2011

On Greenberg's question: an algebraic and computational approach

David H. Chapman
Louisiana State University and Agricultural and Mechanical College

ON GREENBERG'S QUESTION: AN ALGEBRAIC AND COMPUTATIONAL APPROACH

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
David H. Chapman
B.S. in Math., Iowa State University, 2001
M.A. in Math, University of Northern Iowa, 2005
August 2011

To my wife Rachel and my daughter Aleah.

Acknowledgments

I would like to thank Dr. Litherland, Dr. Morales, Dr. Richardson, and Dr. Oporowski for taking the time to sit on my committee. Thank you to both Jeff Sheldon and Chris Penn for taking time to help with the computational side of this work along with Jim Stark whose Gap4 code made the computations in this dissertation possible. Finally, I would like to thank Dr. Perlis for serving as my advisor. Your guidance, patience, and wisdom have been priceless.

Table of Contents

List of Tables

List of Figures

Abstract

Greenberg asked whether arithmetically equivalent number fields share the same Iwasawa invariants. In this dissertation it is shown that the problem naturally breaks up into four cases, depending on properties of Galois groups. This analysis is then used to give a positive answer to Greenberg's question in some nontrivial examples.

Introduction

Take $\bar{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$. Galois proved that two number fields K and K' are isomorphic if and only if the Galois groups $Gal(\bar{\mathbb{Q}}/K)$ and $Gal(\bar{\mathbb{Q}}/K')$ are conjugate in $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. We also know that K and K' share the same zeta function if and only if $Gal(\bar{\mathbb{Q}}/K)$ and $Gal(\bar{\mathbb{Q}}/K')$ are locally conjugate in $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. While standard conjugation (which we call global conjugation) is a single action on the entire group, local conjugation is a distinct action for each individual element. Because the zeta function is an invariant of a number field it follows that:

$$K \cong K' \Rightarrow \zeta_K = \zeta_{K'}$$

In 1926, Gassmann showed that the converse need not hold [7]. If $K = \mathbb{Q}(\sqrt[8]{a})$ and $K' = \mathbb{Q}(\sqrt[8]{16a})$ with a a square free integer not in the set $\{1, 2, -1, -2\}$, then $\zeta_K = \zeta_K$ but $K \ncong K'$. Traditionally, two fields with the same zeta function are called arithmetically equivalent. This, however, depends on whether or not we consider $\mathbb{Q}$ to be our base field. We will use a broader definition; for our purposes, if the Galois groups $Gal(N/K)$ and $Gal(N/K')$ are locally conjugate in $Gal(N/F)$, with N a normal closure of $K \cdot K'$ over some field F, then we say K and K' are Gassmann equivalent over F. So Gassmann equivalent fields are arithmetically equivalent. There are four different fields that we need to consider: the base field F, the two arithmetically equivalent fields K and K', and their common normal closure N.

Any number field K has a class group that we denote $\mathfrak{Cl}(K)$. This group is invariant under isomorphism. The class group of K is trivial if and only if the ring of integers $\mathscr{O}_K$ is a unique factorization domain. So the order of the class group could be considered a measure of how far a ring of integers is from being a unique factorization domain. The class group is known to be finite and abelian. So the class group is a direct product of finite p groups where $p \in \mathbb{Z}$ is prime. The p part of the class group will be denoted $\mathfrak{Cl}_p(K)$. In 1994 Perlis and de Smit showed if $K = \mathbb{Q}(\sqrt[8]{a})$ and $K' = \mathbb{Q}(\sqrt[8]{16a})$ and $a \in \{-15, -31, -33, -63, 65, 66, -65, -66\}$, then $\mathfrak{Cl}_2(K) \neq \mathfrak{Cl}_2(K')$ [3].

Fix a prime $p \in \mathbb{Z}$. A $\mathbb{Z}_p$-tower is an infinite chain of number fields $\{K \subset \cdots \subset K_n \subset \cdots\}$ where the Galois group $Gal(K_n/K)$ is cyclic of order p^n. This tower will be denoted K_∞/K. A number field K has degree $n = r_1 + 2 \cdot r_2$ where r_1 and r_2 are the number of real embedding and complex embeddings respectively. For any number field K there are exactly $1 + r_2$ independent $\mathbb{Z}_p$-towers. This is a corollary of Leopoldt's conjecture which P. Mihălescu proved in 2009 [10]. What this indicates is that a number field K will have at least one $\mathbb{Z}_p$-tower, namely the cyclotomic tower. When K is real this tower is totally real $\mathbb{Z}_p$-tower.

Iwasawa studied the class groups of these $\mathbb{Z}_p$-towers and showed that for any $\mathbb{Z}_p$-tower there are integer values λ, μ, ν and n_0 such that for any $n \geq n_0$ we have:

$$p^{\lambda n + \mu p^n + \nu} = \mid \mathfrak{Cl}_p(K_n) \mid$$

These values are called the four Iwasawa invariants. Often they are referred to as the three Iwasawa invariants. This is understandable since the formula is a statement about an infinite tail. One might ask if two towers share the same Iwasawa invariants. The towers could have different values of n_0, but the p parts of the class group could still coincide on a tail. So the values λ,μ and ν may initially be different, but after a given shift (see lemma 3.2), we will see the values λ,μ and ν are the same.

Iwasawa's student R. Greenberg asked whether the Iwasawa invariants of cyclotomic $\mathbb{Z}_p$-towers over a pair of arithmetically equivalent fields are the same. We generalize his question in chapter 3 as follows:

Do parallel towers over Gassmann equivalent fields share the same Iwasawa invariants?

J. Oh has shown that the λ invariants of parallel cyclotomic towers over Gassmann equivalent feilds are the same [11].

Take F_∞/F to be our $\mathbb{Z}_p$-tower. We see in lemma 3.4 part a) that $F_n \subset K$ implies that $F_n \subset N$ and by part b) that $F_n \subset K$ if and only if $F_n \subset K'$. So there are two values: c which is the largest integer such that $F_c \subset K$, and d which is the largest integer such that $F_d \subset N$.

We call the value d the lag of the tower. This is because there is a canonical isomorphism from the Galois groups of K_d and K'_d over F_d to the Galois groups of K_j and K'_j over F_j for all $j \geq d$. So once we establish Gassmann equivalence at level d then Gassmann equivalence will be preserved for the remainder of the tower.

But K_d and K'_d might not be Gassmann equivalent over F_d. By Theorem 3.8 part a), K_d and K'_d will be Gassmann equivalent over F_0. So take $b \leq c$ to be maximal such that K_d and K'_d are Gassmann equivalent over F_b. By lemma 3.7 K_d and K'_d are Gassmann equivalent over F_d if and only if they are Gassmann equivalent over F_c. Thus, we have Gassmann equivalence at level d (hence for the rest of the tower) if and only if $b = c$. We call $c - b$ the obstruction and we call c the quasi-obstruction. This is because $c = 0$ implies that $c - b = 0$. Based on these values, the tower can fall in to one of four possible categories.

The first category is called the trivial category. If $d \geq 0$ and K_d and K'_d are isomorphic, then we call the tower trivial. This is because isomorphic fields will always share Iwasawa invariants. In this category the answer to Greenberg's question is always yes.

The second category is called the simple category. We call the tower simple when $d = 0$. There are two reasons for this name. The first reason is that whenever $Gal(N/F)$ is simple, $d = 0$ automatically. The second reason is that when $d = 0$ there is a canonical map from $Gal(N/F)$ to $Gal(N_j/F_j)$ for any given $j \geq d$.

The third category is called the reducible category. In this case, K_d and K'_d are Gassmann equivalent over F_d, but are not isomorphic. Any reducible tower over base field F_0 can be considered as a simple tower over base field F_d. So we can consider the reducible case and the simple case to be the same.

Both the simple case and the reducible case will yield the canonical isomorphism. So in answering Greenberg's question these two categories could be considered as the same category.

Our last category will be the latent category. This is when K_d and K'_d are not Gassmann equivalent over F_d. This holds if and only if $c \neq b$. We use the word latent because a local conjugation is hidden within the group $Gal(N/F_d)$ but revealed when we expand to group $Gal(N/F_0)$.

When H and H' are locally conjugate in G, we can call (G, H, H') a Gassmann triple. The local conjugation is a bijection. So $\mid H \mid = \mid H' \mid$ and $[G : H] = [G : H']$. We call the value $[G : H]$ the degree of the triple. There are exactly 19 Gassmann triples of degree at most 15, up to isomorphism [1]. Using GAP4, we determine the possible properties these 19 groups could have if realized as Galois groups. Any Gassmann triple will be the base of some simple tower. One example would be when p does not divide the order of G. But could any of these groups fall into the other three tower categories? We answer this question in chapter 4.

We note that these definitions give a possible blue print for the Galois groups, but give no indicatioin if such groups can be realized as Galois groups within an actual $\mathbb{Z}_p$-tower. At this point, we do have examples of “latent groups,” but we do not have examples of “latent towers.”

We have a specific result in chapter 5. If $K = \mathbb{Q}(\sqrt[8]{a})$ and $K' = \mathbb{Q}(\sqrt[8]{16a})$ with a a square free integer not in the set $\{1, 2, -1, -2\}$, then $K_i \cong K'_i$ when $i \geq 1$ for any $\mathbb{Z}_p$-towers K and K'. This result follows from an analysis of the Galois groups, and in all cases the towers must fall into the trivial category. Thus the answer to Greenberg's question will be yes.

Suppose that $H = Gal(N/K)$ and $H' = Gal(N/K')$ are locally conjugate in $G = Gal(N/\mathbb{Q})$. The modules $\mathbb{Q}(G/H)$ and $\mathbb{Q}(G/H')$ are isomorphic. We can take any matrix mapping $\mathbb{Q}(G/H)$ to $\mathbb{Q}(G/H')$ and by canceling denominators, we can construct a matrix with entries in $\mathbb{Z}$. Possibly the determinant will change. This will yield a homomorphism from $\mathbb{Z}(G/H)$ to $\mathbb{Z}(G/H')$. This matrix could be an isomorphism from $\mathbb{Z}_p(G/H)$ to $\mathbb{Z}_p(G/H')$ as $\mathbb{Z}_p(G)$-modules. If this is the case then $\mathfrak{Cl}_p(K) = \mathfrak{Cl}_p(K')$ [14].

To determine if $\mathfrak{Cl}_p(K) = \mathfrak{Cl}_p(K')$ we need to compute invertible matrices mapping $\mathbb{Z}(G/H)$ to $\mathbb{Z}(G/H')$. A matrix is called doublly stochastic if the sum of the entries in every row is equal to the sum of the entries in every column. We show that these matrices are doubly stochastic. However, we show something even stronger. If a matrix has entries in ring R we call the matrix general doubly stochastic if any value $r \in R$ occurs the same number of times in every row and the same number of times in every column. We see that general doubly stochastic implies doubly stochastic. By proposition 6.11 all these matrices will be generally doubly stochastic. If p does not divide the determinant of such a matrix then we will be able to map $\mathfrak{Cl}_p(K)$ to $\mathfrak{Cl}_p(K')$.

We apply these matrices to our four types of towers. We skip the trivial case because the answer to Greenberg's question is yes. Both the simple and reducible cases will yeild the canonical group isomorphism. The canonical map will preserve the same doubly stochastic matrices from level d to level j with $j \geq d$.

In the latent case, all matrices at level d will have determinant 0. Thus p will divide the determinant for all primes p. However we can construct matrices for K_d and K'_d over F_b with nonzero determinant and matrices for K_j and K'_j over F_b with nonzero determinant, where $j \geq d$. But the base field F_b will remain fixed. So as j tends to infinity, the dimension of the matrix will also tend to infinity. However, by lemma 6.18, if there is a matrix A at level (K_d, K'_d, F_0) such that $p \nmid det(A)$ then there is a matrix A^* at level (K_j, K'_j, F_0) such that $p \nmid det(A^*)$.

So the algorithm for solving Greenberg's question will be as follows. Take K and K' Gassmann equivalent over F. Compute our values c,b and d.

Step 1) If $K_d \cong K'_d$ then $K_j \cong K'_j$ for all $j \geq d$ and the answer to Greenberg's question is yes.

Step 2) If $b = c$ we have either a simple or reducible tower. In both of these cases there will be a doubly stochastic matrix A at level (K_d, K'_d, F_d). If $p \nmid det(A)$ then $\mathfrak{Cl}_p(K_d) \cong \mathfrak{Cl}_p(K'_d)$. But $\mathfrak{Cl}_p(K_j) \cong \mathfrak{Cl}_p(K'_j)$ using the same matrix A, and the answer to Greenberg's question is yes.

Step 3) If $b \neq c$ we have a latent tower and such a matrix can not be constructed for K_d and K'_d over F_d. However such a matrix A can be constructed for K_d and K'_d over F_b. If $p \nmid det(A)$ then $\mathfrak{Cl}_p(K_d) \cong \mathfrak{Cl}_p(K'_d)$ and $\mathfrak{Cl}_p(K_j) \cong \mathfrak{Cl}_p(K'_j)$ using a larger matrix A^* with $p \nmid det(A)$. The answer to Greenberg's question is yes.

Note if all such matrices are divisible by p then our method will not yield an answer to Greenberg's Question.

In the final chapter, we attempt to better understand these matrices by realizing the Gassmann triples under geometric construction. Listing the triples in order by index, the first four triples will have indices of 7,8,8 and 11. Note that the matrices will map $\mathbb{Z}_p(G/H)$ to $\mathbb{Z}_p(G/H')$. So the index of the triple is the dimension of the matrix. The parent group G acts transitively on both G/H and G/H'. Thus G acts transitively on the rows and the columns of the matrix.

The parent group of the triple with index 7 will be the simple group of order 168. This is the automorphism group of Fano plane. The subgroup that fixes a single vertex will be H and the subgroup that fixes a line of the Fano plane will be H'.

The parent group of one triple of index 8 is $GL(3,2)$, which has order 48. Geometrically, we construct the affine geometry of order 3, which has 9 vertices and 12 lines. By omitting one vertex we will have 8 vertices and 8 lines. Again, the group that fixes a vertex will be H and the group that fixes a line will be H'.

The other triple of index 8 will have a parent group of order 32. To construct this group geometrically, take the 8 vertices to be the corners of a cube. Then twist the top face 45 degrees. Instead of considering lines with 3 vertices, we need to consider planes with 4 vertices. Under this construction, there are 10 planes determined by 4 vertices, but this includes the top and bottom planes which will only map to each other. Group G will act transitively on the remaining 8 planes. Thus, the group that fixes a vertex will be H, and the group that fixes one of these 8 planes will be H'.

All three of these constructions are matroids. The subgroup H fixes an element of the matroid, while H' fixes a circuit. Can this construction be generalized to any Gassmann triple?

The parent group of the triple with index 11 will be the simple group of order 660 and the subgroups H and H' are both isomorphic to the automorphism group of a buckyball (or soccer ball). But can we realize this group of order 660 as the automorphism group on 11 vertices? There is such a construction in E. Brown's "Fabulous $(11, 5, 2)$ Biplane". Again H will fix a vertex, but H' will fix a collection of 5 vertices which Brown calls a "block" or a "variety."

As a matroid, this block would be a circuit of order 5. That would make the dimension $5 - 1 = 4$ and geometric constructions are difficult to visualize in dimensions greater than 3. So although this matroid construction is interesting, it is not practical for our purposes. There is more information in a matroid than we actually need. Simply taking vertices and blocks as in Brown's construction will be enough.

The appendix has a matrix for each of the 19 triples with degree less than 16. DeSmit has listed the values of p for which $\mathbb{Z}_p(G/H) \ncong \mathbb{Z}_p(G/H')$ as $\mathbb{Z}_p(G)$-modules for each of the 19 triples. We do not verify his result here. However for each triple we construct a matrix with determinant that is a power of a single prime. And in each case the prime is the value p for which $\mathbb{Z}_p(G/H) \ncong \mathbb{Z}_p(G/H')$ according to deSmit.

Chapter 1

Local Conjugacy and Gassmann Equivalent Groups

Unless stated otherwise the letters $B, B', G, H, H', M, V, W, X$ and Y will be groups, possibly infinite, with the following properties.

- H and H' are subsets of G
- $W \lhd G$ such that $W \subset H \cap H'$
- $M \lhd G$ such that $M \not\subset H \cap H'$
- $B = H \cap M$ and $B' = H' \cap M$
- The groups V, X and Y will be arbitrary

Let $x, y \in G$. The notation $x \sim_G y$ denotes that x is conjugate to y in G.

Definition 1.1. Let $H \leq G$ and $H' \leq G$. We say that H and H' are **locally conjugate** in G (or LC in G) if there is a bijection $\phi : H \to H'$ such that for each $h \in H$ then $\phi(h) \sim_G h$. The map ϕ is called a **local conjugation**. We denote that H and H' are locally conjugate in G by $H \sim_G^{loc} H'$.

Definition 1.2. If there is an element $g \in G$ for which $gHg^{-1} = H'$ then H and H' are **globally conjugate** denoted $H \sim_G H'$.

Notice that $H \sim_G H$ implies the $H \sim_G^{loc} H'$. This follows directly from the definitions. But does the converse hold? If $H \sim_G^{loc} H'$ there may not be a single element in G that conjugates H and H'.

Definition 1.3. If H and H' are locally conjugate in G but not conjugate in G, then we say that H and H' are **non trivial local conjugates** in G (or NTLC).

Example 1.4. Let $G = \{\alpha^a\beta^b\gamma^c\}$ with the following relations.

$$o(\alpha) = 8 \qquad \beta\alpha = \alpha^3\beta$$
$$o(\beta) = 2 \qquad \gamma\alpha = \alpha^5\gamma$$
$$o(\gamma) = 2 \qquad \beta\gamma = \gamma\beta$$

This group can also be denoted $(C_8 \rtimes V_4)$. If we let $H =< \beta, \gamma >= \{e, \beta, \gamma, \beta\gamma\}$ and $H' =< \beta, \alpha^4\gamma >= \{e, \beta, \alpha^4\gamma, \alpha^4\beta\gamma\}$ then by taking ϕ to be the bijection taking each displayed element of H to the corresponding displayed element of H' we have

$$eee^{-1} = e = \phi(e)$$
$$e\beta e^{-1} = \beta = \phi(\beta)$$
$$\alpha\gamma\alpha^7 = \alpha\alpha^{35}\gamma = \alpha^4(\alpha^8)^4\gamma = \alpha^4\gamma = \phi(\gamma)$$
$$\alpha^2\beta\gamma\alpha^6 = \alpha^2\beta\alpha^{30}\gamma = \alpha^2\beta\alpha^6(\alpha^8)^4\gamma =$$
$$= \alpha^2\beta\alpha^6\gamma = \alpha^2\alpha^{18}\beta\gamma =$$
$$= \alpha^4(\alpha^8)^2\beta\gamma = \alpha^4\beta\gamma = \phi(\beta\gamma)$$

Thus $H \sim_G^{loc} H'$. By way of contradiction suppose there is some $g = \alpha^n\beta^m\gamma^k \in G$ for which $gHg^{-1} = H'$. Now $gHg^{-1} = \alpha^n(\beta^m\gamma^k)H(\beta^m\gamma^k)^{-1}\alpha^{-n} = \alpha^n H\alpha^{-n}$. So without loss of generality we can suppose $m = k = 0$. Now:

$$\alpha^n e \alpha^{-n} = e$$

$$\alpha^n \beta \alpha^{-n} = \alpha^n \alpha^{-3n} \beta = \alpha^{-2n} \beta = \alpha^{6n} \beta$$

$$\alpha^n \gamma \alpha^{-n} = \alpha^n \alpha^{-5n} \gamma = \alpha^{-4n} \gamma = \alpha^{4n} \gamma$$

$$\alpha^n \beta \gamma \alpha^{-n} = \alpha^n \beta \alpha^{-5n} \gamma = \alpha^n \alpha^{-15n} \beta \gamma =$$

$$= \alpha^{-14n} \beta \gamma = \alpha^{2n} \beta \gamma$$

This forces:

a) $\alpha^{6n}\beta = \beta$ and $6n \equiv 0 \pmod 8$

b) $\alpha^{4n}\gamma = \alpha^4\gamma$ and $4n \equiv 4 \pmod 8$

But a) implies that n is even while b) implies that n is odd. This is a contradicition.

This example shows us that $H \sim_G^{loc} H' \not\Rightarrow H \sim_G H'$.

Lemma 1.5. *Let X be a subgroup of Y. Then H and H' are locally conjugate in G iff $H \times X$ and $H' \times X$ are locally conjugate in $G \times Y$.*

Proof: $\Rightarrow$ Let $\phi : H \to H'$ be a local conjugation in G. Define $\psi : H \times X \to H' \times X$ by $\psi(h,x) = (\phi(h), x)$. There is $g \in G$ such that $\phi(h) = ghg^{-1}$. Hence $(g, 1_Y)(h,x)(g,1_Y)^{-1} = (ghg^{-1}, 1_Y x 1_Y) = (\phi(h), x)$ and ψ is a local conjugation.

$\Leftarrow$ Let $\psi : (H \times X) \to (H' \times X)$ be a local conjugation in $G \times Y$. Notice for any $h \in H$ there is some $(g,y) \in G \times Y$ so that $\psi(h \times 1_X) = (g \times y)(h \times 1_X)(g \times y)^{-1} = (ghg^{-1}, yy^{-1}) = (ghg^{-1}, 1_Y)$ and we have that $\psi(h, 1_X) \sim_{G \times Y} (h, 1_X)$. Hence ψ induces a bijection from $H \times \{1_X\}$ to $H' \times \{1_X\}$. So define $\phi : H \to H'$ to be the bijection induced by ψ. Thus $h \sim_G \phi(h)$ for each $h \in H$. Therefore ϕ is a local conjugation. $\square$

Recall that g^G denotes the conjugacy class of g in G.

Definition 1.6. (Gassmann's condition) The triple (G, H, H') is called a **Gassmann triple** if for all $g \in G$ we have that $\mid g^G \cap H \mid = \mid g^G \cap H' \mid$

Lemma 1.7. *If (G, H, H') is a Gassmann Triple then H and H' are locally conjugate in G.*

Proof: Suppose $|g^G \cap H| = |g^G \cap H'|$ for all g in G. Fix $\alpha \in H$ and take any bijection $\psi_\alpha : \alpha^G \cap H \to \alpha^G \cap H'$. Note as α varies over H the sets $\alpha^G \cap H$ partition H. Now define $\psi : H \to H'$ so that its restriction to each equivalence class $\alpha^G \cap H$ is ψ_α. So ψ is a bijection. If $h \in H$ then $\psi(h) \in h^G \cap H'$. Thus $\psi(h) \in H'$ with $\psi(h) \sim_G h$. Therefore ψ is a local conjugation from H to H'. □

We will see in Proposition 1.15 that the converse also holds.

Proposition 1.8. *The following are equivalent.*

a) H and H' are locally conjugate in G

b) H/W and H'/W are locally conjugate in G/W

c) $H \times V$ and $H' \times V$ are locally conjugate in $G \times V$

d) $H \times 1_V$ and $H' \times 1_V$ are locally conjugate in $G \times V$

Proof: Note that a) ⇔ c) is an application of Lemma 1.5 where $Y = X = V$ and a) ⇔ d) is an application of Lemma 1.5 where $Y = V$ and $X = 1_V$. We need only check that a) ⇔ b).

a) ⇒ b)

Let $\phi : H \to H'$ be a local conjugation in G. Fix $g \in G$. We want to show that $\mid (gW)^{G/W} \cap H/W \mid = \mid (gW)^{G/W} \cap H'/W \mid$. So let $S_g = \{x \in G \mid xW \sim_{G/W} gW\}$. Let $T_g = H \cap S_g$ and let $T'_g = H' \cap S_g$.

If $h \in T_g$ then $\phi(h) \in H'$. But $\phi(h) \sim_G h$ and $hW \sim_{G/W} gW$ thus $\phi(h)W \sim_{G/W} gW$. So $\phi(h) \in S_g$ and $\phi(h) \in T'_g$. Thus the image of T_g through ϕ is a subset of T'_g. By symmetry the image of T'_g through ϕ^{-1} is a subset T_g. Since ϕ and ϕ^{-1} are both injective it follows that ϕ is a bijection from T_g to T'_g. And $\mid T_g \mid = \mid T'_g \mid$.

If $yW = xW$ and $x \in T_g$ then $y \in T_g$ by construction. Thus for all $x \in T_g$ we have that $xW \subseteq T_g$; likewise for all $x \in T'_g$ we have that $xW \subseteq T'_g$. But by construction

$$x \in T_g \text{ iff } xW \in \{gW^{G/W} \cap H/W\}.$$

Thus if we take $x \in T_g$ then

$$\mid gW^{G/W} \cap H/W \mid = \mid T_g \mid / \mid xW \mid = \mid T_g \mid / \mid W \mid .$$

Likewise

$$x \in T'_g \text{ iff } xW \in \{gW^{G/W} \cap H'/W\}$$

and $\mid gW^{G/W} \cap H'/W \mid = \mid T'_g \mid / \mid W \mid$.

Hence

$$\mid gW^{G/W} \cap H/W \mid = \mid T_g \mid / \mid W \mid = \mid T'_g \mid / \mid W \mid = \mid gW^{G/W} \cap H'/W \mid .$$

So $(G/W, H/W, H'/W)$ forms a Gassmann triple. Therefore by Lemma 1.7 it follows that H/W and H'/W are locally conjugate in G/W.

b) $\Rightarrow$ a)

Let $\psi : H/W \to H'/W$ be a local conjugation in G/W. Fix $g \in G$. We want to show that $\mid g^G \cap H \mid = \mid g^G \cap H' \mid$. It is enough to show that $\mid g^G \cap hW \mid = \mid g^G \cap \psi(hW) \mid$ for all $h \in H$.

Fix $h \in H$. Since ψ is a local conjugation there is $\alpha W \in G/W$ so that $(\alpha W)(hW)(\alpha^{-1}W) = \psi(hW)$. Let $T = g^G \cap hW$. Notice $\alpha T \alpha^{-1} \subseteq g^G \cap \psi(hW)$ and $\alpha T \alpha^{-1} \subseteq g^G$ with $\mid T \mid = \mid \alpha T \alpha^{-1} \mid$. Hence $\mid g^G \cap hW \mid \leq \mid g^G \cap \psi(hW) \mid$. By symmetry

$| g^G \cap hW | \geq | g^G \cap \psi(hW) |$ and

$$| g^G \cap hW | = | g^G \cap \psi(hW) |$$

This holds for all $h \in H$. And since:

$$H = \bigcup_{hW \in H/W} hW \text{ and } H' = \bigcup_{hW \in H/W} \psi(hW)$$

We have that:

$$| g^G \cap H | = | \bigcup_{hW \in H/W} g^G \cap hW | = \sum_{hW \in H/W} | g^G \cap hW |$$

and

$$| g^G \cap H' | = | \bigcup_{hW \in H/W} g^G \cap \psi(hW) | = \sum_{hW \in H/W} | g^G \cap \psi(hW) |$$

So it follows that $| g^G \cap H | = | g^G \cap H' |$. Hence (G, H, H') forms a Gassmann Triple. Therefore by Lemma 1.7 it follows that H and H' are locally conjugate in G. □

We can make a similar statement about global conjugacy.

Proposition 1.9. *The following are equivalent.*

a) H and H' are globally conjugate in G

b) H/W and H'/W are globally conjugate in G/W

c) $H \times V$ and $H' \times V$ are globally conjugate in $G \times V$

d) $H \times 1_V$ and $H' \times 1_V$ are globally conjugate in $G \times V$

Proof: Suppose $(g \times v) \in G \times V$. Then $(g \times v)(H \times V)(g \times v)^{-1} = H' \times V$ will imply that $(g \times v)(H \times 1_V)(g \times v)^{-1} = H' \times 1_V$ which in turn implies that $gHg^{-1} = H'$.

So a) $\Rightarrow$ c) $\Rightarrow$ d). If $gHg^{-1} = H'$ then $(g \times 1_V)(H \times V)(g \times 1_V)^{-1} = H' \times V$. Thus d) $\Rightarrow$ a). We now need to check that *a)* $\Leftrightarrow$ *b)*

a) $\Rightarrow$ *b)* Suppose there is $g \in G$ so that $gHg = H'$. So because $W \lhd G$ we have $(gW)(h \cdot W)(g^{-1}W) = (ghg^{-1})W$ with $ghg^{-1} \in H'$. Thus $(gW)(H/W)(g^{-1}W) \subset (H'/W)$. Hence $(gW)(H/W)(g^{-1}W) \supset (H'/W)$ by symmetry and $(gW)(H/W)(g^{-1}W) = (H'/W)$.

b) $\Rightarrow$ *a)* Suppose $(gW)(H/W)(g^{-1}W) = (H'/W)$ with $g \in G$. Fix $h \in H$. So $(gW)(h{\cdot}W)(g^{-1}W) \in (H'/W)$. So because $W \lhd G$ we have that $(ghg^{-1})W \in H'/W$ and $gHg^{-1} \subset H'$. By symmetry $g^{-1}H'g \subset H$ and $gHg^{-1} = H'$ completing the proof. $\square$

Remark 1.10. Suppose X and Y are subgroups of G. Then for all $\gamma \in G$ we have that $\gamma(X \cap Y) = \gamma X \cap \gamma Y$

Proof: Take $g \in \gamma(X \cap Y)$. So $\gamma^{-1}g \in X \cap Y \subset X$ thus $g \in \gamma X$. Likewise $g \in \gamma Y$. Hence $g \in \gamma X \cap \gamma Y$

Now take $g \in \gamma X \cap \gamma Y$. So $g \in \gamma X$ and $\gamma^{-1}g \in X$. Likewise $\gamma^{-1}g \in Y$. Hence $\gamma^{-1}g \in X \cap Y$ and $g \in \gamma(X \cap Y)$. $\square$

This remark will apply when acting on the right or the left.

Lemma 1.11. *If H and H' are locally conjugate in G, then B and B' are locally conjugate in G.*

Proof: Suppose that $\phi : H \to H'$ is a local conjugation in G. Take $b \in B$ arbitrary. Since $b \in H$ there is $g \in G$ such that $gbg^{-1} = \phi(b)$. Since $b \in M$ and $M \lhd B'$ we have for some $g \in G$ that $\phi(b) \in gMg^{-1} = M$. But $\phi(b) \in H'$ and thus $\phi(b) \in M \cap H' = B'$. Thus $\phi(B) \subseteq B'$. By a symmetric argument $\phi^{-1}(B') \subseteq B$. But ϕ is a bijection. Hence $\phi(B) = B'$ and B and B' are locally conjugate in G. $\square$

Lemma 1.12. *If H and H' are globally conjugate in G, then B and B' are globally conjugate in G.*

Proof: Suppose $gHg = H'$ for $g \in G$. Thus

$$gBg^{-1} = g(H \cap M)g^{-1} = (gHg^{-1}) \cap (gMg^{-1}) = H' \cap M = B'. \; \Box$$

Definition 1.13. Let H be any subgroup of G. The we define the **fixed point character** of H in G as follows:

$$\chi_{G/H} = \#\{\gamma H \in G/H \mid g\gamma H = \gamma H\}.$$

Lemma 1.14. *Assume $\mid H \mid$ is finite. Let $g \in G$, let $C_G(g)$ denote the centralizer of g. Then:*

$$\chi_{G/H}(g) = \frac{\mid g^G \cap H \mid\mid C_G(g) \mid}{\mid H \mid}$$

Proof: Notice for all g and γ in G we have that

$$g\gamma H = \gamma H \text{ iff } \gamma^{-1} g\gamma H = H \text{ iff } g \in \gamma H\gamma^{-1}.$$

But $\gamma H = \gamma h_i H$ for all $h_i \in H$. Thus

$$\begin{aligned}\chi_{G/H}(g) &= \#\{\gamma H \mid g\gamma H = \gamma H\} = \#\{\gamma H \mid g\gamma \in \gamma H\} \\ &= \#\{\gamma H \mid g \in \gamma H\gamma^{-1}\} = \frac{\#\{\gamma \in G \mid g \in \gamma H\gamma^{-1}\}}{\mid H \mid}\end{aligned}$$

So it is enough to show that

$$\#\{\gamma \in G \mid g \in \gamma H\gamma^{-1}\} = \mid C_G(g) \mid\mid g^G \cap H \mid$$

We have two cases:

Case 1) $g^G \cap H = \emptyset$ iff $\forall \gamma \in G$, $g \notin \gamma H\gamma^{-1}$ and our formula holds.

Case 2) If $g^G \cap H \neq \emptyset$ let $\mid g^G \cap H \mid = m$ with $m \geq 1$. Take $g^G \cap H = \{h_1, \ldots, h_m\}$. For each h_i there is $\gamma_i \in G$ so that $\gamma_i g \gamma_i^{-1}$. Let $S = \{\gamma_1, \ldots, \gamma_m\}$. Note that

$\mid S \mid = m$ since the γ_i's are pairwise distinct. Suppose $\gamma \in G$ with $\gamma^{-1} g \gamma \in H$. So $\gamma_i^{-1} g \gamma = h_i$ for some i. Thus $\gamma^{-1} g \gamma = \gamma_i^{-1} g \gamma_i$. Now set $\alpha = \gamma \gamma_i^{-1}$. Then $\gamma = \alpha \gamma_i$ and $\gamma_i^{-1}(\alpha^{-1} g \alpha)\gamma = \gamma^{-1} g \gamma = \gamma^{-1} g \gamma_i$. By cancelation $g = \alpha^{-1} g \alpha$ and $\alpha \in C_G(g)$.

Conversely take $\gamma_i \in S$ and $\alpha \in C_G(g)$. Thus $\gamma_i \alpha \in \{\gamma \in G \mid g \in \gamma H \gamma^{-1}\}$. Therefore

$$\#\{\gamma \in G \mid g \in \gamma H \gamma^{-1}\} = \mid S \mid\mid C_G(g) \mid = m \cdot \mid C_G(g) \mid =$$
$$= \mid g^G \cap H \mid\mid C_G(g) \mid = \mid g^G \cap H \mid\mid C_G(g) \mid$$

completing the proof. □

Proposition 1.15. *Assume $\mid H \mid$ is finite. The following are equivalent:*

a) $\chi_{G/H} = \chi_{G/H'}$

b) (G, H, H') *forms a Gassmann triple*

c) H *and* H' *are locally conjugate in* G

Proof: a) $\Rightarrow$ *b)* Suppose that $\chi_{G/H}(g) = \chi_{G/H'}(g)$ for all $g \in G$. We want to show Gassmann's condition holds. By our formula in lemma 1.14 we need only show that $\mid H \mid = \mid H' \mid$. Notice $\chi_{G/H}(1_g) = \chi_{G/H'}(1_g)$ and $[G : H] = [G : H']$. So $\mid H \mid = \mid H' \mid$ and by lemma 1.14 $\mid g^G \cap H \mid = \mid g^G \cap H' \mid$ for all $g \in G$.

b) $\Rightarrow$ *c)* Lemma 1.7.

c) $\Rightarrow$ *a)* Suppose that $\psi : H \to H'$ is a local conjugation in G. Fix g in G. With out loss of generality suppose that $|g^G \cap H| \geq |g^G \cap H'|$. If $|g^G \cap H| = 0$ then $|g^G \cap H'| = 0$ and we are done. So take $h \in g^G \cap H$. Hence $h \sim_G g$ and $\psi(h) \sim_G h$ thus $\psi(h) \sim_G g$ with $\psi(h) \in H'$. So ψ induces a map from $g^G \cap H$ to $g^G \cap H'$. Therefore $|g^G \cap H| \leq |g^G \cap H'|$ and $|g^G \cap H| = |g^G \cap H'|$. It follows that $\mid H \mid = \mid H' \mid$. Thus by lemma 1.14 *a)* holds. □

Remark 1.16. Suppose G/M is abelian. If $H \sim_G^{loc} H$, then $MH = MH'$.

Proof: Let $\phi : H \to H'$ be a local conjugation in G. Now suppose $h \in H$ and $\gamma \in G$ such that $\phi(h) = \gamma h \gamma^{-1}$. Because G/M is abelian we have:

$$\begin{aligned} M\phi(h) &= M\gamma h \gamma^{-1} = (M\gamma)(Mh)(M\gamma^{-1}) \\ &= (M\gamma)(M\gamma^{-1})(Mh) = \gamma\gamma^{-1}Mh = Mh \end{aligned}$$

The remark follows directly. □

Remark 1.17. Suppose $G = MH$. For any $m_1 \in M$ it follows that $m_1 gH = gH'$ if and only if $m_1 mB = mB$ where $g = mh$ with $m \in M$ and $h \in H$.

Proof of remark: Suppose $m_1 mB = mB$. Then $m_1 gH = m_1 mhH = m_1 mH = mbH$ for some $b \in B$. But $B \subset H$ thus $m_1 gH = mH = mhH = gH$.

Suppose $m_1 gH = gH$, so $m_1 gH = mhH = mH$. So there is $h_2 \in H$ so that $mh_2 = m_1 gh^{-1} = m_1 m$. Thus $h_2 = m^{-1} m_1 m$, but $m^{-1} m_1 m \in M$. Hence $h_2 \in M \cap H$ and $m_1 m \in m(M \cap H) = mB$. Therefore $m_1 mB = mB$. This proves the remark. □

Proposition 1.18. *Suppose H and H' are locally conjugate in G and G/M is abelian.*

a) B and B' are locally conjugate in M if and only if B and B' are locally conjugate in MH

b) If $G = MH$ then B and B' are locally conjugate in M.

Proof of a): $\Rightarrow$ If $\phi : H \to H'$ is a local conjugation in M, then $\phi : H \to H'$ is also a local conjugation in MH.

$\Leftarrow$ Because G/M is ableian $MH = MH'$ Because $H \sim_G^{loc} H'$ if follows that $\mid H \mid = \mid H' \mid, \mid B \mid = \mid B' \mid$ and $[H : B] = [H' : B']$.

Suppose that $\chi_{MH/B} = \chi_{MH'/B'}$ in $MH = MH'$. Fix $m \in M$. We want to show that $\chi_{M/B}(m) = \chi_{M/B'(m)}$. So fix $m_j B \in M/B$ and $h_i B \in H/B$. Notice $B \lhd H$ and so we have that $h_i B = Bh_i$. Thus:

$$m(m_j B) = m_j B \Leftrightarrow m(m_j B)h_i = m_j Bh_i \Leftrightarrow m(m_j h_i B) = m_j h_i B$$

But this is true for all $h_i B \in H/B$. So $\chi_{MH/B}(m) = [H : B]\chi_{M/B}$. By symmetry $\chi_{MH'/B}(m) = [H' : B']\chi_{M/B'}$. Hence $[H : B]\chi_{M/B} = [H' : B']\chi_{M/B'}$ and $\chi_{M/B} = \chi_{M/B'}$. Therefore $B \sim_{MH}^{loc} B'$ implies that $B \sim_{M}^{loc} B'$.□

Proof of b): By lemma 1.11, $H \sim_G^{loc} H'$ implies that $B \sim_G^{loc} B'$. By our work in part a), $B \sim_{MH}^{loc} B'$ if and only if $B \sim_M^{loc} B'$. By assumption $MH = G$. Therefore our claim holds. □

Remark 1.19. The group G acts on G/H by left translation. This gives the group homomorphism $\pi : G \to Sym(G/H)$. Recall that this action is called **faithful** if $| ker(\pi) |= 1_G$.

Remark 1.20. Notice also G acts on G/H' by $\pi' : G \to Sym(G/H')$. We will see in Proposition 1.25 if H and H' are local conjugate in G then that $ker(\pi) = ker(\pi')$. So if H and H' are locally conjugate, π is faithful if and only if π' is faithful.

For our purposes we assume that $ker(\pi)$ and $ker(\pi')$ are both finite.

Lemma 1.21. $ker(\pi) = \bigcap_{g \in G}(gHg^{-1})$

Proof: Let $\gamma \in ker(\pi)$ and $g \in G$ be arbitrary. We want to show that $\gamma \in gHg^{-1}$; that is $\gamma g \in gH$. Since $\gamma \in ker(\pi)$ it follows that $\gamma gH = \pi(\gamma)(gH) = gH$. But $\gamma g \in \gamma gH$. Thus $\gamma g \in gH$ and $\gamma \in gHg^{-1}$. Since g was arbitrary in G, it follows that $\gamma \in \bigcap_{g \in G}(gHg^{-1})$

Now take $\gamma \in \bigcap_{g \in G}(gHg^{-1})$. Let $xH \in G/H$. Notice $\gamma \in xHx^{-1}$ so $\gamma x \in xH$ thus $\gamma xH = xH$. Therefore $\gamma \in ker(\pi)$. □

Corollary 1.22. *If $x \in ker(\pi)$ and $x \sim_G y$ then $y \in ker(\pi)$.*

Proof: Let $x \in ker(\pi)$ and $x \sim_G y$. So there is $z \in G$ such that $zxz^{-1} \in \bigcap_{zg\in G}(zgHg^{-1}z^{-1}) = \bigcap_{g\in G}(gHg^{-1}) = ker(\pi)$ □

Remark 1.23. If $V \lhd G$ and $V \leq H$ then $V \leq ker(\pi)$.

Proof of remark: By definition $gVg^{-1} = V$ for all $g \in G$. Thus $gVg^{-1} = V$ for all $g \in G$. Applying lemma 1.22 completes the proof. □

We will make use of this remark in Chapter 2. For 1.24 and 1.25 we assume that $H \sim_G H'$.

Lemma 1.24. *$ker(\pi) \leq H \bigcap H'$*

Proof: By lemma 1.21 $ker(\pi) = \bigcap_{g\in G}(gHg^{-1}) \leq H$. So we need only check that $ker(\pi) \subseteq H'$. Take $\phi : H \to H'$ a local conjugation in G. So by lemma 1.21 $\phi(ker(\pi)) \subseteq H'$. Take $h' \in \phi(ker(\pi))$ There is $h \in ker(\pi)$ such that $\phi(h) = h'$. Hence $h' \sim_G h$ with $h \in ker(\pi)$.

By lemma 1.22 we have that $h' \in ker(\pi)$. Thus $\phi(ker(\pi)) \subseteq ker(\pi)$. Therefore $ker(\pi) = \phi(ker(\pi)) \subseteq ker(\pi) = \phi(ker(\pi)) \subseteq H'$. □

Proposition 1.25. *$ker(\pi) = ker(\pi')$*

Proof: Let $a \in ker(\pi)$ and $\gamma \in G$ be arbitrary. Notice $\gamma a \gamma^{-1} \sim_G a$. By lemma 1.22 we have that $\gamma^{-1} a \gamma \in ker(\pi)$. But $\gamma \in G$ was arbitrary. Thus $a \in \bigcap_{\gamma\in G} \gamma H' \gamma^{-1} = ker(\pi')$ and $ker(\pi) \subseteq ker(\pi')$. And by symmetry $ker(\pi) = ker(\pi')$. □

Lemma 1.26. *Suppose $V \lhd GV$ such that $G \cap V$ is trivial. If we fix $\alpha_i H, \alpha_j H \in G/H$ and $g \in G$ then*

$$g(\alpha_i H) = \alpha_j H \Leftrightarrow gv(\alpha_i HV) = \alpha_j HV \text{ for all } v \in V$$

Proof: $\Rightarrow$ Suppose $g(\alpha_i H) = \alpha_j H$. Fix $v \in V$. Because V is normal in G we have $g(v\alpha_i HV) = (g\alpha_i H)V = \alpha_j HV$.

$\Leftarrow$ Suppose $gv\alpha_i HV = \alpha_j HV$ for all $v \in V$. Take $v = 1_V$ and $\alpha_j^{-1} g\alpha_i \in HV$. So there is $h \in H$ such that $h^{-1}\alpha_j^{-1} g\alpha_i \in V$. But $h^{-1}\alpha_j^{-1} g\alpha_i \in G$ with $G \cap V$ trivial. So $h^{-1}\alpha_j^{-1} g\alpha_i = 1_G \in H$. Thus $\alpha_j^{-1} g\alpha_i = 1_G \in hH = H$. Therefore $\alpha_j^{-1} g\alpha_i H = H$ and $g\alpha_i H = \alpha_j H$. $\square$

Remark 1.27. Suppose $W = ker(\pi)$. Let $H/W \sim_{G/W} H'/W$ with $\pi : G \to (G/H')$ be defined as above. Suppose also that $[H : W] < \infty$. Then $(G/W,\, H/W,\, H'/W)$ is a faithful Gassmann triple.

Proof of remark: From Proposition 1.8 $H/W \sim_{G/W} H'/W$. Let $G_* = G/W$, $H_* = H/W$ and $H'_* = H'/W$.

Let $\pi_* : G_* \to Sym(G_*/H_*)$ be defined so $\pi_*(g_*)(y_* H_*) = g_* y_* H_*$ for each $g_* \in G_*$ and $x_* H_* \in G_*/H_*$.

By lemma 1.21 $ker(\pi_*) = \bigcap_{g_* \in G} g_* H_* g_*^{-1}$. So let $yW \in \bigcap_{g_* \in G} g_* H_* g_*^{-1}$be arbitrary. Hence for each $g \in G$ there is $h \in H$ so that $yW = (gW)(hW)(gW)^{-1} = ghg^{-1}W$.

Thus $y \in \bigcap_{g \in G}(gHg^{-1}W) = (ker(\pi))W = W$ and $yW = W$. But $W = 1_{G_*}$. Therefore $ker(\pi_*) = ker(\pi'_*)$ is trivial and $(G/W, H/W, H'/W)$ is a faithful G.T. $\square$

Let W_H denote the smallest normal subgroup of G that contains H. Note that the intersection of all normal subgroups of G containing H is again a normal subgroup containing H. Thus W_H can be consider to be this intersection. So if $H \leq W \trianglelefteq G$ then $W_H \leq W$.

Lemma 1.28. *Suppose $H \sim_G^{loc} H'$. Then $W_H = W'_H$.*

Proof: Let W be any normal subgroup of G containing H. Let $h' \in H'$. So there are $h \in H$ and $g \in G$ such that $h' = ghg^{-1}$. Since W is normal in G and $h \in H \leq W$ we have that $h' = ghg^{-1} \in gWg^{-1} = W$.

So W'_H is a normal subgroup containing H. Thus $W'_H \geq W_H$ and $W_H = W'_H$. □

Suppose $H \leq X \leq Y$ and $H' \leq X$. The following two statement will follow directly from the definitions.

Remark 1.29. If $H \sim_X H'$ then $H \sim_Y H'$.

Remark 1.30. If $H \sim_X^{loc} H'$ then $H \sim_Y^{loc} H'$.

Take groups G_{big} and G_{small} such that $G_{big} \geq G_{mid} \geq G_{small} \geq H$ and $G_{small} \geq H'$. Suppose H and H' are NTLC in G_{mid}. Is it possible to construct such groups so that $H \sim_{G_{big}} H'$ but $H \not\sim_{G_{small}}^{loc} H'$? To clarify this question observe the table below.

TABLE 1.1: Comparing global conjugacy and local conjugacy

H and H' within this group are	locally conjugate	globally conjugate
G_{big}	yes	?
G_{mid}	yes	no
G_{small}	?	no

So our assumption is that $H \sim_{G_{mid}}^{loc} H'$ while $H \not\sim_{G_{mid}} H'$. So in row G_{mid} the entries are yes in the first column and no in the second column. By remark 1.30 and the contrapositive of 1.29 there is a yes in the upper left and a no in the lower right. But what answers can be placed in the upper right and lower left?

Example 1.31. Let $G_{mid} = \{\alpha^a\beta^b\gamma^c\}$ be a finite group where the relations of α, β and γ are defined in example **??**. Now define an element δ so that:

$$o(\delta) = 2 \qquad\qquad \delta\alpha = \alpha\delta$$

$$\beta\delta = \delta\beta \qquad\qquad \gamma\delta = \alpha^4\delta\gamma$$

Let $G_{big} =< \alpha, \delta, \beta, \gamma >$ and $G_{small} =< \alpha^4, \beta, \gamma >$. As in example **??** take $H =< \beta, \gamma >= \{e, \beta, \gamma, \beta\gamma\}$ and $H' =< \beta, \alpha^4\gamma >= \{e, \beta, \alpha^4\gamma, \alpha^4\beta\gamma\}$. So $G_{big} \geq G_{mid} \geq G_{small} \geq H$ and $G_{small} \geq H'$. Since H and H' forms a NTLC in G_{mid} it follows that $H \sim_{G_{big}} H'$ and that $H \not\sim^{loc}_{G_{small}} H'$. Now:

$$\delta e\delta = \delta^2 e = e$$

$$\delta\beta\delta = \delta^2\beta = \beta$$

$$\delta\gamma\delta = \delta\alpha^4\delta\gamma = \delta^2\alpha^4\gamma$$

$$\delta\beta\gamma\delta = \delta\beta\alpha^4\delta\gamma = \delta^2\beta\alpha^4\gamma = \alpha^4\beta\gamma$$

Hence $H \sim_{G_{big}} H'$. But G_{small} is an ableian group. Thus $xyx^{-1} = y$ for all x, $y \in G_{small}$. Therefore $H \not\sim^{loc}_{G_{small}} H'$. Thus $G_{big}, G_{mid}, G_{small}, H$ and H' satisfy the table below.

TABLE 1.2: Potential global conjugacy and local conjugacy

H and H' within this group are	locally conjugate	globally conjugate
G_{big}	yes	yes
G_{mid}	yes	no
G_{small}	no	no

This example leads to a definition that will play an important role in subsequent chapters.

Definition 1.32. Suppose H and H' are both subgroups of G_{small}. We say that H and H' forms a **latent triple** if H and H' are not locally conjugate in G_{small}, but there is some $G_{big} \supseteq G_{small}$ such that H and H' are locally conjugate in G_{big}.

We use the word latent, which means hidden, because there is a local conjugation that is not realized in G_{small}. The group G_{small} does not have enough elements to make H and H' locally conjugate, but G_{big} does. We can now apply this definition to example 1.31

TABLE 1.3: Latent triple

triple	locally conjugate	globally conjugate	type
(G_{big}, H, H')	yes	yes	trivial
(G_{mid}, H, H')	yes	no	reducible
(G_{small}, H, H')	no	no	latent

For our purposes G_{big} will be clear in context. Additionally we will be concerned with the case when G_{small} is a normal subgroup of G_{big}

Chapter 2
Gassmann Equivalent Fields

All fields in this chapter are number fields. We let E, F, K, K', L, N, X and Y be the number fields such that

- Let $F \subset N$ is a normal extension
- $F \subset E \cap L$
- $F \subset K \cap K'$
- $KK' \subset N$
- X and Y are arbitrary

The following lemma is a standard result from Galois theory.

Lemma 2.1. *Suppose X is normal over $X \cap Y$. Then under the restriction map $Gal(XY/Y) \cong Gal(X/(X \cap Y))$ and this map is a canonical isomorphism.*

Proof: Let $[X : (X \cap Y)] = n$. It follows from the primitive element theorem that there is some $\alpha \in X$ so that $(X \cap Y)[\alpha] = X$. Thus $XY = Y[\alpha]$. Let $f(x)$ be the characteristic polynomial of α in $X \cap Y$. So f is monic and irreducible in $X \cap Y$. Since X is normal in $X \cap Y$ we can take, $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq X$ to be roots of f.

By way of contradiction, suppose there exist $g(x)$, $h(x) \in Y[x]$ monic such that $0 < \deg(g) < \deg(h)$ and $f(x) = g(x)h(x)$. So there are $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_m} \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. So that $g(x) = (x - \alpha_{i_1}) \cdots (x - \alpha_{i_m}) \in Y[x]$. Hence $g(x) \in \mathbb{Z}[\alpha_1, \alpha_2, \cdots, \alpha_n]$ and the coefficients of g are in X. Thus $g(x) \in (X \cap Y)[x]$ and $g(x)$ is a nontrivial divisor of $f(x)$. This is a contradiction. Therefore f is irreducible

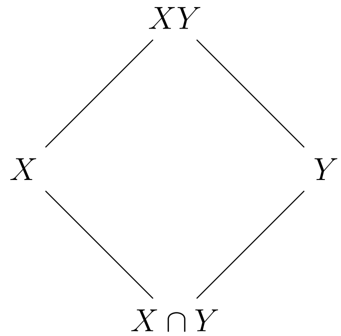

FIGURE 2.1: Canonical Isomorphism: Proposition 2.1

in Y. Thus $Gal(XY/Y) \cong Gal(X/(X \cap Y))$ since they are determined by the same characteristic polynomial $f(x)$.

Now let $\phi : Gal(XY/Y) \to Gal(X/(X \cap Y))$ be the restriction map $\phi(\sigma) = \sigma \mid_X$. Take σ_1,$\sigma_2 \in Gal(XY/Y)$ distinct. So there is some $\beta \in XY$ such that $\sigma_1(\beta) \neq \sigma_2(\beta)$. Take $\beta = \Sigma n_j a_j$ with $n_j \in X$ and $a_j \in Y$ for all j. Then $\sigma_t(\beta) = \Sigma \sigma_t(n_j) a_j$ for $t \in 1, 2$. But σ_1 and σ_2 both fix Y and $\sigma_1(\beta) \neq \sigma_2(\beta)$, Hence there is some $n_s \in X$ such that $\sigma_1(n_s) \neq \sigma_2(n_s)$. Thus $\phi(\sigma_1)(n_s) \neq \phi(\sigma_2)(n_s)$ and ϕ is injective.

Take $\tilde{\phi} : Gal(X/(X \cap Y)) \to Gal(XY/Y)$ with $\tilde{\phi}(\sigma) = \tilde{\sigma}$ such that if $\alpha \in X$ and $\beta \in Y$ then $\tilde{\sigma}(\alpha\beta) = \sigma(\alpha)\beta$. Thus $\tilde{\sigma}(Y) = Y$ and $\tilde{\phi} \in Gal(XY/Y)$. So $\phi(\tilde{\phi}(\sigma))(\alpha\beta) = \phi(\tilde{\sigma}(\alpha\beta)) = \tilde{\sigma}(\alpha\beta) \mid_X = \sigma(\alpha)$. This shows ϕ is surjective and thus bijective. Therefore $Gal(XY/Y)$ and $Gal(X/X \cap Y)$ are canonically isomorphic. □

We now note the relationship between global conjugacy and isomorphic subfields of normal number field.

Lemma 2.2. *Let $G = Gal(N/\mathbb{Q})$. Let $H = Gal(N/K)$. Then for any $\sigma \in G$ we have that $\sigma H \sigma^{-1} = Gal(N/\sigma(K))$*

Proof: (Lemma 2.8.7 in Weintraub[16]) Let $\beta \in K$ and $\tau \in H$. Then $\sigma\tau\sigma^{-1}(\sigma(\beta)) = \sigma\tau(\beta) = \sigma(\beta)$. So all elements that fix $\sigma(K)$ are also in $\sigma H \sigma^{-1}$. Thus $\sigma H \sigma^{-1} \geq Gal(N/\sigma(K))$.

Now take $\phi \in Gal(N/\sigma(K))$. Fix $\beta \in K$ and $(\sigma^{-1}\phi\sigma)(\beta) = \sigma^{-1}(\phi(\sigma(\beta)))$. But ϕ fixes $\sigma(K)$. So $(\sigma^{-1}\phi\sigma)(\beta) = \sigma^{-1}\sigma(\beta) = \beta$. Thus $\sigma^{-1}\phi\sigma$ fixes K. By definition $\sigma^{-1}\phi\sigma \in H$. Therefore $Gal(N/\sigma(K)) \leq \sigma^{-1}H\sigma$.

Therefore $\sigma H \sigma^{-1} = Gal(N/\sigma(K))$. □

Theorem 2.3. *Let $H = Gal(N/K)$ and $H' = Gal(N/K')$. Then $K \cong K'$ if and only if H and H' are globally conjugate in $G = Gal(N/\mathbb{Q})$.*

Proof: ⇒ Let $\sigma : K \to K'$ be an isomorphism. So by lemma 2.2 we have $H' = Gal(N/K') = Gal(N/\sigma(K)) = \sigma H \sigma^{-1}$. Thus H and H' are conjugate.

⇐ Suppose $H' = \sigma H \sigma^{-1}$ for some $\sigma \in G$. So by lemma 2.2 we have $Gal(N/K') = H' = \sigma H \sigma^{-1} = Gal(N/\sigma(K))$. So $\sigma : K \to K'$ is an isomorphism. □

Definition 2.4. We say K and K' are **Gassmann equivalent** fields in F, denoted GE over F, if $Gal(N/K)$ and $Gal(N/K')$ are locally conjugate in $Gal(N/F)$.

When $F = \mathbb{Q}$ we call the fields arithmetically equivalent. But for our purposes is more appropriate to use this broader definition.

Notice this is a definition with respect to our base field F and our normal extension N is suppressed. There is no ambiguity here in light of the following remark.

Remark 2.5. Let N_1 and N_2 be any finite normal extensions of base fields containing both K and K'. Let $G_1 = Gal(N_1/F)$ and $G_2 = Gal(N_2/F)$. Then

$$Gal(N_1/K) {\sim_{G_1}}^{loc} Gal(N_1/K') \Leftrightarrow Gal(N_2/K) {\sim_{G_2}}^{loc} Gal(N_2/K')$$

Proof of remark: Take N_3 to be the normal closure of N_1N_2 with respect to F. Let $G = Gal(N_3/F)$,$W_1 = Gal(N_3/N_1)$,$W_2 = Gal(N_3/N_2)$. So $G_1 = G/W_1$ and $G_2 = G/W_2$. Let $H = Gal(N_2/K)$ and $H' = Gal(N_2/K')$. So by proposition 1.8 the remark follows. □

Proposition 2.6. *Let $G = Gal(N/F)$ and $H = Gal(N/K)$. Take $\pi : G \to Sym(G/H)$ defined in remark 1.19 and let $\tilde{N}$ be the field fixed by $ker(\pi)$. Then $\tilde{N}$ is a normal closure of K with respect to F.*

Proof: Take N_1 a field such that $K \subseteq N_1 \subseteq \tilde{N}$ and N_1 is normal with respect to F. So notice $N_1 \subseteq \tilde{N} \subseteq N$ both normal. Thus $Gal(N/\tilde{N})$ and $Gal(N/N_1)$ are both normal in G. So by remark 1.23 we have $Gal(N/N_1) \leq ker(\pi) = Gal(N/\tilde{N})$. Thus $\tilde{N} \subseteq N_1$ but $N_1 \subseteq \tilde{N}$ and $\tilde{N} = N_1$. Therefore $\tilde{N}$ is a normal closure of K with respect to F. □

Corollary 2.7. *If K and K' are GE over F then they share a normal closure .*

Proof: By Lemma 1.22 $ker(\pi) = ker(\pi')$. So the field $\tilde{N}$ fixed by $ker(\pi)$ contains both K and K'. But by prop 2.6 $\tilde{N}$ is a shared normal closure of both K and K'. □

Proposition 2.8. *a)If K and K' are GE over F then KE and $K'E$ are GE over F.*
b)Suppose that E and NL are both normal over L so that $E \cap NL = L$. The following are equivalent

i) KL and $K'L$ are GE over L

ii) KE and $K'E$ are GE over L

iii) KE and $K'E$ are GE over E

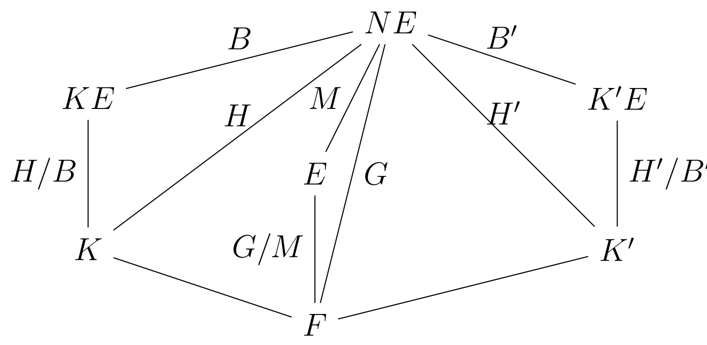

FIGURE 2.2: Gassmann equivalent fields: Proposition 2.8 part a)

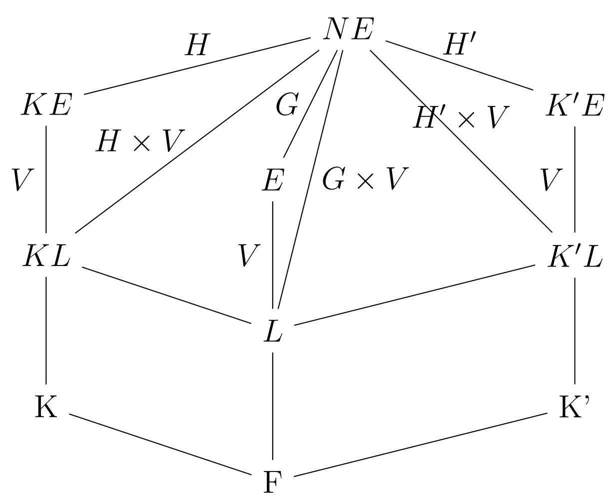

FIGURE 2.3: Gassmann equivalent fields: Proposition 2.8 part b)

Proof of a): Let $G = Gal(NE/F)$, $H = Gal(NE/K)$, $H = Gal(NE/K')$ and $M = Gal(NE/E)$. So $B = H \cap M = Gal(NE/KE)$, $B' = H' \cap M = Gal(NE/K'E)$. Part a) follows from lemma 1.11.

Proof of b): Let $G = Gal(NE/E)$, $H = Gal(NE/KE)$,$H' = Gal(NE/K'E)$ and $V = Gal(E/L)$. So $G \times V = Gal(NE/L)$. By lemma 2.1 $Gal(K'E/K'L) \cong Gal(KE/KL) \cong Gal(NE/NL) = V$ and the fixed fields of $H \times V$ and $H' \times V$ are KL and $K'L$ respectively. So $H \times 1_V = Gal(NE/KE)$ and $H' \times 1_V = Gal(NE/K'E)$. Thus by 1.8 part b) follows. □

Definition 2.9. We say K and K' are **isomorphic fields over** F denoted $K \cong_F K'$ if $Gal(N/K)$ and $Gal(N/K')$ are globally conjugate in $Gal(N/F)$.

Notice $K \cong_F K' \Rightarrow K \cong_{\mathbb{Q}} K' \Leftrightarrow K \cong K'$. We use this definition to clarify the Galios groups in which $Gal(N/K)$ and $Gal(N/K')$ are conjugate.

Proposition 2.10. *a)If K and K' are isomorphic over F then KE and $K'E$ are isomorphic over F.*

b)Suppose that E and NL are both normal over L so that $E \cap NL = L$. The following are equivalent

i) KL and $K'L$ are isomorphic over L

ii) KE and $K'E$ are isomorphic over L

iii) KE and $K'E$ isomorphic over E

Proof: The Galois groups are constructed as in Proposition 2.8. Part a) follows from 1.12 and part b) follows from 1.9.

Proposition 2.11. *Suppose $L \subseteq N$. Then $KE \cong_F K'E$ if and only if $KL \cong_F K'L$*

Proof: Take $G = Gal(NE/F)$, $H = Gal(NE/KL)$,$H' = Gal(NE/K'L)$ and $W = Gal(NE/NL)$. So $H/W = Gal(NL/KL)$ and $H'/W = Gal(NL/K'L)$. Thus by Proposition 1.9 $KE \cong_F K'E$ if and only if $KL \cong_F K'L$.

Remark 2.12. If $KE = K$ then $NE = N$.

Proof of remark: If $KE = K$ then $NE = (NK)E = N(KE) = NK = N$ □

The following proposition is similar to Theorem 1.6 of Chapter 3 in Klingen **??**.

Proposition 2.13. *Let E be a normal extension of F such that $Gal(E/F)$ is abelian and let K and K' be GE over F. Then*

a) $K \cap E = K' \cap E$

b)$KE = K$ iff $K'E = K'$

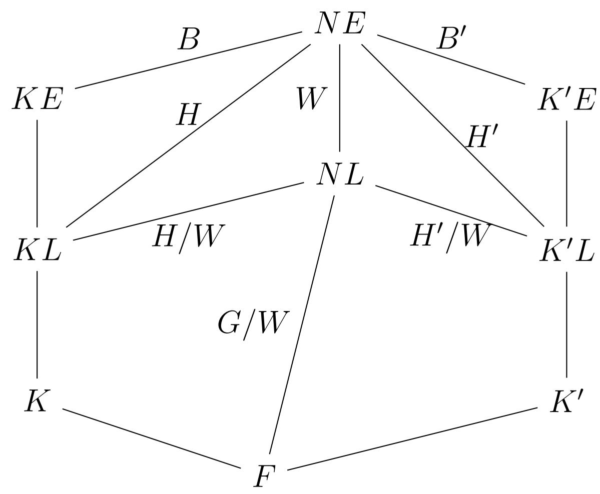

FIGURE 2.4: Isomorphic fields over K and K'

Proof of a): Let N be the common normal closure of K and K'. If $NE \neq N$ then then by remark 2.12 above $KE \neq K$ and $K'E \neq K'$. So suppose $NE = N$. Thus $E \subseteq N$. Take $G = Gal(N/F)$, $H = Gal(N/K)$ and $H' = Gal(N/K')$. So by remark 1.16, $MH = MH'$. Thus:

$$\begin{aligned} Gal(N/(K \cap E)) &= Gal(N/K)Gal(N/E) = MH \\ &= MH' = Gal(N/K')Gal(N/E) = Gal(N/(K' \cap E)) \end{aligned}$$

and part a) holds. □

Proof of b): Note that $KE = K$ iff $K \cap E = K$ and $K'E = K$ iff $K \cap E = K'$. Thus part b) follows directly from part a). □

Proposition 2.14. *Let E be a normal extension of F such that $[E : F] = p$ a prime and let K and K' be GE over F. Then KE and $K'E$ are G.E. over F.*

Proof: Let $G = Gal(NE/F)$, $H = Gal(NE/K)$,$H' = Gal(NE/K')$ and $M = Gal(NE/E)$. So $M \lhd G$ with $H \cap M = B = Gal(NE/KE)$ and $H' \cap M = B' = Gal(NE/K'E)$. Thus by Lemma 1.11, $Gal(NE/KE)$ and $Gal(NE/KE)$ are locally conjugate in $Gal(NE/F)$ and our claim holds. □

Chapter 3

$\mathbb{Z}_p$ Towers

We assume that N is common normal closure of K and K' over F

Definition 3.1. Let p be a prime. A $\mathbb{Z}_p$ **tower** over F denoted F_∞/F is a sequence of fields

$\{F = F_0 \subset F_1 \subset \cdots \subset F_\infty = \cup F_n\} = F_\infty/F$

for which $Gal(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$ for each n.

Proposition 3.2. *Let F_∞/F be a $\mathbb{Z}_p$ tower. Let p^{e_n} be the exact power dividing the p part of the class group of F_n the nth step of the $\mathbb{Z}_p$ tower with $F_0 = F$. Then there are integers $\lambda \geq 0$, $\mu \geq 0$,ν and n_0 all independent of n such that*

$e_n = \lambda n + \mu p^n + \nu \;\forall n \geq n_0$

Proof: by Iwasawa[15]

Proposition 3.3. *Let F_∞/F be a $\mathbb{Z}_p$ tower. Fix an $i \geq 0$. Take $E = E_0 = F_i$, set $E_j = F_{i+j}$ and $E_\infty = \cup E_j = F_\infty$. If λ_F,μ_F and ν_F are the Iwasawa invariants of F and λ_E,μ_E and ν_E are the Iwasawa invariants of E then,*

$\lambda_E = \lambda_F, \mu_E = \mu_F p^i$ *and* $\nu_E = \nu_F + \lambda_F i$.

Proof:

$$\begin{aligned} e_{(n+i)} &= \lambda(n+i) + \mu p^{(n+i)} + \nu \\ &= \lambda n + \mu p^{(n+i)} + (\nu + \lambda i) \\ &= \lambda n + \mu p^n p^i + (\nu + \lambda i) \\ &= (\lambda)n + (\mu p^i)p^n + (\nu + \lambda i) \end{aligned}$$

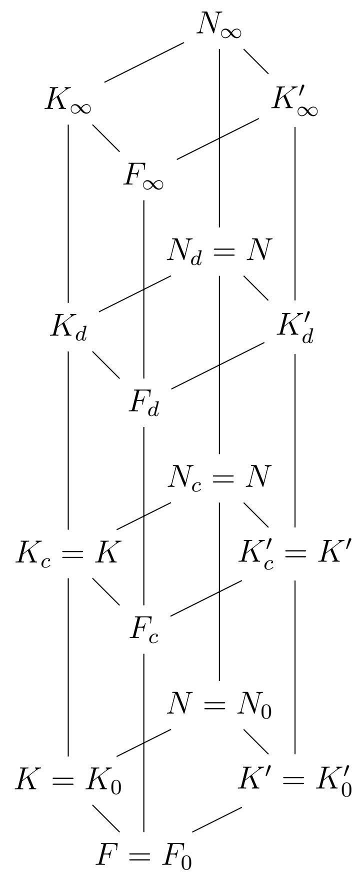

FIGURE 3.1: Towers over K and K'

Thus $\lambda_E = \lambda_F$,$\mu_E = \mu_F p^i$ and $\nu_E = \nu_F + \lambda_F i$. □

Proposition 3.4. *Let F_∞/F be a $\mathbb{Z}_p$ tower. Let c, c' and d be the integers such that $c = max\{i \mid F_i \subseteq K\}$,$c' = max\{i \mid F_i \subseteq K'\}$ and $d = max\{i \mid F_i \subseteq N\}$. Then:*

a) $c \leq d$

b) If K and K' are GE over F then $c = c'$.

Proof: For part a), note by remark 2.12 if $F_n \subseteq K$ then $F_n \subseteq N$. Thus $c \leq d$. As for part b), note that $Gal(N/F_n)$ is cyclic and thus abelain. So by Theorem 2.13 $F_n \subseteq K$ iff $F_n \subseteq K'$. Thus $c = c'$ completing the proof. □

We now define a third integer value b related to c and d.

Definition 3.5. Let $b \leq c$ be maximal such that K_d and K_d are GE over F_b. Then d will be the **lag** of the tower and $c - b$ will be **obstruction** of the tower. We call c the **quasi-obstruction**.

Notice when $c = 0$, our obstruction is zero. The converse might not hold. This is why we call c the quasi-obstruction. The next two lemmas will clarify the names lags and obstruction

Lemma 3.6. *For all $i \geq d$ there is a canonical isomorphism under which*

a) $Gal(N_i/F_i) \cong Gal(N_d/F_d)$

b) $Gal(N_i/K_i) \cong Gal(N_d/K_d)$

c) $Gal(N_i/K_i') \cong Gal(N_d/K_d')$

Proof: Part a) follows from lemma 2.1 with $X = N_d$ and $Y = F_i$. Part b) follows from lemma 2.1 with $X = N_d$ and $Y = K_i$. Part c) follows from lemma 2.1 with $X = N_d$ and $Y = K_i'$. □

Lemma 3.7. *a) K_d and K_d are GE over F_c if and only if K_d and K_d are GE over F_d.*

b) If $c = 0$ then K_d and K_d are GE over F_d.

Proof: Let $M = Gal(N/F_d)$, $H = Gal(N/K)$, $H = Gal(N/K)$ and $G = Gal(N/F)$. So $H \sim_G^{loc} H'$ and G/M is abelian. Thus by lemma 1.18 part a) holds.

Now suppose $c = 0$. Hence $F_c = F_0$. But $F_d \cap K = F_d \cap K_c = F_c = F_0$. So $G = Gal(N/F) = Gal(N/(F_d \cap K)) = Gal(N/F_d)Gal(N/K) = MH$. Thus from part b) follows from proposition 1.18. □

We notice if $i < d$ then $[N_i : F_i] = [N : F_i] = [N : F_d][F_d : F_i] = [N_d : F_d][F_d : F_i]$. So $Gal(N_i/K_i) \ncong Gal(N_d/K_d)$ because the indices are different. So the lag d is the finite number of steps until we have this canonical isomorphism in lemma 6.

If $b \neq c$ then by assumption K_d and K'_d are not GE over K_c. But if $b = c$ then K_d and K'_d are GE over K_c and by lemma 3.7 K_d and K'_d are not GE over K_d. So the obstruction is zero if and only if there is a Gassmann triple at level d. And by lemma 6 this triple is canonically isomorphic to the triple at level i for all $i \geq d$.

Theorem 3.8. *Let p to be any prime and F_∞/F be a $\mathbb{Z}_p$ tower over F. For each $n \geq 0$ take $K_n = KF_n$ and $K'_n = K'F_n$.*

a) If $i \geq 0$ then K and K' are G.E. over F iff K_i and K'_i are G.E. over F

b) Suppose $j \geq i \geq d$. Then the following are equivalent

i)K_i and K'_i are G.E. over F_i

ii)K_j and K'_j are G.E. over F_i

iii)K_j and K'_j are G.E over F_j

Proof of a): Take $E = F_i$. Part a) follows from Proposition 2.8 part a). □

Proof of b): Take $E = F_j$ and take $L = F_i$. Notice E and NL is a normal extension of L and $E \cap NL = F_j \cap NF_i = F_j \cap N_i = F_i = L$. Thus by part b) follows from Theorem 2.8 part b). □

Corollary 3.9. *The following are equivalent*

a) K_d and K'_d are G.E. over F_d

b) There are $j \geq i \geq d$ for which K_j and K'_j are G.E. over F_i

c) For all $j \geq i \geq d$ it follows that K_j and K'_j are G.E. over F_i

Proof: c) $\Rightarrow$ a) Take $i = j = d$.

a) $\Rightarrow$ b) Take $i = j = d$.

b) $\Rightarrow$ c) Suppose there exist $j_1 \geq i_1 \geq d$ such that K_{j_1} and K_{j_1} are G.E. over F_{i_1}. Fix j_2 and i_2 so that $j_2 \geq i_2 \geq d$. So by theorem 3.8 part b) we have that:

$$K_{j_1} \text{ and } K_{j_1} \text{ are G.E. over } F_{i_1}.$$

which implies

$$K_{i_1} \text{ and } K_{i_1} \text{ are G.E. over } F_{i_1} \text{ (from ii}\Rightarrow\text{i)}$$

which implies

$$K_{i_2} \text{ and } K_{i_2} \text{ are G.E. over } F_{i_2} \text{ (from i}\Rightarrow\text{iii or iii}\Rightarrow\text{i)}$$

which implies

$$K_{j_2} \text{ and } K_{j_2} \text{ are G.E. over } F_{i_2} \text{ (from i}\Rightarrow\text{ii).}$$

Since i_2 and j_2 were arbitrary, part c) holds □

Remark 3.10. If $K_i \cong K'_i$ for some i then for any $j \geq i$ we have that $K_j \cong K'_j$.

Proof of remark: Note that for any $j \geq i$ we have that $K_j = K_i F_j$ and $K'_j = K'_i F_j$. The remark follows directly. □

Theorem 3.11. *Let p to be any prime and F_∞/F be a $\mathbb{Z}_p$ tower over F. For each $n \geq 0$ take $K_n = KF_n$ and $K'_n = K'F_n$.*

a) If $i \geq 0$ then K and K' are isomorphic over F iff K_i and K'_i are isomorphic over F

b) Suppose $j \geq i \geq d$. Then the following are equivalent

i)K_i and K'_i are isomorphic over F_i

ii)K_j and K'_j are isomorphic over F_i

iii)K_j and K'_j are isomorphic over F_j

Proof of a): Take $E = F_i$. Part a) follows from Proposition 2.10 part a). □

Proof of b): Take $E = F_j$ and take $L = F_i$. Notice E and NL is a normal extension of L and $E \cap NL = F_j \cap NF_i = F_j \cap N_i = F_i = L$. Thus by part b) follows from Theorem 2.10 part b). □

We now have the tools in place to state Greenberg's Question. Take p prime and let K_∞/K and K'_∞/K' be $\mathbb{Z}_p$ towers. These towers are **parallel towers** if there is a $\mathbb{Z}_p$ tower F_∞/F so that for each i we set $K_i = KF_i$ and $K'_i = K'F_i$.

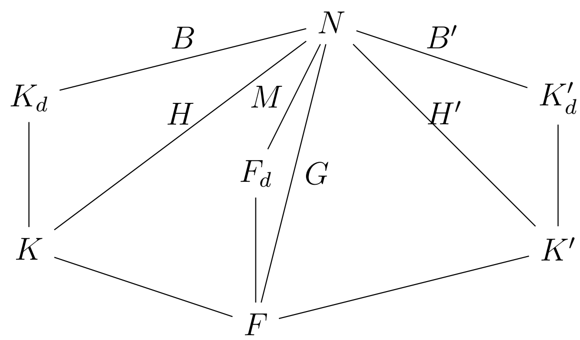

FIGURE 3.2: Galois groups over K_d and K'_d

Greenberg's Question: Do parallel towers over G.E. fields share the same Iwasawa invariants?

For Greenberg's question to be interesting we need to make two assumptions:

- Assumption A) $K \not\cong K'$
- Assumption B) K and K' are not G.E. over F

We break Greenberg's question into four cases using the definitions of global conjugation, local conjugation and latent triples, along with the value d. We set $M = Gal(N/F_d)$. Thus $B = Gal(N/K_d)$ and $B' = Gal(N/K'_d)$. By lemma 3.7, $B \sim_M^{loc} B$ if and only if $b = c$.

Observing table 3.1, we will have one of four different case which we will call our **tower types**.

TABLE 3.1: The four types of towers

	$d = 0$	$d \neq 0$
$B \sim_M B'$	violates assumption A)	1) trivial tower
$B \not\sim_M B'$ But $B \sim_M^{loc} B'$	2) simple tower	3) reducible tower
$B \not\sim_M^{loc} B'$	violates assumption B)	4) latent tower

Tower type 1: trivial tower We define a tower to be a **trivial tower** if $B \sim_M B'$. Notice this in this the case in remark 3.10. So for all $j \geq d$ we have

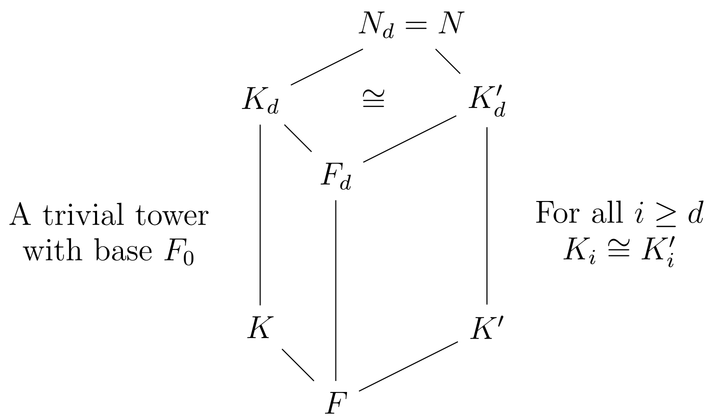

FIGURE 3.3: Trivial tower

that $K_j \cong K'_j$. This would imply the Iwasawa invariants of K and K' would be the same. So for any trivial tower the answer to Greenberg's question is yes.

Tower type 2: simple tower Suppose that $d = 0$. Notice that $Gal(N/F_d)$ will be a normal subgroup of $Gal(N/F_0)$. If $d = 0$ these groups are equal. And when $Gal(N/F_0)$ is a simple group $d = 0$ necessarily. Thus we define a tower to be a **simple tower** if $d = 0$. Also $B \sim_M B'$ will violate Assumption A) and $B \not\sim_M^{loc} B'$ will violate Assumption B). A simple parent group must have a simple tower. However a simple tower need not have simple parent group .

Tower type 3: reducible tower We define a tower to be a **reducible tower** if (M, B, B') is a nontrivial Gassmann triple. Notice since d is not zero we have that $\mid M \mid$is a proper divisor of $\mid G \mid$ and $[G : B]$ is a proper divisor of $[G : H]$. Thus we can consider the question for reducible towers in the following way.

Consider a list of ordered pairs $[a, b]$ where a is the index of a nontrivial Gassman triple and b is the order of the parent group of the corresponding triple. We list all possible Gassman triples in order first by the index and then by the parent group order. If it is know that a certain entry on our list will have a reducible

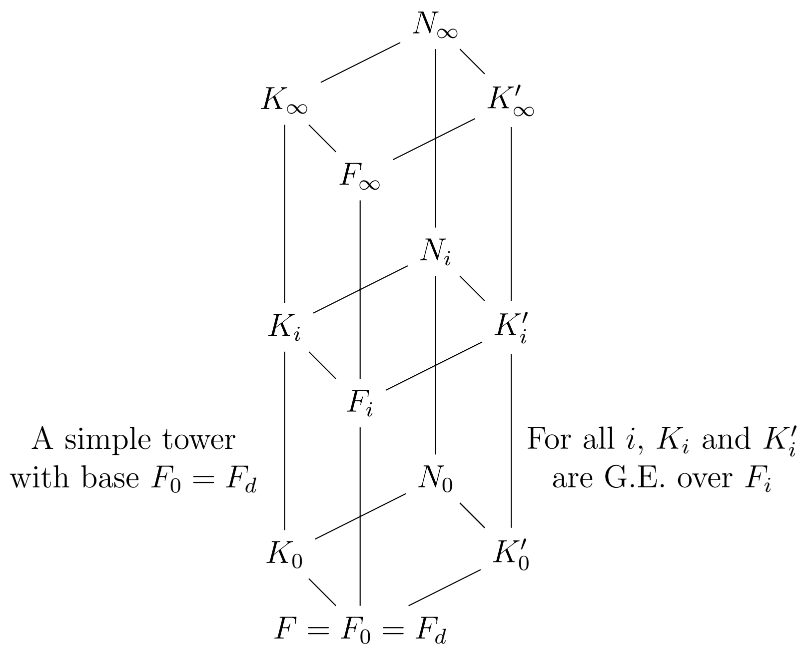

FIGURE 3.4: Simple tower

tower and it is know that all towers with lesser index or group order are known to have an affirmative answer to Greenberg's question, then for the reducible tower in question the answer to Greenberg is yes.

We give a hypothetical example. The index values and parent group orders for the first 7 Gassman triples are as follows:

$[7, 168], [8, 32], [8, 48], [11, 660], [12, 48], [12, 72], [12, 96]$

We will go into more details on these values in the next chapter. Suppose there was a reducible tower of index 12 and order 96. We do not claim at this point to know what all towers corresponding to these triples look like, let alone the answer to Greenberg's question. But hypothetically suppose it is known that all towers corresponding to the first six triples will an affirmative answer to Greenberg's question. Then it would follow that for any reducible tower with index 12 and order 96 the answer to Greenberg's question will be yes.

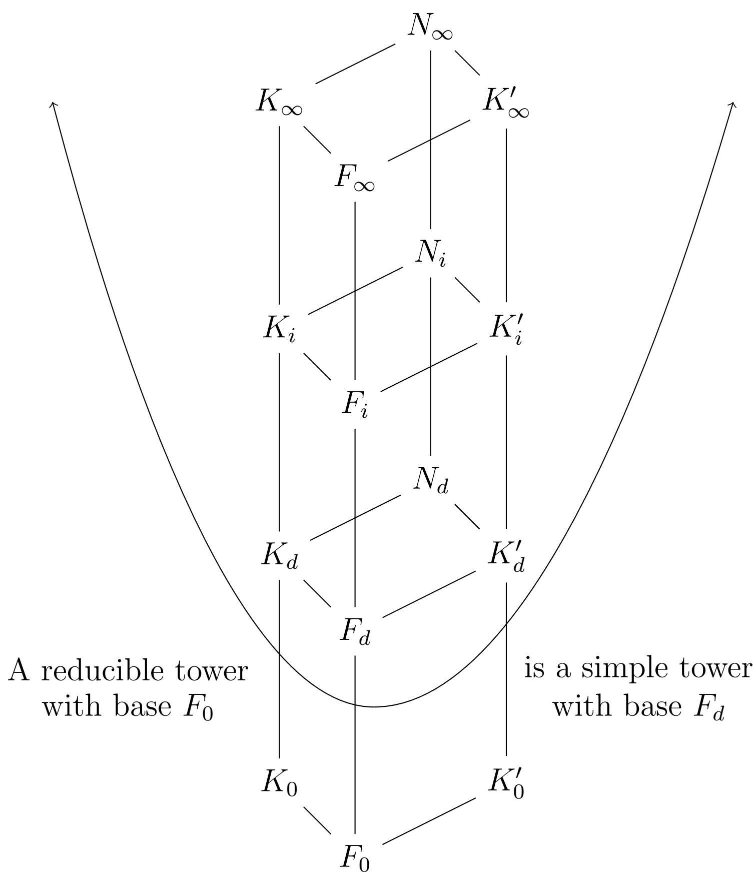

FIGURE 3.5: Reducible tower

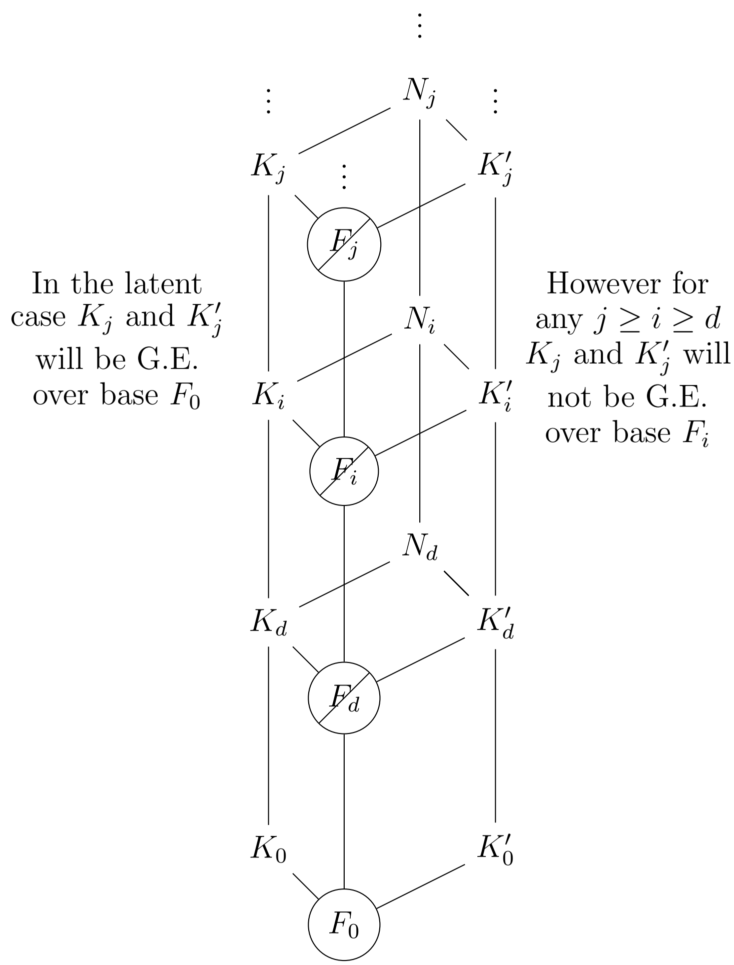

FIGURE 3.6: Latent tower

We can even go one step further. Since 7 and 8 do not divide 12 and 72 does not divide 96. We need only check whether the towers corresponding to the pair[12,48] will have an affirmative answer to Greenberg's question. This will simplify the question for this particular case. But this method describes a rigorous way to search for counterexamples to the statement of Greenberg's question.

Tower type 4: latent tower We define a tower to be a **latent tower** if (M, B, B') is a latent triple. Note that by lemma 1.11 (G, B, B') is a Gassman triple and $M \subset G$. So if (M, B, B') is not a Gassman triple then it will be a latent triple.

It follows from 3.8 part a) that K_i and K_i' will be G.E. over F. But notice in the latent case K_d and K_d will not be G.E. over F_d. It follows from Corollary 3.9 that for all $j \geq i \geq d$ we have K_j and K_j' will not be G.E. over F_i. All towers will fall into one of these four categories. Note in each category the tower type is determined by the triple (M, B, B') which are the three Galois groups at the level d. This is because d is the level that the lag ends.

Chapter 4

Categorization of Galois Groups with Low Index

In this chapter we will refer to the "Gap4" numbers of finite groups with low degree by a pair $[a, b]$ where a indicates the order and b indicates the entry in the Gap4 library. For example $[4, 1]$ is the cyclic group of order 4 and $[4, 2]$ is the Klein 4 group. Note that most groups of order greater than 1000 are not listed in the Gap library and will not have a Gap number.

Let K, K' be Gassman Equivalent over base field F with common normal closure N over F. If $G = Gal(N/F)$,$H = Gal(N/K)$ and $H' = Gal(N/K')$ then (G, H, H') forms a Gassman triple. By de Smit we know there are 19 faithful Gassman triples with index less than 16. We fix a prime p and choose a $\mathbb{Z}_p$-tower F_∞/F. We now want to find the values of c and d as in Proposition 3.4.

Let $M = Gal(N/F_d)$, $B = Gal(N/K_d)$ and $B' = Gal(N/K'_d)$. What properties will these groups have? There are two necessary conditions.

1) $[G : M]$ is the power of a prime

2) G/M is cyclic

Note $p^d = [G : M]$ so the value d will be completely determined by G and M. If $c > 0$ then $gcd(|H|, [G : M]) > 0$. The value p^c will be a divisor of $gcd(\mid H \mid, [G : M])$. For our purposes we call the pair $(p^d, gcd(p^d, \mid H \mid))$ our **lag type**. For the 19 triples there will only be 5 possible cases for these values, 4 of which correspond to non-simple towers. These values are listed in table 4.1.

The first two values in this table will be the lag type, that is $(1, 1)$, $(2, 2)$, $(3, 1)$, $(3, 3)$ or $(4, 4)$. The pair $(1, 1)$ indicates a simple tower.

TABLE 4.1: Lag types

$[G:M]=p^d$	$gcd(p^d, \mid H \mid)$	p	d	possible values for c
1	1	-	0	0
2	2	2	1	0 or 1
3	1	3	1	0
3	3	3	1	0 or 1
4	4	2	2	0,1 or 2

Notice if $M = G$ then $p^d = 1$ and $d = 0$. This would mean there is no lag. Likewise if $M =< e >$ then $G/M = G$ won't be cyclic. As we mentioned in the previous chapter this indicates when the parent group G is simple there will be no lag. This will yield a simple tower. Six of de Smit's 19 triples will fall into this category which are listed in table 4.2.

TABLE 4.2: Simple towers of low index

G	H	H'	$[G:H]$	M	B	B'	tower type	lag type	b=c=d
[168,42]	[24,12]	[24,12]	7	$--$	$--$	$--$	simple	$(1,1)$	0
[660,13]	[60,5]	[60,5]	11	$--$	$--$	$--$	simple	$(1,1)$	0
[5616,?]	[432,732]	[432,732]	13	$--$	$--$	$--$	simple	$(1,1)$	0
[168,42]	[12,3]	[12,3]	13	$--$	$--$	$--$	simple	$(1,1)$	0
[2520,?]	[168,42]	[168,42]	14	$--$	$--$	$--$	simple	$(1,1)$	0
[20160,?]	[1344,?]	[1344,?]	15	$--$	$--$	$--$	simple	$(1,1)$	0

In the 12 non-simple triples we may still have a simple tower. But if $M \lhd G$ is not trivial, this will indicate we have the trivial, reducible or latent case. According to de Smit there are two distinct triples with $[G:H] = [G:H] = 14$ and $|G| = 336$. Although the parent groups are isomorphic the triples themselves are not. Thus we are left with 12 possible parent groups that yeild a lag in our tower. The following table determines all possible M's for a lag in a $\mathbb{Z}_p$ tower over a triple with one of the 12 non-simple parent groups

There are two non-isomorphic parent groups with order 96 and index 12. Since there are two non-isomorphic triples with isomorphic parent groups of order 336.

TABLE 4.3: Lag types of low index

Gap number for G	$[G : H]$	$(p^d, gcd(p^d, \mid H \mid)$	# of Ms	Gap number for M(s)
[32,43]	8	(2,2)	7	see chapter 5
[48,29]	8	(2,2)	1	[24,3]
[48,49]	12	(2,2)	1	[24,13]
[48.49]	12	(3,1)	1	[16,14]
[72,23]	12	(2,2)	3	[36,12][36,13][36,3]
[96,195]	12	(2,2)	3	[48,30][48,48][48,49]
[96,3]	12	(3,1)	1	[32,2]
[192,194]	12	(2,2)	1	[96,3]
[192,194]	12	(3,1)	1	[64,73]
[240,91]	12	(2,2)	1	[120,35]
[240,91]	12	(4,4)	1	[60,5]
[336,209]	14	(2,2)	1	[168,42]
[56448,?]	14	(2,2)	1	[28224,?]
[180,19]	15	(3,3)	1	[60,5]
[360,120]	15	(2,2)	1	[180,19]

Since d is determined by the parent groups and there normal subgroups, this group of order 336 only occurs in table 4.3.

Also within these 12 parent groups there are other possible $M \triangleleft G$. We may have lag type $(4, 2)$ with G/M not cyclic. Also lag type $(6, 2)$ with G/M cyclic will occur, but 6 is not a prime power. These will never be the lag in a $\mathbb{Z}_p$ tower.

We now want to determine the tower types of these normal groups. Note there may or may not be a tower of number fields with $Gal(N/F) = G$ and $Gal(N/F_d) = M$. But if there is such a tower, it will have the properties listed.

In table 4.4 we are only concerned with isomorphic copies of a particular normal subgroup M. So for an example in the parent group with gap number [32,43] there are two normal subgroups with gap number [16,6] in both cases B and B' are cyclic of order 2. Thus $M = [16, 6]$ is only listed once.

TABLE 4.4: Non-simple towers of low index

G	$H \cong H'$	$[G:H]$	M	$B \cong B'$	tower type	lag type	d	c	$c-b$
[32,43]	[4,2]	8	[16,6]	[2,1]	trivial	$(2,2)$	1	0	0
[32,43]	[4,2]	8	[16,7]	[2,1]	trivial	$(2,2)$	1	0	0
[32,43]	[4,2]	8	[16,8]	[2,1]	trivial	$(2,2)$	1	0	0
[32,43]	[4,2]	8	[16,13]	[2,1]	trivial	$(2,2)$	1	0	0
[32,43]	[4,2]	8	[16,11]	[4,2]	latent	$(2,2)$	1	1	1
[48,29]	[6,1]	8	[24,3]	[3,1]	trivial	$(2,2)$	1	0	0
[48,49]	[4,2]	12	[24,13]	[2,1]	trivial	$(2,2)$	1	0	0
[72,23]	[6,1]	12	[36,12]	[3,1]	trivial	$(2,2)$	1	0	0
[72,23]	[6,1]	12	[36,16]	[6,1]	latent	$(2,2)$	1	1	1
[96,3]	[8,2]	12	[32,2]	[8,2]	latent	$(3,1)$	1	1	1
[96,195]	[8,3]	12	[48,30]	[4,1]	trivial	$(2,2)$	1	0	0
[96,195]	[8,3]	12	[48,48]	[4,2]	trivial	$(2,2)$	1	0	0
[96,195]	[8,3]	12	[48,49]	[4,2]	reducible	$(2,2)$	1	0	0
[192,194]	[16,11]	12	[64,73]	[16,11]	latent	$(3,1)$	1	1	1
[192,194]	[16,11]	12	[96,3]	[8,2]	reducible	$(2,2)$	1	0	0
[240,91]	[20,3]	12	[120,35]	[10,1]	trivial	$(2,2)$	1	0	0
[240,91]	[20,3]	12	[60,5]	[5,1]	trivial	$(4,4)$	2	0	0
[336,209]	[24,12]	14	[168,42]	[12,3]	reducible	$(2,2)$	1	0	0
[336,209]	[24,12]	14	[168,42]	[24,12]	reducible	$(2,2)$	1	1	0
[56448,?]	[4032,?]	14	[28224,?]	[4032,?]	reducible*	$(2,2)$	1	1	0
[180,9]	[12,3]	15	[60,5]	[4,2]	trivial	$(3,3)$	1	0	0
[360,120]	[24,12]	15	[180,9]	[12,3]	reducible	$(2,2)$	1	0	0

* Note that in the group of order 56448, (M, B, B) represents the Gassman triple of index 7. But the triple has a nontrivial kernel which is isomorphic to the simple group of order 168.

Chapter 5
Result for Fields of Degree 8

In this chapter we focus on the group $(C_8 \rtimes V_4)$ with order 32 from example 1.4. Set:

$$\alpha = (1,2,3,4,5,6,7,8) \qquad \beta = (2,4)(3,7)(6,8)$$

$$\gamma = (2,6)(4,8)$$

So $G = \langle \alpha, \beta, \gamma \rangle$,$H = \langle \beta, \gamma \rangle$ and $H' = \langle \beta, \alpha^4\gamma \rangle$. We want to determine all possible $M \triangleleft G$. We list the number of such M's in table 5.1.

TABLE 5.1: Normal subgroups of $C_8 \rtimes V_4$

[G:M]	# of M's
32	1
16	1
8	3
4	7
2	7
1	1

If $[G : M]$ is either 32 or 16 then G/M will not be cyclic. Otherwise G would contain an element of order at least 16. This is not possible since G can be considered as a subgroup of S_8. So $M = \langle e \rangle$ or $M = \langle \alpha^4 \rangle$ will not represent the lag of an Iwasawa tower.

For $[G : M] = 4$ we have the 3 cases, which are listed in table 5.2. In all three cases G/M is not cyclic.

For $[G : M] = 8$ we have the seve cases, which are listed in table 5.2. In all seven cases $G/M \cong V_4$ is not cyclic.

TABLE 5.2: Quotient groups when M has index 4

M	Gap #	cosets	Gap # of G/M
$\langle \alpha^2\gamma \rangle$	[4,1] $C4$	$M, \alpha M, \alpha^2 M, \alpha^3 M, \beta M, \alpha\beta M, \alpha^2\beta M, \alpha^3\beta M$	[8,3] $D8$
$\langle \alpha^4, \gamma \rangle$	[4,2] $V4$	$M, \alpha M, \alpha^2 M, \alpha^3 M, \beta M, \alpha\beta M, \alpha^2\beta M, \alpha^3\beta M$	[8,3] $D8$
$\langle \alpha^2 \rangle$	[4,1] $C4$	$M, \alpha M, \gamma M, \alpha\gamma M, \beta M, \alpha\beta M, \beta\gamma M, \alpha\beta\gamma M$	[8,5] $C2 \times C2 \times C2$

TABLE 5.3: Quotient groups when M has index 8

M	Gap #	cosets
$\langle \alpha^2, \alpha\beta \rangle$	[8,4] $Q8$	$M, \beta M, \gamma M, \beta\gamma M$
$\langle \alpha^2, \beta \rangle$	[8,3] $D8$	$M, \alpha M, \gamma M, \alpha\gamma M$
$\langle \alpha^2, \alpha\beta\gamma \rangle$	[8,3] $D8$	$M, \beta M, \gamma M, \beta\gamma M$
$\langle \alpha^2, \beta\gamma \rangle$	[8,4] $Q8$	$M, \alpha M, \gamma M, \alpha\gamma M$
$\langle \alpha\gamma \rangle$	[8,1] $C8$	$M, \alpha M, \beta M, \alpha\beta M$
$\langle \alpha^2, \gamma \rangle$	[8,2] $C4 \times C2$	$M, \alpha M, \beta M, \alpha\beta M$
$\langle \alpha \rangle$	[8,1] $C8$	$M, \beta M, \gamma M, \beta\gamma M$

Finally when $[G : M] = 16$ we have seven cases, six trivial cases and one latent case. These cases are listed in table 5.4. This verifies our table from Section 4 and leads to the following result.

Theorem 5.1. *Let K and K' be Gassman equivalent over base field F. Suppose $K = F(\theta)$, $K' = F(\theta\eta)$ with θ and η algebraic and $[K : F] = [K' : F] = 8$. Suppose also that F_∞/F is a $\mathbb{Z}_2$-tower with $F_0 = F$ and $F_1 = F(\eta)$. Suppose also that $[N : F] = 32$.Then with respect to K and K' and our tower F_∞/F we have the following:*

a) $d \geq 1$ (that is to say there is a lag)

b) If $K_0 \ncong K_0'$ then $c = 0$.

c) $K_1 \cong K_1'$.

d) K and K' will share the same Iwasawa invariants.

Proof: a) Recall $d \geq 1$ iff $F_1 \subseteq N$. But $KK' \subseteq N$, so $\theta \in N$ and $\eta\theta \in N$. Since $\theta^{-1}\theta\eta = \eta \in N$ we have that $F(\eta) = F_1 \subseteq N$ and $d \geq 1$.

TABLE 5.4: Quotient groups when M has index 16

M	Gap #	cosets	$\#(M \cap H)$	type
$\langle \alpha\beta, \alpha\gamma \rangle$	[16,8]	$M, \alpha M$	2	trivial
$\langle \alpha^2, \alpha\beta, \gamma \rangle$	[16,13]	$M, \alpha M$	2	trivial
$\langle \beta, \alpha\gamma \rangle$	[16,7]	$M, \alpha M$	2	trivial
$\langle \alpha, \beta \rangle$	[16,8]	$M, \gamma M$	2	trivial
$\langle \alpha, \gamma \rangle$	[16,6]	$M, \beta M$	2	trivial
$\langle \alpha, \beta\gamma \rangle$	[16,7]	$M, \gamma M$	2	trivial
$\langle \alpha^2, \beta, \gamma \rangle$	[16,11]	$M, \alpha M$	4	latent

b) By way of contradiction suppose that $c \geq 1$. Thus $F_1 \subseteq K_0$ and $F_1 \subseteq K'_0$. So $\eta \in K$ and $\eta \in K'$. Hence $\eta\theta \in K$ and $(\eta)^{-1}(\eta\theta) = \theta \in K'$. This implies that $K = K'$. But $K \ncong K'$ by assumption. This is a contradiction. Therefore $c = 0$

c) If $K \cong K'$ then by construction $K_1 \cong K_1$. So suppose $K \ncong K'$. By part b) $c = 0$. But by part a) there is a lag. By our table the only possible lag in a tower that is not trivial will be when $c = d = 1$. So we have trivial case with $K_d \cong K'_d$ and since $d = 1$, $K_1 \cong K'_1$.

d) In light of remark 3.10, for all $i \geq 1$ we have that $K_i \cong K'_i$. Thus after the first step all invariants will be the same. □

Theorem 5.2. *Let $K = \mathbb{Q}(\sqrt[8]{t})$ and $K' = \mathbb{Q}(\sqrt[8]{16t})$ where $t \in \mathbb{Z}$ with the absolute value of the square free part of t strictly greater than 2. Then K and K' will share the same Iwasawa invariants.*

Proof: Let $\theta = \sqrt[8]{t}$ and $\eta = \sqrt{2}$. We notice that $K = \mathbb{Q}(\theta)$, $K = \mathbb{Q}(\theta\eta)$, $F = \mathbb{Q}$ and $F_1 = \mathbb{Q}(\eta)$ where F_∞/F is a $\mathbb{Z}_2$-tower. By theorem 5.1 our claim holds. □

Chapter 6

G-action on Cosets and Matrix Entries

All groups and fields will have the same properties as in previous chapters. We add a few assumptions.

- $[G : H] = [G : H'] < \infty$
- $\alpha, \beta \in G$ arbitrary and $\gamma = \alpha^{-1}\beta$
- R will be an arbitrary ring

Let ${}^{g}X$ denote gXg^{-1} for any $X \leq G$ and $g \in G$.

Lemma 6.1. *Fix $y, z \in G$ then*

$$Stab_{({}^{y}H)}(zH') = Stab_{({}^{z}H')}(yH) = ({}^{y}H) \cap ({}^{z}H')$$

Proof: By symmetry it is enough to show that $Stab_{({}^{y}H)}(zH') = ({}^{y}H) \cap ({}^{z}H')$. Take $x \in Stab_{({}^{y}H)}(zH')$. So $x \in {}^{y}H$ and $xzH' = zH'$. Thus $xz \in zH'$ and $x \in zH'z^{-1}$. Therefore $x \in ({}^{y}H) \cap ({}^{z}H')$.

Now take $x \in ({}^{y}H) \cap ({}^{z}H')$. Since $x \in {}^{z}H'$, $x = zh'z^{-1}$ for some $h' \in H'$. Hence $xzH' = zh'z^{-1}zH' = zh'H' = zH'$. But $x \in ({}^{y}H) \cap ({}^{z}H') \subset {}^{y}H$.Therefore $x \in Stab_{({}^{y}H)}(zH')$. □

Corollary 6.2. *If $(\alpha H, \beta H')$ is an element of $G/H \times G/H'$ and $g \in G$ acts on $(\alpha H, \beta H')$ component wise then ${}^{\alpha}H \cap {}^{\beta}H' = \{g \in G \mid g(\alpha H, \beta H') = (\alpha H, \beta H')\}$*

Proof: Note ${}^{\alpha}H \cap {}^{\beta}H' \subset \{g \in G \mid g(\alpha H, \beta H') = (\alpha H, \beta H')\}$ follows directly from lemma 6.1. So suppose that $g \in G$ so that $g(\alpha H, \beta H') = (\alpha H, \beta H')$. Hence $g\alpha \in$

αH and $g \in \alpha H \alpha^{-1} = {}^{\alpha}H$. Likewise $g\beta \in \beta H'$. Thus $g \in {}^{\beta}H'$ and $g(\alpha H, \beta H') = (\alpha H, \beta H')$. Therefore $g \in {}^{\alpha}H \cap {}^{\beta}H'$. □

Remark 6.3. For all $g \in G$, if $Y \leq X \leq G$ then $[X : Y] = [{}^{g}X : {}^{g}Y]$

Proof: Note ${}^{g}Y \supset {}^{g}X$, $\mid X \mid = \mid {}^{g}X \mid$ and $\mid Y \mid = \mid {}^{g}Y \mid$. The remark follows directly. □

Theorem 6.4. *The following sets have the same order*

a) the ${}^{\alpha}H$-orbit of $\beta H'$ in G/H'.

b) the ${}^{\beta}H'$-orbit of αH in G/H

c) the H-orbit of $\gamma H'$ in G/H'

d) the H'-orbit of $\gamma^{-1}H$ in G/H

Proof: Take $S = H \cap {}^{\gamma}H'$. Notice by lemma 6.1 the ${}^{\alpha}H$-stablizer of βH is equal to the ${}^{\beta}H$-stablizer of αH. The order of the orbit is the index of the stabilizer. So by lemma 6.1 a) and b) have order $[{}^{\alpha}H : {}^{\alpha}S]$, the order of set c) is $[H : S]$ and the order of set d) is $[{}^{\gamma^{-1}}H : {}^{\gamma^{-1}}S]$. In light of remark 6.3 $[H : S] = [{}^{\gamma^{-1}}H : {}^{\gamma^{-1}}S] = [{}^{\alpha}H : {}^{\alpha}S]$ and the theorem follows. □

We now apply these G-actions to entries a matrix. Let $\rho_1, \cdots, \rho_n$ and $\rho'_1, \cdots, \rho'_n$ be representatives for the left cosets in G of H and H' respectively with $\rho_1 = \rho'_1 = 1_G$. We define homomorphisms π and π' from G into S_n in the following way:

$$\pi_g(i) = j \text{ where } g\rho_i H = \rho_j H \text{ and } \pi'_g(i) = j \text{ where } g\rho'_i H = \rho'_j H$$

for all $g \in G$.

Let $\mathscr{A}$ be the set of all invertible n by n matrices with integral entries such that if $(a_{ij}) = A \in \mathscr{A}$ then $a_{ij} = a_{\pi_g(i)\pi'_g(j)}$ for all $g \in G$.

Definition 6.5. Let $(a_{ij}) = A$ be an n by n matrix with entries in R. We say that A is a G**-action matrix** on the pair H and H' over R if $a_{ij} = a_{\pi_g(i)\pi'_g(j)}$ for all $g \in G$.

We will let $\mathscr{A}$ denote the **family** of all such G**-action matrices** .

Set $\nu = gcd\{det(A)|A \in \mathscr{A}\}$. Note that A, $\mathscr{A}$ and the value ν depend on G, H, H' and our ring R. We want to look at these matrices in general form.

Definition 6.6. Let $A = (a_{ij})$ be a G-action matrix on H and H' over R. Suppose the G-action on $(G/H, G/H')$ has exactly k distinct orbits. Let $Y = y_{ij}$ be a G-action matrix on H and H' over the polynomial ring $R[x_1, \cdots, x_k]$ where $\{x_1, \cdots, x_k\}$ are distinct indeterminants. Then Y is called a **general form of** A if the following three conditions hold:

1) For each pair i, j, $y_{i,j} = 1_R x_t$ for some $x_t \in \{x_1, \cdots, x_k\}$

2) $y_{ij} = y_{st}$ if an only if there is $g \in G$ such that $s = \pi_g(i)$ and $t = \pi'_g(j)$.

3) There is a map $\psi_A : \{x_1, \cdots, x_k\} \to R$ such that for each pair i, j $\psi_A(y_{ij} = a_{ij}$

We will now borrow a definition from probability

Definition 6.7. In probability a square matrix is **doubly stochastic** if every entry of the matrix is nonnegative, the sum of every row is the same and the sum of every column is the same.

For our purposes we will need to relax the specifications of this definition.

Definition 6.8. (definition 6.7 revised)Let A be a square matrix with entries in a ring with unity R. Then A is **doubly stochastic** if there is some $\alpha \in R$ such that the sum of each row in A and the sum of each column in A is equal to α.

Notice this revised definition can apply to a ring that has no order relation (such as $\mathbb{C}$) where the terms positive and negative won't apply.

Definition 6.9. Let $(a_{ij}) = A$ be a G-action matrix off H and H' with entries in R. Then A is a **general doubly stochastic matrix** if A has some general form matrix Y that is double stochastic in the ring $R[x_i, \cdots, x_k]$

Theorem 6.10. *Let $Y = (y_{ij})$ be a general form of matrix A Fix the pair i, j and suppose $y_{i,j} = x_t \in \{x_1, \cdots x_k\}$. Then the following sets have the same order:*

a) the set of entries in row i equal to x_t.

b) the set of entries in column j equal to x_t.

c) the set of entries in row 1 equal to x_t.

d) the set of entries in column 1 equal to x_t.

Proof: Take $\alpha = \rho_i$, and $\beta = \rho'_j$. Thus $\gamma = \alpha^{-1}\beta = \rho_i^{-1}\rho'_j$. Entry y_{ij} corresponds to the element $(\rho_i H, \rho'_j H') = (\alpha H, \beta H')$.The entries of row i equal to x_t will correspond to all elements of $(G/H, G/H')$ in the orbit of $(\alpha H, \beta H')$ that fix the first element αH. By lemma 6.1 this is the ${}^{\alpha}H$-orbit of $\beta H'$ in G/H' which is set a) in Theorem 6.4,

By similar arguments the sets b) , c) and d) will have the same order as sets b), c) and d) of Theorem 6.4. This completes the proof. □

Corollary 6.11. *The matrix A is general doubly stochastic matrix.*

Proof: The sum of row t is $\Sigma t_i x_i$ and the sum of column s is $\Sigma s_i x_i$ where t_i is the number of x_i's in row t and s_i is the number of x_i's in column s. But by Theorem 6.10 $s_i = t_i$ for any fixed i and any row s and t. Thus $\Sigma s_i x_i = \Sigma t_i x_i$ and our claim holds. □

Take $0 \leq i \leq j$. Let $G_{ji} = Gal(N_j/F_i)$, $H_{ji} = Gal(N_j/K_i)$ and $H'_{ji} = Gal(N_j/K'_j)$. We let $\mathscr{A}_{ij}$ denote the family of all G_{ji}-action matrices over H_{ji} and H'_{ji}.

We let $\nu_{ji} = gcd\{det(A) \mid A \in \mathscr{A}_{ij}\}$.

Proposition 6.12. *This construction of ν_{ji} with respect to K_j and K'_j over F_i is independent of the normal subfield we choose for constructing our Galois groups.*

Proof: Suppose $j \leq j_1 \leq j_2$. It is enough to show show that when our construction in N_{j_1} and N_{j_2} will both yield the same value ν_{ji}

Take $V = Gal(N_{j_2}/N_{j1})$, $G = Gal(N_{j_1}/F_0)$, $H = Gal(N_{j_1}/K'_i)$ and $H' = Gal(N_{j_1}/K_i)$. Define $\tilde{\pi}$ and $\tilde{\pi}'$ so that $\tilde{\pi}_{gv}(i) = j$ where $gv\rho_i HV = \rho_j HV$ and $\pi'_{gv}(i) = j$ where $gv\rho'_i HV = \rho'_j HV$ for all $gv \in GV$ and all cosets $\rho_i HV, \rho_j \in GV/HV$ and $\rho'_i H'V, \rho'_j HV \in GV/H'V$.

Notice $V \lhd G$ and $G \cap V = (Gal(N_{j_1}/F_0)) \cap (Gal(N_{j_2}/N_{j_1}))$ So if $\sigma \in (G \cap V)$ then $\sigma(N_{j_1}) = N_{j_1}$ with $\sigma \in G$. Thus $\sigma = 1_G$.

Because V is a normal subgroup, V acts trivially on GV/HV. So by proposition 1.26, for any a_{ij} in our matrix and for any $g \in G$, we have that $a_{ij} = a_{\pi_g(i)\pi'_g(j)} = a_{\tilde{\pi}_g(i)\tilde{\pi}'_g(j)}$, completing our proof. □

Proposition 6.13. *Let $\mathscr{A}$ be the family of all G-action matrices on H and H' and let $\nu = gcd\{det(A) \mid A \in \mathscr{A}\}$. Let $G = Gal(N/F)$,$H = Gal(N/K)$ and $H' = Gal(N/K')$ for some common normal closure N of fields K and K'. For any prime p*

a) $p \nmid \nu$ if and only if $\mathbb{Z}_p(G/H) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/H')$

b) If $p \nmid \nu$ then $\mathfrak{Cl}_p(K) \cong \mathfrak{Cl}_p(K')$.

c) $H \sim^{loc}_G H'$ if and only if $\nu \neq 0$

Proof: Part a) is Lemma 3 in Perlis [14] and part b) is Theorem 3 Perlis [14] and part c) is Lemma 2 in Perlis [14]. □

In Perlis' paper the Galios groups are assumed to be over the base field $\mathbb{Q}$. However there in nothing about the base field itself that enters into the proof apart from the Galois groups themselves. Thus by assuming an arbitrary base F_0 these results will still apply.

Proposition 6.14. *For all $j \geq d$ we have $\nu_{dd} = \nu_{jj}$.*

Proof: By lemma , $G_{dd} \cong G_{jj}$, $H_{dd} \cong H_{jj}$ and $H'_{dd} \cong H'_{jj}$ canonically. Hence $G_{dd}/H_{dd} \cong G_{jj}/H_{jj}$ and $G_{dd}/H'_{dd} \cong G_{jj}/H'_{jj}$ canonically. Thus $\nu_{dd} = \nu_{jj}$ follows by definition. □

Lemma 6.15. *If $\mathbb{Z}_p(G/(B\times V)) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/(B'\times V))$ then $\mathbb{Z}_p(G/(B\times 1_V)) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/(B' \times 1_V))$*

Proof: Notice $(B \times V)/(B \times 1_V) \cong V \cong (B' \times V)/(B' \times 1_V)$. So if $\mathbb{Z}_p(G/(B \times V)) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/(B' \times V))$ then

$$\mathbb{Z}_p(G/(B \times 1_V)) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/(B \times V)) \otimes \mathbb{Z}_p(V) \cong_{\mathbb{Z}_p(G)}$$

$$\mathbb{Z}_p(G/(B' \times V)) \otimes \mathbb{Z}_p(V) \cong_{\mathbb{Z}_p(G)} \mathbb{Z}_p(G/(B' \times 1_V)).\Box$$

Proposition 6.16. *If $\mathbb{Z}_p(G_{j0}/H_{jd}) \cong_{\mathbb{Z}_p(G_{j0})} \mathbb{Z}_p(G_{j0}/H'_{jd})$*
then $\mathbb{Z}_p(G_{j0}/H_{jj}) \cong_{\mathbb{Z}_p(G_{j0})} \mathbb{Z}_p(G_{j0}/H'_{jj})$.

Proof: Take $G = G_{j0} = Gal(N_j/F_i)$, $V = Gal(N_j/N_d)$, $B = Gal(N_d/K_d)$ and $B' = Gal(N_d/K'_d)$.

Thus $B \times V = Gal(N_d/K_d) \times Gal(N_j/N_d) = Gal(N_j/K_d) = H_{jd}$, $B' \times V = Ga(N_j/K'_d) \times Gal(N_j/N_d) = Gal(N_j/K'_d) = H'_{jd}$, $B \times 1_V = Gal(N_d/K_d) \times Gal(N_j/N_j) = Gal(N_j/K_d) = H_{jj}$ and $B \times 1_V = Gal(N_d/K_d) \times Gal(N_j/N_j) = Gal(N_j/K_d) = H'_{jj}$. Hence by Proposition 6.15 our claim follows. □

Definition 6.17. The **support** of an integer α, denoted $supp(\alpha)$ is the set of all primes $p \in \mathbb{Z}$ such that $p \mid \alpha$.

Proposition 6.18. *For all $j \geq d$, we have $supp(\nu_{j0}) \subset supp(\nu_{d0})$*

Proof: Applying proposition 6.13 to proposition 6.12 we have

$$\mathbb{Z}_p(G_{j0}/H_{jd}) \cong_{\mathbb{Z}_p(G_{j0})} \mathbb{Z}_p(G_{j0}/H'_{jd}) \Leftrightarrow \mathbb{Z}_p(G_{d0}/H_{dd}) \cong_{\mathbb{Z}_p(G_{d0})} \mathbb{Z}_p(G_{d0}/H'_{dd})$$

By proposition 6.13 applied to proposition 6.16, we have $p \nmid \nu_{d0} \Rightarrow p \nmid \nu_{j0}$. Therefore $supp(\nu_{j0}) \subset supp(\nu_{d0})$. □

We can now formulate an algorithm for answering Greenberg's question. Suppose that K and K' are Gassmann equivalent over F. Compute our values b, c and d:

Step 1) If K_d and K_d are Isomorphic then the answer to Greenberg's question is yes. Note that it is sufficient to check for the trivial case at level d since from Theorem 3.11 $K_d \cong_F K'_d$ if and only if $K_j \cong_F K'_j$ for some $j \geq d$.

Step 2) Suppose that $b = c$. Then we have either the simple or reducible case. Either way applying lemma 6.13 part c), we have ν_{dd} is nonzero. By Proposition 6.16 it follows that $\nu_{dd} = \nu_{jj}$ for all $j \geq d$. So if $p \nmid \nu_{dd}$ then the answer to Greenberg's question is yes. Otherwise our algorithm will not yield an answer to Greenberg's question.

Step 3) Suppose that $b \neq c$. Thus we have the latent case. So $\nu_{ii} = 0$ for all $i \geq d$. However $\nu_{i0} = 0$ for all $i \geq 0$. And by lemma 6.18, $supp(\nu_{jj}) \subset supp(\nu_{dd})$ for all $j \geq d$. Thus if $p \nmid \nu_{d0}$ then $p \nmid \nu_{j0}$. So if $p \nmid \nu_{d0}$ then the answer to Greenberg's question is yes. Otherwise our algorithm will not yield an answer to Greenberg's question.

In light of proposition 6.13 to check that $\mathfrak{Cl}_p K \cong (Cl)_p K'$ it is sufficient to show that $p \nmid \nu_{00}$. According to Bosma and de Smit [1]for each of the 19, the support of ν_{00} contains exactly one prime. In the appendix we construct matrices verifying that $supp(\nu_{00})$ contains at most one prime. In each case the prime is the prime stated by by Bosma and de Smit

Chapter 7

Geometric Constructions and Gassmann Equivalence

In this chapter we have the following:

- (G, H, H') will be a Gassmann triple
- $[G : H] = [G : H'] = n < \infty$

As we have seen the parent group G of Gassmann triple acts on G/H and G/H' by left composition. This action determines the structure of our matrices in chapter 6. The purpose of this chapter is to attempt to generate these matrices from geometric constructions. Because this action transitively permutes the row and columns of an $n \times n$ matrix we hope to realize our parent groups as transitive subgroups of S_n. We will only focus on the four cases when $[G, H] \leq 11$. So there is one triple with index 7, two triples with index 8 and one triple with index 11.

The first triple has as its parent group the unique simple group of order 168. It is well known that this group is the automorphism group of Fano plane.

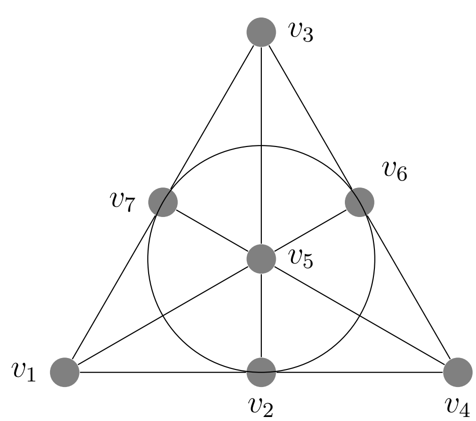

FIGURE 7.1: Fano Plane

Recall the order of the orbit is the index of the stabilizer. So the stabilizer of vertex v_1 has index 7. This stabilizer is actually the subgroup H. But how do we construct the other subgroup H'? We define a **block** to be a subset of the vertices on which G is acting. If we can construct a block in such a way that this block has an orbit of n distinct blocks, then the stabilizer of this block will have index n in G. This subgroup is H'

If we were to take this construction to be a matroid we could consider this block to be either a circuit or a hyperplane. However a matroid has more structure than we actually need. So using the language of vertices and block will be enough for our purposes.

The blocks we need are the lines of the Fano plane. Six of these are the collinear triples of vertices. The seventh line will be the three points lying on the constructed circle, $\{v_2, v_6, v_7\}$. The following is the list of the 7 transitive blocks.

$$\text{Block } 1 = \{v_3, v_4, v_6\}$$
$$\text{Block } 2 = \{v_2, v_3, v_5\}$$
$$\text{Block } 3 = \{v_1, v_2, v_4\}$$
$$\text{Block } 4 = \{v_1, v_3, v_7\}$$
$$\text{Block } 5 = \{v_2, v_6, v_7\}$$
$$\text{Block } 6 = \{v_1, v_2, v_6\}$$
$$\text{Block } 7 = \{v_4, v_5, v_5\}$$

In our matrix, rows will correspond to vertices and the columns will correspond to blocks. The value is A whenever the row vertex is in the column block and B otherwise. This will yield the general doubly stochastic matrix for our triple (G, H, H'). We have chosen our vertices and blocks so that the matrix is diagonally symmetric, which highlights the doubly stochastic property.

$$\begin{pmatrix} B & B & A & A & B & A & B \\ B & A & A & B & A & B & B \\ A & A & B & A & B & B & B \\ A & B & A & B & B & B & A \\ B & A & B & B & B & A & A \\ A & B & B & B & A & A & B \\ B & B & B & A & A & B & A \end{pmatrix}$$

The second triple has index 8 and as its parent group $C_8 \rtimes V_4$ which has order 32. In this construction instead of taking lines in two space we are taking planes in three space. We take the 8 vertices of a cube. Then we rotate the top face 45 degrees. From above, the vertices will appear as in figure 7.2.

Blocks will be four vertex sets that are coplanar and the parent group G will take coplanar blocks to coplanar blocks. There are ten such blocks. Two are the top face $\{v_1, v_3, v_5, v_7\}$ and the bottom face $\{v_2, v_4, v_6, v_8\}$ which are in one orbit. The remaining eight blocks are in another orbit. These blocks are:

$$\text{Block } 1 = \{v_1, v_2, v_4, v_5\}$$
$$\text{Block } 2 = \{v_1, v_3, v_4, v_8\}$$
$$\text{Block } 3 = \{v_2, v_3, v_7, v_8\}$$
$$\text{Block } 4 = \{v_1, v_2, v_6, v_7\}$$
$$\text{Block } 5 = \{v_1, v_5, v_6, v_8\}$$
$$\text{Block } 6 = \{v_4, v_5, v_7, v_8\}$$
$$\text{Block } 7 = \{v_3, v_4, v_6, v_7\}$$
$$\text{Block } 8 = \{v_2, v_3, v_5, v_6\}$$

Each block contains two points from the top face and two points from the bottom face. So in the case of block 1 notice that the line through vertices v_1 and v_5 will

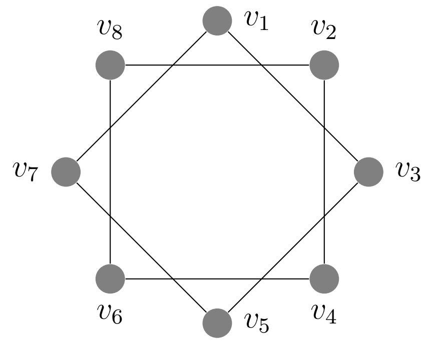

FIGURE 7.2: Construction of $C_8 \rtimes V_4$

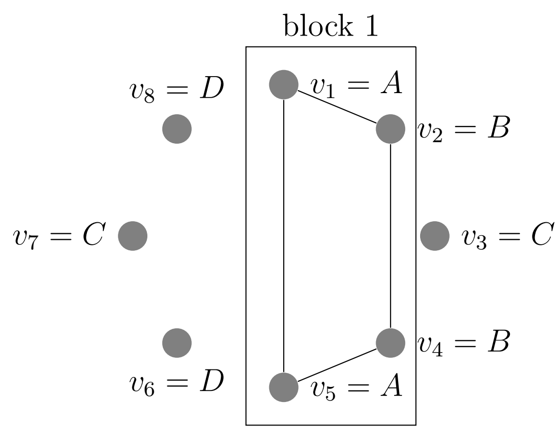

FIGURE 7.3: Block construction in $C_8 \rtimes V_4$

be parallel to the line through vertices v_2 and v_4. Thus these four points will be coplanar. As in the previous case, H will fix vertex v_1 and H' will fix block 1.

- $A =$ in the block, opposing vertices
- $B =$ in the block, not opposing vertices
- $C =$ not in the block, opposing vertices
- $D =$ not in the block, not opposing vertices

Within block 1 the points v_1 and v_5 will be on opposite corners of the top face, but v_2 and v_4 are not on opposite corners of the bottom face. Thus H' will not act transitively on the points of block 1. There are two H'-orbits within block 1 which we label A and B. There will be two H'-orbits outside block 1 which we label C and D. This yeilds our general doubly stochastic matrix:

$$\begin{pmatrix} A & B & C & B & A & D & C & D \\ B & C & B & A & D & C & D & A \\ C & B & A & D & C & D & A & B \\ B & A & D & C & D & A & B & C \\ A & D & C & D & A & B & C & B \\ D & C & D & A & B & C & B & A \\ C & D & A & B & C & B & A & D \\ D & A & B & C & B & A & D & C \\ A & B & C & B & A & D & C & D \end{pmatrix}$$

The third triple has index 8 and as it's parent group $GL(3,2)$ which has order 48. As in the Fano plane we take lines in two space. We take the affine plane of order 3 which has 9 points and 12 line. By omitting a single point we have 8 points and 8 lines.

The eight blocks will be:

$$\text{Block } 1 = \{v_1, v_2, v_7\}$$

$$\text{Block } 2 = \{v_1, v_6, v_8\}$$

$$\text{Block } 3 = \{v_5, v_7, v_8\}$$

$$\text{Block } 4 = \{v_4, v_6, v_7\}$$

$$\text{Block } 5 = \{v_3, v_5, v_6\}$$

$$\text{Block } 6 = \{v_2, v_4, v_5\}$$

$$\text{Block } 7 = \{v_1, v_3, v_4\}$$

$$\text{Block } 8 = \{v_2, v_3, v_8\}$$

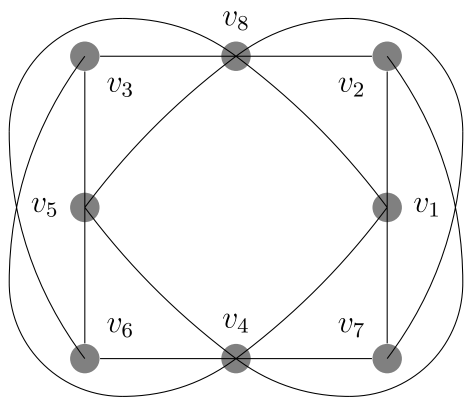

FIGURE 7.4: Construction of $GL(3, 2)$

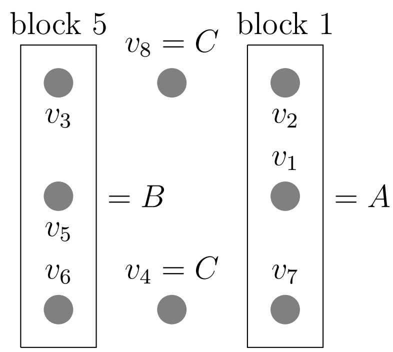

FIGURE 7.5: Block construction in $GL(3, 2)$

Subgroup H will fix vertex v_1 and H' will fix block 1. But notice block 1 has a vertex in common with every block except block 5. Thus any group element of H' must fix the block 1 and it must fix block 5. Since there are two remaining vertices which are in the same H'-orbit it follows that there are three orbits in H'. These orbits are constructed as follows:

- $A =$ in the block
- $B =$ in the opposing block
- $C =$ in neither block

This yields our generally doubly stochastic matrix:

$$\begin{pmatrix} A & A & B & C & B & B & A & C \\ A & B & C & B & B & A & C & A \\ B & C & B & B & A & C & A & A \\ C & B & B & A & C & A & A & B \\ B & B & A & C & A & A & B & C \\ B & A & C & A & A & B & C & B \\ A & C & A & A & B & C & B & B \\ C & A & A & B & C & B & B & A \end{pmatrix}$$

The fourth triple has index 11 and as it's parent group is the unique simple group of order 660. Both H and H' will be isomorphic to the automorphism group of the Bucky-ball (or soccer ball). The construction is by E. Brown [2].

The blocks have order 5. Thus to construct this as a matroid using either circuits or hyper planes would require more than three dimensions. Here are the blocks:

$$\text{Block } 1 = \{v_3, v_7, v_8, v_9, v_{11}\}$$

$$\text{Block } 2 = \{v_2, v_6, v_7, v_8, v_{10}\}$$

$$\text{Block } 3 = \{v_1, v_5, v_6, v_7, v_9\}$$

$$\text{Block } 4 = \{v_4, v_5, v_6, v_8, v_{11}\}$$

$$\text{Block } 5 = \{v_3, v_4, v_5, v_7, v_{10}\}$$

$$\text{Block } 6 = \{v_2, v_3, v_4, v_6, v_9\}$$

$$\text{Block } 7 = \{v_1, v_2, v_3, v_5, v_8\}$$

$$\text{Block } 8 = \{v_1, v_2, v_4, v_7, v_{11}\}$$

$$\text{Block } 9 = \{v_1, v_3, v_6, v_{10}, v_{11}\}$$

$$\text{Block } 10 = \{v_2, v_5, v_9, v_{10}, v_{11}\}$$

$$\text{Block } 11 = \{v_1, v_4, v_8, v_9, v_{10}\}$$

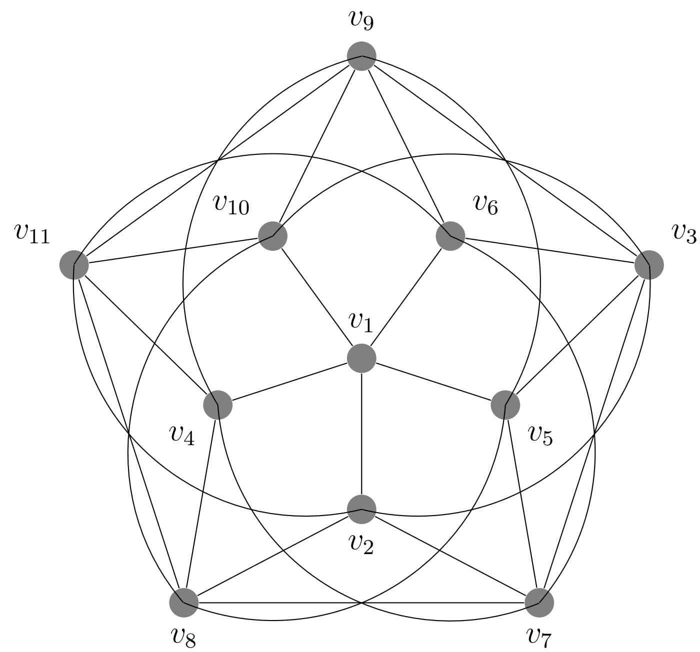

FIGURE 7.6: Construction of the $(11, 5, 2)$-biplane

Under this construction there are 3 types of blocks with order 5. Block 1 is of the first type, which is the outer ring of 5 vertices. Block 1 is the only block of this type. Block 9 is of the second type which will contain two vertices in the outer ring, two vertices in the inner ring and the center vertex. Five blocks have this type and rotating this block about the center will yield the remaining four blocks. Block 2 is of the third type and will contain three vertices from the inner ring and two vertices from the outer ring. Again five blocks have this third type and rotating this block about the center will yield the remaining four blocks.

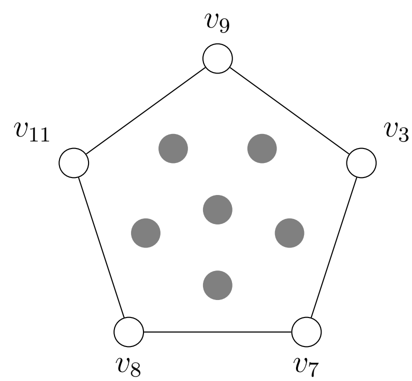

FIGURE 7.7: Block 1

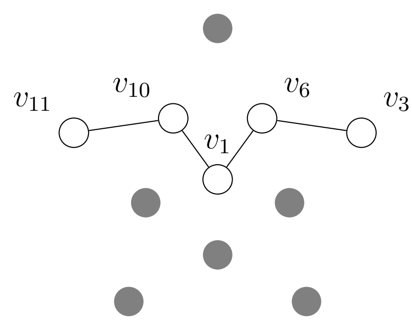

FIGURE 7.8: Block 9

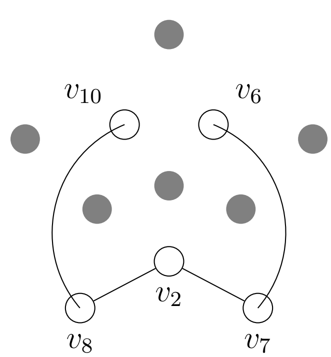

FIGURE 7.9: Block 2

The group H will fix v_1 and the group H' will fix block 1. This will yield the following matrix:

$$\begin{pmatrix}
B & B & A & B & B & B & A & A & A & B & A \\
B & A & B & B & B & A & A & A & B & A & B \\
A & B & B & B & A & A & A & B & A & B & B \\
B & B & B & A & A & A & B & A & B & B & A \\
B & B & A & A & A & B & A & B & B & A & B \\
B & A & A & A & B & A & B & B & A & B & B \\
A & A & A & B & A & B & B & A & B & B & B \\
A & A & B & A & B & B & A & B & B & B & A \\
A & B & A & B & B & A & B & B & B & A & A \\
B & A & B & B & A & B & B & B & A & A & A \\
A & B & B & A & B & B & B & A & A & A & B
\end{pmatrix}$$

This last construction is also known as the $(11, 5, 2)$-biplane. The first value indicates the total number of vertices. The second number is the number of vertices in each block. The third value indicates the number of vertices contained in the intersection of any two distinct blocks. The Fano plane is also known as the $(7, 3, 1)$-biplane. Can the other two constructions be considered as biplanes? The answer is no. In our triple with $G \cong C_8 \rtimes V_4$, block 1 intersect block 2 will contain two vertices but block 1 intersect block 3 will contain one vertex. In our triple with $G \cong GL(3, 2)$, block 1 intersect block 5 will be empty, but block 1 intersect any other block will contain one vertex.

References

[1] Bosma, W. and B. de Smit, *On arithmetically equivalent number fields of small degree*, in Algorithmic number theory (Sydney, 2002), 6779, Lecture Notes in Comput. Sci., 2369, Springer, Berlin, 2002

[2] Brown, E. *The Fabulous (11,5,2)-Biplane*, Mathematics Magazine 77 (April 2004), 87-100

[3] De Smit, B. and R. Perlis, *Zeta functions do not determine class numbers*, Bull. Amer. Math. Soc. 31 (1994), 213215.

[4] De Smit, B.,*On arithmetically equivalent fields with distinct p-class numbers* Juornal of Algebra 272 (2004) 417-424

[5] Dummit, D. S. and R. M. Foote *Abstract Algebra* John Wiley & Sons 2004

[6] Garling, D.J.H *A Course in Galois Theory* Cambridge University Press 1986

[7] Gassmann, F., *Bemerkungen zu der vorstehenden Arbeit von Hurwitz* , Math. Z. 25 (1926), 124143.

[8] Greenberg, R.,*On the Iwasawa invariants of totally real number fields* American Journal Math,98(1976),263-284

[9] Klingen, N. *Arithmetical Similarities:Prime Decomposition and Finite Group Theory* Clarendon Press, Oxford 1998

[10] Mihailescu, P.,*The T and T^* components of* Λ *- modules and Leopoldt's conjecture*, Preprint, arXiv:0905.1274

[11] Oh, J.,*Iwasawa Invariants* Trends in Mathematics, Volume 3 (2000) 108-111

[12] Oxley, J. G. *Matroid theory.* Oxford University Press, New York. 1992

[13] Perlis, R.*On the equation* $K(s) = K'(s)$, J. Number Theory 9, 1977, 342360

[14] Perlis, R. *On the class numbers of arithmetically equivalent fields*, J. Number Theory 10 (1978), 489509.

[15] Washington, L.C *Introduction to Cyclotomic Fields Second Edition* Springer-Verlag 1997

[16] Weintraub, S. *Galois Theory* Springer-Verlag, New York, 2005.

Appendix: Matrices

We construct matrices verifying the support of ν_{00} as defined in section 6. Matrix $M_{(n,t)}$ will indicate a matrix for triple with index n and parent group of order t. There are two distinct triples of index 12, order 96, and two distinct triples of index 14, order 336. In each case we construct matrix MA and MB for the two triples. We also construct two matrices MA and MB for the triple of index 12 and order 72. There is only one such triple. The reason we construct two matrices is that two distinct primes divide the determinant of each matrix. Both 3 and 19 divide $det(MA)$, and both 2 and 3 divide $det(MB)$. Since no other primes divide the determinants of either matrix, the support will contain at most one prime, namely 3. The results here coincide with the results of Bosma and de Smit [1].

$$M_{(7,168)} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

$det(M_{(7,168)}) = -512 = -1 * 2^9$

$$M_{(8,32)} = \begin{pmatrix} 2 & 2 & 3 & 2 & 2 & 1 & 3 & 1 \\ 2 & 3 & 2 & 2 & 1 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 & 3 & 1 & 2 & 2 \\ 2 & 2 & 1 & 3 & 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 1 & 2 & 2 & 3 & 2 \\ 1 & 3 & 1 & 2 & 2 & 3 & 2 & 2 \\ 3 & 1 & 2 & 2 & 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 & 2 & 2 & 1 & 3 \end{pmatrix}$$

$det(M_{(8,32)}) = 1024 = 2^{10}$

$$M_{(8,48)} = \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$det(M_{(8,48)}) = 243 = 3^5$

$$M_{(11,660)} = \begin{pmatrix} 2 & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 3 \\ 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 \\ 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 \\ 3 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 2 \\ 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 2 & 2 \\ 3 & 3 & 3 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 2 \\ 3 & 3 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 3 \\ 3 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 \\ 3 & 2 & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 \end{pmatrix}$$

$det(M_{(11,660)}) = 6561 = 3^8$

$$M_{(12,48)} = \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 3 & 1 & 3 & 3 & 1 & 3 & 1 \\ 1 & -1 & -1 & 1 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 1 & -1 & -1 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 \\ -1 & 1 & 1 & -1 & 3 & 1 & 3 & 1 & 1 & 3 & 1 & 3 \\ 3 & 3 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 & 1 & -1 & -1 & 1 & 1 & 1 & 3 & 3 \\ 1 & 3 & 1 & 3 & 1 & 1 & 3 & 3 & -1 & 1 & 1 & -1 \\ 3 & 1 & 3 & 1 & 1 & 1 & 3 & 3 & 1 & -1 & -1 & 1 \\ 1 & 3 & 1 & 3 & 3 & 3 & 1 & 1 & 1 & -1 & -1 & 1 \\ 3 & 1 & 3 & 1 & 3 & 3 & 1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 3 & 1 & 1 & 1 & -1 & -1 & 1 & 3 & 3 & 1 & 1 \\ 1 & 1 & 3 & 3 & -1 & 1 & 1 & -1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

$det(M_{(12,48)}) = 268435456 = 2^{28}$

$$MA_{(12,72)} = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}$$

$det(MA_{(12,72)}) = 1539 = 3^4 * 19$

$$MB_{(12,72)} = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 2 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 2 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$det(MB_{(12,72)}) = 20736 = 2^8 * 3^4$
$gcd(1539, 20736) = 81 = 3^4$

$$MA_{(12,96)} = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$det(MA_{(12,96)}) = 131072 = 2^{26}$

$$M_{(12,192)} = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2 & 0 & 2 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{pmatrix}$$

$$det(M_{(12,192)}) = 4194304 = 2^{22}$$

$$M_{(12,240)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$det(M_{(12,240)}) = -625 = -1 * 5^4$$

$$M_{(13,5616)} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$det(M_{(13,5616)}) = -6615 = -3^8$$

$$M_{(14,168)} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 & 2 & -1 & 2 & 0 & -1 & -1 & 2 & -1 & 0 \\ 2 & 2 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 2 & 0 & -1 & -1 & 2 \\ 2 & -1 & 2 & -1 & 0 & 0 & 2 & -1 & -1 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 2 & -1 & -1 & 2 & -1 & 0 & 2 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1 & 0 & 2 \\ -1 & 2 & 2 & 0 & -1 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 2 & 0 \\ 0 & -1 & 2 & -1 & 2 & 0 & -1 & 0 & 2 & -1 & -1 & 0 & -1 & 2 \\ 0 & 2 & -1 & 0 & 2 & -1 & 2 & -1 & 0 & -1 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & -1 & 2 & 2 & 0 & -1 & -1 & 2 & 0 & -1 & 2 & -1 \\ 2 & 0 & -1 & 2 & 0 & -1 & 0 & -1 & 2 & -1 & -1 & 0 & 2 & -1 \\ -1 & 0 & 2 & 2 & -1 & 0 & -1 & -1 & 0 & 2 & 0 & 2 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 & 2 & -1 & -1 & 2 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$det(M_{(14,168)}) = 1073741824 = 2^{30}$

$$MA_{(14,336)} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & -1 & -1 & 0 & 0 & -1 & -1 & 2 & 0 & -1 & -1 & 2 & 0 \\ -1 & 2 & 2 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 0 & -1 & -1 & 2 \\ 2 & -1 & 0 & 2 & -1 & 0 & -1 & 2 & 0 & -1 & -1 & 0 & -1 & 2 \\ 2 & 0 & -1 & -1 & 2 & 0 & 2 & 0 & -1 & -1 & 0 & 2 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 2 & 0 & 2 & -1 & -1 & 2 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 & 2 & -1 & 0 & 2 & -1 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & 2 & -1 & 0 & -1 & 2 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 2 \\ -1 & 0 & 2 & 0 & 2 & -1 & -1 & 2 & -1 & 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 2 & -1 & 2 & 2 & -1 & -1 & 0 & 0 & -1 & 2 & -1 \\ -1 & 2 & 0 & 2 & 0 & -1 & -1 & 0 & -1 & 2 & -1 & 2 & 0 & -1 \\ 2 & -1 & 2 & 0 & -1 & 0 & 0 & -1 & -1 & 2 & 2 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$det(MA_{(14,336)}) = 1073741824 = 2^{30}$

$$MB_{(14,336)} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 2 \\ 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 2 & 3 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 2 & 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 3 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\ 3 & 2 & 3 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}$$

$$det(MB_{(14,336)}) = -8192 = -1 * 2^{13}$$

$$M_{(14,56448)} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 \\ -1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$det(M_{(14,56448)}) = -262144 = -1 * 2^{18}$$

$$M_{(15,180)} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$det(M_{(15,180)}) = -262144 = -1 * 2^{18}$$

$$M_{(15,360)} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$det(M_{(15,360)}) = -262144 = -1 * 2^{18}$$

$$
M_{(15,2520)} = \begin{pmatrix}
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1
\end{pmatrix}
$$

$det(M_{(15,2520)}) = 268435456 = 2^{28}$

$$
M_{(15,20160)} = \begin{pmatrix}
-1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
$$

$det(M_{(15,20160)}) = 268435456 = 2^{28}$

Vita

David H. Chapman was born on June 14, 1978, in Glendale, California. He finished his undergraduate studies at Iowa State University in December of 2001. After teaching in public schools, he earned a master's degree in mathematics from the University of Northern Iowa in December 2005. In August 2006 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2011.