

8-2018

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Recommended Citation

Tugaut, Julian (2018) "Exit-Time of Granular Media Equation Starting in a Local Minimum," *Communications on Stochastic Analysis*: Vol. 12 : No. 1 , Article 3.

DOI: 10.31390/cosa.12.1.03

Available at: <https://digitalcommons.lsu.edu/cosa/vol12/iss1/3>

EXIT-TIME OF GRANULAR MEDIA EQUATION STARTING IN A LOCAL MINIMUM

JULIAN TUGAUT

ABSTRACT. We are interested in a nonlinear partial differential equation: the granular media one. Thanks to some of our previous results [10, 11], we know that under easily checked assumptions, there is a unique steady state. We point out that we consider a case in which the confining potential is not globally convex. According to recent articles [8, 9], we know that there is weak convergence towards this steady state. However, we do not know anything about the rate of convergence. In this paper, we make a first step to this direction by proving a deterministic Kramers' type law concerning the first time that the solution of the granular media equation leaves a local well. In other words, we show that the solution of the granular media equation is trapped around a local minimum during a time exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H\right\}$, H being the so-called exit-cost.

1. Introduction

In this paper, we are interested in the following so-called granular media equation:

$$\frac{\partial}{\partial t}\mu_t^\sigma(x) = \frac{\partial}{\partial x} \left\{ \frac{\sigma^2}{2} \frac{\partial}{\partial x} \mu_t^\sigma(x) + \mu_t^\sigma(x) \left(V'(x) + F' * \mu_t^\sigma(x) \right) \right\}, \quad (1.1)$$

where the confining potential V is nonconvex (double-wells) and the interacting potential F is convex. The exact assumptions will be given subsequently.

This partial differential equation has a natural interpretation in terms of stochastic processes. Indeed, let us consider the following so-called McKean-Vlasov diffusion:

$$\begin{cases} X_t^\sigma = X_0 + \sigma B_t - \int_0^t (W_s^\sigma)'(X_s^\sigma) ds \\ W_s^\sigma = V + F * \mathcal{L}(X_s^\sigma) \end{cases}. \quad (1.2)$$

Here, $*$ denotes the convolution. Since the law of the process intervenes in the drift, this equation is nonlinear - in the sense of McKean. By μ_t^σ , we denote the law at time t of the process X^σ . It is well-known that the family of probability measures $\{\mu_t^\sigma; t \geq 0\}$ satisfies the granular media equation starting from $\mathcal{L}(X_0)$.

Received 2017-11-25; Communicated by the editors.

2010 *Mathematics Subject Classification*. Primary 35B40, 60F10; Secondary 60J60, 60G10, 60H10.

Key words and phrases. Granular media equation, McKean-Vlasov diffusion, double-well potential, exit-time, large deviations.

We will use the recent results about the exit-problem of the McKean-Vlasov diffusion (see [12, 13]) in order to prove a deterministic Kramers' type law for any σ sufficiently small:

$$\exp \left[\frac{2}{\sigma^2} (H_0 - \delta) \right] < T_\kappa(\sigma) < \exp \left[\frac{2}{\sigma^2} (H_0 + \delta) \right],$$

H_0 being the associated exit-cost (which will be described later), δ being an arbitrarily small constant and

$$\begin{aligned} T_\kappa(\sigma) &:= \inf \left\{ t \geq 0 : \mathbb{E} \left[(X_t^\sigma - b)^2 \right] > \kappa^2 \right\} \\ &= \inf \left\{ t \geq 0 : \int_{\mathbb{R}} (x - b)^2 \mu_t^\sigma(dx) > \kappa^2 \right\}, \end{aligned}$$

where b is a minimizer of V corresponding to a local and non global minimum.

We now give the assumptions on V and F .

Assumption 1.1. *The potentials V and F satisfy the following hypotheses:*

- *The coefficients V' and F' are locally Lipschitz, that is, for each $R > 0$ there exists $K_R > 0$ such that*

$$|V'(x) - V'(y)| + |F'(x) - F'(y)| \leq K_R |x - y|,$$

for $x, y \in \{z \in \mathbb{R} : |z| < R\}$.

- *The function V is continuously differentiable.*
- *The potential V is convex at infinity: $\lim_{|x| \rightarrow +\infty} V''(x) = +\infty$.*
- *The potential V has two wells ($a < 0$ and $b > 0$) and a local maximum located in 0.*
- *The function V'' is convex.*
- *$F(x) := \frac{\alpha}{2} x^2$ with $\alpha > 0$.*

[Exit-time of granular media equation]Exit-time of granular media equation starting in a local minimum An example of such potential is $V(x) := \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}$. In this case $a = -\frac{1+\sqrt{5}}{2} < 0 < \frac{-1+\sqrt{5}}{2} = b$.

If the initial law is a Dirac measure, we know that there exists a unique strong solution X^σ to Equation (1.2), see [5, Theorem 2.13]. Moreover, we have: $\sup_{t \in \mathbb{R}_+} \mathbb{E} \left\{ |X_t^\sigma|^{2p} \right\} < \infty$ for any $p \in \mathbb{N}^*$.

From now on, we consider the potential $W_b := V + F * \delta_b$. Indeed, by classical large deviations result, for any $T > 0$, in the small-noise limit, the diffusion $(X_t^\sigma)_{0 \leq t \leq T}$ starting at $X_0 = b$ is close to the diffusion $(Y_t^\sigma)_{0 \leq t \leq T}$ defined like so:

$$Y_t^\sigma = b + \sigma W_t - \int_0^t W_b'(Y_s^\sigma) ds.$$

An important tool to understand the long-time behaviour of μ_t^σ is the set of invariant probabilities. This set has been precisely described in [6, 7, 10, 11]. From these works, we know that there exists an invariant probability near - in the small-noise limit - the distribution δ_b if and only if b is the unique global minimizer of W_b .

Assumption 1.2. *There exists $y \neq b$ such that $W_b(y) < W_b(b)$.*

Immediately, from Assumption 1.2, we deduce that W_b has another minimizer than b , that is here denoted as a' and a unique local maximizer (since V'' is convex) denoted as c . From now on, we consider the following exit-cost:

$$H_0 := W_b(c) - W_b(b). \quad (1.3)$$

The long-time behaviour of μ_t^σ has been solved in the convex case (see [1, 2, 3, 4]) and in the non-convex case (see [8, 9]).

An important and remaining question is the one of the rate of convergence. In [3], a rate of convergence has been obtained if V is convex but not uniformly strictly convex. Here, with double-wells potential, we can not use this result. It is an easy exercise to show that μ^σ stays a long time (that does depend on σ) close to δ_b in the small-noise limit. The result of the paper is a characterization of this time.

According to [11], with Assumption 1.1 and Assumption 1.2, there exists - if the noise σ is sufficiently small - a unique steady state for Equation (1.1). Consequently, if $\mu_0 = \delta_{x_0}$ where $x_0 \in]0; +\infty[$, we know that μ_t^σ converges weakly towards the unique invariant probability.

The aim of the current work is to study what happens if $x_0 := b$. For doing so, we use the recent results about the exit-time of the associated McKean-Vlasov diffusion in [12, 13].

From now on, we consider the *deterministic* time

$$T_\kappa(\sigma) := \inf \left\{ t \geq 0 : \int_{\mathbb{R}} (x - b)^2 \mu_t^\sigma(dx) \geq \kappa^2 \right\}$$

for any $\kappa > 0$. In the following, κ is arbitrarily small. In particular, we assume that

$$\kappa^2 \leq \frac{1}{2}(c - b)^2.$$

We consider an additional assumption on the interaction:

Assumption 1.3. *We have $\alpha < \frac{V''(b)}{\sqrt{2}}$.*

This last assumption is used in order to be able to apply the results in [13].

We now give the result of the article.

Theorem 1.4. *For any $\kappa \in]0; \frac{1}{\sqrt{2}}|c - b|[$, for any $\delta > 0$, there exists $\sigma(\kappa, \delta)$ such that for all $0 < \sigma < \sigma(\kappa, \delta)$:*

$$\exp \left[\frac{2}{\sigma^2} (H_0 - \delta) \right] < T_\kappa(\sigma) < \exp \left[\frac{2}{\sigma^2} (H_0 + \delta) \right]. \quad (1.4)$$

2. Proof of Theorem 1.4

The lower-bound has already been proved in [13, Proposition C]. Indeed, in [13], the constant T_κ does correspond to the first time t such that $\mathbb{E} \left[(X_t - b)^2 \right] < \kappa^2$, which here is 0 since $X_0 = b$.

Consequently, we have

$$\sup_{0 \leq t \leq \exp\left[\frac{2}{\sigma^2}(H_0 - \delta)\right]} \mathbb{E} \left[(X_t - b)^2 \right] < \kappa^2,$$

if σ is sufficiently small. We deduce $T_\kappa(\sigma) > \exp\left[\frac{2}{\sigma^2}(H_0 - \delta)\right]$ if σ is small enough.

We now prove the upper-bound by proceeding by a *reductio ad absurdum*. Set $\delta > 0$. We assume that there exists a sequence $(\sigma_n)_n$ which goes to 0 as n goes to infinity such that, for any $n \in \mathbb{N}$, we have:

$$\exp\left[\frac{2}{\sigma_n^2}(H_0 + \delta)\right] \leq T_\kappa(\sigma_n), \quad (2.1)$$

We now introduce the two diffusions $X^{+, \kappa}$ and $X^{-, \kappa}$ by

$$X_t^{\pm, \kappa} = b + \sigma_n B_t - \int_0^t \nabla V(X_s^{\pm, \kappa}) ds - \alpha \int_0^t (X_s^{\pm, \kappa} - (b \pm \kappa)) ds \quad (2.2)$$

From now on, κ is arbitrarily small. By b_κ^\pm , we denote the positive critical point (close to b) of the potential $x \mapsto V(x) + \frac{\alpha}{2}(x - (b \pm \kappa))^2$. By a simple computation, we get:

$$b_\kappa^\pm = b \pm \frac{\alpha}{V''(b) + \alpha} \kappa + o(\kappa).$$

Now, if κ is small enough, we know that the Freidlin-Wentzell theory may be applied to Diffusion $X^{\pm, \kappa}$ and domain $]c; +\infty[$. So, we deduce that

$$\tau_{]c; +\infty[}^\pm(\sigma_n) := \inf \{t \geq 0 : X_t^{\pm, \kappa} \leq c\}$$

satisfies a Kramers' type law. In particular, we have

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left(\exp\left[\frac{2}{\sigma_n^2}(H_\kappa^\pm(c) - \delta)\right] \leq \tau_{]c; +\infty[}^\pm(\sigma_n) \leq \exp\left[\frac{2}{\sigma_n^2}(H_\kappa^\pm(c) + \delta)\right] \right) = 0,$$

for any $\delta > 0$. Here, $H_\kappa^\pm(c) := V(c) - V(b_\kappa^\pm) + \frac{\alpha}{2}(c - b \pm \kappa)^2$.

The main idea now is to compare the exit-time of X with the ones of $X^{\pm, \kappa}$. We have

$$\sup_{0 \leq t \leq \exp\left[\frac{2}{\sigma_n^2}(H_0 - \delta)\right]} \mathbb{E} \left[|X_t - b|^2 \right] < \kappa^2.$$

Consequently, for any $t \in \left[0; \exp\left[\frac{2}{\sigma_n^2}(H_0 - \delta)\right]\right]$, we have $X_t^{-, \kappa} \leq X_t \leq X_t^{+, \kappa}$. As a consequence, if we put $\tau(\sigma_n) := \inf \{t \geq 0 : X_t \leq c\}$, we have

$$\tau_\kappa^-(\sigma_n) \leq \tau(\sigma_n) \leq \tau_\kappa^+(\sigma_n).$$

However, a Kramers' type law is satisfied by $\tau_\kappa^\pm(\sigma_n)$. So, for any $\xi > 0$, we have

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left(\exp\left[\frac{2}{\sigma_n^2}(H_\kappa^-(c) - \xi)\right] \leq \tau(\sigma_n) \leq \exp\left[\frac{2}{\sigma_n^2}(H_\kappa^+(c) + \xi)\right] \right) = 1.$$

Consequently, by taking κ sufficiently small, we obtain that for any $\delta > 0$, we have

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left(\exp\left[\frac{2}{\sigma_n^2}(H_0 - \delta)\right] \leq \tau(\sigma_n) \leq \exp\left[\frac{2}{\sigma_n^2}(H_0 + \delta)\right] \right) = 1. \quad (2.3)$$

By $T_c(\sigma_n)$, we denote the first time that X^{σ_n} returns to $]c; +\infty[$. By proceeding similarly, we have the following inequality:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(T_c(\sigma_n) \leq \exp \left[\frac{2}{\sigma_n^2} (H_0 + \frac{\delta}{2}) \right] \right) = 0. \quad (2.4)$$

Indeed, the exit-cost for going from the left to the right is $W_b(c) - W_b(a') > W_b(c) - W_b(b)$. We recall that a' is the global minimizer of W_b .

Inequalities (2.3) and (2.4) imply the following limit:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(X^{\sigma_n} \exp \left[\frac{2}{\sigma_n^2} (H_0 + \frac{\delta}{2}) \right] \geq c \right) = 0.$$

In particular:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| X^{\sigma_n} \exp \left[\frac{2}{\sigma_n^2} (H_0 + \frac{\delta}{2}) \right] - b \right|^2 \right] \geq (c - b)^2 \geq 2\kappa^2 > \kappa^2.$$

Last limit means that $T_\kappa(\sigma_n) < \exp \left[\frac{2}{\sigma_n^2} (H_0 + \frac{\delta}{2}) \right]$ if n is large enough, which is absurd according to (2.1).

We deduce that Hypothesis (2.1) was wrong. Consequently, we obtain the upper-bound:

$$\exp \left[\frac{2}{\sigma^2} (H_0 + \delta) \right] > T_\kappa(\sigma),$$

if σ is small enough. This achieves the proof.

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