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STOCHASTIC REPRESENTATION OF TAU FUNCTIONS WITH AN APPLICATION TO THE KORTEWEG–DE VRIES EQUATION

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ABSTRACT. In this paper we express the tau functions considered by Pöppe in [23] for the Korteweg de Vries (KdV) equation, as the Laplace transforms of iterated Skorohod integrals. Our main tool is the notion of Fredholm determinant of an integral operator. A stochastic representation of tau functions for the N-soliton solutions of KdV has been proved by Ikeda and Taniguchi in [14]. They express the N-soliton solutions as the Laplace transform of a quadratic functional of N independent Ornstein-Uhlenbeck processes. Our first step is to provide the Wiener chaos decomposition of the underlying functional and to identify the Fredholm determinant of an integral operator in their representation. Our general result goes beyond the N-soliton case and enables us to consider a non soliton solution of KdV associated to a Gaussian process with Cauchy covariance function.

1. Introduction

The solutions of different families of partial differential equations (PDE) can be expressed as the mean of functionals of a stochastic process by the Feynman-Kac formula (cf. [7], [22], [26]). We are interested in Korteweg–de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations and KdV (resp. KP) hierarchies (cf. [18]) which are families of PDE in infinitely many variables that contain KdV (resp. KP). For these PDE no Feynman-Kac formula exists to represent their solutions.

The KdV equation, used to model the time evolution of waves in shallow waters, is one of the simplest and most useful nonlinear equations admitting solitary waves. Two forms of KdV are used classically,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.1}$$

and

$$u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}. \tag{1.2}$$

One can go from one form to the other by a linear change of variables. The KP equation is a generalization to two spatial dimensions x and y of KdV equation

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and is written as

$$\frac{3}{4}u_{yy} = \partial_x \left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx} \right). \tag{1.3}$$

Equation (1.1) is the most encountered in the literature; equation (1.2) is the form that appears when one is interested in the KdV hierarchy (cf. [18]). Let us describe what is meant by the KdV and KP hierarchies. Our reference is the book [18]. Consider a smooth function u of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, the differential operator $P = \frac{\partial^2}{\partial x^2} + u(t,x)$ and fix k. We look for an eigenfunction w satisfying $Pw = k^2 w$. Note first that if u depends only on x, it is possible to find w using the formal expansion

$$w = \mathrm{e}^{kx} \sum_{j \in \mathbb{N}} \frac{w_j}{k^j}. \tag{1.4}$$

By considering that k is an independent variable, one can determine the w_j iteratively given u.

Note moreover that if u depends on x and on t, then w should depend on t also. Suppose that w evolves according to $\frac{\partial w}{\partial t} = Bw$ where B is a linear differential operator. Lax (cf. [16]) proved that the compatibility condition that w must satisfy implies the following identity

$$\frac{\partial P}{\partial t} = [B, P]. \tag{1.5}$$

Identity (1.5) is an identity between differential operators. If we set $B = \frac{\partial^3}{\partial x^3} + f_2 \frac{\partial}{\partial x} + f_3$ and use the fact that $\frac{\partial P}{\partial t} = \frac{\partial u}{\partial t}$, then $f_2 = \frac{3}{2}u$ and $f_3 = \frac{3}{4}u_x$ and (1.5) is equivalent to require that u satisfies KdV (cf. [18]).

Consider now the operator L such that $L^2 = P$. It is proved in [18] that L is a pseudo-differential operator of the form $\sum_{k \in \mathbb{N}} f_k \partial^{1-k}$, with $f_0 = 1$, $f_1 = 0$, $f_2 = \frac{1}{2}u$ and $f_3 = -\frac{1}{4}u_x$. It is also proved there that for any odd integer ℓ the Lie bracket $[P, (L^\ell)_+]$ is a polynomial in u and its derivatives w.r.t. x where the positive part of a pseudo-differential operator $A = \sum_{k \in \mathbb{N}} f_k \partial^{n-k}$ is the classical differential operator $(A)_+ = \sum_{k=0}^{n} f_k \partial^{n-k}$. Note that for $\ell = 3$, $(L^\ell)_+ = B$ and equation (1.5) can be written as

$$\frac{\partial P}{\partial t} = [(L^\ell)_+, P].$$

Given an odd integer ℓ let us introduce an additionnal variable x_l. We still denote by u a function which now depends on t, x and x_l and $P = \frac{\partial^2}{\partial x^2} + u$. The compatibility condition for the linear system of PDEs

$$\begin{cases} Pw = k^2 w, \\ \frac{\partial w}{\partial x_\ell} = (L^\ell)_+ w, \end{cases}$$

takes the Lax form (see [16])

$$\frac{\partial P}{\partial x_\ell} = [(L^\ell)_+, P]. \tag{1.6}$$

Since $\frac{\partial P}{\partial x_\ell} = \frac{\partial u}{\partial x_\ell}$ another form of (1.6) is $\frac{\partial u}{\partial x_\ell} = [(L^\ell)_+, P]$. Assume that u depends on the infinite family of variables (x_ℓ). Then the family of PDEs (1.6) in u obtained when ℓ describes the set of odd integers is called the KdV hierarchy. If we take $\ell = 3$ and set $x_3 = t$ equation (1.6) coincides with KdV equation. Let us set $K_\ell(u) = [(L^\ell)_+, P]$. Besides the fact that they express compatibility conditions of linear systems of PDEs as stated above, the KdV hierarchy (1.6) is an infinite family of commuting symmetries of the KdV equation since $\frac{\partial K_\ell(u)}{\partial x_j} = \frac{\partial K_j(u)}{\partial x_\ell}$ holds for any pair $\ell \neq j$ (see [18]). The above results can be extended to a pseudo-differential operator

$$L = \partial + \sum_{p\in\mathbb{N}^*} f_p\, \partial^{-p} \tag{1.7}$$

more general than $(\partial^2 + u)^{\frac{1}{2}}$. One looks for a solution to $Lw = kw$ of the form

$$w = \mathrm{e}^{\xi(k,\mathbf{x})} \sum_{j\in\mathbb{N}^*} \frac{w_j}{k^j}, \tag{1.8}$$

where $\mathbf{x} = (x_j,\, j \in \mathbb{N})$ is an infinite vector of variables and $\xi(k,\mathbf{x}) = \sum_{j\in\mathbb{N}^*} x_j\, k^j$, assuming for any integer j that the evolution $\frac{\partial w}{\partial x_j} = (L^j)_+ w$ is satisfied. The compatibility condition for the system

$$\begin{cases} Lw = kw, \\ \frac{\partial w}{\partial x_j} = (L^j)_+ w, \end{cases}$$

takes the form

$$\frac{\partial L}{\partial x_j} = [(L^j)_+, L]. \tag{1.9}$$

The family of PDEs (1.9) when $j \in \mathbb{N}^*$ is called the KP hierarchy.

Remark 1.1. The KP hierarchy is an infinite set of PDEs in the infinite set of functions $f_p, p \in \mathbb{N}^*$ and the infinite set of variables x_j.

Remark 1.2. If the operator L in (1.7) satisfies $(L^2)_- = 0$ the family of PDEs (1.9) reduces to the KdV hierarchy, where the negative part of a pseudo-differential operator A is defined by $(A)_- := A - A_+$.

In the study of KdV, KP and their hierarchies, the so-called *tau functions* such that derivatives of their logarithm are solutions, play a key role. Let us introduce them on a fundamental example. A similar result, that we detail at the end of this introduction, holds for KP. The famous method of Inverse Scattering Transform introduced by Gardner, Greene, Kruskal and Miura in [9] and [10] produces, for each integer $N \geq 1$, solutions to the KdV equation (1.2) of the form

$$u(x,t) = 2\frac{d^2}{dx^2} \log\det(I + G(x,t)), \tag{1.10}$$

where

$$G_{ij}(x,t) := \frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x-(\eta_i^3+\eta_j^3)t} \tag{1.11}$$

is a $N \times N$-matrix and $m_i, \eta_i > 0$, $1 \leq i \leq N$ are constants called *scattering data*. In this case

$$\tau(x,t) := \det(I + G(x,t)), \tag{1.12}$$

is a tau function and the solutions of the form (1.10) are called N-solitons. For more details we refer the reader to [6].

Another method to obtain tau functions for KdV has been developed by Hirota in [12] (see details in Section 3 and also [13]). The class of solutions obtained by this method is larger than the family of N-solitons described in (1.10)-(1.11). In [23] (see also [24]) Pöppe has proved that these tau functions can be expressed as Fredholm determinants of some integral operators as a consequence of the dressing method introduced by Zakharov and Shabat in [29]. Fredholm determinants are an extension of determinants of finite size matrices to integral operators (see Section 3.2 and [27]).

In this paper we prove a stochastic representation of the tau functions obtained by Pöppe in [23]. The question of finding a stochastic representation of tau functions has been addressed by Ikeda and Taniguchi in [14] for KdV. Their result was later extended to the KP hierarchy by Aihara, Akahori, Fujii and Nitta in [1]. Both papers consider tau functions that generate N-solitons and can be written as the determinant of a finite size matrix (cf. equation (1.12) for KdV). In both papers a tau function is expressed as the Laplace transform of a quadratic functional of N independent Gaussian processes: Ornstein–Uhlenbeck processes in [14] and Lévy areas in [1].

Fredholm determinant is the key tool to obtain our results. It appears in Probability Theory in the change of measure formulas on Wiener spaces (cf. for instance [4] and [28]). Moreover it is proved in [11] and [25] that in some cases a Fredholm determinant coincides with the Laplace transform of a random variable belonging to a Wiener chaos (see [20]) of order lower than 2. To start with, we identify the Fredholm determinant of an integral operator (see Theorem 4.1) in the result obtained in [14]. We express this operator using the scattering data and we provide the Wiener chaos decomposition of the underlying variable (cf. Theorem 4.3). Then we prove that the tau functions of [23] admit a stochastic representation as Laplace transforms of iterated Skorohod integrals of order 2 (see Theorem 4.6). This enables us to show that in some cases the tau function is the Laplace transform of the integral of the square of a Gaussian process with respect to some measure (see Theorem 4.8). When this measure is a linear combination of Dirac massses the associated tau function generates a N-soliton (1.10)-(1.11) (see Corollary 4.10). Thus our representation extends the result of [14] to solutions of KdV that are not necessarily N-solitons. We extend our results to the KP hierarchy in Theorem 4.12.

Let us now detail the analogue of (1.10)-(1.11) for the KP hierarchy. In this case the N-soliton tau function has the form

$$\tau(x_1, x_2, \dots) = \det(I + G(x_1, x_2, \dots)), \tag{1.13}$$

where G is the square matrix

$$G(x_1, x_2, \dots) := \left(\frac{\sqrt{m_i m_j}}{p_i - q_j} e^{-\frac{1}{2}\sum_{l=1}^{\infty}\{(p_i^l - q_i^l) + (p_j^l - q_j^l)\}x_l} \right)_{1 \le i,j \le N}, \tag{1.14}$$

and $p_i, q_i \in \mathbb{R}$ with $p_i \neq q_i$ and $m_i > 0$, $1 \le i \le N$ are constants.

In particular if we choose $x_1 = x$, $x_2 = y$, $x_3 = t$ and $x_k = 0$ for all $k \geq 4$, the function

$$u(x, y, t) := -2\partial_x^2 \log \tau(x, y, t, 0, \ldots),$$

is a solutions of the KP equation (1.3). Moreover if we consider positive constants η_i, $i = 1, \ldots, N$ and set $p_i = \eta_i$ and $q_i = -\eta_i$ for $i = 1, \ldots, N$, then by choosing $x_1 = x$, $x_3 = t$, $x_k = 0$ for $k \geq 4$, matrix (1.14) reduces to matrix (1.11), (1.13) becomes (1.12) and we retrieve the solution (1.10) of the KdV equation (1.2).

The paper is organized as follows. In Section 2 we present the stochastic representations of soliton tau functions obtained in [14] (Section 2.1) and [1] (Section 2.2). Section 3 is devoted to Hirota's method (Section 3.1), a short introduction to Fredholm determinants (Section 3.2) and the related results of Pöppe in [23] (Section 3.3). Our results are stated and proved in Section 4.

2. Previous Results in the N-soliton Case

2.1. Stochastic representation of the N-soliton tau function of the KdV hierarchy. The following theorem proved in [14] provides a stochastic representation of the N-soliton tau functions for the KdV hierarchy. For simplicity we present it for the KdV equation.

Let $N \in \mathbb{N}^\star$, $p \in \mathbb{R}^N$ with $p_i \neq p_j$ for $i \neq j$, $D := \operatorname{diag}\{p_1, \ldots, p_N\}$. We denote by ξ_p the $\mathbb{R}^N$-valued Ornstein-Uhlenbeck process defined as the unique solution of the stochastic differential equation

$$d\xi_p(s) = dW(s) + D\xi_p(s)ds, \quad \xi_p(0) = 0, \tag{2.1}$$

where W is a N-dimensional Wiener process. Given a positive vector $c \in \mathbb{R}^N$ we define the process $X_{p,c}$ by

$$X_{p,c}(s) := \langle c, \xi_p(s)\rangle, \quad s \geq 0, \tag{2.2}$$

and for $a > 0$ we introduce the symmetric $N \times N$-matrix

$$E(a) := D^2 + a^2 c \otimes c, \tag{2.3}$$

which can be written as UR^2U^{-1}, where R (resp. U) is a diagonal (resp. orthogonal) matrix. Then we set

$$\zeta(z, t) := zR + tR^3, \tag{2.4}$$

for $(z, t) \in \mathbb{R}^2$.

Theorem 2.1. *(cf. [14]) Let x and t be positive real numbers. Let*

$$\beta_{a,x,t}(y) := -((\partial_z \phi_a)\phi_a^{-1})(x - y, t), \tag{2.5}$$

where

$$\phi_a(z, t) := U\{\cosh(\zeta(z, t)) - \sinh(\zeta(z, t))R^{-1}U^{-1}DU\}U^{-1}. \tag{2.6}$$

If we set

$$I_{p,c,a}(x, t) := \mathbb{E}\left[e^{-\frac{a^2}{2}\int_0^x X_{p,c}(y)^2 dy + \frac{1}{2}\langle(\beta_{a,x,t}(x) - D)\xi_p(x), \xi_p(x)\rangle}\right], \tag{2.7}$$

then $u(x, t) := -4\,\partial_x^2 \log I_{p,c,a}(x, t)$ satisfies the KdV equation (1.2).

Theorem 2.1 is inspired by a result of Cameron and Martin (cf. [3]). The key argument in its proof is the identity

$$I_{p,c,a}(x,t) = (\det \phi_a(0,t)/\det \phi_a(x,t))^{1/2} \cdot e^{-\frac{x}{2}\mathrm{Tr}\ D}, \tag{2.8}$$

which is a consequence of Girsanov Theorem and of the fact that ϕ_a satisfies $\partial_{xx}\phi_a - E(a)\phi_a = 0$.

2.2. Stochastic representation of the N-soliton tau function of the KP hierarchy. In [1] the N-soliton tau functions of the KP hierarchy (cf. (1.13)-(1.14)) are related to the Laplace transform of some generalized stochastic area functionals. This paper provides an extension of the results in [14] since the tau functions in [14] are particular cases of those in [1].

Let $W^l = (W^{l,1}, W^{l,2})$, $l = 1, \ldots, N$ be mutually independent two-dimensional Brownian motions starting at the origin. The stochastic area of W^l is

$$S^l := \int_0^1 (W_s^{l,2} dW_s^{l,1} - W_s^{l,1} dW_s^{l,2}).$$

Let $\Lambda := \mathrm{diag}\ \{\lambda_1, \ldots, \lambda_N\}$ where λ_l, $l = 1, 2, \ldots, N$ are positive numbers. Let C be a $N \times N$ real matrix and $C^{\pm}$ be its symmetric and skew-symmetric parts $C^{\pm} = (C \pm C^{\star})/2$. Let us set $\mathbf{W}_t^i = (W_t^{1,i}, \ldots, W_t^{N,i})$ for $i = 1, 2$, and define for any complex number σ

$$\hat{S}(\sigma) := \sigma \sum_{l=1}^{N} \lambda_l S^l + \sigma \langle \Lambda^{1/2} C^- \Lambda^{1/2} \mathbf{W}_1^1, \mathbf{W}_1^2 \rangle - \frac{\sigma^2}{2} \sum_{k=1,2} \langle \Lambda^{1/2} C^+ \Lambda^{1/2} \mathbf{W}_1^k, \mathbf{W}_1^k \rangle. \tag{2.9}$$

In the following we denote by $\mathbf{i}$ the complex square root of -1. The result of [1] is to prove a relationship between the N-soliton tau function given in (1.13)-(1.14) and $\mathbb{E}\ [e^{\hat{S}(\mathbf{i})}]$ for some $\hat{S}(\mathbf{i})$.

Theorem 2.2. *(cf. [1]) Let $p_k \in \mathbb{R}$, $q_k \in \mathbb{R}$ and $m_k > 0$, $1 \le k \le N$, such that $p_k \neq q_\ell$ for all $k \neq \ell$. Let $\tau(x_1, x_2, \ldots)$ be the N-soliton tau function defined by (1.13)-(1.14).*
Define $P := (1/(p_k - q_\ell))_{1 \le k,\ell \le N}$ and assume that $\min_{k,\ell} |p_k - q_\ell|$ is sufficiently large so that $I + P$ is invertible.
Define $\xi_k := \sum_{n=1}^{\infty} (p_k^n - q_k^n) x_n$, $\lambda_k := \frac{1}{2}(\xi_k - \log m_k)$ for $1 \le k \le N$ and $C := (I - P)(I + P)^{-1}$.

Consider $\hat{S}(\mathbf{i})$ associated to $(\lambda_k; 1 \le k \le N)$ and C by (2.9) with $\sigma = \mathbf{i}$. Then

$$\frac{\exp\left(-\frac{1}{2}\sum_{k=1}^{N} \xi_k\right)}{\mathbb{E}\ [\exp \hat{S}(\mathbf{i})]},$$

is a N-soliton tau function of the KP hierarchy proportional to $\tau(x_1, x_2, \ldots)$.

Proof. Since G given by (1.14) satisfies $G = e^{-\Lambda}Pe^{-\Lambda}$, we have $\tau = \det(I + e^{-\Lambda}Pe^{-\Lambda}) = \det(I + Pe^{-2\Lambda})$. Moreover, if we set $M := \cosh\Lambda + C\sinh\Lambda$ we have

$$\begin{aligned}\det(M) &= \det\left(\frac{e^{\Lambda}+e^{-\Lambda}}{2} + C\frac{e^{\Lambda}-e^{-\Lambda}}{2}\right)\\ &= 2^{-n}\det((I+C)e^{\Lambda} + (I-C)e^{-\Lambda})\\ &= 2^{-n}\det((I+C)e^{\Lambda})\det(I + (I-C)(I+C)^{-1}e^{-2\Lambda})\\ &= 2^{-n}\det(I+C)\exp\left(\sum_{k=1}^{N}(\xi_k + \log m_k)/2\right)\det(I+Pe^{-2\Lambda}).\end{aligned}$$

The latter equality holds true since $C = (I-P)(I+P)^{-1}$ is equivalent to $P = (I-C)(I+C)^{-1}$. The main step comes from the result of Theorem 2 in [1] that shows the identity

$$\mathbb{E}[\exp\hat{S}(\mathbf{i})] = \det(\cosh\Lambda + C\sinh\Lambda)^{-1}, \tag{2.10}$$

for $\max_k|\lambda_k|$ and $||C^+||$ sufficiently small. □

Corollary 2.3. *Let η_k and m_k, $k = 1,\dots,N$, be positive constants. Define $P = (1/(\eta_k+\eta_\ell))_{1\le k,\ell\le N}$ and $\lambda_k := \eta_k x + \eta_k^3 t - \frac{1}{2}\log m_k$. If $\min_{k,\ell}(\eta_k+\eta_\ell)$ is sufficiently large so that $I+P$ is invertible and let $C := (I-P)(I+P)^{-1}$, then the N-soliton solution (1.10)-(1.11) of the KdV equation is equal to $-2\partial_x^2\log\mathbb{E}\,[\exp\hat{S}(\mathbf{i})]$ where $\hat{S}(\mathbf{i})$ is associated to $(\lambda_k;\, 1\le k\le N)$ and C by (2.9) with $\sigma = \mathbf{i}$.*

Proof. Corollary 2.3 is deduced from Theorem 2.2 by choosing $p_i = \eta_i$ and $q_i = -\eta_i$ for $i = 1,\dots,N$, $x_1 = x$, $x_3 = t$, $x_k = 0$ for $k \ge 4$. □

Note that Theorem 2.2 provides a tau function valid for a whole family of PDE whereas Corollary 2.3 expresses a tau function for a given equation in this family.

3. Tau Functions of KdV: Hirota's Method and Fredholm Determinants

3.1. Hirota's bilinear operator and tau function. It is easy to check that if $u = -2\partial_x^2\log\tau$ is a solution of (1.1) then τ solves

$$\tau\tau_{xt} - \tau_x\tau_t + \tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3\tau_{xx}^2 = 0. \tag{3.1}$$

Hirota's method (cf. [13]) to construct solutions to the KdV equation consists in solving (3.1). In order to do so Hirota introduced a bilinear operator $(a,b)\mapsto \mathbf{D}_t^m\mathbf{D}_x^n(a\cdot b)$ defined for two sufficiently smooth functions a and b of two variables (x,t) and $m,n\in\mathbb{N}$ by

$$\mathbf{D}_t^m\mathbf{D}_x^n(a\cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t)b(x',t')\Big|_{x'=x,\ t'=t}. \tag{3.2}$$

For instance, when $n = m = 1$,

$$\mathbf{D}_t\mathbf{D}_x(a\cdot b) = a_{xt}b + ab_{xt} - a_tb_x - a_xb_t.$$

With this operator, equation (3.1) takes the simple form

$$\mathbf{D}_x(\mathbf{D}_t + \mathbf{D}_x^3)(\tau\cdot\tau) = 0. \tag{3.3}$$

One obtains N-soliton solutions of KdV by looking for solutions of (3.3) as power series in a small parameter ϵ,

$$\tau = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \cdots . \tag{3.4}$$

If τ satisfies (3.3) and (3.4) and if moreover we choose $\tau_1 = \sum_{\ell=1}^{N} c_\ell e^{\kappa_\ell x - \kappa_\ell^3 t}$ for positive parameters c_ℓ, κ_ℓ, we find that τ_ℓ must vanish for $\ell > N$ and we can solve (3.3) for $\tau_2, \dots, \tau_N$ iteratively. The solution τ thus obtained coincides with the determinant of the $N \times N$ matrix (cf. [6])

$$A_{k\ell}(x,t) = \delta_{k\ell} + c_i^2 \frac{e^{-(\kappa_k+\kappa_\ell)x}}{\kappa_k + \kappa_\ell} e^{8\kappa_k^3 t}. \tag{3.5}$$

For $N = 1$, $\tau(x,t) = 1 + e^{8t-2x}$ solves (3.3) and the corresponding solution of KdV is the 1-soliton solution $u = -2\partial_x^2 \log \tau(x,t) = -2\mathrm{sech}^2(x - 4t)$.

3.2. Fredholm determinants. Let us now introduce briefly the Fredholm determinants. We refer to [27] for more details. Consider an integral operator

$$\mathcal{A}f(x) := \int_a^b K(x,y) f(y) dy, \tag{3.6}$$

with continuous kernel K and $a < b$. The integral equation $(I + z\mathcal{A})f = g$ where g is a given continuous function admits a unique continuous solution f if and only if $D(z) \neq 0$ where D is the entire function

$$D(z) := 1 + \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell!} \int_a^b \cdots \int_a^b \begin{vmatrix} K(s_1,s_1) & \cdots & K(s_1,s_\ell) \\ \vdots & \ddots & \vdots \\ K(s_\ell,s_1) & \cdots & K(s_\ell,s_\ell) \end{vmatrix} ds_1 \dots ds_\ell, \quad z \in \mathbb{C}. \tag{3.7}$$

This result was proved by Fredholm in [8].

Definition 3.1. $D(z)$ is called the *Fredholm determinant* of $I + z\mathcal{A}$ and is written $\det(I + z\mathcal{A})$.

Let us moreover denote by $\mathcal{J}_\infty(H)$ the set of compact operators on a complex separable Hilbert space H. If $\mathcal{A} \in \mathcal{J}_\infty(H)$, the operator $|\mathcal{A}| := (\mathcal{A}^\star\mathcal{A})^{1/2}$ is positive semidefinite and we denote by $(\lambda_n(\mathcal{A}))_{n=1}^N$, $N \leq \infty$, the set of its eigenvalues. Let

$$\mathcal{J}_p(H) := \{\mathcal{A} \in \mathcal{J}_\infty(H);\ \sum_{n=1}^{N} \lambda_n(\mathcal{A})^p < \infty\}, \quad 1 \leq p < \infty.$$

The elements of $\mathcal{J}_1(H)$ (resp. $\mathcal{J}_2(H)$) are called *trace class* (resp. *Hilbert-Schmidt*) operators. If $\mathcal{A} \in \mathcal{J}_1(H)$,

$$\mathrm{Tr}\ \mathcal{A} := \sum_{n=1}^{N} \lambda_n(\mathcal{A}), \tag{3.8}$$

$$\det(I + z\mathcal{A}) = \prod_{n=1}^{N} (1 + z\lambda_n(\mathcal{A})). \tag{3.9}$$

If $\mathcal{A} \in \mathcal{J}_2(H)$ the right-hand side of (3.9) does not necessarily converge when $N = +\infty$. However in that case we can define the Carleman-Fredholm determinant

$$\det{}_2(I + z\mathcal{A}) := \prod_{n=1}^{N} (1 + z\lambda_n(\mathcal{A}))e^{-z\lambda_n(\mathcal{A})}, \tag{3.10}$$

which satisfies

$$\det{}_2(I + z\mathcal{A}) = \det(I + z\mathcal{A})e^{-z\mathrm{Tr}\ \mathcal{A}}. \tag{3.11}$$

Note that if $\mathcal{A} \in \mathcal{J}_1(H)$ is given by (3.6),

$$\mathrm{Tr}\ \mathcal{A} = \int_a^b K(x,x)dx. \tag{3.12}$$

3.3. Tau functions of the KdV equation as Fredholm determinants. In [23] Pöppe has expressed the tau functions for KdV obtained by Hirota's method as Fredholm determinants of integral operators of the form (3.6) with $b = +\infty$. The interval of integration being unbounded, we need to specify the set of functions f for which $\mathcal{A}f$ is well defined and to provide assumptions on their kernel so that their trace (3.8) and their determinant (3.7) exist (cf. [19]).

Let $1/2 < \nu \leq 1$ and define

$$C_\nu := \{f \in C([0,+\infty),\mathbb{C});\ ||f||_\nu := \sup_{s\geq 0} |f(s)|(1+s)^\nu < \infty\},$$

where $C([0,+\infty[,\mathbb{C})$ denotes the set of continuous functions from $[0,+\infty[$ to $\mathbb{C}$. We denote by LC_ν the set of integral operators of the form (3.6) defined on C_ν such that

$$\sup_{s,t\geq 0} (1+s)^\nu(1+t)^\nu|K(s,t)| < \infty. \tag{3.13}$$

Then Tr $\mathcal{A}$ and $\det(I+z\mathcal{A})$ for $\mathcal{A} \in LC_\nu$ are given by (3.7) and (3.12) respectively, with $b = +\infty$. Moreover $\lambda \mapsto p(\lambda) := \det(I + \lambda\mathcal{A})$ is analytic on $\mathbb{C}$.

If K depends on a parameter x and is differentiable with respect to x, then p satisfies

$$\partial_x\ p(\lambda) = p \cdot \mathrm{Tr}\ (\partial_x(\lambda\mathcal{A})(I + \lambda\mathcal{A})^{-1}), \tag{3.14}$$

for all $\lambda \in \mathbb{C}$ such that $(I + \lambda\mathcal{A})$ is invertible.

Theorem 3.2. *(cf. [23]) Consider a kernel $F(t,x)$ such that F and its derivatives up to order 4 (resp. 2) in x (resp. t) are in $C_{2\nu}$. Assume moreover that F solves the linearized KdV equation*

$$F_t + 8F_{xxx} = 0. \tag{3.15}$$

Let $\mathcal{F}_{(x,t)}$ be the integral operator defined by

$$\mathcal{F}_{(x,t)}f(s) := \int_0^{+\infty} F(s + u + 2x, t)f(u)du. \tag{3.16}$$

Then,

$$\tau(x,t;\lambda) := \det(I + \lambda\mathcal{F}_{(x,t)}), \tag{3.17}$$

solves Hirota equation (3.3) for every $\lambda \in \mathbb{C}$. Moreover, if the function τ is nowhere vanishing,

$$u(x,t) := -2\frac{\partial}{\partial x}\left(\frac{\tau_x}{\tau}\right),$$

is a solution of the KdV equation (1.1).

The proof of Theorem 3.2 is interesting since it is rather simple and can be adapted to other nonlinear equations such as the KP equation. It makes use of the continuous functional $[\cdot]$ defined on LC_ν by

$$[\mathcal{A}] := -K(0,0).$$

If we denote for simplicity $\mathcal{F}_{(x,t)}$ by $\mathcal{F}$, then $\mathcal{F}$ belongs to LC_ν and satisfies

$$[\mathcal{F}] = \mathrm{Tr}\ (\partial_x\ \mathcal{F}) \quad \text{and} \quad [(I+\mathcal{F})^{-1}\mathcal{F}] = \mathrm{Tr}\ (\partial_x\ \mathcal{F})(I+\mathcal{F})^{-1}.$$

From (3.14) we recognize in the latter identity the derivative with respect to x of $\log\det(I+\mathcal{F})$.

4. Main Results

4.1. Identification of a Fredholm determinant in the representation of the N-soliton tau function of KdV. In this section we keep the notations of Section 2.1. The stochastic representation of a tau function of KdV given in [14] actually involves a Fredholm determinant as stated in the following theorem.

Theorem 4.1. *Let us define the Volterra integral operator V on the space of continuous functions from $[0,x]$ to $\mathbb{R}^d$ by*

$$V_{(x,t)}(f)(y) := \int_0^y \phi_a'(u,t)\phi_a^{-1}(u,t)f(u)du, \quad y\in[0,x]. \tag{4.1}$$

Then $u(x,t) := -4\ \partial_x^2 \log\det(I - V_{(x,t)}/2)$ satisfies the KdV equation (1.2).

Proof. Let $I_{p,c,a}$ defined in (2.7). We prove that

$$\det(I - V_{(x,t)}/2) = I_{p,c,a}(x,t).$$

Then the result follows from Theorem 2.1.
The kernel of V,

$$K_{(x,t)}(v,u) := \phi_a'(u,t)\phi_a(u,t)^{-1}\mathbf{1}_{0\le u\le v\le x},$$

being continuous, Tr $V_{(x,t)}$ is well defined and given by Tr $V_{(x,t)} = \int_0^x K_{(x,t)}(s,s)ds$. From [27], the Fredholm determinant of $I-\lambda V_{(x,t)}$, $\lambda\in\mathbb{R}$ can be expressed using the trace of $V_{(x,t)}$ and of its powers as follows

$$\det(I-\lambda V_{(x,t)}) = \exp\left(-\sum_{n=1}^{\infty}\frac{\mathrm{Tr}\ V_{(x,t)}^n}{n}\lambda^n\right).$$

Moreover Tr $V_{(x,t)}^n = 0$ for all $n>1$ as a consequence of Lemma 4.2 below. Using Jacobi's formula, we obtain

$$\begin{aligned}\det(I-\lambda V_{(x,t)}) &= \exp\left(-\lambda \mathrm{Tr}\ \int_0^x K_{(x,t)}(s,s)ds\right)\\ &= \exp\left(-\lambda\int_0^x \mathrm{Tr}\ (\phi_a'(s,t)\phi_a^{-1}(s,t))ds\right) = \left(\frac{\det(\phi_a(0,t))}{\det(\phi_a(x,t))}\right)^\lambda.\end{aligned}$$

We conclude by choosing $\lambda = 1/2$ and thanks to identity (2.8). □

Lemma 4.2. *(cf. [15]) Let T_1, $T_2, \ldots, T_\ell$ be Hilbert-Schmidt operators with respective kernels G_1, G_2, ..., G_ℓ in $L^2(\mathbb{R}^2; \mathbb{C}^{N\times N})$. Then we have for $\ell \geq 2$*

$$Tr\ (T_1 T_2 \cdots T_\ell) = Tr \int_{\mathbb{R}^\ell} G_1(s_1, s_2) G_2(s_2, s_3) \cdots G_\ell(s_\ell, s_1) ds_\ell \cdots ds_1.$$

4.2. Fredholm determinant and scattering data. The operator defined in (4.1) depends on ϕ_a. It would be preferable to exhibit an integral operator whose kernel depends explicitly on the parameters a, c_i, p_i in (2.1), (2.2) and (2.3) that play the role of scattering data, and such that its Fredholm determinant coincides with $I_{p,c,a}$.

Let $\Delta := [0, x] \times \{1, \ldots, N\}$ equipped with $\mu = \lambda \otimes \nu$ the tensor product of the Lebesgue measure on $[0, x]$ with the counting measure on $\{1, \ldots, N\}$. We use the notations of [11] and [20] for the stochastic integration $I_1(f_1)$ of a function $f_1 \in L^2(\Delta, \mu)$

$$I_1(f_1) = \int_\Delta f_1(\eta) dW_\eta = \sum_{i=1}^N \int_0^x f_1(u, k) dW_u^i,$$

and the double stochastic integral $I_2(f_2)$ of a symmetric function $f_2 \in L^2(\Delta^2, \mu^2)$,

$$\begin{aligned} \frac{1}{2} I_2(f_2) &= \int_{\Delta_2} f_2(\eta_1, \eta_2) dW_{\eta_1} dW_{\eta_2} \\ &= \sum_{i_1, i_2 = 1}^N \int_0^x \int_0^{s_2} f_2((s_1, i_1), (s_2, i_2)) dW_{s_1}^{i_1} dW_{s_2}^{i_2}, \end{aligned}$$

where $(W^i)_{1 \leq i \leq N}$ are N independent Brownian motions and

$$\Delta_2 := \{((s_1, i_1), (s_2, i_2)) \in \Delta^2 \,;\, 0 \leq s_1 \leq s_2 \leq x\}.$$

Theorem 4.3. *Let $\mathcal{C}$ denote the integral operator $\mathcal{C}\psi(\cdot) := \int_\Delta f_2(\cdot, \eta)\psi(\eta) d\mu(\eta)$, where*

$$\begin{aligned} f_2((v, \ell), (u, k)) :=& a^2 \frac{c_k c_\ell}{p_k + p_\ell} \left(e^{(p_k + p_\ell)x} - e^{(p_k + p_\ell) u \vee v} \right) e^{-p_k u} e^{-p_\ell v} \\ &- (\beta_{a,x,t}(x) - D)_{k,\ell} e^{(p_k + p_\ell)x - u p_k - v p_\ell}, \end{aligned}$$

$u \vee v$ denotes $\max(u, v)$ *and $\beta_{a,x,t}$ was defined in (2.5). Then*

$$u(x, t) := -4\ \partial_x^2 \log \left(\det(I + \mathcal{C}) \right)^{-1/2}$$

satisfies the KdV equation (1.2).

Proof. Let $I_{p,c,a}$ defined in (2.7). We prove that $\left(\det(I + \mathcal{C})\right)^{-1/2} = I_{p,c,a}(x, t)$. For simplicity we write the proof in detail only for the case $t = 0$. The arguments are similar for $t > 0$. When $t = 0$, $\beta_{a,x,t}(x) = D$ (cf. (2.5)-(2.6)). Let f be the $\mathbb{R}^N$-valued function given by $f_k(u) := c_k e^{p_k(y-u)} \mathbf{1}_{[0,y]}(u)$. Then, $X_{p,c}$ defined in (2.2) satisfies

$$X_{p,c}(y) = \sum_{k=1}^N c_k \int_0^y e^{p_k(y-u)} dW_u^k = I_1(f).$$

Denoting by D the Malliavin derivative operator, we have for $(u,k)\in\Delta$,

$$D_{(u,k)}X^2_{p,c}(y) = 2\ D_{(u,k)}X_{p,c}(y)\cdot X_{p,c}(y) = 2\ \mathbf{1}_{[0,y]}(u)c_k e^{p_k(y-u)}X_{p,c}(y).$$

Hence,

$$\begin{aligned}
D_{(u,k)}&\int_0^x X^2_{p,c}(y)dy \\
&= \int_0^x 2\ \mathbf{1}_{[0,y]}(u)c_k e^{p_k(y-u)}X_{p,c}(y)dy \\
&= \int_0^x 2c_k e^{p_k(y-u)}\mathbf{1}_{[0,y]}(u)\sum_{\ell=1}^N\int_0^y c_\ell e^{p_\ell(y-v)}dW^\ell_v dy \\
&= 2\sum_{\ell=1}^N\int_0^x \frac{c_kc_\ell}{p_k+p_\ell}\left(e^{(p_k+p_\ell)x}-e^{(p_k+p_\ell)u\vee v}\right)e^{-p_ku}e^{-p_\ell v}dW^\ell_v.
\end{aligned}$$

Thus, $D_{(u,k)}\int_0^x X^2_{p,c}(y)dy = 2I_1(f_2(\cdot,(u,k)))$ with

$$f_2((v,\ell),(u,k)) := \frac{c_kc_\ell}{p_k+p_\ell}\left(e^{(p_k+p_\ell)x}-e^{(p_k+p_\ell)u\vee v}\right)e^{-p_ku}e^{-p_\ell v}.$$

Note that f_2 is symmetric with respect to its two variables (u,k) and (v,ℓ). We then obtain the following expansion of $\int_0^x X_{p,c}(y)^2dy$ into a finite sum of multiple stochastic integrals of a symmetric function

$$\int_0^x X_{p,c}(y)^2dy = \sum_{k=1}^N \frac{c_k^2}{4p_k^2}\left(e^{2p_kx}-2p_kx-1\right)+I_2(f_2). \tag{4.2}$$

We can apply Proposition 4.4 below to $I_2(f_2)$. Indeed thanks to (4.2) we know that $\mathbb{E}\left[e^{-I_2(f_2)}\right]$ is well defined. Moreover Remark 4.5 following Proposition 4.4 ensures that $(I+\mathcal{C})$ has positive spectrum. Therefore (4.3) is valid and we have

$$\mathbb{E}\left[e^{-I_2(f_2)}\right] = \left(\det_2(I+\mathcal{C})\right)^{-1/2}.$$

This implies

$$\mathbb{E}\left[e^{-\frac{a^2}{2}\int_0^x X_{p,c}(y)^2dy}\right] = \left(\det_2(I+\mathcal{C})\right)^{-1/2} e^{-\frac{a^2}{2}\sum_k \frac{c_k^2}{4p_k^2}\left(e^{2p_kx}-2p_kx-1\right)}.$$

The determination of the trace

$$\mathrm{Tr}(\mathcal{C}) = \sum_{k=1}^N\int_0^x f_2((u,k),(u,k))du = \sum_{k=1}^N\frac{c_k^2}{4p_k^2}\left(e^{2p_kx}-2p_kx-1\right),$$

and identity (3.11) enable us to conclude. □

Proposition 4.4. *(cf. [11]) Let $B\in L^2(\Delta,\mu)$ and let C be the kernel of a symmetric Hilbert-Schmidt operator $\mathcal{C}$ on $L^2(\Delta,\mu)$ such that $(I+\mathcal{C})$ has positive spectrum. Let Y be a random variable admitting the following Wiener chaos decomposition*

$$Y = \int_\Delta B(\eta)dW_\eta + \int_{\Delta_2} C(\eta_1,\eta_2)dW_{\eta_1}\ dW_{\eta_2}.$$

Then $\mathbb{E}\left[e^{-Y}\right]$ *is well defined and satisfies*

$$\mathbb{E}\left[e^{-Y}\right] = [\det_2(I+\mathcal{C})]^{-1/2} \exp\left[\frac{1}{2}\int_{\Delta_2} B(\eta_1)(I+\mathcal{C})^{-1}(\eta_1,\eta_2)B(\eta_2)d\eta_1\ d\eta_2\right], \tag{4.3}$$

where $\det_2$ *is the Carleman-Fredholm determinant and we use the same notation for the operator* $(I+\mathcal{C})^{-1}$ *and its kernel.*

Remark 4.5. It has been proved in [5] and [11] that the existence of the expectation (4.3) is actually equivalent to the assumption that $(I+\mathcal{C})$ has positive spectrum.

4.3. Stochastic representation of general tau functions. In this section we consider the form (1.1) of the KdV equation. We express the tau functions obtained by Pöppe in [23] (cf. Theorem 3.2) as the Laplace transforms of some second order Wiener functionals. The fact that these tau functions are Fredholm determinants is a key tool. The tau functions which produce soliton solutions are retrieved as a particular case.

We denote by δ the Skorohod integral defined as the adjoint of the Malliavin derivative D. For properties of this integral we refer the reader to the Section 1.3 of [20]. In the proposition below, the notation $\delta(\delta(\phi))$ denotes the iterated Skorohod integral of ϕ which is the adjoint of D^2 (cf. [21]).

Theorem 4.6. *Let* F *be a solution of (3.15) that satisfies the assumptions of Theorem 3.2 and set* $\phi_{(x,t)}(a,b) := F(a+b+2x,t)$*. Then for all* x *and* t *such that* $||\phi_{(x,t)}||_{L^2(\mathbb{R}^2_+)} < 1$,

$$\tau(x,t) := \mathbb{E}[e^{-\frac{1}{2}\delta(\delta(\phi_{(x,t)}))-\frac{1}{4}\int_0^\infty F(s+2x,t)ds}]^{-2}, \tag{4.4}$$

is a tau function of the KdV equation (1.1) associated to the solution $u(x,t) = -2\partial_x^2 \log\tau(x,t)$.

Proof. For simplicity we omit the subscripts (x,t) in $\phi_{(x,t)}$. Using Proposition 4.7 below and (3.11) we obtain the equality

$$\mathbb{E}[e^{-\frac{1}{2}\delta(\delta(\phi))-\frac{1}{2}\mathrm{Tr}\ \phi}]^{-2} = \det(I+\phi).$$

Moreover $\mathrm{Tr}\ \phi = \int_0^\infty \phi(s,s)ds = \frac{1}{2}\int_0^\infty F(s+2x,t)ds$. Then Theorem 3.2 implies that

$$\tau(x,t) := \mathbb{E}[e^{-\frac{1}{2}\delta(\delta(\phi_{(x,t)}))-\frac{1}{4}\int_0^\infty F(s+2x,t)ds}]^{-2}$$

is a tau function of KdV with corresponding solution $u = -2\partial_x^2 \log\tau$. □

Proposition 4.7. *(cf.* [25]*) Let* $\phi \in L^2(\mathbb{R}^2_+)$ *such that* $||\phi||_{L^2(\mathbb{R}^2_+)} < 1$*. Then*

$$\mathbb{E}[e^{-\frac{1}{2}\delta(\delta(\phi))}] = \frac{1}{\sqrt{\det_2(I+\phi)}}, \tag{4.5}$$

where $\det_2(I+\phi)$ *is the Carleman-Fredholm determinant of* $I+\phi$ *defined in (3.10).*

The assumption $||\phi_{(x,t)}||_{L^2(\mathbb{R}_+^2)} < 1$ defines a set of (x,t) such that (4.4) is well defined. A similar condition can be found in [1] and [14] for the N-soliton case. In particular in [14] the solution is given only for $x \geq 0$. However Proposition 4.7 remains true if we replace the assumption $||\phi||_{L^2(\mathbb{R}_+^2)} < 1$ by the assumption that the eigenvalues of the integral operator with kernel ϕ are greater than -1 (cf. [5] and [11]).

We now consider a particular solution of (3.15) defined by

$$F(x,t) := \int_0^\infty e^{8\kappa^3 t - \kappa x} d\mu(\kappa), \tag{4.6}$$

where μ is a σ-finite measure such that $\mu(\{0\}) = 0$, and such that there exists $I \subset \mathbb{R} \times \mathbb{R}_+$ satisfying

$$\int_0^1 \frac{1}{\kappa} e^{8\kappa^3 t - 2\kappa x} d\mu(\kappa) < \infty \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{\kappa}} e^{8\kappa^3 t - 2\kappa x} d\mu(\kappa) < \infty, \quad \forall (x,t) \in I. \tag{4.7}$$

Theorem 4.8. *Let F be given by (4.6) where μ satisfies (4.7). Let $(X_\kappa)_{\kappa > 0}$ be a centered Gaussian process with covariance function $\mathbb{E}[X_{t_1} X_{t_2}] = 1/(t_1 + t_2)$. Then for all $(x,t) \in I$ such that $||\phi_{(x,t)}||_{L^2(\mathbb{R}_+^2)} < 1$,*

$$\tau(x,t) := \mathbb{E} \exp\left\{ -\frac{1}{2} \int_0^\infty e^{8\kappa^3 t - 2\kappa x} X_\kappa^2 d\mu(\kappa) \right\}, \tag{4.8}$$

is a tau function of the KdV equation (1.1): the function $u(x,t) := 4\partial_x^2 \log \tau(x,t)$ is a solution of (1.1).

Proof. Let $\phi_{(x,t)}(a,b) := F(a+b+2x, t)$. Then

$$\delta(\delta(\phi_{(x,t)})) = \int_0^\infty \int_0^\infty \int_0^\infty e^{8\kappa^3 t - \kappa(a+b+2x)} d\mu(\kappa)\ dW_a\ dW_b.$$

Applying twice the Fubini Theorem 4.9 below we obtain,

$$\begin{aligned}
\delta(\delta(\phi_{(x,t)})) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{8\kappa^3 t - \kappa(a+b+2x)} dW_a\ dW_b\ d\mu(\kappa) \\
&= \int_0^\infty e^{8\kappa^3 t - 2\kappa x} \left\{ \left(\int_0^\infty e^{-\kappa s} dW_s \right)^2 - \int_0^\infty e^{-2\kappa a} ds \right\} d\mu(\kappa) \\
&= \int_0^\infty e^{8\kappa^3 t - 2\kappa x} X_\kappa^2 d\mu(\kappa) - \int_0^\infty \frac{1}{2a} e^{8\kappa^3 t - 2\kappa x} d\mu(\kappa),
\end{aligned}$$

where $X_\kappa := \int_0^\infty e^{-\kappa s} dW_s$. The result follows from Theorem 4.6. $\square$

Theorem 4.9. *(cf. [20]) Consider a random field $\{u_t(x), 0 \leq t < \infty, x \in G\}$, where $G \subset \mathbb{R}$, such that for all $x \in G$, $u_\cdot(x) \in Dom\ \delta$. Let μ be a σ-finite measure on G. Suppose that $\int_G \left(\mathbb{E} \int_0^\infty |u_t(x)|^2 dt \right)^{1/2} d\mu(x) < \infty$, and $\int_G \mathbb{E}[\delta(u_\cdot(x))^2]^{1/2} d\mu(x) < \infty$, then the process $\{\int_G u_t(x) d\mu(x), 0 \leq t < \infty\}$ is Skorohod integrable and*

$$\delta\left(\int_G u_t(x) d\mu(x) \right) = \int_G \delta(u_\cdot(x)) d\mu(x).$$

Proof. A Fubini Theorem between the Skorohod integral δ and an integral with respect to a measure μ can be found in [20] (Ex. 3.2.7). It is not difficult to prove the present version where the integration domain is not necessarily bounded. We leave the proof to the reader. $\square$

The solutions of KdV provided by Theorems 4.6 and 4.8 are different from N-soliton solutions. They are constructed from a Gaussian process with a covariance function which is an infinite dimensional extension of a Cauchy matrix. When μ in (4.8) is a sum of Dirac distributions, we retrieve as a corollary the N-soliton solution of KdV.

Corollary 4.10. *Let c_n and κ_n, $n = 1, \dots, N$ be positive constants. Let F be defined by*

$$F(x,t) := \sum_{n=1}^{N} c_n^2 e^{8\kappa_n^3 t - \kappa_n x}. \tag{4.9}$$

Then a N-soliton solution to the KdV equation (1.1) is given by

$$u(x,t) = 4\partial_x^2 \log \mathbb{E} \exp \left\{ -\frac{1}{2} \sum_{n=1}^{N} c_n^2 e^{8\kappa_n^3 t} \left(\int_0^\infty e^{-\kappa_n (x+s)} dW_s \right)^2 \right\}. \tag{4.10}$$

Remark 4.11. 1) Let us mention another route to obtain (4.10). Let us write the right-hand side of (4.10) as

$$\tau(x,t) := \mathbb{E} \exp \left\{ -\frac{1}{2} \sum_{n=1}^{N} c_n^2 e^{8\kappa_n^3 t - 2\kappa_n x} \left(\int_0^\infty e^{-\kappa_n s} dW_s \right)^2 \right\}.$$

Let

$$R := \operatorname{diag}\{c_n^2 e^{8\kappa_n^3 t - 2\kappa_n x}, n = 1, \dots, N\}$$

and let X be the N-dimensional vector whose components are

$$X_n := \int_0^\infty e^{-\kappa_n a} dW_a.$$

Then X is Gaussian with mean 0 and covariance matrix Λ given by the Cauchy matrix

$$\Lambda_{m,n} = \mathbb{E}(X_m X_n) = \int_0^\infty e^{-(\kappa_m + \kappa_n)s} ds = \frac{1}{\kappa_m + \kappa_n}.$$

Finally,

$$\tau(x,t) = \mathbb{E} e^{-\frac{1}{2} X^\star R X} = \det(I + R\Lambda)^{-1/2},$$

and $u(x,t) = -2\partial_x^2 \log \det(I + R\Lambda)$. Using the identity $\det(I+AB) = \det(I+BA)$ we retrieve the determinant of the matrix (3.5). Hence u is the N-soliton solution of KdV (1.1).
2) There is some similarity between the covariance function of the process $\left(e^{-\kappa x} X_\kappa\right)_\kappa$ in Theorem 4.8 and the matrix (3.5).
3) The solution obtained in Corollary 4.10 reminds us of equation (2.7) proposed in [14] where t acts as a parameter and the processes $\left(\int_0^\infty e^{-\kappa_n (x+s)} dW_s\right)_x$ correspond to the Ornstein-Uhlenbeck processes considered in [14].

Theorem 4.6 and Theorem 4.8 can be extended to the KP hierarchy. Indeed an extension of Theorem 3.2 to the KP hierarchy is proved by Pöppe and Sattinger in [24]. Let us recall the notation for the hierarchy variables $\mathbf{x} := (x_1, x_2, \ldots)$ and $\mathbf{z} := (z_1, x_2, \ldots)$. Then Pöppe and Sattinger showed that a tau function for the KP hierarchy can be written as the Fredholm determinant of the integral operator

$$\mathcal{F}_{\mathbf{x}}\psi(y) = \int_0^\infty F((x_1 + y, x_2, x_3, \ldots), (x_1 + z, x_2, x_3, \ldots))\psi(z)dz, \tag{4.11}$$

where $F(\mathbf{x}, \mathbf{z})$ satisfies the following system of linear PDEs in infinitely many variables

$$\frac{\partial}{\partial x_n}F - \frac{\partial^n}{\partial x_1^n}F + (-1)^n \frac{\partial^n}{\partial z_1^n}F = 0, \quad n = 2, 3, \ldots, \tag{4.12}$$

and $\mathcal{F}_{\mathbf{x}}$ is assumed to belong to LC_ν, $1/2 < \nu \leq 1$ for all $\mathbf{x}$.

System (4.12) is the analogous of (3.15) for the KP hierarchy. The variables $x_2, x_3, \ldots$ in equation (4.11) act as parameters exactly like the time variable t does in equation (4.4). Defining $\xi(\mathbf{x}, k) := \sum_{j=1}^\infty x_j k^j$, system (4.12) admits the solution

$$F(\mathbf{x}, \mathbf{z}) = e^{\xi(\mathbf{x},p) - \xi(\mathbf{z},q)}.$$

More general solutions can then be obtained by superposition of such fundamental solutions as follows

$$F(\mathbf{x}, \mathbf{z}) = \int_{\mathbb{C}^2} e^{\xi(\mathbf{x},p) - \xi(\mathbf{z},q)} d\mu(p, q),$$

where μ is some measure on $\mathbb{C}^2$ with support included in $\{(p, q) \in \mathbb{C}^2, \mathrm{Re}\ p < 0 < \mathrm{Re}\ q\}$.

Theorem 4.12. *Let $1/2 < \nu \leq 1$ and let $\mathcal{F}_{\mathbf{x}} \in LC_\nu$ be an integral operator defined by (4.11) where F is real valued and satisfies (4.12).*
Let $\phi_{\mathbf{x}}(y, z) := F((x_1 + y, x_2, x_3, \ldots), (x_1 + z, x_2, x_3, \ldots))$. Then for all $\mathbf{x}$ such that $||\phi_{\mathbf{x}}||_{L^2(\mathbb{R}_+^2)} < 1$,

$$\tau(\mathbf{x}) := \mathbb{E}[e^{-\frac{1}{2}\delta(\delta(\phi_{\mathbf{x}})) - \frac{1}{2}\int_0^\infty \phi_{\mathbf{x}}(y,y)dy}]^{-2} \tag{4.13}$$

is a tau function for the KP hierarchy.

Proof. The proof of this theorem follows the same lines as the proof of Theorem 4.6. □

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