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SYMMETRIC WEIGHTED ODD-POWER VARIATIONS OF FRACTIONAL BROWNIAN MOTION AND APPLICATIONS

DAVID NUALART AND RAGHID ZEINEDDINE*

Abstract. We prove a non-central limit theorem for the symmetric weighted odd-power variations of the fractional Brownian motion with Hurst parameter $H < 1/2$. As applications, we study the asymptotic behavior of the trapezoidal weighted odd-power variations of the fractional Brownian motion and the fractional Brownian motion in Brownian time $Z_t := X_{Y_t},\ t \geq 0$, where X is a fractional Brownian motion and Y is an independent Brownian motion.

1. Introduction

Let $X = (X_t)_{t\geq 0}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1/2)$. The purpose of this paper is to prove a non-central limit theorem for symmetric weighted odd-power variations of X and derive some applications.

For any integers $n \geq 1$ and $j \geq 0$ we will make use of the notation $\Delta_{j,n}X := X_{(j+1)2^{-n}} - X_{j2^{-n}}$ and $\beta_{j,n} := \frac{1}{2}(X_{j2^{-n}} + X_{(j+1)2^{-n}})$. The main result of the paper is the following theorem.

Theorem 1.1. *Let X be a fBm with Hurst parameter $H < 1/2$. Fix an integer $r \geq 1$. Assume that $f \in C^{2r-1}(\mathbb{R})$. Then, as $n \to \infty$, we have*

$$\left(2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} f(\beta_{j,n}) \big(2^{nH}(\Delta_{j,n}X)\big)^{2r-1}\right)_{t\geq 0} \xrightarrow{\text{Law}} \left(\sigma_r \int_0^t f(X_s) dW_s\right)_{t\geq 0}, \tag{1.1}$$

where W is a standard Brownian motion independent of X, σ_r is the constant given by

$$\sigma_r^2 = E[X_1^{4r-2}] + 2\sum_{j=1}^{\infty} E[(X_1(X_{1+j} - X_j))^{2r-1}], \tag{1.2}$$

and the convergence holds in the Skorokhod space $D([0, \infty))$.

The proof of this result is based on the methodology of big blocks-small blocks, used, for instance, in [5, 6] and the following stable convergence of odd-power

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variations of the fBm

$$\left(2^{-n/2}\sum_{j=0}^{\lfloor 2^n t\rfloor-1}\left(2^{nH}\Delta_{j,n}X\right)^{2r-1}, X_t\right)_{t\geq 0} \xrightarrow[n\to\infty]{\text{Law}} (\sigma_r W_t, X_t)_{t\geq 0}\,, \tag{1.3}$$

where σ_r is defined in (1.2) and in the right-hand side of (1.3), the process W is a Brownian motion independent of X. The proof of the convergence (1.3) for a fixed t follows from the Breuer-Major Theorem (we refer to [15, Chapter 7] and [6] for a proof of this result based on the Fourth Moment theorem).

A rather complete analysis of the asymptotic behavior of weighted power variations of the fBm was developed in [13, Corollary 3]. However, the case of symmetric weighted power variations was not considered in this paper. On the other hand, motivated by applications to the asymptotic behavior of symmetric Riemann sums for critical values of the Hurst parameter, Theorem 1.1 was proved in [2, Proposition 3.1] when $H=\frac{1}{4r-2}$ for a function of the form $f^{(2r-1)}$ and assuming that $f\in C^{20r-15}(\mathbb{R})$ is such that f and its derivatives up to the order $20r-15$ have moderate growth. The proof given here, inspired by the recent work of Harnett, Jaramillo and Nualart [8], allows less derivatives and no growth condition.

In the second part of the paper we present two applications of Theorem 1.1 First, we deduce the following convergence in law of the trapezoidal weighted odd-power variations of the fBm with Hurst parameter $H<1/2$.

Proposition 1.2. *Let X be a fBm with Hurst parameter $H<1/2$. Fix an integer $r\geq 1$. Then, if $f\in C^M(\mathbb{R})$, where $M>2r-2+\frac{1}{2H}$, as $n\to\infty$, we have*

$$\left(2^{-n/2}\sum_{j=0}^{\lfloor 2^n t\rfloor-1}\frac{1}{2}\left(f(X_{j2^{-n}})+f(X_{(j+1)2^{-n}})\right)\left(2^{nH}\Delta_{j,n}X\right)^{2r-1}\right)_{t\geq 0}$$
$$\xrightarrow{Law}\left(\sigma_r\int_0^t f(X_s)dW_s\right)_{t\geq 0},$$

in the Skorokhod space $D([0,\infty))$, where W is a Brownian motion independent of X.

In the particular case $r=2$ and $H=1/6$, this result has been proved in [16] with longer arguments and using in a methodology introduced in [12]. The limit in this case, that is $\sigma_2\int_0^t f(X_s)dW_s$, is the correction term in the Itô-type formula in law proved in [16].

The asymptotic behavior of weighted odd-power variations of fBm with Hurst parameter $H<1/2$ has been already studied (see [13] and the references therein). More precisely, it is proved that for $H<1/2$, for any integer $r\geq 2$, and for a sufficiently smooth function f, we have

$$2^{nH-n}\sum_{j=0}^{\lfloor 2^n t\rfloor-1} f(X_{j2^{-n}})\left(2^{nH}\Delta_{j,n}X\right)^{2r-1}\xrightarrow[n\to\infty]{L^2} -\frac{\mu_{2r}}{2}\int_0^t f'(X_s)ds,$$

where $\mu_{2r} := E[N^{2r}]$ with $N \sim \mathscr{N}(0,1)$. By similar arguments, one can show that

$$2^{nH-n} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} f(X_{(j+1)2^{-n}})\big(2^{nH}\Delta_{j,n}X\big)^{2r-1} \xrightarrow[n\to\infty]{L^2} \frac{\mu_{2r}}{2}\int_0^t f'(X_s)ds,$$

which implies that

$$2^{nH-n} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}\big(f(X_{j2^{-n}}) + f(X_{(j+1)2^{-n}})\big)\big(2^{nH}\Delta_{j,n}X\big)^{2r-1} \xrightarrow[n\to\infty]{L^2} 0. \tag{1.4}$$

Thus, a natural question is to know whether it is possible to replace the normalization 2^{nH-n} by another one in order to get a non-degenerate limit in the convergence (1.4)? Proposition 1.2 gives us the answer to this question.

Our second application of Theorem 1.1 deals with the asymptotic behavior of weighted odd-power variations of the so-called *fractional Brownian motion in Brownian time* (fBmBt in short) when $H < 1/2$. The fBmBt is defined as

$$Z_t = X_{Y_t}, \quad t \geq 0,$$

where X is a two-sided fractional Brownian motion, with Hurst parameter $H \in (0,1)$, and Y is a standard (one-sided) Brownian motion independent of X. The process Z is self-similar of order $H/2$, it has stationary increments but it is not Gaussian. In the case $H = \frac{1}{2}$, where X is a *standard* Brownian motion, one recovers the celebrated *iterated Brownian motion* (iBm). This terminology was coined by Burdzy in 1993 (see [3]), but the idea of considering the iBm is actually older than that. Indeed, Funaki [7] discovered in 1979 that iBm may be used to represent the solution of the following parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{8}\left(\frac{\partial u}{\partial x}\right)^4, \ (t,x) \in (0,\infty)\times\mathbb{R}.$$

We refer the interested reader to the research works of Nane (see, e.g., [11] and the references therein) for many other interesting relationships between iterated processes and partial differential equations.

In 1998, Burdzy and Khoshnevisan [4] showed that iBm can be somehow considered as the canonical motion in an independent Brownian fissure. As such, iBm reveals to be a suitable candidate to model a diffusion in a Brownian crack. To support their claim, they have shown that the two components of a reflected two-dimensional Brownian motion in a Wiener sausage of width $\epsilon > 0$ converge to the usual Brownian motion and iterated Brownian motion, respectively, when ϵ tends to zero.

Let us go back to the second application of Theorem 1.1, we have the following theorem on the convergence in law of modified weighted odd-power variations of the fBmBt.

Theorem 1.3. *Suppose that $H < \frac{1}{2}$ and fix an integer $r \geq 1$. Let $f \in C^M(\mathbb{R})$, where $M > 2r - 2 + \frac{1}{2H}$. Then, we have*

$$\left(2^{-\frac{n}{4}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}\big(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}})\big)\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})\big)^{2r-1}\right)_{t\geq 0}$$
$$\xrightarrow[n\to\infty]{\text{Law}} \left(\sigma_r \int_0^{Y_t} f(X_s)dW_s\right)_{t\geq 0}, \tag{1.5}$$

in the Skorokhod space $D([0,\infty))$, where for $u \in \mathbb{R}$, $\int_0^u f(X_s)dW_s$ is the Wiener-Itô integral of $f(X)$ with respect to W defined in (4.12) and $\{T_{k,n} : 1 \leq k \leq 2^n t\}$ is a collection of stopping times defined in (4.3) that approximates the common dyadic partition $\{k2^{-n} : 1 \leq k \leq 2^n t\}$ of order n of the time interval $[0, t]$.

Theorem 1.3 completes the study of the asymptotic behavior of the modified weighted odd-power variations of the fBmBt in [19], where the case $H \leq 1/6$ was missing. In addition, in Theorem 1.3 we have convergence in the Skorokhod space $D([0,\infty))$, whereas in [19] we only proved the convergence of the finite dimensional distributions.

We remark that in many papers (see, for instance [2]) the authors use the uniform partition, but in this paper we work with dyadic partitions. Actually, Theorem 1.1 and Proposition 1.2 hold also with the uniform partition. However, the dyadic partition plays a crucial role in Theorem 1.3.

The paper is organized as follows. In Section 2 we give some elements of Malliavin calculus and some preliminary results. In Section 3, we prove Theorem 1.1 and finally in Section 4 we prove Proposition 1.2 and Theorem 1.3.

2. Elements of Malliavin Calculus

In this section, we gather some elements of Malliavin calculus we shall need in the sequel. The reader in referred to [17, 15] for details and any unexplained result.

Suppose that $X = (X_t)_{t\in\mathbb{R}}$ a two-sided fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, X is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathscr{A}, P)$, with covariance function,

$$C_H(t,s) = E(X_t X_s) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \quad s,t \in \mathbb{R}.$$

We suppose that $\mathscr{A}$ is the σ-field generated by X. For all $n \in \mathbb{N}^*$, we let $\mathscr{E}_n$ be the set of step functions on $[-n, n]$, and $\mathscr{E} := \cup_n \mathscr{E}_n$. Set $\varepsilon_t = \mathbf{1}_{[0,t]}$ (resp. $\mathbf{1}_{[t,0]}$) if $t \geq 0$ (resp. $t < 0$). Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\langle \varepsilon_t, \varepsilon_s \rangle_{\mathfrak{H}} = C_H(t,s), \quad s,t \in \mathbb{R}. \tag{2.1}$$

The mapping $\varepsilon_t \mapsto X_t$ can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space $\mathbb{H}_1$ associated with X. We will denote this isometry by $\varphi \mapsto X(\varphi)$.

Let $\mathscr{F}$ be the set of all smooth cylindrical random variables of the form

$$F = \phi(X_{t_1}, \ldots, X_{t_l}),$$

where $l \in \mathbb{N}^*$, $\phi : \mathbb{R}^l \to \mathbb{R}$ is a C^∞-function such that f and all its partial derivatives have at most polynomial growth, and $t_1 < \cdots < t_l$ are some real numbers. The derivative of F with respect to X is the element of $L^2(\Omega; \mathfrak{H})$ defined by

$$D_s F = \sum_{i=1}^{l} \frac{\partial \phi}{\partial x_i}(X_{t_1}, \dots, X_{t_l}) \varepsilon_{t_i}(s), \qquad s \in \mathbb{R}.$$

In particular $D_s X_t = \varepsilon_t(s)$. For any integer $k \geq 1$, we denote by $\mathbb{D}^{k,2}$ the closure of $\mathscr{F}$ with respect to the norm

$$\|F\|_{k,2}^2 = E(F^2) + \sum_{j=1}^{k} E[\|D^j F\|_{\mathfrak{H}^{\otimes j}}^2].$$

The Malliavin derivative D satisfies the chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $F_1, \dots, F_n$ are in $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) DF_i.$$

We denote by δ the adjoint of the derivative operator D, also called the divergence operator. A random element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of the divergence operator δ, denoted $\mathrm{Dom}(\delta)$, if and only if it satisfies

$$|E\langle DF, u\rangle_{\mathfrak{H}}| \leq c_u \sqrt{E(F^2)} \text{ for any } F \in \mathscr{F}.$$

If $u \in \mathrm{Dom}(\delta)$, then $\delta(u)$ is defined by the duality relationship

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big), \tag{2.2}$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geq 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathscr{A}, P)$ generated by the random variables $\{H_n(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. Recall that $H_0 = 0$, $H_p(x) = (-1)^p \exp(\frac{x^2}{2}) \frac{d^p}{dx^p} \exp(-\frac{x^2}{2})$ for $p \geq 1$. The mapping

$$I_n(h^{\otimes n}) := H_n(X(h)) \tag{2.3}$$

provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ and $\mathbb{H}_n$. The relation (2.2) extends to the multiple Skorokhod integral δ^q ($q \geq 1$), and we have

$$E\big(F\delta^q(u)\big) = E\big(\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\big), \tag{2.4}$$

for any element u in the domain of δ^q, denoted $\mathrm{Dom}(\delta^q)$, and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(u) = I_q(u)$ for any $u \in \mathfrak{H}^{\odot q}$.

For any Hilbert space V, we denote $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables (see [17, page 31]). The operator δ^q is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$, for any $p > 1$ and every integers $k \geq q \geq 1$, that is, we have

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq C_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})},$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ and some constant $C_{k,p} > 0$. These estimates are consequences of Meyer inequalities (see [17, Proposition 1.5.7]). We need the following result (see [12, Lemma 2.1]) on the Malliavin calculus with respect to X.

Lemma 2.1. *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let u be a symmetric element in $\operatorname{Dom} \delta^q$. Assume that, for any $0 \leq r+j \leq q$, $\left\langle D^r F, \delta^j(u) \right\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega; \mathfrak{H}^{\otimes q-r-j})$. Then, for any $r = 0, \dots, q-1$, $\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of δ^{q-r} and we have*

$$F \delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}(\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}).$$

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, for every $r = 0, \dots, n \wedge m$, the contraction of f and g of order r is the element of $\mathfrak{H}^{\otimes(n+m-2r)}$ defined by

$$f \otimes_r g = \sum_{k_1, \dots, k_r = 1}^{\infty} \langle f, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

2.1. Preliminary results.

We will make use of the following notation:

$$\partial_{j2^{-n}} = \mathbf{1}_{[j2^{-n},(j+1)2^{-n}]}, \quad \varepsilon_t = \mathbf{1}_{[0,t]}, \quad \widetilde{\varepsilon}_{j2^{-n}} = \frac{1}{2}\left(\varepsilon_{j2^{-n}} + \varepsilon_{(j+1)2^{-n}}\right).$$

We need the following preliminary results.

Lemma 2.2. *We fix two integers $n > m \geq 2$, and for any $j \geq 0$, define $k := k(j) = \sup\{i \geq 0 : i2^{-m} \leq j2^{-n}\}$. The following inequality holds true for some constant C_T depending only on T:*

$$\sum_{j=0}^{\lfloor 2^n T \rfloor - 1} \left| \left\langle \partial_{j2^{-n}}, \widetilde{\varepsilon}_{k(j)2^{-m}} \right\rangle_{\mathfrak{H}} \right| \leq C_T 2^{m(1-2H)}. \tag{2.5}$$

Proof. See Lemma 2.2, inequality (2.11), in the paper by Binotto Nourdin and Nualart [2]. In this paper the inequality is proved for $\varepsilon_{k(j)2^{-m}}$ but the case $\widetilde{\varepsilon}_{k(j)2^{-m}}$ can be proved by the same arguments. □

Lemma 2.3. *Let $0 \leq s < t$. Then*

$$\sum_{j=\lfloor 2^n s \rfloor}^{\lfloor 2^n t \rfloor - 1} \left| \left\langle \partial_{j2^{-n}}, \widetilde{\varepsilon}_{j2^{-n}} \right\rangle_{\mathfrak{H}} \right| = \frac{1}{2} 2^{-2nH} \left(\lfloor 2^n t \rfloor - \lfloor 2^n s \rfloor \right)^{2H}. \tag{2.6}$$

Proof. We can write

$$\begin{aligned}
&\sum_{j=\lfloor 2^n s \rfloor}^{\lfloor 2^n t \rfloor - 1} \left| \left\langle \partial_{j2^{-n}}, \widetilde{\varepsilon}_{j2^{-n}} \right\rangle_{\mathfrak{H}} \right| \\
&\quad = \frac{1}{2} \sum_{j=\lfloor 2^n s \rfloor}^{\lfloor 2^n t \rfloor - 1} \left| \mathbb{E}\left[\left(X_{(j+1)2^{-n}} - X_{j2^{-n}} \right) \left(X_{(j+1)2^{-n}} + X_{j2^{-n}} \right) \right] \right| \\
&\quad = 2^{-2nH} \frac{1}{2} \sum_{j=\lfloor 2^n s \rfloor}^{\lfloor 2^n t \rfloor - 1} \left[(j+1)^{2H} - j^{2H} \right],
\end{aligned}$$

which gives the desired result. □

3. Proof of Theorem 1.1

In this section we provide the proof of Theorem 1.1. We will make use of the following notation:

$$\Phi_n(t) = 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} f(\beta_{j,n}) \big(2^{nH} \Delta_{j,n} X\big)^{2r-1} \tag{3.1}$$

and $Z_t = \sigma_r \int_0^t f(X_s) dW_s$, where we recall that W is a Brownian motion independent of X. In order to prove Theorem 1.1, we need to show the following two results:
(A) *Convergence of the finite dimensional distributions:* Let $0 \le t_1 < \cdots < t_d$ be fixed. Then, we have

$$(\Phi_n(t_1), \ldots, \Phi_n(t_d)) \stackrel{Law}{\to} (Z_{t_1}, \ldots, Z_{t_d}).$$

(B) *Tightness:* The sequence Φ_n is tight in $D([0,\infty))$. That is, for every $\varepsilon, T > 0$, there is a compact set $K \subset D([0,T])$, such that

$$\sup_{n \ge 1} P\left[\Phi_n \in K^c\right] < \varepsilon.$$

The proof of statements **(A)** and **(B)** will be done in several steps.

Step 1: Reduction to compact support functions. As in [8] in the proof of **(A)** and **(B)** we can assume that f has compact support. Indeed, fix $L \ge 1$ and let $f_L \in C^{2r-1}(\mathbb{R})$ be a compactly supported function, such that $f_L(x) = f(x)$ for all $x \in [-L, L]$. Define

$$\Phi_n^L(t) = 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} f_L(\beta_{j,n}) \big(2^{nH} \Delta_{j,n} X\big)^{2r-1} \tag{3.2}$$

and $Z_t^L = \sigma_r \int_0^t f_L(X_s) dW_s$. For **(B)**, we choose L such that $P(\sup_{t\in[0,T]} |X_t| > L) < \frac{\varepsilon}{2}$. Then, if $K_L \subset D([0,T])$ is a compact set such that $\sup_{n\ge1} P\left[\Phi_n^L \in K_L^c\right] < \frac{\varepsilon}{2}$, we obtain

$$P\left[\Phi_n \in K_L^c\right] \le P\left[\Phi_n^L \in K_L^c, \sup_{t\in[0,T]} |X_t| \le L\right] + P(\sup_{t\in[0,T]} |X_t| > L) < \varepsilon.$$

With a similar argument, we can show that given a compactly supported function $\phi \in C(\mathbb{R}^d)$, the limit

$$\lim_{n\to\infty} E[\phi(\Phi_n^L(t_1), \ldots, \Phi_n^L(t_d)) - \phi(Z_{t_1}^L, \ldots, Z_{t_d}^L)] = 0$$

implies the same limit with $\Phi_n^L(t_i)$ replaced by $\Phi_n(t_i)$ and $Z_{t_i}^L$ replaced by Z_{t_i}.

Step 2: Proof **(A)** *assuming that f has compact support.* The proof is based on the small blocks-big blocks approach. Fix $m \le n$ and for each $j \ge 0$ we write $k := k(j) = \sup\{i \ge 0 : i2^{-m} \le j2^{-n}\}$, that is, $k(j)$ is the largest dyadic number in the mth generation which is less or equal than $j2^{-n}$. Define

$$\widetilde{\Phi}_{n,m}(t) = 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor -1} f(\beta_{k(j),m}) \big(2^{nH} \Delta_{j,n} X\big)^{2r-1}. \tag{3.3}$$

This term can be decomposed as follows

$$\begin{aligned}\widetilde{\Phi}_{n,m}(t) &= 2^{-n/2} \sum_{k=0}^{\lfloor 2^m t \rfloor -1} f(\beta_{k,m}) \sum_{j=k2^{n-m}}^{(k+1)2^{n-m}-1} \big(2^{nH} \Delta_{j,n} X\big)^{2r-1} \\ &\quad + 2^{-n/2} f(\beta_{\lfloor 2^m t \rfloor, m}) \sum_{j=\lfloor 2^m t \rfloor 2^{n-m}}^{\lfloor 2^n t \rfloor -1} \big(2^{nH} \Delta_{j,n} X\big)^{2r-1}\end{aligned}$$

The convergence (1.3) implies that for any $\mathscr{A}$-measurable and bounded random variable η, the random vector $(\widetilde{\Phi}_{n,m}(t_1), \dots, \widetilde{\Phi}_{n,m}(t_d), \eta)$ converges in law, as n tends to infinity, to the vector $(Y_m^1, \dots, Y_m^d, \eta)$, where

$$Y_m^i = \sigma_r \sum_{k=0}^{\lfloor 2^m t_i \rfloor -1} f(\beta_{k,m}) \big(\Delta_{k,m} W\big) + \sigma_r f(\beta_{\lfloor 2^m t_i \rfloor, m}) \big(W_{t_i} - W_{(\lfloor 2^m t_i \rfloor) 2^{-m}}\big)$$

for $i = 1, \dots, d$. Clearly, Y_m^i converges in $L^2(\Omega)$, as m tends to infinity, to Z_{t_i} for $i = 1, \dots, d$.

Then, it suffices to show that

$$\lim_{m \to \infty} \limsup_{n \to \infty} \sum_{i=1}^{d} \|\Phi_n(t_i) - \widetilde{\Phi}_{n,m}(t_i)\|_{L^2(\Omega)} = 0. \tag{3.4}$$

Let $c_{1,r}, \dots, c_{r,r}$ will denote the coefficients of the Hermite expansion of x^{2r-1}, namely,

$$x^{2r-1} = \sum_{u=1}^{r} c_{u,r} H_{2(r-u)+1}(x).$$

Then, we can write

$$\begin{aligned}(2^{nH} \Delta_{j,n} X)^{2r-1} &= \sum_{u=1}^{r} c_{u,r} H_{2(r-u)+1}\left(2^{nH} \Delta_{j,n} X\right) \\ &= \sum_{u=1}^{r} c_{u,r} 2^{nH(2(r-u)+1)} \delta^{2(r-u)+1}\left(\partial_{j2^{-n}}^{\otimes 2(r-u)+1}\right).\end{aligned} \tag{3.5}$$

Set $w := w(u) = 2(r-u)+1$. Substituting (3.5) into (3.1), yields

$$\Phi_n(t_i) = \sum_{u=1}^{r} c_{u,r} \sum_{j=0}^{\lfloor 2^n t_i \rfloor -1} f(\beta_{j,n}) 2^{-n/2+wnH} \delta^w \left(\partial_{j2^{-n}}^{\otimes w}\right).$$

On the other hand, (3.3) can be also written as

$$\widetilde{\Phi}_{n,m}(t_i) = \sum_{u=1}^{r} c_{u,r} \sum_{j=0}^{\lfloor 2^n t_i \rfloor - 1} f(\beta_{k(j),m}) 2^{-n/2+wnH} \delta^w \left(\partial_{j2^{-n}}^{\otimes w} \right).$$

With the help of Lemma 2.1 we can express these terms as linear combinations of Skorokhod integrals:

$$\Phi_n(t_i) = \sum_{u=1}^{r} c_{u,r} \sum_{\ell=0}^{w} \binom{w}{\ell} \Theta_{u,\ell}^n(t_i)$$

and

$$\widetilde{\Phi}_{n,m}(t_i) = \sum_{u=1}^{r} c_{u,r} \sum_{\ell=0}^{w} \binom{w}{\ell} \widetilde{\Theta}_{u,\ell}^{n,m}(t_i)$$

where

$$\Theta_{u,\ell}^n(t_i) = 2^{-\frac{n}{2}+wnH} \sum_{j=0}^{\lfloor 2^n t_i \rfloor - 1} \delta^{w-\ell} \left(f^\ell(\beta_{j,n}) \partial_{j2^{-n}}^{\otimes(w-\ell)} \langle \widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}} \rangle_{\mathfrak{H}}^\ell \right),$$

and

$$\widetilde{\Theta}_{u,\ell}^{n,m}(t_i) = 2^{-\frac{n}{2}+wnH} \sum_{j=0}^{\lfloor 2^n t_i \rfloor - 1} \delta^{w-\ell} \left(f^\ell(\beta_{k(j),m}) \partial_{j2^{-n}}^{\otimes(w-\ell)} \langle \widetilde{\varepsilon}_{k(j)2^{-m}}, \partial_{j2^{-n}} \rangle_{\mathfrak{H}}^\ell \right).$$

Then, it suffices to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{d} \| \Theta_{u,\ell}^n(t_i) - \widetilde{\Theta}_{u,\ell}^{n,m}(t_i) \|_{L^2(\Omega)} = 0$$

for all $1 \le u \le r$ and $0 \le \ell \le w$. We can decompose the difference $\Theta_{u,\ell}^n(t_i) - \widetilde{\Theta}_{u,\ell}^{n,m}(t_i)$ as follows

$$\Theta_{u,\ell}^n(t_i) - \widetilde{\Theta}_{u,\ell}^{n,m}(t_i) = 2^{-\frac{n}{2}+wnH} \sum_{j=0}^{\lfloor 2^n t_i \rfloor - 1} \delta^{w-\ell} \left(F_{j,\ell}^{n,m} \partial_{j2^{-n}}^{\otimes(w-\ell)} \right) =: T_{i,\ell}^{n,m}$$

where

$$F_{j,\ell}^{n,m} = f^\ell(\beta_{j,n}) \langle \widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}} \rangle_{\mathfrak{H}}^\ell - f^\ell(\beta_{k(j),m}) \langle \widetilde{\varepsilon}_{k(j)2^{-m}}, \partial_{j2^{-n}} \rangle_{\mathfrak{H}}^\ell.$$

By Meyer's inequality

$$\begin{aligned}
&\|T_{i,\ell}^{n,m}\|_2^2 \\
&\le C2^{-n+2nwH}\sum_{h=0}^{w-\ell}\left\|\sum_{j=0}^{\lfloor 2^n t_i\rfloor-1} D^h F^{n,m}_{k(j),j,\ell}\otimes \partial_{j2^{-n}}^{\otimes(w-\ell)}\right\|^2_{L^2(\Omega;\mathfrak{H}^{\otimes(w-l+h)})} \\
&= C2^{-n+2nwH}\sum_{h=0}^{w-\ell}\sum_{j_1,j_2=0}^{\lfloor 2^n t_i\rfloor-1}\mathbb{E}\left[\langle D^h F^{n,m}_{j_1,\ell}, D^h F^{n,m}_{j_2,\ell}\rangle_{\mathfrak{H}^{\otimes h}}\right]\langle \partial_{j_1 2^{-n}}, \partial_{j_2 2^{-n}}\rangle^{w-\ell} \\
&\le C2^{-n+2n\ell H}\sum_{h=0}^{w-\ell}\sum_{j_1,j_2=0}^{\lfloor 2^n t_i\rfloor-1}\|D^h F^{n,m}_{j_1,\ell}\|_{L^2(\Omega;\mathfrak{H}^{\otimes h})} \\
&\qquad\times\|D^h F^{n,m}_{j_2,\ell}\|_{L^2(\Omega;\mathfrak{H}^{\otimes h})}|\rho_H(j_1-j_2)|^{w-\ell}.
\end{aligned}$$

We will consider two different cases:

Case $w-\ell\ge 1$: We can make the decomposition

$$\begin{aligned}
F^{n,m}_{j,\ell} &= f^{(\ell)}(\beta_{j,n})\langle \widetilde{\varepsilon}^{\otimes\ell}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes\ell}_{k(j)2^{-m}}, \partial^{\otimes\ell}_{j2^{-n}}\rangle_{\mathfrak{H}^{\otimes\ell}} \\
&\quad + \left(f^{(\ell)}(\beta_{j,n}) - f^{(\ell)}(\beta_{k(j),m})\right)\langle \widetilde{\varepsilon}_{k(j)2^{-m}}, \partial_{j2^{-n}}\rangle^{\ell}_{\mathfrak{H}},
\end{aligned}$$

and hence, we have

$$\begin{aligned}
D^h F^{n,m}_{j,\ell} &= f^{(\ell+h)}(\beta_{j,n})\widetilde{\varepsilon}^{\otimes h}_{j2^{-n}}\langle \widetilde{\varepsilon}^{\otimes\ell}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes\ell}_{k(j)2^{-m}}, \partial^{\otimes\ell}_{j2^{-n}}\rangle_{\mathfrak{H}^{\otimes\ell}} \\
&\quad + f^{(\ell+h)}(\beta_{j,n})\left(\widetilde{\varepsilon}^{\otimes h}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes h}_{k(j)2^{-m}}\right)\langle \widetilde{\varepsilon}_{k(j)2^{-m}}, \partial_{j2^{-n}}\rangle^{\ell}_{\mathfrak{H}} \\
&\quad + \left(f^{(\ell+h)}(\beta_{j,n}) - f^{(\ell+h)}(\beta_{k(j),m})\right)\widetilde{\varepsilon}^{\otimes h}_{k(j)2^{-m}}\langle \widetilde{\varepsilon}_{k(j)2^{-m}}, \partial_{j2^{-n}}\rangle^{\ell}_{\mathfrak{H}}.
\end{aligned}$$

From the previous equality, and the compact support condition of f, we deduce that there exists a constant $C>0$, such that

$$\begin{aligned}
\left\|D^h F^{n,m}_{j,\ell}\right\|_{L^2(\Omega;\mathfrak{H}^{\otimes h})} &\le C\left\|\widetilde{\varepsilon}_{j2^{-n}}\right\|^h_{\mathfrak{H}}\left\|\widetilde{\varepsilon}^{\otimes\ell}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes\ell}_{k(j)2^{-m}}\right\|_{\mathfrak{H}^{\otimes\ell}}\left\|\partial^{\otimes\ell}_{j2^{-n}}\right\|_{\mathfrak{H}^{\otimes\ell}} \\
&\quad + C\left\|\widetilde{\varepsilon}^{\otimes h}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes h}_{k(j)2^{-m}}\right\|_{\mathfrak{H}^{\otimes h}}\left\|\widetilde{\varepsilon}_{k(j)2^{-m}}\right\|^{\ell}_{\mathfrak{H}}\left\|\partial_{j2^{-n}}\right\|^{\ell}_{\mathfrak{H}} \\
&\quad + \left\|f^{(\ell+h)}(\beta_{j,n}) - f^{(\ell+h)}(\beta_{k(j),m})\right\|_2 \\
&\qquad \times\left\|\widetilde{\varepsilon}_{k(j)2^{-m}}\right\|^{h+\ell}_{\mathfrak{H}}\left\|\partial_{j2^{-n}}\right\|^{\ell}_{\mathfrak{H}}.
\end{aligned}$$

Using Cauchy-Schwarz inequality we get, for any natural number $p\ge 1$

$$\begin{aligned}
\left\|\widetilde{\varepsilon}^{\otimes p}_{j2^{-n}} - \widetilde{\varepsilon}^{\otimes p}_{k(j)2^{-m}}\right\|_{\mathfrak{H}^{\otimes p}} &\le \left\|\widetilde{\varepsilon}_{j2^{-n}} - \widetilde{\varepsilon}_{k(j)2^{-m}}\right\|_{\mathfrak{H}}\sum_{i=0}^{p-1}\left\|\widetilde{\varepsilon}_{j2^{-n}}\right\|^i_{\mathfrak{H}}\left\|\widetilde{\varepsilon}_{k(j)2^{-m}}\right\|^{p-1-i}_{\mathfrak{H}} \\
&\le C\left\|\widetilde{\varepsilon}_{j2^{-n}} - \widetilde{\varepsilon}_{k(j)2^{-m}}\right\|_{\mathfrak{H}}.
\end{aligned}$$

Therefore

$$\begin{aligned}\left\|D^h F_{j,\ell}^{n,m}\right\|_{L^2(\Omega;\mathfrak{H}^{\otimes h})} &\leq C\left\|\partial_{j2^{-n}}\right\|_{\mathfrak{H}}^{\ell}\Big(\left\|\widetilde{\varepsilon}_{j2^{-n}}-\widetilde{\varepsilon}_{k(j)2^{-m}}\right\|_{\mathfrak{H}}\\ &\qquad+\left\|f^{(\ell+h)}(\beta_{j,n})-f^{(\ell+h)}(\beta_{k(j),m})\right\|_2\Big)\\ &\leq C2^{-\ell nH}\Big(\sup_{|t-s|\leq 2^{-m}}\|X_t-X_s\|_2\\ &\qquad+\left\|f^{(\ell+h)}(\beta_{j,n})-f^{(\ell+h)}(\beta_{k(j),m})\right\|_2\Big).\end{aligned}$$

Because $f^{(\ell+h)}$ is uniformly continuous, for any given $\varepsilon>0$ there exists $\delta>0$ such that $|x-y|<\delta$ implies $|f^{(\ell+h)}(x)-f^{(\ell+h)}(y)|<\varepsilon$. Therefore, we can write

$$\left\|f^{(\ell+h)}(\beta_{j,n})-f^{(\ell+h)}(\beta_{k(j),m})\right\|_2\leq\varepsilon+\frac{2}{\delta}\|f^{(\ell+h)}\|_\infty\|\beta_{j,n}-\beta_{k(j),m}\|_2$$

and this leads to the estimate

$$\left\|D^h F_{j,\ell}^{n,m}\right\|_{L^2(\Omega;\mathfrak{H}^{\otimes h})}\leq C2^{-\ell nH}\left(\sup_{|t-s|\leq 2^{-m}}\|X_t-X_s\|_2+\varepsilon\right),$$

which implies

$$\|T_{i,\ell}^{n,m}\|_2^2 \quad\leq\quad C\left(\sup_{|t-s|\leq 2^{-m}}\sum_{i=0}^{w-\ell}\|X_t-X_s\|_2+\varepsilon\right)^2\sum_{j=0}^{\lfloor 2^n t_i\rfloor-1}|\rho_H(j)|^{w-\ell}.$$

Then, the series $\sum_{j=0}^{\infty}|\rho_H(j)|^{w-\ell}$ is convergent because $w-\ell\geq 1$ and $H<1/2$, and we obtain

$$\lim_{m\to\infty}\sup_n\|T_{i,\ell}^{n,m}\|_2^2=0,$$

because ε is arbitrary.

Case $\ell=w$. in this case we have

$$\begin{aligned}\|T_{i,w}^{n,m}\|_2^2 &\leq 2^{-n+2wnH}\left(\sum_{j=0}^{\lfloor 2^n t_i\rfloor-1}\|F_{j,w}^{n,m}\|_2\right)^2\\ &\leq C2^{-n+2wnH}\left(\sum_{j=0}^{\lfloor 2^n t_i\rfloor-1}|\langle\widetilde{\varepsilon}_{j2^{-n}},\partial_{j2^{-n}}\rangle_{\mathfrak{H}}^w|+|\langle\widetilde{\varepsilon}_{k(j)2^{-m}},\partial_{j2^{-n}}\rangle_{\mathfrak{H}}^w|\right)^2\\ &\leq C2^{n(2H-1)}\left(\sum_{j=0}^{\lfloor 2^n t_i\rfloor-1}|\langle\widetilde{\varepsilon}_{j2^{-n}},\partial_{j2^{-n}}\rangle_{\mathfrak{H}}|+|\langle\widetilde{\varepsilon}_{k(j)2^{-m}},\partial_{j2^{-n}}\rangle_{\mathfrak{H}}|\right)^2.\end{aligned}$$

Finally, using (2.5) and (2.6), we obtain

$$\|T_{i,w}^{n,m}\|_2^2\leq C2^{n(2H-1)}2^{2m(1-2H)},$$

which implies

$$\lim_{m\to\infty}\limsup_{n\to\infty}\|T_{i,w}^{n,m}\|_2^2=0.$$

Step 3: Proof **(B)** *assuming that f has compact support.* We claim that for every $0 \le s \le t \le T$, and $p > 2$, there exists a constant $C > 0$, such that

$$E\left[|\Phi_n(t) - \Phi_n(s)|^p\right] \le C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{pH}. \tag{3.6}$$

Then, by the 'Billingsley criterion' (see [1, Theorem 13.5]), (3.6) implies the desired tightness property. From the computations in the proof of **(A)**, we need to show that for any $1 \le u \le r$ and for any $0 \le \ell \le w$, where $w = 2(r-u)+1$,

$$\|\Theta^n_{u,\ell}(t) - \Theta^n_{u,\ell}(s)\|_p \le C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{1}{2}} + C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{H}. \tag{3.7}$$

By Meyer's inequalities,

$$\begin{aligned}
&\|\Theta^n_{u,\ell}(t) - \Theta^n_{u,\ell}(s)\|_p \\
&= 2^{-\frac{n}{2}+wnH}\left\|\sum_{j=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor -1} \delta^{w-\ell}\left(f^{(\ell)}(\beta_{j,n})\partial_{j2^{-n}}^{\otimes(w-\ell)}\langle\widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}}\rangle_{\mathfrak{H}}^{\ell}\right)\right\|_p \\
&\le C2^{-\frac{n}{2}+wnH} \\
&\quad\times\sum_{h=0}^{w-\ell}\left\|\sum_{j=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor -1} f^{(\ell+h)}(\beta_{j,n})\widetilde{\varepsilon}_{j2^{-n}}^{\otimes h}\otimes\partial_{j2^{-n}}^{\otimes(w-\ell)}\langle\widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}}\rangle_{\mathfrak{H}}^{\ell}\right\|_{L^p(\Omega;\mathfrak{H}^{\otimes(w-\ell+h)})} \\
&= C2^{-\frac{n}{2}+wnH} \\
&\quad\times\sum_{h=0}^{w-\ell}\left\|\left\|\sum_{j=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor -1} f^{(\ell+h)}(\beta_{j,n})\widetilde{\varepsilon}_{j2^{-n}}^{\otimes h}\otimes\partial_{j2^{-n}}^{\otimes(w-\ell)}\langle\widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}}\rangle_{\mathfrak{H}}^{\ell}\right\|^2_{\mathfrak{H}^{\otimes(w-\ell+h)}}\right\|^{\frac{1}{2}}_{\frac{p}{2}}.
\end{aligned}$$

As a consequence, since f has compact support, applying Minkowski inequality, there is a constant C such that

$$\begin{aligned}
&\|\Theta^n_{u,\ell}(t) - \Theta^n_{u,\ell}(s)\|_p^2 \\
&\le C2^{-n+2wnH}\sum_{h=0}^{w-\ell}\Bigg\|\sum_{j,k=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor -1} f^{(\ell+h)}(\beta_{j,n})f^{(\ell+i)}(\beta_{k,n}) \\
&\quad\times\left\langle\widetilde{\varepsilon}_{j2^{-n}}, \widetilde{\varepsilon}_{k2^{-n}}\right\rangle^h\left\langle\delta_{j2^{-n}}, \delta_{k2^{-n}}\right\rangle^{w-\ell}\left\langle\widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}}\right\rangle_{\mathfrak{H}}^{\ell}\langle\widetilde{\varepsilon}_{k2^{-n}}, \partial_{k2^{-n}}\rangle_{\mathfrak{H}}^{\ell}\Bigg\|_{\frac{p}{2}} \\
&\le C2^{-n+2\ell nH}\sum_{j,k=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor -1}|\rho_H(j-k)|^{w-\ell}\left|\left\langle\widetilde{\varepsilon}_{j2^{-n}}, \partial_{j2^{-n}}\right\rangle_{\mathfrak{H}}\right|^{\ell}\left|\langle\widetilde{\varepsilon}_{k2^{-n}}, \partial_{k2^{-n}}\rangle_{\mathfrak{H}}\right|^{\ell}.
\end{aligned}$$

We will consider two different cases:

Case $w-\ell \geq 1$: In this case, we obtain

$$\begin{aligned} \|\Theta^n_{u,\ell}(t)-\Theta^n_{u,\ell}(s)\|_p^2 &\leq C2^{-n}\sum_{j,k=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor-1}|\rho_H(j-k)|^{w-\ell} \\ &\leq C\frac{\lfloor 2^n t\rfloor-\lfloor 2^n s\rfloor}{2^n}\sum_{h=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor-1}|\rho_H(h)|^{w-\ell} \\ &\leq C\frac{\lfloor 2^n t\rfloor-\lfloor 2^n s\rfloor}{2^n}. \end{aligned}$$

because the series $\sum_{j=0}^{\infty}|\rho_H(j)|^{w-\ell}$ is convergent because $w-\ell\geq 1$ and $H<1/2$. This implies the inequality (3.7) in this case.

Case $\ell = w$: We have

$$\begin{aligned} \|\Theta^n_{u,w}(t)-\Theta^n_{u,w}(s)\|_p^2 &\leq C2^{-n+2wnH}\left(\sum_{j=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{j2^{-n}},\partial_{j2^{-n}}\right\rangle_{\mathfrak{H}}\right|^w\right)^2 \\ &\leq C2^{-n+2nH}\left(\sum_{j=\lfloor 2^n s\rfloor}^{\lfloor 2^n t\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{j2^{-n}},\partial_{j2^{-n}}\right\rangle_{\mathfrak{H}}\right|\right)^2. \end{aligned}$$

Finally, applying (2.6) and the fact that $2^{-n+2nH}\leq 1$, we obtain

$$\|\Theta^n_{u,w}(t)-\Theta^n_{u,w}(s)\|_p^2 \leq C\left(\frac{\lfloor 2^n t\rfloor-\lfloor 2^n s\rfloor}{2^n}\right)^{2H}.$$

This completes the proof of part **(B)**.

4. Applications

4.1. The trapezoidal weighted odd-power variations of fractional Brownian motion. The trapezoidal weighted odd-power variations of the fBm is given in Proposition 1.2. We give its proof below.

Proof. By a localization argument similar to that used in the proof of Theorem 1.1, we can assume that f has compact support. Choose an integer N such that $\frac{1}{2H}-1<N\leq M-(2r-1)$, which is possible because $M>2r-2+\frac{1}{2H}$. Since $f\in C^M(\mathbb{R})$, by Taylor expansion, we have for all $x,y\in\mathbb{R}$ and $N\leq M-(2r-1)$,

$$\begin{aligned} f(y) &= f(\frac{1}{2}(x+y))+\frac{1}{2}f'(\frac{1}{2}(x+y))(y-x) \\ &\quad +\sum_{k=2}^{N}\frac{1}{2^k}\frac{1}{k!}f^{(k)}(\frac{1}{2}(x+y))(y-x)^k+R_N^{(1)}, \\ f(x) &= f(\frac{1}{2}(x+y))+\frac{1}{2}f'(\frac{1}{2}(x+y))(x-y) \\ &\quad +\sum_{k=2}^{N}\frac{1}{2^k}\frac{1}{k!}f^{(k)}(\frac{1}{2}(x+y))(x-y)^k+R_N^{(2)}, \end{aligned}$$

where the residual terms $R_N^{(1)}$ and $R_N^{(2)}$ are bounded by $C|y-x|^{N+1}$. We deduce that, for all integer $N \geq 1$,

$$\frac{1}{2}(f(x)+f(y)) = f(\frac{1}{2}(x+y)) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2^k}\frac{1}{k!} f^{(2k)}(\frac{1}{2}(x+y))(y-x)^{2k} + R_N(x,y), \quad (4.1)$$

where $R_N(x,y) \leq C|y-x|^{N+1}$. Recall that $\beta_{j,n} := 1/2(X_{j2^{-n}} + X_{(j+1)2^{-n}})$ and we also write $\Delta_{j,n} f(X) := \frac{1}{2}(f(X_{j2^{-n}}) + f(X_{(j+1)2^{-n}}))$. Set

$$\Psi_n(t) = 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} \Delta_{j,n} f(X) \big(2^{nH} \Delta_{j,n} X\big)^{2r-1}$$

and let $\Phi_n(t)$ be defined in (3.1). Then, in view of Theorem 1.1, it suffices to show that the difference $\Psi_n - \Phi_n$ converges to zero in probability in the Skorokhod space as $n \to \infty$. Using the expansion (4.1), we obtain

$$\begin{aligned}
\Psi_n(t) - \Phi_n(t) &= 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} \big(\Delta_{j,n} f(X) - f(\beta_{j,n})\big)\big(2^{nH} \Delta_{j,n} X\big)^{2r-1} \\
&= 2^{-n/2-2nHk} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2^k}\frac{1}{k!} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} f^{(2k)}(\beta_{j,n})\big(2^{nH} \Delta_{j,n} X\big)^{2k+2r-1} \\
&\quad + 2^{-n/2} \sum_{j=0}^{\lfloor 2^n t \rfloor - 1} R_N(X_{j2^{-n}}, X_{(j+1)2^{-n}})\big(2^{nH} \Delta_{j,n} X\big)^{2r-1} \\
&=: A_n(t) + B_n(t)
\end{aligned}$$

Thanks to Theorem 1.1, and taking into account that $f^{(2k)} \in C^{2k+2r-1}(\mathbb{R})$ for all $k \leq \lfloor N/2 \rfloor$ because $N + 2r - 1 \leq M$, we deduce that $A_n(\cdot)$ converges to 0 in probability as $n \to \infty$ in $D([0,\infty))$. Therefore, it is enough to prove the convergence in probability to 0 of $B_n(\cdot)$ in $D([0,\infty))$. This follows from the following estimates

$$\begin{aligned}
E\left[\sup_{0 \leq t \leq T} |B_n(t)|\right] &\leq C 2^{-\frac{n}{2}} 2^{-nH(N+1)} \sum_{j=0}^{\lfloor 2^n T \rfloor - 1} E[|2^{nH} \Delta_{j,n} X|^{N+2r}] \\
&\leq C_T 2^{\frac{n}{2} - nH(N+1)},
\end{aligned}$$

taking into account that $H(N+1) > \frac{1}{2}$. □

4.2. The weighted power variations of fractional Brownian motion in Brownian time. The so-called *fractional Brownian motion in Brownian time* (fBmBt in short) is defined as

$$Z_t = X_{Y_t}, \quad t \geq 0,$$

where X is a two-sided fractional Brownian motion, with Hurst parameter $H \in (0,1)$, and Y is a standard (one-sided) Brownian motion independent of X. The process Z_t is not a Gaussian process and it is self-similar (of order $H/2$) with stationary increments. When $H = 1/2$, one recovers the celebrated *iterated Brownian motion.*

Let $f : \mathbb{R} \to \mathbb{R}$. Then, for any $t \geq 0$ and any integer $p \geq 1$, the weighted p-variation of Z is defined as

$$M_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}\big(f(Z_{k2^{-n}}) + f(Z_{(k+1)2^{-n}})\big)(\Delta_{k,n} Z)^p.$$

where, as before, $\Delta_{k,n} Z = Z_{(k+1)2^{-n}} - Z_{k2^{-n}}$. After proper normalization we may expect the convergence (in some sense) to a non-degenerate limit (to be determined) of

$$N_n^{(p)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}\big(f(Z_{k2^{-n}}) + f(Z_{(k+1)2^{-n}})\big)\big[(\Delta_{k,n} Z)^p - E[(\Delta_{k,n} Z)^p]\big], \tag{4.2}$$

for some κ to be discovered. Due to the fact that one cannot separate X from Y inside Z in the definition of $N_n^{(p)}$, working directly with (4.2) seems to be a difficult task (see also [10, Problem 5.1]). That is why, following an idea introduced by Khoshnevisan and Lewis [9] in the study of the case $H = 1/2$, we introduce the following collection of stopping times (with respect to the natural filtration of Y), denoted by

$$\mathscr{T}_n = \{T_{k,n} : k \geq 0\}, \quad n \geq 0, \tag{4.3}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geq 0$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geq 0$, the stopping times $T_{k,n}$, appearing in (4.3), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\big\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y_{T_{k-1,n}}\}\big\}, \quad k \geq 1.$$

As shown in [9], as n tends to infinity the collection $\{T_{k,n} : 1 \leq k \leq 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leq k \leq 2^n t\}$ of order n of the time interval $[0, t]$ (see [9, Lemma 2.2] for a precise statement). Based on this fact, one can introduce the counterpart of (4.2) based on $\mathscr{T}_n$, namely,

$$\tilde{N}_n^{(p)}(t) = 2^{-n\tilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}\big(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}})\big)\big[\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})\big)^p - \mu_p\big],$$

with $\mu_p := E[N^p]$, where $N \sim \mathscr{N}(0,1)$ and for some $\tilde{\kappa} > 0$ to be discovered. At this stage, it is worthwhile noting that we are dealing with a modified weighted p-variation of Z. In fact, the collection of stopping times $\{T_{k,n} : 1 \leq k \leq 2^n t\}$ will play an important role in our analysis as we will see in Lemma 4.2.

4.2.1. *Known results about the weighted power variations of fBmBt.* The asymptotic behavior of $\tilde{N}_n^{(p)}(t)$, as n tends to infinity, has been studied in [14] when $H = 1/2$. For $H = 1/2$, one can deduce the following finite dimensional distributions (f.d.d.) convergence in law from [14, Theorem 1.2].

1) For $f \in C_b^2(\mathbb{R})$ and for any integer $r \geq 2$, we have

$$\left(2^{-\frac{n}{4}}\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\frac{1}{2}\big(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}})\big)\big(2^{\frac{n}{4}}(Z_{T_{k+1,n}}-Z_{T_{k,n}})\big)^{2r-1}\right)_{t\geq 0}$$
$$\xrightarrow[n\to\infty]{\text{f.d.d.}}\left(\int_0^{Y_t}f(X_s)(\mu_{2r}d^\circ X_s+\sqrt{\mu_{4r-2}-\mu_{2r}^2}\,dW_s)\right)_{t\geq 0}, \tag{4.4}$$

with $\mu_n := E[N^n]$, where $N \sim \mathscr{N}(0,1)$, for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^\circ X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in probability of $2^{-\frac{nH}{2}}W_n^{(1)}(f,t)$ as $n \to \infty$, with $W_n^{(1)}(f,t)$ defined in (4.9), W is a two-sided Brownian motion independent of (X,Y) and for $u \in \mathbb{R}$, $\int_0^u f(X_s)dW_s$ is the Wiener-Itô integral of $f(X)$ with respect to W defined in (4.12).

For $H \neq 1/2$, the second author of this paper has proved in [19] the following result with $f \in C_b^\infty(\mathbb{R})$ (f is infinitely differentiable with bounded derivatives of all orders),

2) For $\frac{1}{6} < H < \frac{1}{2}$ and for any integer $r \geq 2$, we have

$$\left(2^{-\frac{n}{4}}\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\frac{1}{2}\big(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}})\big)\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}}-Z_{T_{k,n}})\big)^{2r-1}\right)_{t\geq 0}$$
$$\xrightarrow[n\to\infty]{\text{f.d.d.}}\left(\beta_{2r-1}\int_0^{Y_t}f(X_s)dW_s\right)_{t\geq 0}, \tag{4.5}$$

where for $u \in \mathbb{R}$, $\int_0^u f(X_s)dW_s$ is the Wiener-Itô integral of $f(X)$ with respect to W defined in (4.12) and $\beta_{2r-1} = \sigma_r$, where σ_r is defined in Theorem 1.1.

3) Fix a time $t \geq 0$, for $H > \frac{1}{2}$ and for any integer $r \geq 1$, we have

$$2^{-\frac{nH}{2}}\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\frac{1}{2}\big(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}})\big)\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}}-Z_{T_{k,n}})\big)^{2r-1}$$
$$\xrightarrow[n\to\infty]{L^2}\frac{(2r)!}{r!2^r}\int_0^{Y_t}f(X_s)d^\circ X_s, \tag{4.6}$$

where for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^\circ X_s$ is defined as in (4.4).

As it has been mentioned in [19], the limit of the weighted $(2r-1)$-variation of Z for $H = \frac{1}{2}$ in (4.4) is intermediate between the limit of the weighted $(2r-1)$-variation of Z for $\frac{1}{6} < H < \frac{1}{2}$ in (4.5) and the limit of the weighted $(2r-1)$-variation of Z for $H > \frac{1}{2}$ in (4.6). A natural question is then to discovered what happens for $H \leq 1/6$. The answer is given in Theorem 1.3.

Remark 4.1. One can remark that, thanks to Theorem 1.3, (4.5) holds true for $H \leq 1/6$.

4.2.2. *Asymptotic behavior of the trapezoidal weighted odd-power variations of the fBmBt for $H < 1/2$.* The asymptotic behavior of the trapezoidal weighted odd-power variations of the fBmBt for $H < 1/2$ is given in Theorem 1.3. Inspired by [9], the proof of Theorem 1.3, given below, will be done in several steps.

Step 1: A key lemma. For each integer $n \geq 1$, $k \in \mathbb{Z}$ and real number $t \geq 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2},(j+1)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y_{T_{k,n}}\}_{k\geq 0}$, that is,

$$\begin{aligned} U_{j,n}(t) = \sharp\big\{k = 0,\dots,\lfloor 2^n t\rfloor - 1 :& \\ & Y_{T_{k,n}} = j2^{-n/2} \text{ and } Y_{T_{k+1,n}} = (j+1)2^{-n/2}\big\}; \\ D_{j,n}(t) = \sharp\big\{k = 0,\dots,\lfloor 2^n t\rfloor - 1 :& \\ & Y_{T_{k,n}} = (j+1)2^{-n/2} \text{ and } Y_{T_{k+1,n}} = j2^{-n/2}\big\}. \end{aligned}$$

The following lemma taken from [9, Lemma 2.4] is going to be the key when studying the asymptotic behavior of the weighted power variation $V_n^{(2r-1)}(f,t)$ of order $r \geq 1$, defined, for $t \geq 0$, as:

$$V_n^{(2r-1)}(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \frac{1}{2}\big(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}})\big)\big[\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})\big)^{2r-1}\big]. \tag{4.7}$$

Its main feature is to separate X from Y, thus providing a representation of $V_n^{(2r-1)}(f,t)$ which is amenable to analysis.

Lemma 4.2. *Fix $f : \mathbb{R} \to \mathbb{R}$, $t \geq 0$ and $r \in \mathbb{N}^*$. Then*

$$\begin{aligned} V_n^{(2r-1)}(f,t) = \sum_{j\in\mathbb{Z}} \frac{1}{2}\Big(f(X_{j2^{-\frac{n}{2}}}) + f(X_{(j+1)2^{-\frac{n}{2}}})\Big) \\ \times\big[\big(2^{\frac{nH}{2}}(X_{(j+1)2^{-\frac{n}{2}}} - X_{j2^{-\frac{n}{2}}})\big)^{2r-1}\big]\big(U_{j,n}(t) - D_{j,n}(t)\big). \end{aligned}$$

Step 2: Transforming the weighted power variations of odd order. By [9, Lemma 2.5], one has

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} \mathbf{1}_{\{0\leq j<j^*(n,t)\}} & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^*(n,t) = 0 \\ -\mathbf{1}_{\{j^*(n,t)\leq j<0\}} & \text{if } j^*(n,t) < 0 \end{cases},$$

where $j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor,n}}$. As a consequence,

$$V_n^{(2r-1)}(f,t) = \sum_{j=0}^{j^*(n,t)-1} \frac{1}{2}\big(f(X^+_{j2^{-n/2}}) + f(X^+_{(j+1)2^{-n/2}})\big)\big(X^{n,+}_{j+1} - X^{n,+}_j\big)^{2r-1}$$

if $j^*(n,t) > 0$, $V_n^{(2r-1)}(f,t) = 0$ if $j^*(n,t) = 0$ and

$$V_n^{(2r-1)}(f,t) = \sum_{j=0}^{|j^*(n,t)|-1} \frac{1}{2}\big(f(X^-_{j2^{-n/2}}) + f(X^-_{(j+1)2^{-n/2}})\big)\big(X^{n,-}_{j+1} - X^{n,-}_j\big)^{2r-1}$$

if $j^*(n,t) < 0$, where $X^+_t := X_t$ for $t \geq 0$, $X^-_{-t} := X_t$ for $t < 0$, $X^{n,+}_t := 2^{\frac{nH}{2}} X^+_{2^{-\frac{n}{2}}t}$ for $t \geq 0$ and $X^{n,-}_{-t} := 2^{\frac{nH}{2}} X^-_{2^{-\frac{n}{2}}(-t)}$ for $t < 0$.

Let us now introduce the following sequence of processes $W^{(2r-1)}_{\pm,n}$:

$$W^{(2r-1)}_{\pm,n}(f,t) = \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} \frac{1}{2}\big(f(X^{\pm}_{j2^{-\frac{n}{2}}})+f(X^{\pm}_{(j+1)2^{-\frac{n}{2}}})\big)(X^{n,\pm}_{j+1}-X^{n,\pm}_{j})^{2r-1}, \quad t\geq 0 \tag{4.8}$$

$$W^{(2r-1)}_{n}(f,t) := \begin{cases} W^{(2r-1)}_{+,n}(f,t) & \text{if } t\geq 0 \\ W^{(2r-1)}_{-,n}(f,-t) & \text{if } t<0 \end{cases}. \tag{4.9}$$

We then have,

$$V^{(2r-1)}_{n}(f,t) = W^{(2r-1)}_{n}(f,Y_{T_{\lfloor 2^n t\rfloor,n}}). \tag{4.10}$$

Step 3: A result concerning the trapezoidal weighted odd-power variations of the fBm. We have the following proposition.

Proposition 4.3. *Let $H<\frac{1}{2}$. Given an integer $r\geq 1$ then, for any $f\in C^M(\mathbb{R})$, where $M>2r-2+\frac{1}{2H}$,*

$$\left(2^{-\frac{n}{4}}W^{(2r-1)}_{n}(f,t)\right)_{t\in\mathbb{R}} \underset{n\to\infty}{\overset{\text{Law}}{\longrightarrow}} \left(\sigma_r\int_0^t f(X_s)dW_s\right)_{t\in\mathbb{R}}, \tag{4.11}$$

in $D(\mathbb{R})$, where $W^{(2r-1)}_{n}(f,t)$ is defined in (4.9), W is a two-sided Brownian motion independent of (X,Y), and $\int_0^t f(X_s)dW_s$ is defined in the following natural way: for $u\in\mathbb{R}$,

$$\int_0^u f(X_s)dW_s := \begin{cases} \int_0^u f(X^+_s)dW^+_s & \text{if } u\geq 0 \\ \int_0^{-u} f(X^-_s)dW^-_s & \text{if } u<0 \end{cases}, \tag{4.12}$$

where $W^+_t=W_t$ if $t>0$ and $W^-_t=W_{-t}$ if $t<0$, X^+ and X^- are defined in Step 2, and $\int_0^u f(X^{\pm}_s)dW^{\pm}_s$ must be understood in the Wiener-Itô sense.

Proof. We define, for all $j,n\in\mathbb{N}$, $\tilde{\beta}^{\pm}_{j,n} := \frac{1}{2}(X^{\pm}_{j2^{-\frac{n}{2}}}+X^{\pm}_{(j+1)2^{-\frac{n}{2}}})$. Let us introduce the following sequence of processes:

$$M_{\pm,n}(f,t) = \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} f\big(\tilde{\beta}^{\pm}_{j,n}\big)(X^{n,\pm}_{j+1}-X^{n,\pm}_{j})^{2r-1}, \quad t\geq 0,$$

$$M_n(f,t) := \begin{cases} M_{+,n}(f,t) & \text{if } t\geq 0 \\ M_{-,n}(f,-t) & \text{if } t<0 \end{cases}. \tag{4.13}$$

Then, by the same arguments that have been used in the proof of Proposition 1.2, we have

$$2^{-\frac{n}{4}}M_n(f,\cdot) - 2^{-\frac{n}{4}}W^{(2r-1)}_{n}(f,\cdot) \underset{n\to+\infty}{\longrightarrow} 0,$$

in probability in $D(\mathbb{R})$. So, in order to prove (4.11) it is enough to prove the following result

$$\left(2^{-\frac{n}{4}}M_n(f,t)\right)_{t\in\mathbb{R}} \underset{n\to\infty}{\overset{\text{Law}}{\longrightarrow}} \left(\sigma_r\int_0^t f(X_s)dW_s\right)_{t\in\mathbb{R}}, \tag{4.14}$$

in $D(\mathbb{R})$. The proof of (4.14) will be done in two steps, first we prove the convergence in law of the finite dimensional distributions and later we prove tightness.

1. Convergence in law of the finite dimensional distributions. Our purpose is to prove that

$$\left(2^{-\frac{n}{4}} M_n(f,t)\right)_{t\in\mathbb{R}} \overset{f.d.d.}{\underset{n\to\infty}{\longrightarrow}} \left(\sigma_r \int_0^t f(X_s)dW_s\right)_{t\in\mathbb{R}},$$

which is equivalent, by (4.13), to prove that

$$\left(2^{-\frac{n}{4}} M_{\pm,n}(f,t)\right)_{t\geq 0} \overset{f.d.d.}{\underset{n\to\infty}{\longrightarrow}} \left(\sigma_r \int_0^t f(X_s^{\pm})dW_s^{\pm}\right)_{t\geq 0}. \quad (4.15)$$

The proof of (4.15) uses arguments similar to those employed in part **(A)** of the proof of Theorem 1.1, the main ingredient being the small blocks/big blocks approach. Fix $m \leq n$ and for each $j \geq 0$ we denote by $k := k(j) = \sup\{i \geq 0 : i2^{-m/2} \leq j2^{-n/2}\}$. Define

$$\widetilde{M}_{n,m}^{\pm}(f,t) = \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} f(\tilde{\beta}_{k(j),m}^{\pm})\big(X_{j+1}^{n,\pm} - X_j^{n,\pm}\big)^{2r-1}.$$

It is known that (see (3.5) in [18] and part **(a)** in the proof of Proposition 5.1 in [19])

$$\left(2^{-n/4}\widetilde{M}_{n,m}^{\pm}(f,t)\right)_{t\geq 0} \overset{f.d.d.}{\underset{n\to\infty}{\longrightarrow}} \left(L_m^{\pm}(t)\right)_{t\geq 0},$$

where

$$\begin{aligned} L_m^{\pm}(t) \quad := \quad & \sigma_r \sum_{k=0}^{\lfloor 2^{m/2}t\rfloor -1} f(\tilde{\beta}_{k,m}^{\pm})\big(W_{(k+1)2^{-m/2}}^{\pm} - W_{k2^{-m/2}}^{\pm}\big) \\ & +\sigma_r f(\tilde{\beta}_{\lfloor 2^{m/2}t\rfloor,m}^{\pm})\big(W_t^{\pm} - W_{(\lfloor 2^{m/2}t\rfloor)2^{-m/2}}^{\pm}\big), \end{aligned}$$

with $W_t^+ = W_t$ if $t > 0$ and $W_t^- = W_{-t}$ if $t < 0$, where W is a two-sided Brownian motion independent of (X,Y). From the theory of stochastic calculus for semimartingales, we deduce that $L_m^{\pm}(t) \overset{L^2}{\longrightarrow} \sigma_r \int_0^t f(X_s^{\pm})dW_s^{\pm}$ as $m \to \infty$. Then, it is enough to prove that, for all $t \geq 0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty} \|2^{-\frac{n}{4}} M_{+,n}(f,t) - 2^{-n/4}\widetilde{M}_{n,m}^{+}(f,t)\|_{L^2(\Omega)} = 0,$$

$$\lim_{m\to\infty}\limsup_{n\to\infty} \|2^{-\frac{n}{4}} M_{-,n}(f,t) - 2^{-n/4}\widetilde{M}_{n,m}^{-}(f,t)\|_{L^2(\Omega)} = 0.$$

The proof of the last claim is similar to the proof of (3.4) and is left to the reader.

2. Proof of Tightness. The distribution of the sequence $\big(2^{-\frac{n}{4}} M_n(f,\cdot)\big)_{n\in\mathbb{N}}$ is tight in $D(\mathbb{R})$. To prove this claim we will show that for any $T > 0$ and for every $-T < s \leq t < T$, and $p > 2$, there exists a constant $C > 0$, such that

$$\begin{aligned} E\left[|2^{-\frac{n}{4}} M_n(f,t) - 2^{-\frac{n}{4}} M_n(f,s)|^p\right] \quad \leq \quad & C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{p}{2}} \\ & +C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{pH}. \quad (4.16) \end{aligned}$$

To do so, we distinguish three cases, according to the sign of $s,t \in \mathbb{R}$:

i). Suppose that $0 \le s \le t$. In this case we can write

$$\begin{aligned}
&E\left[|2^{-\frac{n}{4}}M_n(f,t) - 2^{-\frac{n}{4}}M_n(f,s)|^p\right] \\
&= E\left[|2^{-\frac{n}{4}}M_{+,n}(f,t) - 2^{-\frac{n}{4}}M_{+,n}(f,s)|^p\right] \\
&\le C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{pH},
\end{aligned}$$

where the proof of the last inequality is the same as the proof of (3.6).

ii). Suppose $s \le t \le 0$. Then, we have

$$\begin{aligned}
&E\left[|2^{-\frac{n}{4}}M_n(f,t) - 2^{-\frac{n}{4}}M_n(f,s)|^p\right] \\
&= E\left[|2^{-\frac{n}{4}}M_{-,n}(f,-t) - 2^{-\frac{n}{4}}M_{-,n}(f,-s)|^p\right] \\
&\le C\left(\frac{\lfloor 2^n(-s)\rfloor - \lfloor 2^n(-t)\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n(-s)\rfloor - \lfloor 2^n(-t)\rfloor}{2^n}\right)^{pH} \\
&= C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{pH},
\end{aligned}$$

where the proof of the second inequality is the same as the proof of (3.6) and we get the last equality since for any $x < 0$, $\lfloor -x\rfloor = -\lfloor x\rfloor - 1$.

iii). Suppose $s < 0 < t$. Then, we can write

$$\begin{aligned}
&E\left[|2^{-\frac{n}{4}}M_n(f,t) - 2^{-\frac{n}{4}}M_n(f,s)|^p\right] \\
&\le C\big(E\left[|2^{-\frac{n}{4}}M_n(f,t) - 2^{-\frac{n}{4}}M_n(f,0)|^p\right] \\
&\qquad + E\left[|2^{-\frac{n}{4}}M_n(f,s) - 2^{-\frac{n}{4}}M_n(f,0)|^p\right]\big) \\
&= C\big(E\left[|2^{-\frac{n}{4}}M_{+,n}(f,t) - 2^{-\frac{n}{4}}M_{+,n}(f,0)|^p\right] \\
&\qquad + E\left[|2^{-\frac{n}{4}}M_{-,n}(f,-s) - 2^{-\frac{n}{4}}M_{-,n}(f,0)|^p\right]\big) \\
&\le C\left(\frac{\lfloor 2^n t\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n(-s)\rfloor}{2^n}\right)^{pH} \\
&\le C\left(\frac{\lfloor 2^n t\rfloor + \lfloor 2^n(-s)\rfloor + 1}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n(-s)\rfloor + \lfloor 2^n t\rfloor + 1}{2^n}\right)^{pH} \\
&= C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{\frac{p}{2}} + C\left(\frac{\lfloor 2^n t\rfloor - \lfloor 2^n s\rfloor}{2^n}\right)^{pH},
\end{aligned}$$

where we have the third inequality by i) and ii).

Finally, we have proved (4.16) which proves the tightness of $\big(2^{-\frac{n}{4}}M_n(f,\cdot)\big)_{n\in\mathbb{N}}$ in $D(\mathbb{R})$. $\square$

Step 4: Convergence in law of $Y_{T_{\lfloor 2^n\cdot\rfloor,n}}$. As it was mentioned in [9], $\{2^{n/2}Y_{T_{k,n}} : k \ge 0\}$ is a simple and symmetric random walk on $\mathbb{Z}$. Observe that for all $t \ge 0$, $Y_{T_{\lfloor 2^n t\rfloor,n}} = 2^{-n/2} \times 2^{n/2}Y_{T_{\lfloor 2^n t\rfloor,n}} = 2^{-n/2}\sum_{l=0}^{\lfloor 2^n t\rfloor -1} 2^{n/2}(Y_{T_{l+1,n}} - Y_{T_{l,n}})$, where $\big(2^{n/2}(Y_{T_{l+1,n}} - Y_{T_{l,n}})\big)_{l\in\mathbb{N}}$ are independent and identically distributed random variables following the Rademacher distribution. By Donsker theorem, we get that

$$\big(Y_{T_{\lfloor 2^n t\rfloor,n}}\big)_{t\geq 0} \xrightarrow[n\to\infty]{law} (Y_t)_{t\geq 0} \text{ in } D([0,+\infty)). \tag{4.17}$$

Step 5: Last step in the proof of Theorem 1.3. Thanks to Proposition 4.3, to (4.17), and to the independence of X, W and Y , we have

$$\big(2^{-\frac{n}{4}}W_n^{(2r-1)}(f,\cdot), Y_{T_{\lfloor 2^n\cdot\rfloor,n}}\big) \xrightarrow[n\to+\infty]{law} \big(\sigma_r\int_0^{\cdot} f(X_s)dW_s, Y\big) \text{ in } D(\mathbb{R})\times D([0,+\infty)). \tag{4.18}$$

Let us define $(B_t)_{t\in\mathbb{R}}$ as follows $B_t := \sigma_r\int_0^t f(X_s)dW_s$. Since $(x,y)\in D(\mathbb{R})\times D([0,+\infty))\mapsto x\circ y\in D([0,+\infty))$ is measurable (see M16 at page 249 in [1] for a proof of this result) and since $B\circ Y$ is continuous, then, by (4.18) and Theorem 2.7 in [1], it follows that

$$2^{-\frac{n}{4}}W_n^{(2r-1)}(f, Y_{T_{\lfloor 2^n\cdot\rfloor,n}}) \xrightarrow[n\to+\infty]{law} B\circ Y = \sigma_r\int_0^{Y(\cdot)} f(X_s)dW_s, \text{ in } D([0,+\infty)).$$

The proof of Theorem 1.3 follows from (4.10) and the last convergence in law.

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