5-4-2018

Absence of Point Spectrum for the Self-Dual Extended Harper'S Model

Rui Han
University of California, Irvine

Recommended Citation
Han, R. (2018). Absence of Point Spectrum for the Self-Dual Extended Harper'S Model. *International Mathematics Research Notices, 2018* (9), 2801-2809. https://doi.org/10.1093/imrn/rnw279

R. Han (2017) "Absence of Point Spectrum for the Self-Dual Extended Harper'S Model,"
International Mathematics Research Notices, Vol. 2018, No. 9, pp. 2801–2809
Advance Access Publication January 19, 2017
doi: 10.1093/imrn/rnw279

Absence of Point Spectrum for the Self-Dual Extended Harper'S Model

Rui Han*

Department of Mathematics, University of California, Irvine CA, 92697, USA

Correspondence to be sent to: e-mail: rhan2@uci.edu

We give a simple proof of absence of point spectrum for the self-dual extended Harper's model. We get a sharp result which improves that of [1] in the isotropic self-dual regime.

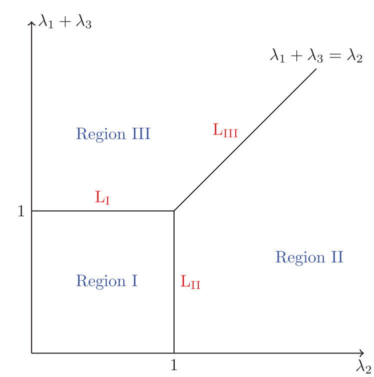
1 Introduction

We study the extended Harper's model on $l^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)_n = c_\lambda(\theta+n\alpha)u_{n+1} + \tilde{c}_\lambda(\theta+(n-1)\alpha)u_{n-1} + v(\theta+n\alpha)u_n, \tag{1.1}$$

where $c_\lambda(\theta) = \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})}$ and $v(\theta) = 2\cos 2\pi\theta$. $\tilde{c}_\lambda(\theta) = \overline{c_\lambda(\theta)}$ for $\theta \in \mathbb{T}$ and its analytic extension when $\theta \notin \mathbb{T}$. We refer to $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ as coupling constants, $\theta \in \mathbb{T} = [0, 1]$ as the phase and α as the frequency.

In [2] the authors partitioned the parameter space into the following three regions.



Received October 7, 2016; Revised October 7, 2016; Accepted October 31, 2016
Communicated by Prof. Svetlana Jitomirskaya

Region I: $0 \le \lambda_1 + \lambda_3 \le 1, 0 < \lambda_2 \le 1$,
Region II: $0 \le \lambda_1 + \lambda_3 \le \lambda_2, 1 \le \lambda_2$,
Region III: $\max\{1, \lambda_2\} \le \lambda_1 + \lambda_3$, $\lambda_2 > 0$.

According to the action of the *duality transformation* $\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \to \hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$, we have the following observation [2]:

Observation 1.1. σ is a bijective map on $0 \le \lambda_1 + \lambda_3$, $0 < \lambda_2$.

(i) $\sigma(\mathrm{I}^\circ) = \mathrm{II}^\circ$, $\sigma(\mathrm{III}^\circ) = \sigma(\mathrm{III}^\circ)$
(ii) Letting $\mathrm{L_I} := \{\lambda_1 + \lambda_3 = 1, 0 < \lambda_2 \le 1\}$, $\mathrm{L_{II}} := \{0 \le \lambda_1 + \lambda_3 \le 1, \lambda_2 = 1\}$, and $\mathrm{L_{III}} := \{1 \le \lambda_1 + \lambda_3 = \lambda_2\}$, $\sigma(\mathrm{L_I}) = \mathrm{L_{III}}$ and $\sigma(\mathrm{L_{II}}) = \mathrm{L_{II}}$. □

As σ bijectively maps $\mathrm{III} \cup \mathrm{L_{II}}$ on to itself, the literature refers to $\mathrm{III} \cup \mathrm{L_{II}}$ as the *self-dual regime*. We further divide III into $\mathrm{III}_{\lambda_1=\lambda_3}$ (*isotropic self-dual regime*) and $\mathrm{III}_{\lambda_1\neq\lambda_3}$ (*anisotropic self-dual regime*).

A complete understanding of the spectral properties of the extended Harper's model for a.e. θ has been established:

Theorem 1.2 ([1]). The following Lebesgue decomposition of the spectrum of $H_{\lambda,\alpha,\theta}$ holds for a.e.θ.

- For all Diophantine α, for Region I, $H_{\lambda,\alpha,\theta}$ has pure point spectrum.
- For all irrational α, for Regions II, $\mathrm{III}_{\lambda_1\neq\lambda_3}$, $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum.
- For all irrational α, for Region $\mathrm{III}_{\lambda_1=\lambda_3}$, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum. □

As pointed out in [1], the main missing link between [2, 3] and Theorem 1.2 is the following theorem, excluding eigenvalues in the self-dual regime. We say θ is α-rational if $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$, otherwise we say θ is α-irrational.

Theorem 1.3 ([1]). For all irrational α,

- for $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cup \mathrm{L_{II}}$, $H_{\lambda,\alpha,\theta}$ has empty point spectrum for all α-irrational θ.
- for $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$, $H_{\lambda,\alpha,\theta}$ has empty point spectrum for a.e. θ. □

In [1] the authors had to exclude more phases than α-rational θ in the isotropic self-dual regime.

In this article, we give a simple proof of the following theorem.

Theorem 1.4. For all irrational α, for $\lambda \in \mathrm{III}$, $H_{\lambda,\alpha,\theta}$ has empty point spectrum for all α-irrational θ. □

Remark 1.1. Our result for the isotropic self-dual regime $\mathrm{III}_{\lambda_1=\lambda_3}$ is sharp. Indeed, according to Proposition 5.1 in [1], for α-rational θ, $H_{\lambda,\alpha,\theta}$ has point spectrum. □

We organize this article in the following way: in Section 2, we include some preliminaries, in Section 3, we present two lemmas that will be used in Section 5, then we deal with $\mathrm{III}_{\lambda_1=\lambda_3}$ and $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3=1\}$ in Section 4 and $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3>1\}$ in Section 5.

2 Preliminaries

2.1 Rational approximation

Let $\{\frac{p_m}{q_m}\}$ be the continued fraction approximants of α, then

$$\frac{1}{2q_{m+1}} \leq \|q_m\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{m+1}}. \tag{2.1}$$

The exponent $\beta(\alpha)$ is defined as follows

$$\beta(\alpha) = \limsup_{m\to\infty} \frac{\ln q_{m+1}}{q_m}. \tag{2.2}$$

It describes how well is α approximated by rationals.

2.2 Self-dual extended Harper's model

Let $|c|_\lambda(\theta) = \sqrt{c_\lambda(\theta)\tilde{c}_\lambda(\theta)}$ be the analytic function that coincides with $|c_\lambda(\theta)|$ when $\theta \in \mathbb{T}$.

The presence of singularities of $c_\lambda(\theta)$ is explicit:

Observation 2.1 (e.g., [1]). The function $c_\lambda(\theta)$ has at most two zeros. Necessary conditions for real roots are $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$ or $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3=\lambda_2\}$. Moreover,

- for $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$, $c_\lambda(\theta)$ has real roots determined by

$$2\lambda_3 \cos 2\pi\left(\theta+\frac{\alpha}{2}\right) = -\lambda_2, \tag{2.3}$$

and giving rise to a double root at $\theta = \frac{1}{2} - \frac{\alpha}{2}$ if $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3} \cap \{\lambda_1+\lambda_3=\lambda_2\}$.

- for $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3=\lambda_2\}$, $c_\lambda(\theta)$ has only one simple real root at $\theta = \frac{1}{2}-\frac{\alpha}{2}$. □

Remark 2.1. By the definition of the duality transformation σ, Observation 2.1 implies that $c_{\hat{\lambda}}(\theta)$ has singular point if and only if $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$ or $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3=1\}$. □

It will be clear in Section 4 that presence of singularities of $c_{\hat{\lambda}}(\theta)$ indeed simplifies the proof of empty point spectrum of $H_{\lambda,\alpha,\theta}$.

3 Lemmas

Lemma 3.1. For $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3>1\}$, when $\lambda_3>\lambda_1$, we have

$$\frac{c_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = e^{-2\pi i(\theta+\frac{\alpha}{2})+if(\theta)} \quad \text{and} \quad \frac{\tilde{c}_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = e^{2\pi i(\theta+\frac{\alpha}{2})-if(\theta)},$$

for a real analytic function $f(\theta)$ on $\mathbb{T}$ with $\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta = 0$. □

Proof. By the definition of $c_{\hat{\lambda}}(\theta)$ we have

$$\begin{aligned} c_{\hat{\lambda}}(\theta) =& \frac{\lambda_3}{\lambda_2}e^{-2\pi i(\theta+\frac{\alpha}{2})} + \frac{1}{\lambda_2} + \frac{\lambda_1}{\lambda_2}e^{2\pi i(\theta+\frac{\alpha}{2})} && (3.1)\\ =& \frac{\lambda_1}{\lambda_2}e^{-2\pi i(\theta+\frac{\alpha}{2})}(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-), && (3.2)\end{aligned}$$

where $y_\pm = \frac{-1\pm\sqrt{1-4\lambda_1\lambda_3}}{2\lambda_1}$. Note that

$$y_+ = \overline{y_-} \text{ with } |y_+| = |y_-| = \sqrt{\frac{\lambda_3}{\lambda_1}} > 1, \quad \text{when } 1 \le 2\sqrt{\lambda_1\lambda_3}, \tag{3.3}$$

$$y_+, y_- \in \mathbb{R} \text{ with } |y_+| > |y_-| = \frac{2\lambda_3}{\lambda_1+\sqrt{1-4\lambda_1\lambda_3}} > 1, \quad \text{when } \lambda_1+\lambda_3 > 1 > 2\sqrt{\lambda_1\lambda_3}. \tag{3.4}$$

Note that

$$\frac{c_{\hat{\lambda}}(\theta)}{|c|_{\hat{\lambda}}(\theta)} = \sqrt{\frac{c_{\hat{\lambda}}(\theta)}{\tilde{c}_{\hat{\lambda}}(\theta)}} = e^{-2\pi i(\theta+\frac{\alpha}{2})}\sqrt{\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)}}. \tag{3.5}$$

By (3.3), we have

$$\int_{\mathbb{T}} \arg\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)}\mathrm{d}\theta = 0, \tag{3.6}$$

and

$$\left|\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)}\right|\equiv 1. \tag{3.7}$$

Thus there exists a real analytic function $g(\theta)$ on $\mathbb{T}$ such that

$$\frac{(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{2\pi i(\theta+\frac{\alpha}{2})}-y_-)}{(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_+)(e^{-2\pi i(\theta+\frac{\alpha}{2})}-y_-)}=e^{ig(\theta)}, \tag{3.8}$$

with $\int_{\mathbb{T}} g(\theta)\mathrm{d}\theta=0$. Taking $f(\theta)=g(\theta)/2$ yields the desired the result.

Lemma 3.2. There is a subsequence $\{\frac{p_{m_l}}{q_{m_l}}\}$ of the continued fraction approximants of α so that for any analytic function f on $\mathbb{T}$ with $\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta=0$, we have

$$\lim_{l\to\infty} f(x)+f(x+\alpha)+\cdots+f(x+q_{m_l}\alpha-\alpha)=0$$

uniformly in $x\in\mathbb{T}$. □

Proof. Suppose f is analytic on $|\mathrm{Im}\theta|\le\delta_0$, then $|\hat{f}(n)|\le ce^{-2\pi\delta_0|n|}$ for some constant $c>0$.

Case 1. If $\beta(\alpha)=0$, then by solving the coholomogical equation we get $f(x)=h(x+\alpha)-h(x)$ for some analytic $h(x)$. Then

$$\begin{aligned}&\lim_{m\to\infty}(f(x)+f(x+\alpha)+\cdots+f(x+q_m\alpha-\alpha))\\&=\lim_{m\to\infty}(h(x+q_m\alpha)-h(x))=0\end{aligned}$$

uniformly in x.

Case 2. If $\beta(\alpha)>0$, choose a sequence m_l such that $q_{m_l+1}\ge e^{\frac{\beta}{2}q_{m_l}}$. Then

$$\begin{aligned}&|f(x)+f(x+\alpha)+\cdots+f(x+q_{m_l}\alpha-\alpha)|\\&=\left|\sum_{|n|\ge 1}\hat{f}(n)(1+e^{2\pi in\alpha}+\cdots+e^{2\pi in(q_{m_l}-1)\alpha})e^{2\pi inx}\right|\\&=\left|\sum_{|n|\ge 1}\hat{f}(n)\frac{1-e^{2\pi inq_{m_l}\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inx}\right|\end{aligned}$$

$$\leq \sum_{1\leq|n|\leq q_{m_l}-1} c\left|\frac{1-e^{2\pi i n q_{m_l}\alpha}}{1-e^{2\pi i n\alpha}}\right| + \sum_{|n|\geq q_{m_l}} ce^{-2\pi\delta_0|n|}q_{m_l}$$

$$\leq c\frac{q_{m_l}^3}{q_{m_{l+1}}} + cq_{m_l}e^{-2\pi\delta_0 q_{m_l}} \to 0 \text{ as } l\to\infty$$

uniformly in x.

4 Consequence of point spectrum

This part follows from [1]. We present the material here for completeness and readers' convenience.

Suppose $\{u_n\}$ is an $l^2(\mathbb{Z})$ solution to $H_{\lambda,\alpha,\theta}u = Eu$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. This means

$$c_\lambda(\theta+n\alpha)u_{n+1} + \tilde{c}_\lambda(\theta+(n-1)\alpha)u_{n-1} + 2\cos(2\pi(\theta+n\alpha))u_n = Eu_n. \tag{4.1}$$

Let $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{2\pi i n x} \in L^2(\mathbb{T})$. Multiplying (4.1) by $e^{2\pi i n x}$ and then summing over n, we get

$$e^{2\pi i\theta}c_{\hat{\lambda}}(x)u(x+\alpha) + e^{-2\pi i\theta}\tilde{c}_{\hat{\lambda}}(x-\alpha)u(x-\alpha) + 2\cos 2\pi x\; u(x) = \frac{E}{\lambda_2}u(x), \tag{4.2}$$

where $\hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, 1, \frac{\lambda_1}{\lambda_2})$. If we multiply (4.1) by $e^{-2\pi i n x}$ and sum over n, we get

$$e^{-2\pi i\theta}c_{\hat{\lambda}}(x)u(-x-\alpha) + e^{2\pi i\theta}\tilde{c}_{\hat{\lambda}}(x-\alpha)u(-x+\alpha) + 2\cos 2\pi x\; u(-x) = \frac{E}{\lambda_2}u(-x). \tag{4.3}$$

Thus writing (4.2), (4.3) in terms of matrices, we get

$$\begin{aligned}&\frac{1}{c_{\hat{\lambda}}(x)}\begin{pmatrix} \frac{E}{\lambda_2} - 2\cos 2\pi x & -\tilde{c}_{\hat{\lambda}}(x-\alpha) \\ c_{\hat{\lambda}}(x) & 0\end{pmatrix}\begin{pmatrix} u(x) & u(-x) \\ e^{-2\pi i\theta}u(x-\alpha) & e^{2\pi i\theta}u(-(x-\alpha))\end{pmatrix} \\ &= \begin{pmatrix} u(x+\alpha) & u(-(x+\alpha)) \\ e^{-2\pi i\theta}u(x) & e^{2\pi i\theta}u(-x)\end{pmatrix}\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta}\end{pmatrix}\end{aligned} \tag{4.4}$$

Let $M_\theta(x) \in L^2(\mathbb{T})$ be defined by

$$M_\theta(x) = \begin{pmatrix} u(x) & u(-x) \\ e^{-2\pi i\theta}u(x-\alpha) & e^{2\pi i\theta}u(-(x-\alpha))\end{pmatrix}.$$

Let

$$A_{\hat{\lambda},E/\lambda_2}(x) = \frac{1}{c_{\hat{\lambda}}(x)}\begin{pmatrix} \frac{E}{\lambda_2} - 2\cos 2\pi x & -\tilde{c}_{\hat{\lambda}}(x-\alpha) \\ c_{\hat{\lambda}}(x) & 0\end{pmatrix}$$

be the transfer matrix associated to $H_{\hat{\lambda},\alpha,\theta}$ and

$$R_\theta = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}$$

be the constant rotation matrix. Then (4.4) becomes

$$A_{\hat{\lambda},E}(x)M_\theta(x) = M_\theta(x+\alpha)R_\theta. \tag{4.5}$$

Taking determinant, we have the following proposition.

Proposition 4.1 ([1]). If θ is α-irrational, then

$$|\det M_\theta(x)| = \frac{b}{|c|_{\hat{\lambda}}(x-\alpha)} \tag{4.6}$$

for some constant $b > 0$ and a.e.$x \in \mathbb{T}$. □

5 Regions $\mathrm{III}_{\lambda_1=\lambda_3}$ and $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 = 1\}$

We will show the following lemma.

Lemma 5.1. If θ is α-irrational, then for $\lambda \in \mathrm{III}_{\lambda_1=\lambda_3}$ or $\lambda \in \mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 = 1\}$, $H_{\lambda,\alpha,\theta}$ has no point spectrum. □

Proof. According to Remark 2.1, we have $c_{\hat{\lambda}}(x_0) = 0$ for some $x_0 \in \mathbb{T}$. Presence of singularity implies $\frac{1}{c_{\hat{\lambda}}(x)} \notin L^1(\mathbb{T})$. Thus by (4.6), $\det M_\theta(x) \notin L^1(\mathbb{T})$. This contradicts with $M_\theta(x) \in L^2(\mathbb{T})$.

6 Regions $\mathrm{III}_{\lambda_1\neq\lambda_3} \cap \{\lambda_1+\lambda_3 > 1\}$

Without loss of generality, we assume $\lambda_3 > \lambda_1$. Fix θ. Denote $\det M_\theta(x) = g(x)$ for simplicity.

Lemma 6.1. If θ is α-irrational, then $H_{\lambda,\alpha,\theta}$ has no point spectrum in the anisotropic self-dual region. □

Proof. Taking determinant in (4.5), we get:

$$\frac{\tilde{c}_{\hat{\lambda}}(x-\alpha)}{c_{\hat{\lambda}}(x)} g(x) = g(x+\alpha).$$

This implies

$$g(x+k\alpha)=\frac{\tilde{c}_{\hat{\lambda}}(x+k\alpha-2\alpha)\cdots\tilde{c}_{\hat{\lambda}}(x)\,\tilde{c}_{\hat{\lambda}}(x-\alpha)}{c_{\hat{\lambda}}(x+k\alpha-\alpha)\cdots c_{\hat{\lambda}}(x+\alpha)c_{\hat{\lambda}}(x)}g(x). \tag{6.1}$$

Taking $k=q_{m_l}$, as in Lemma 3.2, on one hand, since $g(x)$ is an L^1 function, as the determinant of an L^2 matrix, and $\lim_{l\to\infty}\|q_{m_l}\alpha\|_{\mathbb{T}}=0$, we have

$$\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}=0.$$

By (6.1), this implies

$$0=\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}=\lim_{l\to\infty}\int\left|1-\frac{\prod_{j=-1}^{q_{m_l}-2}\tilde{c}_{\hat{\lambda}}(x+j\alpha)}{\prod_{j=0}^{q_{m_l}-1}c_{\hat{\lambda}}(x+j\alpha)}\right|\cdot|g(x)|\mathrm{d}x. \tag{6.2}$$

On the other hand, by Lemma 3.1

$$\begin{aligned}
&\lim_{l\to\infty}\int\left|1-\frac{\prod_{j=-1}^{q_{m_l}-2}\tilde{c}_{\hat{\lambda}}(x+j\alpha)}{\prod_{j=-1}^{q_{m_l}-1}c_{\hat{\lambda}}(x+j\alpha)}\right|\cdot|g(x)|\mathrm{d}x\\
&=\lim_{l\to\infty}\int\left|1-\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{4\pi iq_{m_l}x}e^{2\pi iq_{m_l}(q_{m_l}-1)\alpha}\right|\cdot|g(x)|\mathrm{d}x\\
&\geq\liminf_{l\to\infty}\left(\int|1-e^{4\pi iq_{m_l}x+2\pi iq_{m_l}^2\alpha}||g(x)|\mathrm{d}x\right.\\
&\qquad\left.-\int\left|1-\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{-2\pi iq_{m_l}\alpha}\right|\cdot|g(x)|\mathrm{d}x\right)\\
&:=\liminf_{l\to\infty}(I_1-I_2).
\end{aligned} \tag{6.3}$$

Combining the fact $\|q_{m_l}\alpha\|_{\mathbb{T}}\to 0$ with Lemma 3.2, we get pointwise convergence,

$$\frac{|c|_{\hat{\lambda}}(x-\alpha)}{|c|_{\hat{\lambda}}(x+q_{m_l}\alpha-\alpha)}e^{-i\left(\sum_{j=-1}^{q_{m_l}-2}f(x+j\alpha)+\sum_{j=0}^{q_{m_l}-1}f(x+j\alpha)\right)}e^{-2\pi iq_{m_l}\alpha}\to 1\ \text{ as } l\to\infty.$$

Then by dominated convergence theorem, we get $\lim_{l\to\infty}I_2=0$. Then (6.3) implies that for any small constant $\delta>0$,

$$\begin{aligned}
\lim_{l\to\infty}\|g(x+q_{m_l}\alpha)-g(x)\|_{L^1}&\geq\liminf_{l\to\infty}I_1\\
&\geq\liminf_{l\to\infty}\int_{\|2q_{m_l}x+q_{m_l}^2\alpha\|_{\mathbb{T}}\geq\delta}4\delta|g(x)|\mathrm{d}x,
\end{aligned}$$

where $|\{x : \|2q_{m_l}x + q_{m_l}^2\alpha\| \geq \delta\}| \triangleq |F_{m_l,\delta}| = 1 - 2\delta$. Thus

$$\lim_{l\to\infty} \|g(x+q_{m_l}\alpha) - g(x)\|_{L^1} \geq \liminf_{l\to\infty} (4\delta\|g\|_{L^1} - 4\delta \int_{F^c_{m_l,\delta}} |g(x)|\mathrm{d}x)$$
$$\geq \liminf_{l\to\infty} (4\delta\|g\|_{L^1} - 8\delta^2\|g\|_{L^\infty}).$$

By (4.6) $|g(x)| = \frac{b}{|c|_{\hat{\lambda}}(x-\alpha)}$ for some constant $b > 0$, thus $\|g\|_{L^1}$, $\|g\|_{L^\infty}$ are positive finite numbers, so one can choose $\delta \sim 0$ such that $4\delta\|g\|_{L^1} - 8\delta^2\|g\|_{L^\infty}$ is strictly positive. This contradicts with (6.2).

Funding

This work was partially supported by the National Science Foundation [DMS-1401204].

Acknowledgements

I would like to thank Svetlana Jitomirskaya for useful discussions.

References

[1] Avila, A., S. Jitomirskaya, and C. A. Marx. "Spectral theory of extended Harper's model and a question by Erdős and Szekeres." (2016), preprint arXiv:1602.05111.

[2] Jitomirskaya, S., and C. A. Marx. "Analytic quasi-perodic cocycles with singularities and the Lyapunov exponent of extended Harper's model." *Communications in Mathematical Physics* 316, no. 1 (2012): 237–67.

[3] Jitomirskaya, S., and C. A. Marx. "Erratum to: Analytic quasi-perodic cocycles with singularities and the Lyapunov Exponent of Extended Harper's Model." *Communications in Mathematical Physics* 317, no. 1 (2013): 269–71.