\mathbb{D}_2, e_2) &= \eta f_2(X, Y, Z) + g_2(X, Y, Z) + \nu h_2(X, Y, Z) \end{aligned} \tag{4.1}$$

Then for the pointed parallel connection of these two ribbon graphs,

$$\begin{aligned} &R(P((\mathbb{D}_1, e_1), (\mathbb{D}_2, e_2)), e; X, Y, Z, \eta, \nu) \\ &= \eta f_1 f_2 + (f_1 g_2 + g_1 f_2 + Y g_1 h_2 + Y h_1 g_2 + Y g_1 g_2) + \nu (f_1 h_2 + h_1 f_2 + Y h_1 h_2) \end{aligned} \tag{4.2}$$

Proof. Define $\mathbb{D} := P((\mathbb{D}_1, e_1), (\mathbb{D}_2, e_2))$. We separate the pointed BRT polynomial into three parts by Proposition 3.5.

$$\begin{aligned} R(\mathbb{D}, e; X, Y, Z, \eta, \nu) = \eta &\sum_{\text{Type 1}} X^{r(\mathbb{D}) - r_{\mathbb{D}}(S \cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S \cup e)} \\ + &\sum_{\text{Type 2}} X^{r(\mathbb{D}) - r_{\mathbb{D}}(S \cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S \cup e)} \\ + \nu &\sum_{\text{Type 3}} X^{r(\mathbb{D}) - r_{\mathbb{D}}(S \cup e)} Y^{n_{\mathbb{D}}(S)} Z^{g_{\mathbb{D}}(S \cup e)} \end{aligned} \tag{4.3}$$

We change each $n_{\mathbb{D}}(S)$ to $n_{\mathbb{D}}(S \cup e)$ and break the exponents into two parts by Lemma 4.1.

Notation. We assume that it is clear in which ribbon graphs we evaluate the statistics of subribbon graphs, e.g. $r_{\mathbb{D}_1}(S_1 \cup e_1)$ will be written as $r(S_1 \cup e_1)$.

$$= \eta \sum_{\text{Type 1}} X^{r(\mathbb{D}_1)-r(S_1\cup e_1)+r(\mathbb{D}_2)-r(S_2\cup e_2)} Y^{n(S_1\cup e_1)+n(S_2\cup e_2)} Z^{g(S_1\cup e_1)+g(S_2\cup e_2)}$$
$$+ Y^{-1} \sum_{\text{Type 2}} X^{r(\mathbb{D}_1)-r(S_1\cup e_1)+r(\mathbb{D}_2)-r(S_2\cup e_2)} Y^{n(S_1\cup e_1)+n(S_2\cup e_2)} Z^{g(S_1\cup e_1)+g(S_2\cup e_2)}$$
$$+ \nu Y^{-1} \sum_{\text{Type 3}} X^{r(\mathbb{D}_1)-r(S_1\cup e_1)+r(\mathbb{D}_2)-r(S_2\cup e_2)} Y^{n(S_1\cup e_1)+n(S_2\cup e_2)} Z^{g(S_1\cup e_1)+g_(S_2\cup e_2)}$$

Using the analysis of the combinations of types and remembering that both orders must be considered, this can be written as:

$$= \eta f_1 f_2 + Y^{-1}(Y f_1 g_2 + Y g_1 f_2 + Y^2 g_1 h_2 + Y^2 h_1 g_2 + Y^2 g_1 g_2)$$
$$+ \nu Y^{-1}(Y f_1 h_2 + Y h_1 f_2 + Y^2 h_1 h_2)$$
$$= \eta f_1 f_2 + (f_1 g_2 + g_1 f_2 + Y g_1 h_2 + Y h_1 g_2 + Y g_1 g_2) + \nu(f_1 h_2 + h_1 f_2 + Y h_1 h_2)$$

□

Corollary 4.1. *Using Proposition 3.2, we find that*

$$R((\mathbb{D}_1, e_1) \oplus_2 (\mathbb{D}_2, e_2)) = R(P((\mathbb{D}_1, e_1), (\mathbb{D}_2, e_2)) \smallsetminus e) =$$
$$X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})}[X f_1 f_2 + (f_1 g_2 + g_1 f_2 + Y g_1 h_2 + Y h_1 g_2 + Y g_1 g_2) + Z^{-1}(f_1 h_2 + h_1 f_2 + Y h_1 h_2)],$$

where $\mathbb{D} = P((\mathbb{D}_1, e_1), (\mathbb{D}_2, e_2))$.

5. Example

Here is an example of how the pointed BRT polynomial can be used to compute the two-sum of two ribbon graphs. We first compute the pointed BRT polynomial of the ribbon graph $(\mathbb{D}, \overrightarrow{e})$ pictured in Figure 6. The different spanning subribbon graphs are also pictured in Figure 6, with edge e drawn as a dashed line since it is not included among the states for the pointed BRT polynomial. This ribbon graph has one spanning subribbon graph with zero edges, two with one edge, and one with two edges. Each contributes a term of η, 1, and νYZ, respectively, to the pointed BRT polynomial, so the result is $R(\mathbb{D}, e) = \eta + 2 + \nu YZ$. So, in the language of Proposition 3.5, $f = 1$, $g = 2$, and $h = YZ$.

Now, we take the two-sum of this ribbon graph with itself. According to Corollary 4.1, the BRT polynomial should be

$$\begin{aligned} R((\mathbb{D}, e) \oplus_2 (\mathbb{D}, e)) &= X^{r(\mathbb{D}\smallsetminus e)-r(\mathbb{D})}[X f^2 + 2fg + 2Ygh + Yg^2 + Z^{-1}(2fh + Yh^2)] \\ &= X + 4 + 4Y^2 Z + 4Y + Z^{-1}(2YZ + Y^3 Z^2) \\ &= X + 4 + 6Y + 4Y^2 Z + Y^3 Z \end{aligned}$$

We can check this by actually constructing the two-sum (See Figure 6). For this two sum, sub-ribbon graphs with the same number of edges have identical statistics. This ribbon graph has one spanning sub-ribbon graph with four edges, four with three edges, six with two edges, four with one edge, and one with no edges, each contibuting a term of $Y^3 Z$, $Y^2 Z$, Y, 1, and X, respectively, to the BRT polynomial. So, the BRT polynomial of the two-sum is $X + 4 + 6Y + 4Y^2 Z + Y^3 Z$, which is the same as the one produced by Corollary 4.1.

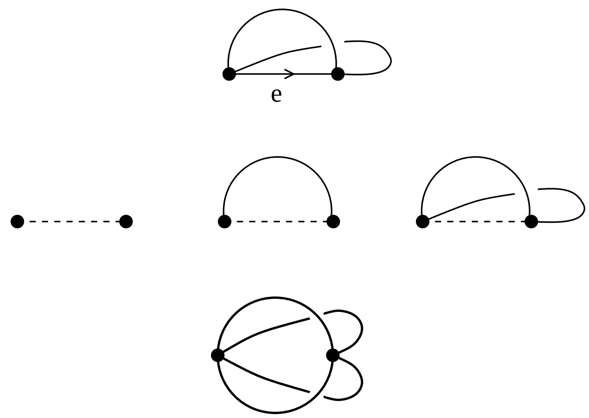

FIGURE 6. *Top*: A pointed dessin $(\mathbb{D}, e)$. *Center*: The states of $(\mathbb{D}, e)$. *Bottom*: The two-sum $(\mathbb{D}, e) \oplus_2 (\mathbb{D}, e)$.

6. Future Direction

Definition 6.1. The *tensor product* of an oriented ribbon graph $\mathbb{D}_1$ and a pointed ribbon graph $\mathbb{D}_2$ is constructed by performing the two-sum operation of $\mathbb{D}_1$ and $\mathbb{D}_2$ at every edge of $\mathbb{D}_1$.

It is well known how the Tutte poynomial of the tensor product is related to the Tutte polynomial of the graph and the pointed Tutte polynomial of the pointed graph [4]. We would like to have similar results for the BRT polynomial of the tensor product of ribbon graphs. Huggett and Moffatt have explored this question without using the pointed BRT polynomial [6]. It would be interesting to explore this question using the pointed BRT polynomial defined in this thesis.

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: `bfarme2@tigers.lsu.edu`