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Gate Transient Equations for Digital BiCMOS Circuits

Beshara Elmufdi

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Beshara Elmufdi
Department of Electrical and Computer Engineering
Louisiana State University

Faculty Advisor
Dr. Ashok Srivastava

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1 Introduction

Complementary Metal Oxide Semiconductor (CMOS) devices are the most popular technology used today in the manufacture of digital integrated circuits. The reasons why CMOS is so popular compared to bipolar are threefold: First, the individual Metal Oxide Semiconductor (MOS) transistor used in CMOS devices, occupies a relatively smaller area than a Bipolar Junction Transistor (BJT). Second, the MOS fabrication process involves fewer steps and as a result achieves fewer defects per unit chip than the bipolar fabrication process. Third, CMOS devices have lower power dissipation than bipolar devices. Typically, power dissipation in CMOS devices is two to three orders of magnitude smaller than that of bipolar devices. As a result of these advantages, circuits using CMOS technology are significantly cheaper to manufacture and consume less power than bipolar circuits of equivalent function. It is for these reasons that CMOS is so popular. On the other hand, the signal propagation delay due to large interconnect capacitances is a major factor which limits the performance of CMOS digital integrated circuits. The system's speed is ultimately dependent on the current driving capabilities of those gates driving large capacitive loads. To drive a large capacitive load with a specified propagation delay using CMOS devices we must design the output buffer with sufficiently large transistor channel width. But as we increase the transistor channel width, the proportionally increasing parasitic capacitances will eventually annihilate the speed improvement to be gained by larger transistor sizing.

In comparison, bipolar junction transistors (BJTs) have large current driving capabilities. Hence, system speed is not affected as much by gates driving large capacitances. But because of larger size, higher complexity of manufacture and higher power dissipation, bipolar devices are not as popular as CMOS devices.

Improved performance in terms of speed and power dissipation can be obtained using with BiCMOS technology. BiCMOS circuits integrate both bipolar and CMOS transistors in their design. BiCMOS circuits have the advantages of both the lower power dissipation of CMOS devices and the larger current driving capabilities of bipolar devices. They offer fast switching and improved driving capabilities as shown in Fig. 1.

The disadvantage of BiCMOS circuits lie in the increased fabrication complexity due to the fact that the fabrication of BJTs requires more steps than CMOS. Typically 3-4 additional masks are needed for the BiCMOS process.

In [1], Raje, Sarawasat and Cham developed a group of equations describing the behavior of the BiCMOS circuit during switching accounting for both high-level and low-level injection. They obtained equations to model the BiCMOS circuit using increasingly accurate modeling of the base pushout effect by increasing the complexity of the instantaneous Q_1 transit time expression. In this work the equations introduced in [1] are derived. In Sections 2 and 3 the differential equations for i_c and V_e are derived from the equivalent model introduced in [1]. In Section 4 the differential equations are then solved for increasingly accurate modeling of the base pushout effect. Finally in Section 5 some discrepancies between the work in this work and [1] are discussed.

A program to model the BiCMOS circuit transient was written using the equations derived in this work (Appendix E). Standard values were used, for all MOSFET and BJT parameters used in the equations. The resulting values obtained from the program were used to study transient response characteristics of the BiCMOS circuit.

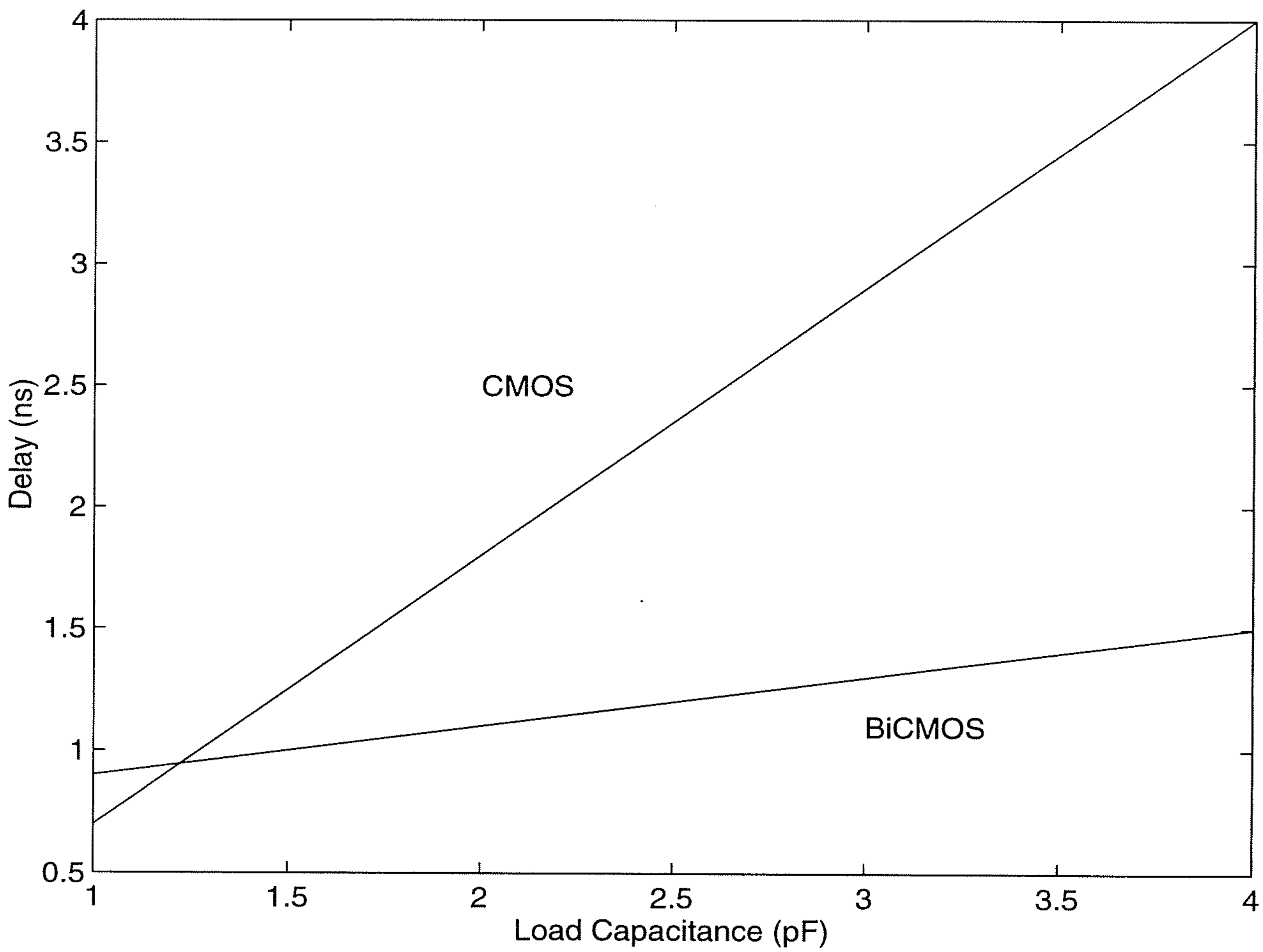

Figure 1: Delay vs. load capacitance for conventional CMOS and BiCMOS buffers obtained from [7]

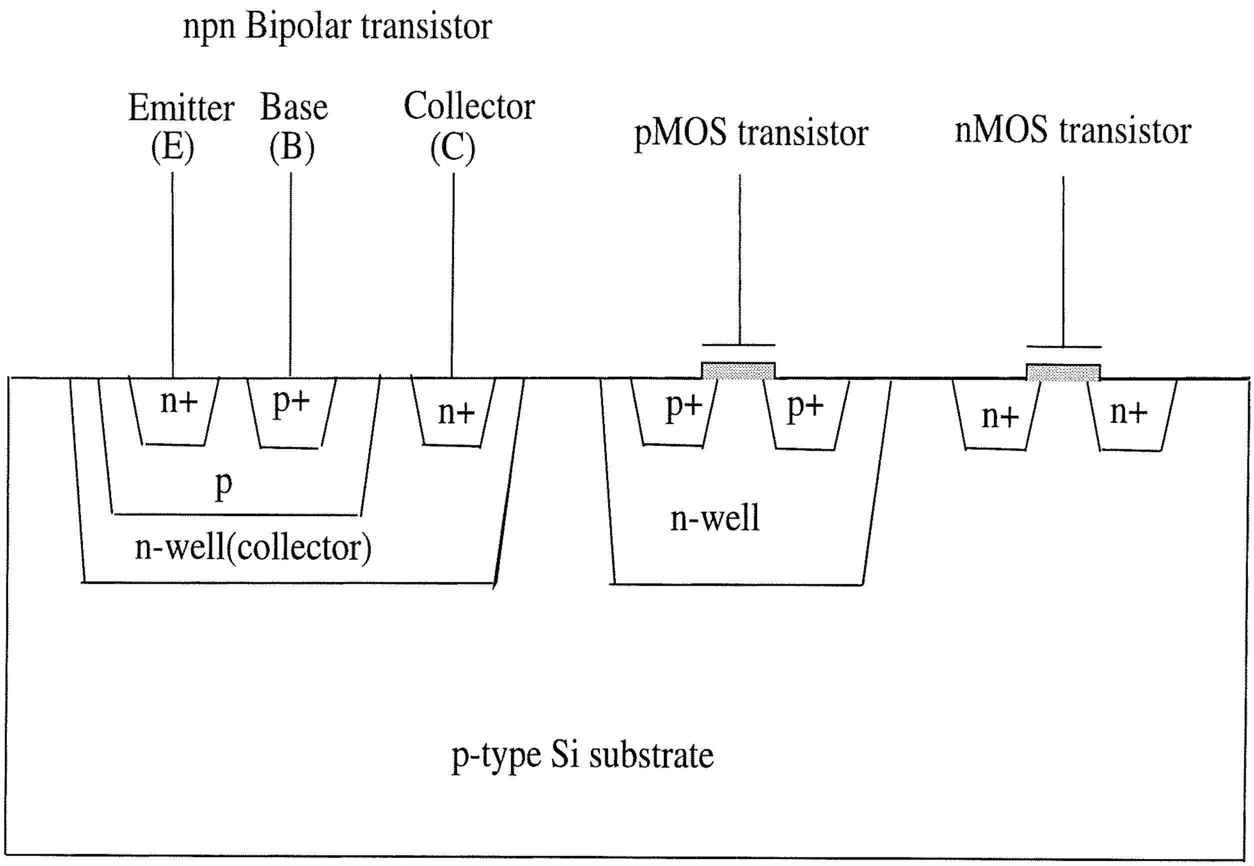

Figure 2: Simplified cross-section of npn bipolar transistor, pMOS transistor and nMOS transistor

2 Equivalent Circuit

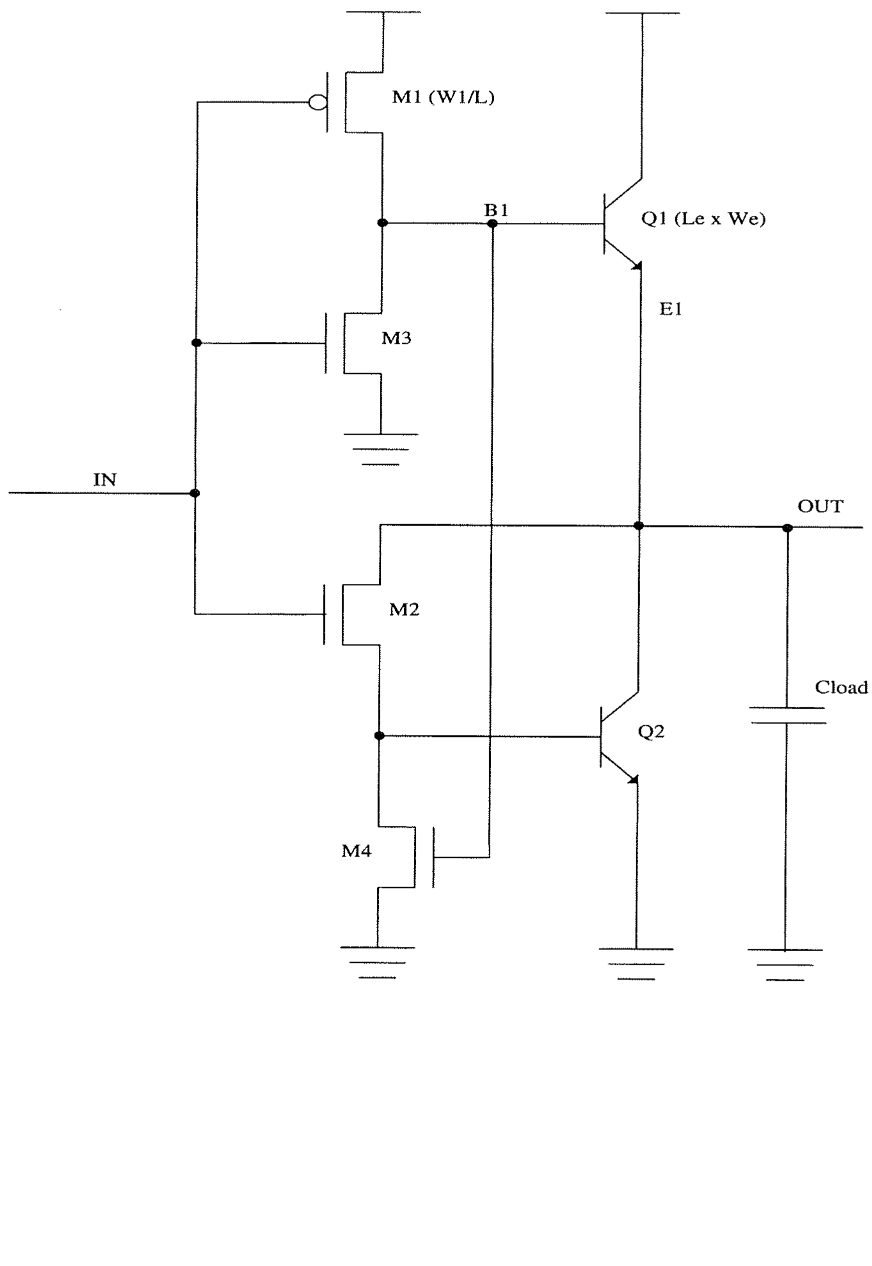

Figure 3: Diagram of BiCMOS circuit

The standard BiCMOS circuit is shown above. The model to be used in this analysis is one for the output rising only. At time t_0 the voltage input falls from high to low. When this happens, the pMOS transistor M_1 is turned on and starts operating in saturation mode. Hence it can be said that M_1 is operating as a current source, where the amount of current is dependent on the $I_m[\ ]$ function. Also when the input falls the nMOS transistors M_3 and M_2 are turned off and hence their influence can be ignored except for their gate-drain overlap capacitances. Before V_{be} reaches V_{diode}, transistor Q_1 is in cut-off mode. When Q_1 is in cut-off mode, $i_b = i_c = 0$ and hence the only parameter that is significant is the base-emitter capacitance V_{be}. When V_{be} reaches V_{diode}, Q_1 is turned on and starts to operate in the forward active mode. In forward active mode the transistor can be modeled with a diode and a collector current source. There is also the base-emitter capacitance to be added. Also there are parasitic capacitances as described in Appendix B. The base resistance on Q_1, R_b does not have a significant impact on the model since it is effectively in series with the much larger MOS resistance. Also using conventional technologies with buried layers, the collector resistance, R_c can be reduced to the point where it is ensured that the base-collector junction does not get forward biased during the transient. Hence the equivalent circuit Fig. 4 is obtained.

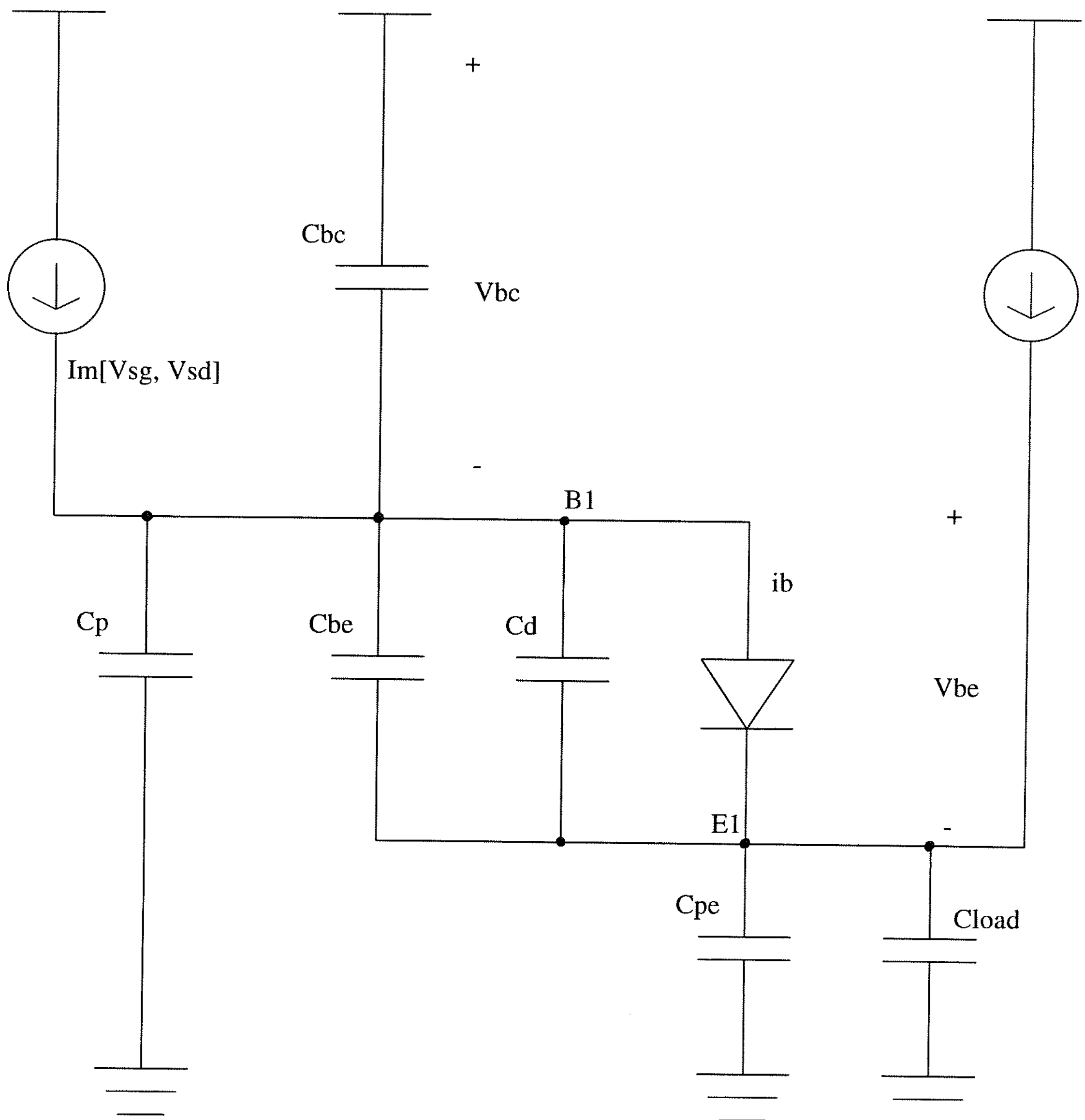

Figure 4: Diagram of equivalent circuit for output rising transient

3 Gate Transient Equations

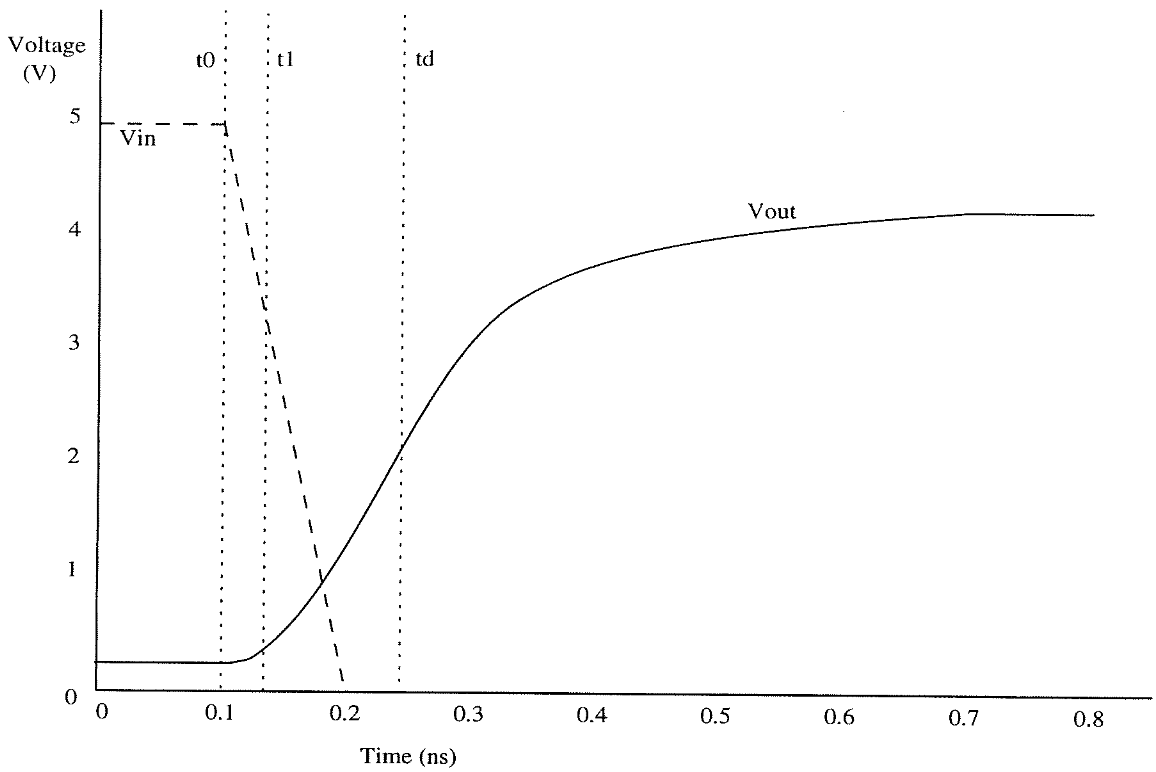

Figure 5: Output voltage waveform during the pull-up transient, and the three time instants used in the analysis

3.1 $t = t_0$ to t_1

Time t_0 is the time at which the input voltage falls from high to low. Time t_1 is when the base emmitter voltage of the BJT Q_1 equals V_{diode}. Initially at time t_0 the input voltage is V_{ih} and the output voltage for the inverter is V_{ol}. Also before t_0 the voltage at B_1 is $0V$. At time t_0 when the input voltage starts to drop below V_{oh}, $V_b = V_b(t_0)$ which is calculated using the equation in the section on voltages for t_0 and t_1 (Appendix D).

After the input voltage drops, the nMOS transistor M_3 is turned off and the pMOS transistor M_1 is turned on and starts to operate in the saturation region. Also just after t_0, Q_1 is off and hence $i_c = i_b = 0V$. C_d is absent since its value is dependent on i_c which is 0. The current from M_1 is used to charge B_1 until the base-emmitter voltage $V_{be} = V_{diode}$ at t_1. Since V_{be} rises it can be assumed that the output voltage also rises. The MOS current (I_{mos1}) can be taken to be the mean current between t_0 and t_1. Hence the diagram can be redrawn to just account for the simplifications.

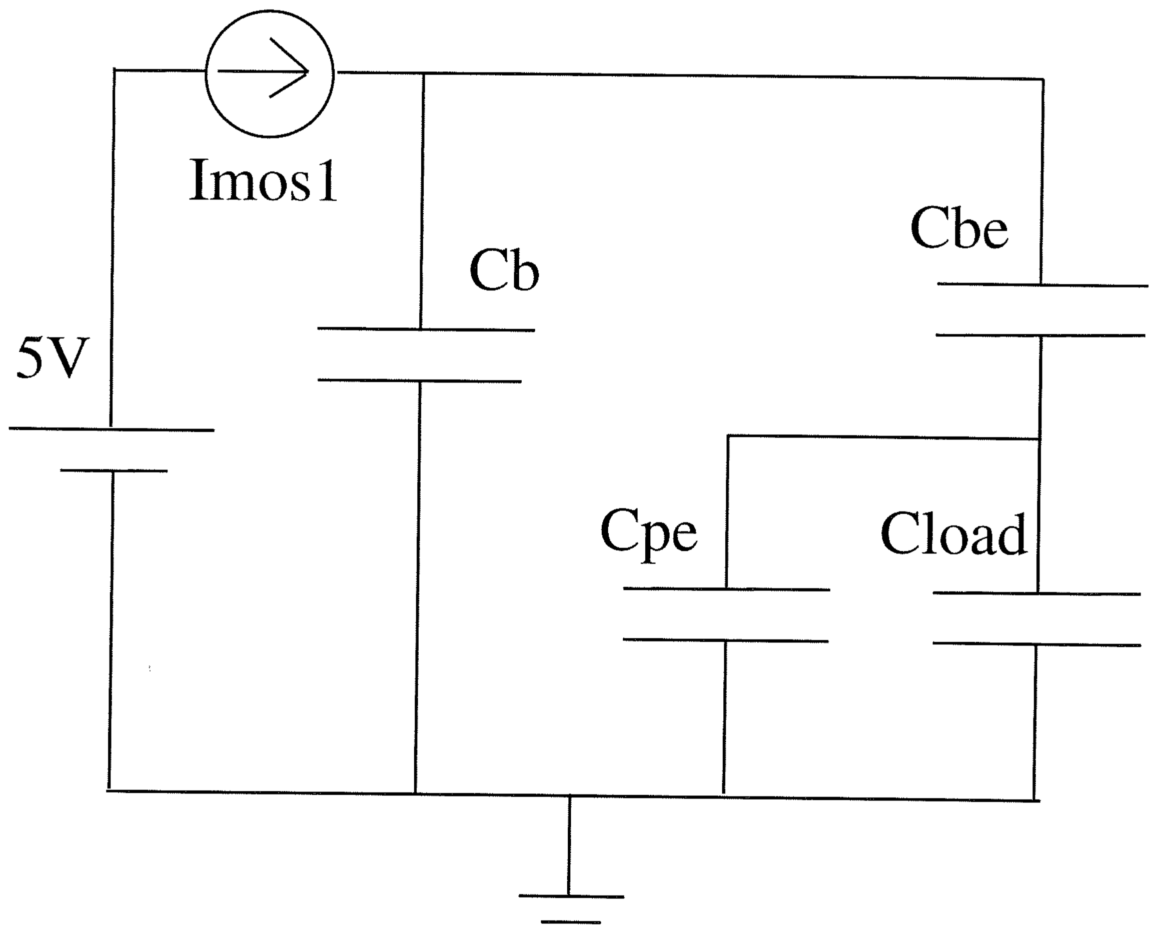

Figure 6: Diagram of equivalent circuit for output rising transient between t_0 and t_1

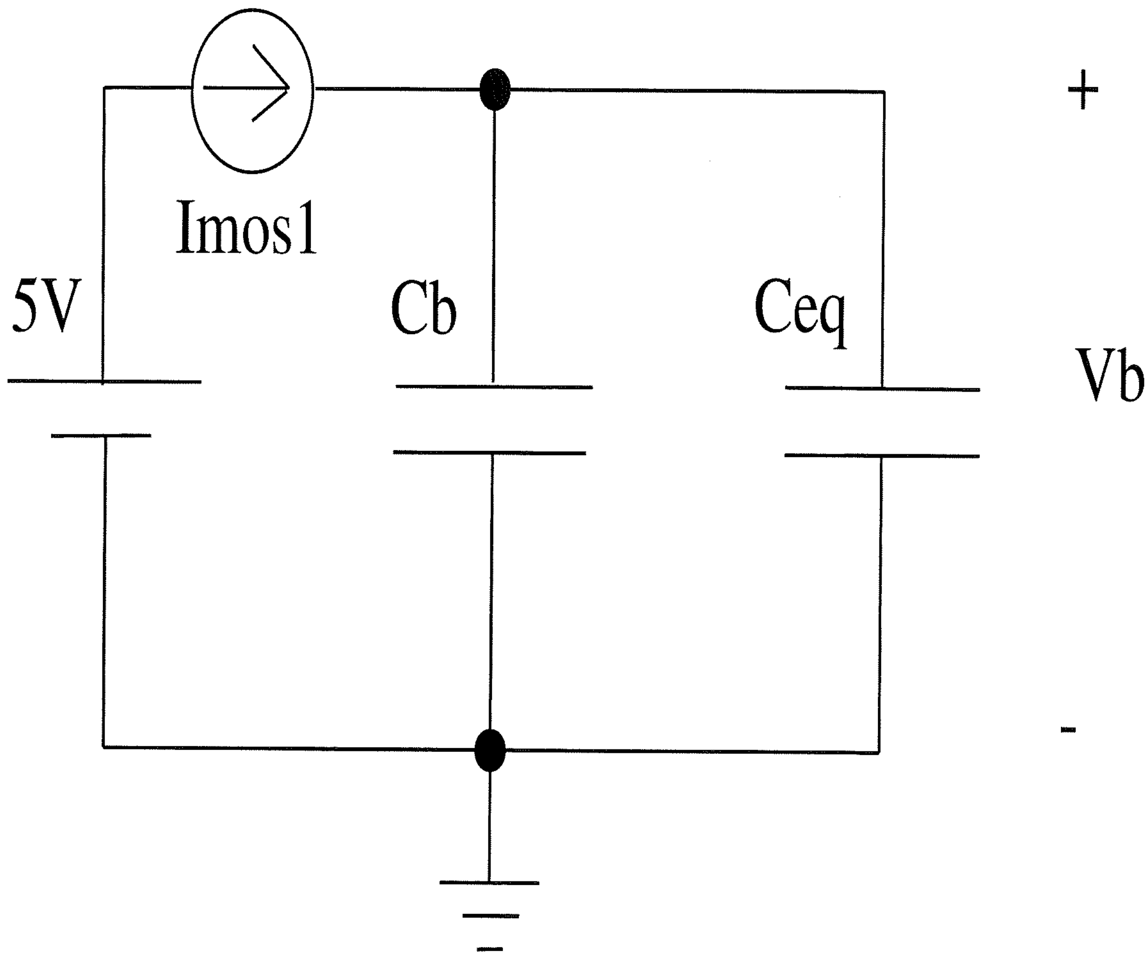

Figure 7: Simplified diagram of equivalent circuit for output rising transient between t_0 and t_1

From Fig. 6, if we let

$$C_L = C_{pe} + C_{load}$$

and

$$\frac{1}{C_{eq}} = \frac{1}{C_{be}} + \frac{1}{C_L}$$

$$C_{eq} = \frac{C_{be}C_L}{C_{be} + C_L}$$

Hence by using Kirchoff's current law on the Fig. 7 simplified diagram,

$$I_{mos1} = C_b \frac{dV_b}{dt} + C_{eq} \frac{dV_b}{dt}$$

$$I_{mos1} = \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) \frac{dV_b}{dt} \tag{1}$$

I_{mos1} was taken as the mean MOS current between times t_0 and t_1. The current at each time is calculated using the MOS current equation I_m (Appendix C). When calculating the MOS current V_{sg} can be assumed to be the supply voltage (V_{dd}) minus the input voltage which happens to always be the input low voltage (V_{il}) whenever M_1 is turned on at saturation. V_{sd} can be assumed to be the supply voltage (V_{dd}) minus the voltage at B_1. Then since at B_1 the voltage at t_0 is $V_b(t_0)$, at t_0 , $V_{sd} = V_{dd} - V_b(t_0)$. Also since at B_1 the voltage at t_1 ia $V_b(t_1)$, at t_1, $V_{sd} = V_{dd} - V_b(t_1)$. Hence it can be said that,

$$\begin{aligned} I_{mos1} = & \quad 0.5(I_m[V_{dd} - V_{il}, V_{dd} - V_b(t_0)] \\ + & \quad I_m[V_{dd} - V_{il}, V_{dd} - V_b(t_1)]) \end{aligned}$$

$$\begin{aligned}
I_{mos1}dt &= \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) dV_b \\
\int_{t_0}^{t_1} I_{mos1}dt &= \int_{V_b(t_0)}^{V_b(t_1)} \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) dV_b \\
I_{mos1} \int_{t_0}^{t_1} dt &= \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) \int_{V_b(t_0)}^{V_b(t_1)} dV_b \\
I_{mos1} \Big[t \Big]_{t_0}^{t_1} &= \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) \Big[V_b \Big]_{V_b(t_0)}^{V_b(t_1)} \\
I_{mos1}(t_1 - t_0) &= \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L} \right) (V_b(t_1) - V_b(t_0))
\end{aligned}$$

Hence,

$$t_1 = t_0 + \left(C_b + \frac{C_{be}C_L}{C_{be} + C_L}\right)\left(\frac{V_b(t_1) - V_b(t_0)}{I_{mos1}}\right) \tag{2}$$

Using equation (2) and data obtained from the program developed for this work (Appendix E), Fig. 8 and 9 were obtained. From Fig. 8 it can be observed that the time delay t_1 decreases sharply for load capacitance $0 \leq C_L < 0.2$pF and then starts increasing more slowly for $C_L > 0.2$pF. The minimum time delay t_1 can be obtained with a load capacitance of approximately 0.2pF. From Fig. 9 it can be observed that the time delay t_1 decreases for gate width $0 \leq W < 4\mu$m and then increases for gate width $W > 4\mu$m. The minimum time delay t_1 can be obtained with a gate width of approximately 4μm.

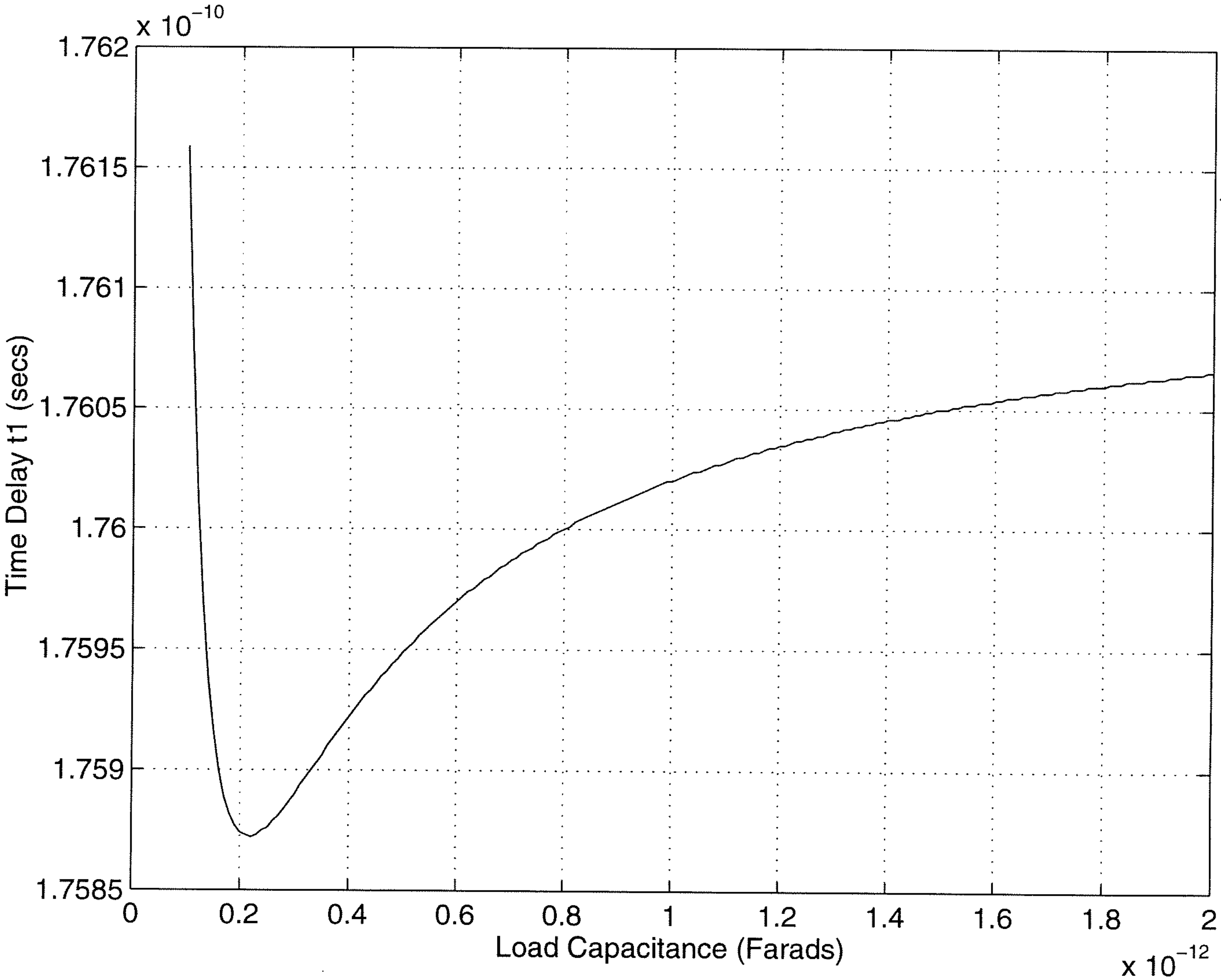

Figure 8: Graph of t_1 V.s. C_L with $W = 5\mu$m

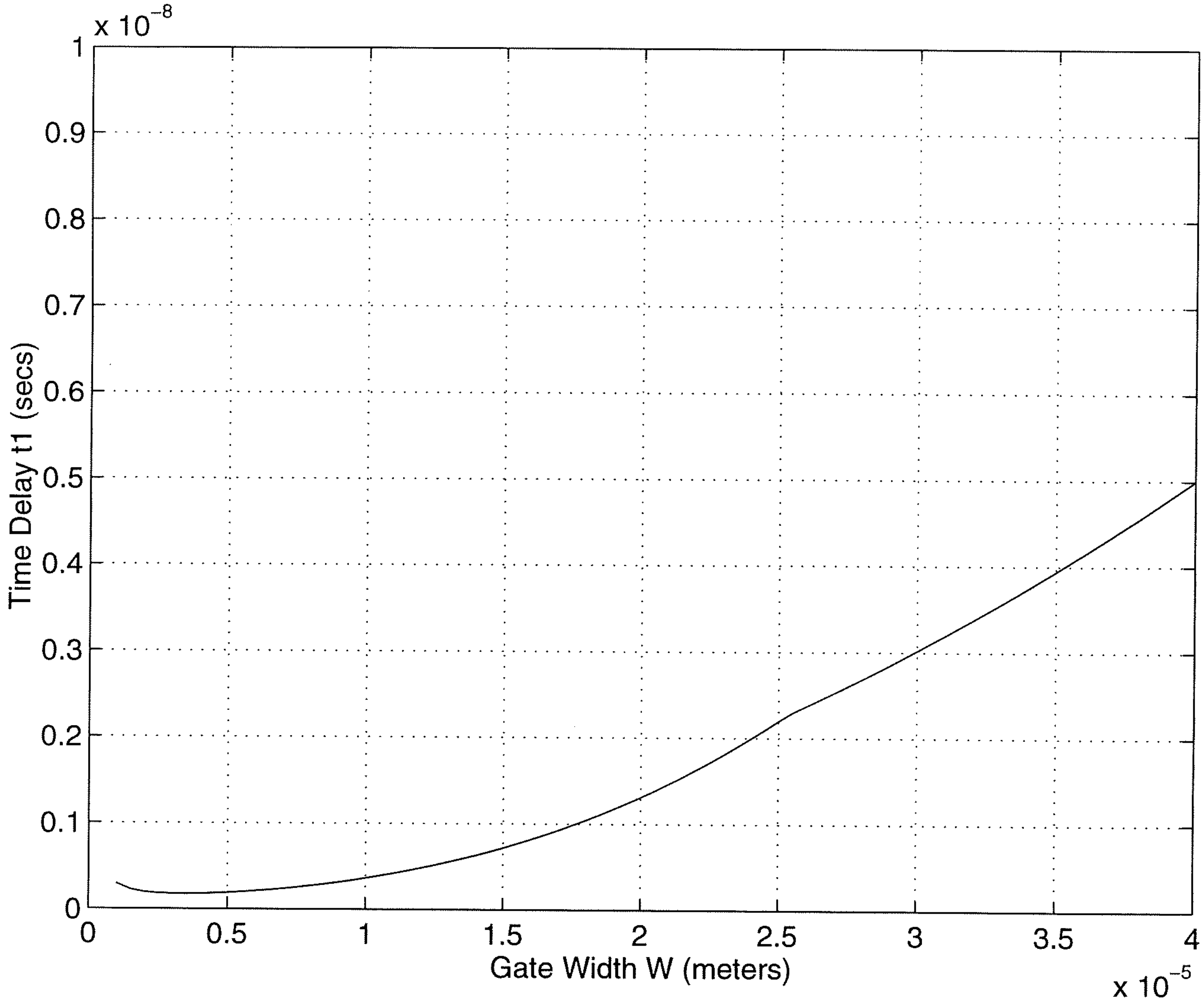

Figure 9: Graph of V_e V.s. t with $C_L = 1$ pF

3.2 $t = t_1$ to t_d

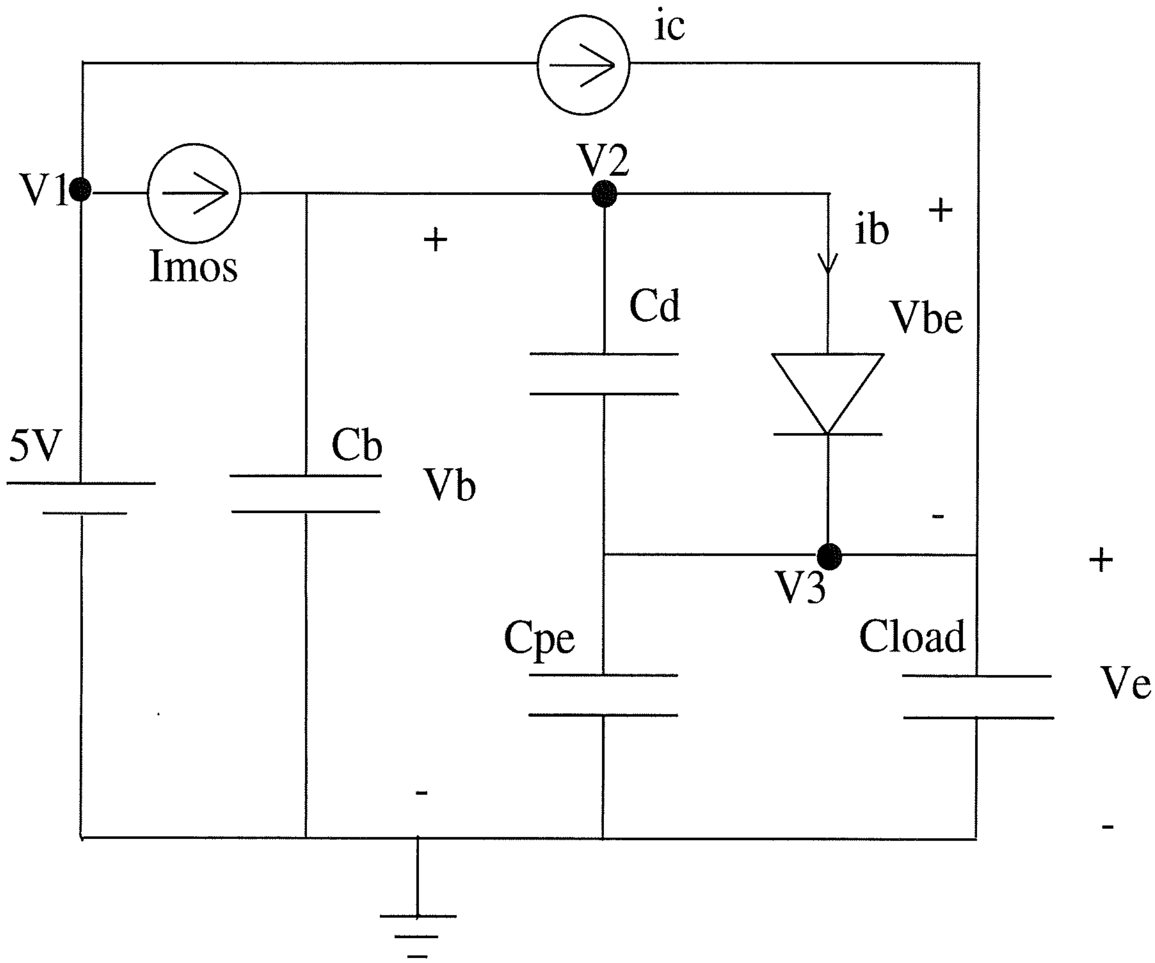

Figure 10: Diagram of equivalent circuit for output rising transient between t_1 and t_d

Time t_d is the time at which the output voltage V_e is equal to half the supply voltage $\frac{V_{dd}}{2}$. Since between t_1 and t_d the BJT Q_1 is on, it is said that the base-emmitter voltage, V_{be} remains constant at V_{diode}. Therefore there is no current through C_{be}. Since the Q_1 is turned on, the collector current of Q_1, i_c begins to rise. As a result of the rise of i_c, the output voltage V_e begins to rise. Also since Q_1 is turned on, the base current i_b begins to rise dependent on the collector current i_c, the BJT current gain β and the knee current on Q_1, I_k. Also the nMOS transistor M_4 is turned ON. As as result it drains the excess base minority carrier charge of Q_2 and assure that Q_2 remains in cut-off mode. Hence the diagram for the equivalent circuit can be redrawn as,

From Fig. 8, by writing Kirchoff's current law at the base B_1 we obtain the following equation:

$$I_{mos} = C_b \frac{dV_2}{dt} + C_d \frac{d(V_2 - V_3)}{dt} + i_b$$

Then we can say $V_2 = V_b$ and $V_2 - V_3 = V_{be}$. Also since $C_d \frac{dV_{be}}{dt} = \frac{d(\tau \cdot i_c)}{dt}$, the equation can then be rewritten as:

$$I_{mos} = C_b \frac{dV_b}{dt} + \frac{d(\tau i_c)}{dt} + i_b \tag{3}$$

The above equation for I_{mos} can be interpreted as saying that the MOS current is used to charge the base parasitic and the base collector capacitances (C_b), toward increasing the stored BJT charge in the base ($\tau \cdot i_c$) and for the Q_1 base current i_b. Independently I_{mos} can be taken as the mean MOS current between times t_1 and t_d.Again the current at each time is calculated using the MOS current equation I_m (Appendix C). When calculating the MOS current V_{sg} can be assumed to be the supply voltage (V_{dd}) minus the input voltage which happens to always be the input low voltage (V_{il}) whenever M_1 is turned on at saturation. V_{sd} can be assumed to be the supply voltage (V_{dd}) minus the voltage at B_1. Then since at B_1 the voltage at t_1 is $V_b(t_1)$, at t_1 , $V_{sd} = V_{dd} - V_b(t_1)$. Also at B_1 the voltage at t_d is $V_e(t_d) + V_{diode}$. Since we know that $V_e(t_d) = \frac{V_{dd}}{2}$, we can say that $\frac{V_{dd}}{2} + V_{diode} = V_b(t_d)$. So at time t_1, $V_{sd} = V_{dd} - V_b(t_d)$. Hence it can be said that,

$$\begin{aligned} I_{mos} = & \quad 0.5(I_m[V_{dd} - V_{il}, V_{dd} - V_b(t_1)] \\ & + \; I_m[V_{dd} - V_{il}, V_{dd} - V_b(t_d)]) \end{aligned}$$

and

$$V_b(t_d) = \frac{V_{dd}}{2} + V_{diode}$$

$$i_b = I_{mos} - C_b \frac{dV_b}{dt} - \frac{d(\tau i_c)}{dt}$$

i_b can be written in terms of the collector current.

$$i_b = \frac{i_c}{\beta}\left(1 + \frac{i_c}{I_k}\right) \tag{4}$$

Now by writing Kirchoff's current law at the output node we obtain:

$$i_c + i_b + C_d\frac{d(V_2 - V_3)}{dt} = C_{pe}\frac{dV_3}{dt} + C_{load}\frac{dV_3}{dt}$$

Then we can say $C_{pe} + C_{load} = C_L$, $V_2 - V_3 = V_{be}$ and $V_3 = V_e$. Also since $C_d\frac{dV_{be}}{dt} = \frac{d(\tau \cdot i_c)}{dt}$, the equation then becomes:

$$\frac{d(\tau i_c)}{dt} + i_b + i_c = C_L\frac{dV_e}{dt} \tag{5}$$

Then since we know that $V_{be} = V_{diode}$ (i.e. V_{be} is constant) between t_1 and t_d, and that $V_b = V_{be} + V_e$, we can then conclude that $\frac{dV_b}{dt} = \frac{dV_e}{dt}$. So by substiting this into (3) we obtain:

$$I_{mos} = C_b\frac{dV_e}{dt} + \frac{d(\tau i_c)}{dt} + i_b$$

$$i_b = I_{mos} - C_b\frac{dV_e}{dt} - \frac{d(\tau i_c)}{dt} \tag{6}$$

Then by substituting (6) into (5) the equation becomes:

$$C_L\frac{dV_e}{dt} = \frac{d(\tau i_c)}{dt} + I_{mos} - C_b\frac{dV_e}{dt} - \frac{d(\tau i_c)}{dt} + i_c$$

$$\frac{dV_e}{dt}(C_b + C_L) = I_{mos} + i_c$$

$$\frac{dV_e}{dt} = \frac{I_{mos} + i_c}{C_b + C_L} \tag{7}$$

Also by substituting (4) and (7) into (3) we obtain:

$$I_{mos} = C_b\left(\frac{I_{mos} + i_c}{C_b + C_L}\right) + \frac{d(\tau i_c)}{dt} + \frac{i_c}{\beta}\left(1 + \frac{i_c}{I_k}\right)$$

$$\frac{d(\tau i_c)}{dt} + \frac{C_b i_c}{C_b + C_L} + \frac{i_c}{\beta} + \frac{{i_c}^2}{\beta I_k} = I_{mos} - \frac{C_b I_{mos}}{C_b + C_L}$$

$$\frac{d(\tau i_c)}{dt} + i_c\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{{i_c}^2}{\beta I_k} = I_{mos}\frac{C_L}{C_b + C_L} \tag{8}$$

4 Solving the i_c and V_e Differential Equations

The i_c and V_e differential equation can be solved using increasingly accurate modeling of the base-pushout effect. This corresponds to increasingly complex expressions for τ. These increasingly complex expressions for τ are the constant, linear and generalized forms for τ. Below the differential equations for i_c and V_e are solved using these increasingly complex expressions for τ.

4.1 Constant τ

4.1.1 *Solving for i_c using Constant τ*

We assume $\tau = \tau_f$, a constant with no dependence on operating conditions. With constant τ, the base pushout effect due to high level injection at the base collector junction is neglected. τ can then be substituted into (8),

$$\tau_f \frac{di_c}{dt} + i_c\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{{i_c}^2}{\beta I_k} = I_{mos}\frac{C_L}{C_b + C_L}$$

$$\tau_f \frac{di_c}{dt} = -i_c\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) - \frac{{i_c}^2}{\beta I_k} + I_{mos}\frac{C_L}{C_b + C_L}$$

$$-\tau_f \beta I_k \frac{di_c}{dt} = i_c^2 + i_c \beta I_k\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) - I_{mos}\beta I_k\frac{C_L}{C_b + C_L} \tag{9}$$

Then if we take the quadratic equation at the Right Hand Side of (9) we can rewrite it as:

$$i_c^2 + i_c \beta I_k\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) - I_{mos}\beta I_k\frac{C_L}{C_b + C_L} = (i_c - \lambda_1)(i_c - \lambda_2) \tag{10}$$

Since this is a quadratic equation we can use the Quadratic Formula $\left(ax^2 + bx + c = 0, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to find i_c. So,

$$\lambda_1, \lambda_2 = \frac{-\beta I_k\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) \pm \sqrt{\left(-I_k\left(\frac{1}{\beta} + \frac{C_b}{C_b+C_L}\right)\right)^2 + 4I_{mos}\beta I_k \frac{C_L}{C_b+C_L}}}{2}$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-I_k\left(1 + \frac{\beta C_b}{C_b + C_L}\right) \pm \sqrt{\left(-I_k\left(1 + \frac{\beta C_b}{C_b + C_L}\right)\right)^2 + 4I_{mos}\beta I_k\frac{C_L}{C_b + C_L}}\right)$$

Where

$$\lambda_1 = \frac{1}{2}\left(-I_k\left(1 + \frac{\beta C_b}{C_b + C_L}\right) + \sqrt{\left(-I_k\left(1 + \frac{\beta C_b}{C_b + C_L}\right)\right)^2 + 4I_{mos}\beta I_k\frac{C_L}{C_b + C_L}}\right) \tag{11}$$

end

$$\lambda_2 = \frac{1}{2}\left(-I_k\left(1+\frac{\beta C_b}{C_b+C_L}\right) - \sqrt{\left(-I_k\left(1+\frac{\beta C_b}{C_b+C_L}\right)\right)^2 + 4I_{mos}\beta I_k \frac{C_L}{C_b+C_L}}\right) \quad (12)$$

So by substituting the more compact quadratic equation form in (10) back into (9) with λ_1 and λ_2 the same as in (11) and (12) respectively, we obtain:

$$-\tau_f \beta I_k \frac{di_c}{dt} = (i_c - \lambda_1)(i_c - \lambda_2)$$

$$-\tau_f \beta I_k \frac{di_c}{(i_c - \lambda_1)(i_c - \lambda_2)} = dt$$

$$-\tau_f \beta I_k \int_{i_c(t_1)}^{i_c(t)} \frac{di_c}{(i_c - \lambda_1)(i_c - \lambda_2)} = \int_{t_1}^{t} dt \quad (13)$$

Then by taking the factor inside the Left Hand Side integral in (13) we can use partial fractions such that:

$$\begin{aligned} \frac{1}{(i_c - \lambda_1)(i_c - \lambda_2)} &= \frac{A}{i_c - \lambda_1} + \frac{B}{i_c - \lambda_2} \\ 1 &= B(i_c - \lambda_1) + A(i_c - \lambda_2) \end{aligned}$$

If we let $i_c = \lambda_1$

$$1 = B(\lambda_1 - \lambda_1) + A(\lambda_1 - \lambda_2)$$

$$A = \frac{1}{(\lambda_1 - \lambda_2)}$$

If we let $i_c = \lambda_2$

$$1 = B(\lambda_2 - \lambda_1) + A(\lambda_2 - \lambda_2)$$

$$B = -\frac{1}{\lambda_1 - \lambda_2}$$

So by substituting A and B back into the equation we obtain:

$$\frac{1}{(i_c - \lambda_1)(i_c - \lambda_2)} = \frac{1}{(\lambda_1 - \lambda_2)(i_c - \lambda_1)} - \frac{1}{(\lambda_1 - \lambda_2)(i_c - \lambda_2)} \quad (14)$$

Then by substituting (14) into (13) and let $i_c(t_1) = 0$ and $i_c(t) = i_c$, (13) becomes:

$$-\tau_f \beta I_k \int_0^{i_c} \left(\frac{1}{(\lambda_1 - \lambda_2)(i_c - \lambda_1)} - \frac{1}{(\lambda_1 - \lambda_2)(i_c - \lambda_2)}\right) di_c = \int_{t_1}^{t} dt$$

$$-\tau_f \beta I_k \frac{1}{\lambda_1-\lambda_2}\int_0^{i_c}\left(\frac{1}{i_c-\lambda_1}-\frac{1}{i_c-\lambda_2}\right)di_c = \int_{t_1}^{t} dt$$

$$-\tau_f \beta I_k \frac{1}{\lambda_1-\lambda_2}\Bigg[\ln(i_c-\lambda_1)-\ln(i_c-\lambda_2)\Bigg]_0^{i_c} = \Bigg[t\Bigg]_{t_1}^{t}$$

$$-\tau_f \beta I_k \frac{1}{\lambda_1-\lambda_2}\left[\ln(i_c-\lambda_1)-\ln(i_c-\lambda_2)-(\ln(-\lambda_1)-\ln(-\lambda_2))\right]_0^{i_c} = t-t_1$$

$$-\tau_f \beta I_k \frac{1}{\lambda_1-\lambda_2}\left(\ln\left(\frac{i_c-\lambda_1}{i_c-\lambda_2}\right)+\ln\left(\frac{\lambda_2}{\lambda_1}\right)\right) = t-t_1$$

$$\ln\left(\left(\frac{i_c-\lambda_1}{i_c-\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)\right) = \frac{(\lambda_1-\lambda_2)(t-t_1)}{-\tau_f\beta I_k}$$

If we let $x = \frac{(\lambda_1-\lambda_2)(t-t_1)}{\tau_f \beta I_k}$ Then

$$\begin{aligned}
\ln\left(\left(\frac{i_c-\lambda_1}{i_c-\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)\right) &= -x \\
\left(\frac{i_c-\lambda_1}{i_c-\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right) &= e^{-x} \\
\frac{i_c-\lambda_1}{i_c-\lambda_2} &= \frac{\lambda_1}{\lambda_2}e^{-x} \\
i_c-\lambda_1 &= (i_c-\lambda_2)\frac{\lambda_1}{\lambda_2}e^{-x} \\
i_c &= \frac{\lambda_1-\lambda_1 e^{-x}}{1-\frac{\lambda_1}{\lambda_2}e^{-x}}
\end{aligned}$$

$$i_c = \frac{\lambda_1\lambda_2(1-e^{-x})}{\lambda_2-\lambda_1 e^{-x}} \tag{15}$$

$$\begin{aligned}
i_c &= \frac{\lambda_1\lambda_2-\lambda_1\lambda_2 e^{-x}}{\lambda_2-\lambda_1 e^{-x}} \\
&= \frac{2(\lambda_1\lambda_2-\lambda_1\lambda_2 e^{-x})}{2(\lambda_2-\lambda_1 e^{-x})} \\
&= \frac{2\lambda_1\lambda_2-2\lambda_1\lambda_2 e^{-x}+(\lambda_2^2-\lambda_2^2)+(\lambda_1^2 e^{-x}-\lambda_1^2 e^{-x})}{2(\lambda_2-\lambda_1 e^{-x})} \\
&= \frac{(\lambda_1\lambda_2-\lambda_1^2 e^{-x}+\lambda_2^2-\lambda_1\lambda_2 e^{-x})+(\lambda_1\lambda_2+\lambda_1^2 e^{-x}-\lambda_2^2-\lambda_1\lambda_2 e^{-x})}{2(\lambda_2-\lambda_1 e^{-x})} \\
&= \frac{(\lambda_1+\lambda_2)(\lambda_2-\lambda_1 e^{-x})+(\lambda_1-\lambda_2)(\lambda_2+\lambda_1 e^{-x})}{2(\lambda_2-\lambda_1 e^{-x})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\left(\frac{\lambda_2+\lambda_1 e^{-x}}{\lambda_2+\lambda_1 e^{-x}}\right) \\
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\left(\frac{1+\frac{\lambda_1}{\lambda_2}e^{-x}}{1-\frac{\lambda_1}{\lambda_2}e^{-x}}\right) \qquad (16)
\end{aligned}
$$

Then we can say

$$
\begin{aligned}
1+\frac{\lambda_1}{\lambda_2}e^{-x} &= 1-\left(-\frac{\lambda_1}{\lambda_2}\right)e^{-x} \\
&= 1-\left(-\frac{\lambda_2}{\lambda_1}\right)^{-1}e^{-x} \\
&= 1-e^{ln\left(-\frac{\lambda_2}{\lambda_1}\right)^{-1}}e^{-x} \\
&= 1-e^{-ln\left(-\frac{\lambda_2}{\lambda_1}\right)}e^{-x} \\
&= 1-e^{-x-ln\left(-\frac{\lambda_2}{\lambda_1}\right)} \qquad (17)
\end{aligned}
$$

We can also say that

$$
\begin{aligned}
1-\frac{\lambda_1}{\lambda_2}e^{-x} &= 1+\left(-\frac{\lambda_1}{\lambda_2}\right)e^{-x} \\
&= 1+\left(-\frac{\lambda_2}{\lambda_1}\right)^{-}1e^{-x} \\
&= 1+e^{ln\left(-\frac{\lambda_2}{\lambda_1}\right)^{-}1}e^{-x} \\
&= 1+e^{-ln\left(-\frac{\lambda_2}{\lambda_1}\right)}e^{-x} \\
&= 1+e^{-x-ln\left(-\frac{\lambda_2}{\lambda_1}\right)} \qquad (18)
\end{aligned}
$$

Then we can take equations (17) and (18) and substitute them back into equation (16).

$$
\begin{aligned}
i_c &= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\left(\frac{1-e^{-x-\ln\left(-\frac{\lambda_2}{\lambda_1}\right)}}{1+e^{-x-\ln\left(-\frac{\lambda_2}{\lambda_1}\right)}}\right) \\
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\left(\frac{e^{\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}-e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}}{e^{\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}+e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}}\right)
\end{aligned}
$$

Then since

$$
e^{\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}-e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)} = \sinh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)
$$

and

$$e^{\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)} + e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)} = \cosh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)$$

It can be seen that:

$$\begin{aligned}
\frac{e^{\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)} - e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}}{e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)} + e^{-\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}} &= \frac{\sinh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)}{\cosh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)} \\
&= \tanh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)
\end{aligned}$$

Hence

$$i_c = \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{1}{2}\left(x+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right)$$

So by substituting $x = \frac{(\lambda_1-\lambda_2)(t-t_1)}{\tau_f \beta I_k}$ The equation is now:

$$\begin{aligned}
i_c &= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{1}{2}\left(\frac{(\lambda_1-\lambda_2)(t-t_1)}{\tau_f\beta I_k}+\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)\right) \\
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{(\lambda_1-\lambda_2)(t-t_1)+\tau_f\beta I_k\ln\left(-\frac{\lambda_2}{\lambda_1}\right)}{2\tau_f\beta I_k}\right) \\
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{(\lambda_1-\lambda_2)\left(t-t_1+\frac{\tau_f\beta I_k}{\lambda_1-\lambda_2}\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}{2\tau_f\beta I_k}\right) \\
&= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{(\lambda_1-\lambda_2)\left(t-t_1+\frac{\tau_f\beta I_k}{\lambda_1-\lambda_2}\ln\left(-\frac{\lambda_2}{\lambda_1}\right)\right)}{2\tau_f\beta I_k}\right)
\end{aligned}$$

Then if we let $\tau' = \frac{2\tau_f\beta I_k}{\lambda_1-\lambda_2}$ the resulting equation will be:

$$i_c = \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{t+\frac{\tau'}{2}\ln\left(-\frac{\lambda_2}{\lambda_1}\right)-t_1}{\tau'}\right)$$

Then if we let $c = \frac{\tau'}{2}\ln\left(-\frac{\lambda_2}{\lambda_1}\right) - t_1$ the resulting equation will be:

$$i_c = \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh\left(\frac{t+c}{\tau'}\right) \tag{19}$$

4.1.2 *Solving for V_e using Constant τ*

Now we can use (19) in (7) to find the output voltage waveform.

$$\frac{dV_e(t)}{dt} = \frac{I_{mos} + i_c(t)}{C_b + C_L}$$

$$\frac{dV_e}{dt} = \frac{1}{C_b + C_L}\left(I_{mos} + i_c(t)\right)$$

$$\frac{dV_e}{dt} = \frac{1}{C_b + C_L}\left[I_{mos} + \left(\frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2}\tanh\left(\frac{t+c}{\tau'}\right)\right)\right]$$

$$\frac{dV_e}{dt} = \frac{1}{C_b + C_L}\left[\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right) + \left(\frac{\lambda_1 - \lambda_2}{2}\tanh\left(\frac{t+c}{\tau'}\right)\right)\right]$$

$$dV_e = \frac{1}{C_b + C_L}\left[\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right) + \left(\frac{\lambda_1 - \lambda_2}{2}\tanh\left(\frac{t+c}{\tau'}\right)\right)\right] dt$$

$$\int_{V_e(t_1)}^{V_e(t)} dV_e = \frac{1}{C_b + C_L}\left[\int_{t_1}^{t}\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right) dt + \int_{t_1}^{t}\left(\frac{\lambda_1 - \lambda_2}{2}\tanh\left(\frac{t+c}{\tau'}\right)\right) dt\right]$$

$$\Big[V_e\Big]_{V_e(t_1)}^{V_e(t)} dV_e = \frac{1}{C_b + C_L}\left(\left[\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right) t\right]_{t_1}^{t} + \left[\left(\frac{\lambda_1 - \lambda_2}{2}\right)\tau' \ln\left(\cosh\left(\frac{t+c}{\tau'}\right)\right)\right]_{t_1}^{t}\right)$$

$$\begin{aligned} V_e(t) - V_e(t_1) &= \frac{1}{C_b + C_L}\Bigg[\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right)(t - t_1) + \\ &\qquad \tau'\left(\frac{\lambda_1 - \lambda_2}{2}\right)\left(\ln\left(\cosh\left(\frac{t+c}{\tau'}\right)\right) - \ln\left(\cosh\left(\frac{t_1+c}{\tau'}\right)\right)\right)\Bigg] \end{aligned}$$

$$\begin{aligned} V_e(t) &= V_e(t_1) + \frac{1}{C_b + C_L}\Bigg[\left(I_{mos} + \frac{\lambda_1 + \lambda_2}{2}\right)(t - t_1) + \\ &\qquad \tau'\left(\frac{\lambda_1 - \lambda_2}{2}\right)\ln\left(\frac{\cosh\left(\frac{t+c}{\tau'}\right)}{\cosh\left(\frac{t_1+c}{\tau'}\right)}\right)\Bigg] \end{aligned} \tag{20}$$

Then since

$$\begin{aligned} \frac{\lambda_1 + \lambda_2}{2} &= \frac{2\left(0.5\left(-I_k\left(1 + \frac{\beta C_b}{C_b + C_L}\right)\right)\right)}{2} \\ &= -\frac{I_k}{2}\left(1 + \frac{\beta C_b}{C_b + C_L}\right) \end{aligned} \tag{21}$$

and

$$\tau' = \frac{2\tau_f \beta I_k}{\lambda_1 - \lambda_2}$$

Then by substituting (21) and τ' into (20) we obtain

$$\begin{aligned}
V_e(t) &= V_e(t_1) + \frac{1}{C_b + C_L}\left[\left(-\frac{I_k}{2}\left(1 + \frac{\beta C_b}{C_b + C_L}\right) + I_{mos}\right)(t - t_1)\right. \\
&\quad \left. + \left(\frac{2\tau_f \beta I_k}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_1 - \lambda_2}{2}\right) \ln\left(\frac{\cosh\left(\frac{t+c}{\tau'}\right)}{\cosh\left(\frac{t_1+c}{\tau'}\right)}\right)\right] \\
&= V_e(t_1) + \frac{1}{C_b + C_L}\left[\left(I_{mos} - \frac{I_k}{2}\left(1 + \frac{\beta C_b}{C_b + C_L}\right)\right)(t - t_1) + \right. \\
&\quad \left. \tau_f \beta I_k \ln\left(\frac{\cosh\left(\frac{t+c}{\tau'}\right)}{\cosh\left(\frac{t_1+c}{\tau'}\right)}\right)\right] \qquad (22)
\end{aligned}$$

Fig. 11 and 12 show the transient i_c and V_e waveforms using the equations developed above. It can be seen that i_c rises asymptotically to a maximum value equal to λ_1 and keeps on rising since I_{mos} is a constant. This is an acceptable approximation between t_1 and t_d since the pMOS is predominantly in saturation and in this interval I_m varies slowly

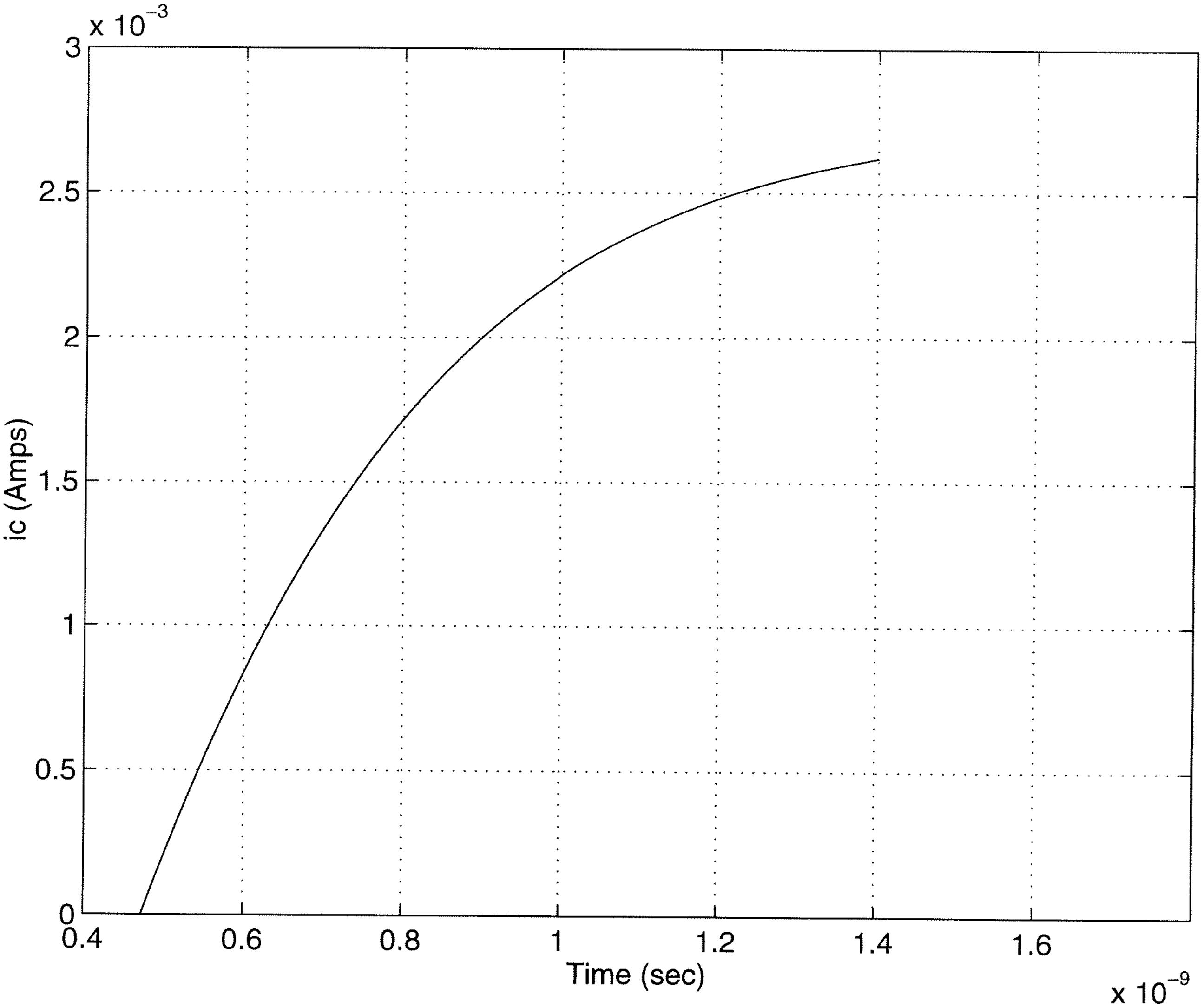

Figure 11: Graph of i_c V.s. t for Constant τ , W $= 5.0\mu m$ and $C_L = 1$ pF

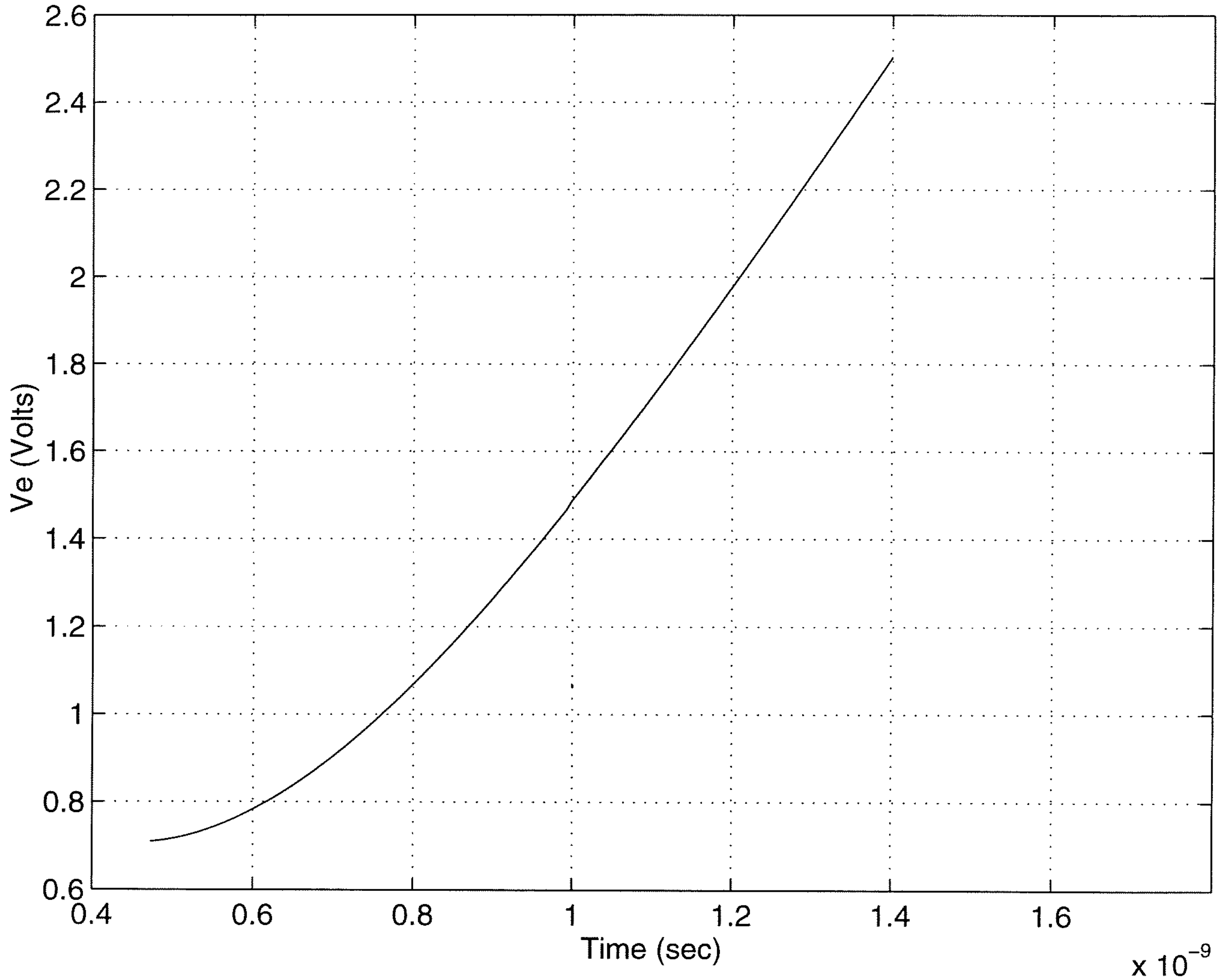

Figure 12: Graph of V_e V.s. t for Constant τ, W = $5.0\mu m$ and $C_L = 1$ pF

4.2 Linear τ

4.2.1 *Solving for i_c using Linear τ*

We assume $\tau = \tau_f\left(1 + \frac{Bi_c}{I_k}\right)$. So τ is linearly dependent on i_c with the factor B representing a proportionality constant. The base widening is accounted for by the increased τ at high collector currents. One shortcoming is that τ is assumed to have no dependence on the base-collector voltage. Another shortcoming is that the base pushout is tied to the value of the knee current(I_k) while it really depends onthe collector doping which has no effect on I_k. This linear τ is substituted into equation. (8). The equation then can be arranged as follows:

$$\tau_f\left(1+\frac{Bi_c}{I_k}\right)\frac{di_c}{dt} + i_c\left(\frac{1}{\beta}+\frac{C_b}{C_b+C_L}\right) + \frac{i_c^2}{\beta I_k} = I_{mos}\frac{C_L}{C_b+C_L}$$

$$\tau_f\left(1+\frac{Bi_c}{I_k}\right)\frac{di_c}{dt} = -\frac{i_c^2}{\beta I_k} - i_c\left(\frac{1}{\beta}+\frac{C_b}{C_b+C_L}\right) + I_{mos}\frac{C_L}{C_b+C_L}$$

$$\tau_f\beta I_k\left(1+\frac{Bi_c}{I_k}\right)\frac{di_c}{dt} = -i_c^2 - i_c\beta I_k\left(\frac{1}{\beta}+\frac{C_b}{C_b+C_L}\right) + I_{mos}\beta I_k\frac{C_L}{C_b+C_L}$$

$$-\tau_f\beta I_k\left(1+\frac{Bi_c}{I_k}\right)\frac{di_c}{dt} = i_c^2 + i_c\beta I_k\left(\frac{1}{\beta}+\frac{C_b}{C_b+C_L}\right) - I_{mos}\beta I_k\frac{C_L}{C_b+C_L}$$

$$-\tau_f\beta I_k\left(1+\frac{Bi_c}{I_k}\right)\frac{di_c}{dt} = (i_c-\lambda_1)(i_c-\lambda_2)$$

Where λ_1, λ_2 are the same as in (11) and (12) respectively.

$$\frac{-\tau_f\beta I_k\left(1+\frac{Bi_c}{I_k}\right)}{(i_c-\lambda_1)(i_c-\lambda_2)}di_c = dt$$

$$-\tau_f\beta I_k\int_{i_c(t_1)}^{i_c(t)}\frac{\left(1+\frac{Bi_c}{I_k}\right)}{(i_c-\lambda_1)(i_c-\lambda_2)}di_c = \int_{t_1}^{t}dt \tag{23}$$

Then by taking the factor inside the L.H.S integral of equation (23) we can use partial fractions such that:

$$\frac{1+\frac{Bi_c}{I_k}}{(i_c-\lambda_1)(i_c-\lambda_2)} = \frac{A}{i_c-\lambda_1} + \frac{C}{i_c-\lambda_2}$$

$$1+\frac{Bi_c}{I_k} = A(i_c-\lambda_2) + C(i_c-\lambda_1)$$

If we let $i_c = \lambda_1$

$$1+\frac{B\lambda_1}{I_k} = A(\lambda_1-\lambda_2) + C(\lambda_1-\lambda_1)$$

$$A = \frac{1 + \frac{B\lambda_1}{I_k}}{\lambda_1 - \lambda_2}$$

If we let $i_c = \lambda_2$

$$1 + \frac{B\lambda_2}{I_k} = A(\lambda_2 - \lambda_2) + C(\lambda_2 - \lambda_1)$$

$$C = -\frac{1 + \frac{B\lambda_2}{I_k}}{\lambda_1 - \lambda_2}$$

So by substituting A and C we obtain:

$$\begin{aligned}
\frac{1 + \frac{Bi_c}{I_k}}{(i_c - \lambda_1)(i_c - \lambda_2)} &= \frac{1 + \frac{B\lambda_1}{I_k}}{(\lambda_1 - \lambda_2)(i_c - \lambda_1)} - \frac{1 + \frac{B\lambda_2}{I_k}}{(\lambda_1 - \lambda_2)(i_c - \lambda_2)} \\
&= \frac{1}{\lambda_1 - \lambda_2}\left(\frac{1 + \frac{B\lambda_1}{I_k}}{i_c - \lambda_1} - \frac{1 + \frac{B\lambda_2}{I_k}}{i_c - \lambda_2}\right) \\
&= \frac{1}{\lambda_1 - \lambda_2}\left(-\left(\frac{1 + \frac{B\lambda_1}{I_k}}{\lambda_1\left(1 - \frac{i_c}{\lambda_1}\right)}\right) + \left(\frac{1 + \frac{B\lambda_2}{I_k}}{\lambda_2\left(1 - \frac{i_c}{\lambda_2}\right)}\right)\right)
\end{aligned} \tag{24}$$

So by substituting (24) into (23) and if we let $i_c(t_1) = 0$ and $i_c(t) = i_c$, (23) becomes:

$$-\tau_f \beta I_k \int_0^{i_c} \frac{1}{\lambda_1 - \lambda_2}\left(-\left(\frac{1 + \frac{B\lambda_1}{I_k}}{\lambda_1\left(1 - \frac{i_c}{\lambda_1}\right)}\right) + \left(\frac{1 + \frac{B\lambda_2}{I_k}}{\lambda_2\left(1 - \frac{i_c}{\lambda_2}\right)}\right)\right) di_c = \int_{t_1}^{t} dt$$

$$-\frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2} \int_0^{i_c} \left(-\left(\frac{1 + \frac{B\lambda_1}{I_k}}{\lambda_1\left(1 - \frac{i_c}{\lambda_1}\right)}\right) + \left(\frac{1 + \frac{B\lambda_2}{I_k}}{\lambda_2\left(1 - \frac{i_c}{\lambda_2}\right)}\right)\right) di_c = \int_{t_1}^{t} dt$$

$$-\frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2}\left[\left(1 + \frac{B\lambda_1}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_1}\right) - \left(1 + \frac{B\lambda_2}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_2}\right)\right]_0^{i_c} = [t]_{t_1}^{t}$$

$$-\frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2}\left[\left(1 + \frac{B\lambda_1}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_1}\right) - \left(1 + \frac{B\lambda_2}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_2}\right)\right] = t - t_1$$

$$t = t_1 - \frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2}\left[\left(1 + \frac{B\lambda_1}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_2}\right) - \left(1 + \frac{B\lambda_2}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_2}\right)\right]$$

$$t = t_1 + \frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2}\left[-\left(1 + \frac{B\lambda_1}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_1}\right) + \left(1 + \frac{B\lambda_2}{I_k}\right)\ln\left(1 - \frac{i_c}{\lambda_2}\right)\right] \tag{25}$$

[1]

[1] See section 5 on discrepancies

4.2.2 *Solving for V_e using Linear τ*

Now we can use equation (25) in (7) to find the output voltage waveform.

$$\frac{dV_e}{dt} = \frac{I_{mos} + i_c(t)}{C_b + C_L}$$

$$dV_e = \left(\frac{I_{mos} + i_c(t)}{C_b + C_L}\right) dt$$

$$\frac{dV_e}{di_c} = \left(\frac{I_{mos} + i_c(t)}{C_b + C_L}\right) \frac{dt}{di_c} \tag{26}$$

Then from (25) it can be seen that:

$$\frac{dt}{di_c} = \frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2} \left(\left(\frac{1 + \frac{B\lambda_1}{I_k}}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)} \right) - \left(\frac{1 + \frac{B\lambda_2}{I_k}}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2}\right)} \right) \right) \tag{27}$$

Then if we let $Y = \left(1 + \frac{B\lambda_1}{I_k}\right)$ and $Z = \left(1 + \frac{B\lambda_2}{I_k}\right)$ equation (27) becomes:

$$\frac{dt}{di_c} = \frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2} \left(\left(\frac{Y}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)} \right) - \left(\frac{Z}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2}\right)} \right) \right) \tag{28}$$

So by substituting (28) into (26) we obtain:

$$\begin{aligned} \frac{dV_e}{di_c} &= \left(\frac{I_{mos} + i_c(t)}{C_b + C_L}\right) \left(\frac{\tau_f \beta I_k}{\lambda_1 - \lambda_2} \left(\left(\frac{Y}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)} \right) - \left(\frac{Z}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2}\right)} \right) \right)\right) \\ &= \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(\left(\frac{Y(I_{mos} + i_c(t))}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)} \right) - \left(\frac{Z(I_{mos} + i_c(t))}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2}\right)} \right) \right) \end{aligned} \tag{29}$$

The if we take $(I_{mos} + i_c(t)) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)}\right)$

$$\begin{aligned} (I_{mos} + i_c(t)) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)}\right) &= \Big(I_{mos} + i_c(t) + (\lambda_1 - \lambda_1)\Big) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)}\right) \\ &= (I_{mos} + \lambda_1) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)}\right) + (i_c - \lambda_1) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1}\right)}\right) \end{aligned}$$

$$
\begin{aligned}
&= (I_{mos}+\lambda_1)\left(\frac{1}{\lambda_1\left(1-\frac{i_c}{\lambda_1}\right)}\right)+(i_c-\lambda_1)\left(\frac{1}{\lambda_1\left(\frac{\lambda_1-i_c}{\lambda_1}\right)}\right)\\
&= (I_{mos}+\lambda_1)\left(\frac{1}{\lambda_1\left(1-\frac{i_c}{\lambda_1}\right)}\right)+(i_c-\lambda_1)\left(\frac{1}{\lambda_1-i_c}\right)\\
&= (I_{mos}+\lambda_1)\left(\frac{1}{\lambda_1\left(1-\frac{i_c}{\lambda_1}\right)}\right)+(i_c-\lambda_1)\left(\frac{1}{i_c-\lambda_1}\right)\\
&= (I_{mos}+\lambda_1)\left(\frac{1}{\lambda_1\left(1-\frac{i_c}{\lambda_1}\right)}\right)-1 \qquad (30)
\end{aligned}
$$

And if we take $(I_{mos}+i_c(t))\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)$

$$
\begin{aligned}
(I_{mos}+i_c(t))\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right) &= \Big(I_{mos}+i_c(t)+(\lambda_2-\lambda_2)\Big)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)\\
&= (I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+(i_c-\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)\\
&= (I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+(i_c-\lambda_2)\left(-\frac{1}{\lambda_2\left(\frac{\lambda_2-i_c}{\lambda_2}\right)}\right)\\
&= (I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+(i_c-\lambda_2)\left(-\frac{1}{\lambda_2-i_c}\right)\\
&= (I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+(i_c-\lambda_2)\left(\frac{1}{i_c-\lambda_2}\right)\\
&= (I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+1 \qquad (31)
\end{aligned}
$$

Then if we substitute (30) and (31) back into (29) we obtain:

$$
\begin{aligned}
\frac{dV_e}{di_c} &= \frac{\tau_f\beta I_k}{(C_b+C_L)(\lambda_1-\lambda_2)}\left(Y\left((I_{mos}+\lambda_1)\left(\frac{1}{\lambda_1\left(\frac{i_c}{\lambda_1}\right)}\right)-1\right)\right.\\
&\left.+Z\left((I_{mos}+\lambda_2)\left(-\frac{1}{\lambda_2\left(1-\frac{i_c}{\lambda_2}\right)}\right)+1\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
dV_e &= \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(Y \left((I_{mos} + \lambda_1) \left(\frac{1}{\lambda_1 \left(\frac{i_c}{\lambda_1} \right)} \right) - 1 \right) \right. \\
&\quad \left. + Z \left((I_{mos} + \lambda_2) \left(- \frac{1}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2} \right)} \right) + 1 \right) \right) di_c
\end{aligned}
$$

$$
\begin{aligned}
\int_{V_e(t_1)}^{V_e(t)=V_e} dV_e &= \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(- Y \int_{i_c(t_1)=0}^{i_c(t)=i_c} \left(1 - (I_{mos} + \lambda_1) \left(\frac{1}{\lambda_1 \left(1 - \frac{i_c}{\lambda_1} \right)} \right) \right) di_c \right. \\
&\quad \left. + Z \int_{i_c(t_1)=0}^{i_c(t)=i_c} \left((I_{mos} + \lambda_2) \left(- \frac{1}{\lambda_2 \left(1 - \frac{i_c}{\lambda_2} \right)} \right) + 1 \right) di_c \right)
\end{aligned}
$$

$$
\begin{aligned}
\Big[V_e \Big]_{V_e(t_1)}^{V_e} &= \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(- Y \left[i_c + (I_{mos} + \lambda_1) \ln \left(1 - \frac{i_c}{\lambda_1} \right) \right]_0^{i_c} \right. \\
&\quad \left. + Z \left[i_c + (I_{mos} + \lambda_2) \ln \left(1 - \frac{i_c}{\lambda_2} \right) \right]_0^{i_c} \right)
\end{aligned}
$$

$$
\begin{aligned}
V_e - V_e(t_1) &= \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(Y \left(-i_c - (I_{mos} + \lambda_1) \ln \left(1 - \frac{i_c}{\lambda_1} \right) \right) \right. \\
&\quad \left. + Z \left(i_c + (I_{mos} + \lambda_2) \ln \left(1 - \frac{i_c}{\lambda_2} \right) \right) \right) \qquad (32)
\end{aligned}
$$

So by substituting $Y = \left(1 + \frac{B\lambda_1}{I_k} \right)$ and $Z = \left(1 + \frac{B\lambda_2}{I_k} \right)$ into (32) we obtain:

$$
\begin{aligned}
V_e &= V_e(t_1) + \frac{\tau_f \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(\left(1 + \frac{\lambda_1 B}{I_k} \right) \left(-i_c - (I_{mos} + \lambda_1) \ln \left(1 - \frac{i_c}{\lambda_1} \right) \right) \right. \\
&\quad \left. + \left(1 + \frac{\lambda_2 B}{I_k} \right) \left(i_c + (I_{mos} + \lambda_2) \ln \left(1 - \frac{i_c}{\lambda_2} \right) \right) \right) \qquad (33)
\end{aligned}
$$

[2]

From the plots of V_e for constant and linear τ (Fig. 12 and 14), it can be seen that t_d is a lower value for constant τ (1.4 ns) than for linear τ (1.68 ns). This is probably due to the fact that with higher collector current, the τ factor increases in the linear case to account for the base widening. Therefore, at all current levels τ is larger in the linear case than in the constant case, resulting in a higher output voltage slope for the constant τ and hence a smaller time delay t_d.

When the plots of i_c for constant and linear τ (Fig. 11 and 13) are compared it can be observed that for constant τ the collector current, i_c, is higher and reaches approximately 2.7 mA at t_d while for linear τ, i_c only reaches 2.3 mA at t_d. Hence the slope of the i_c V.s. t curve for any given time for linear τ is shallower than the slope for the constant τ curve.

[2] See section 5 on discrepancies

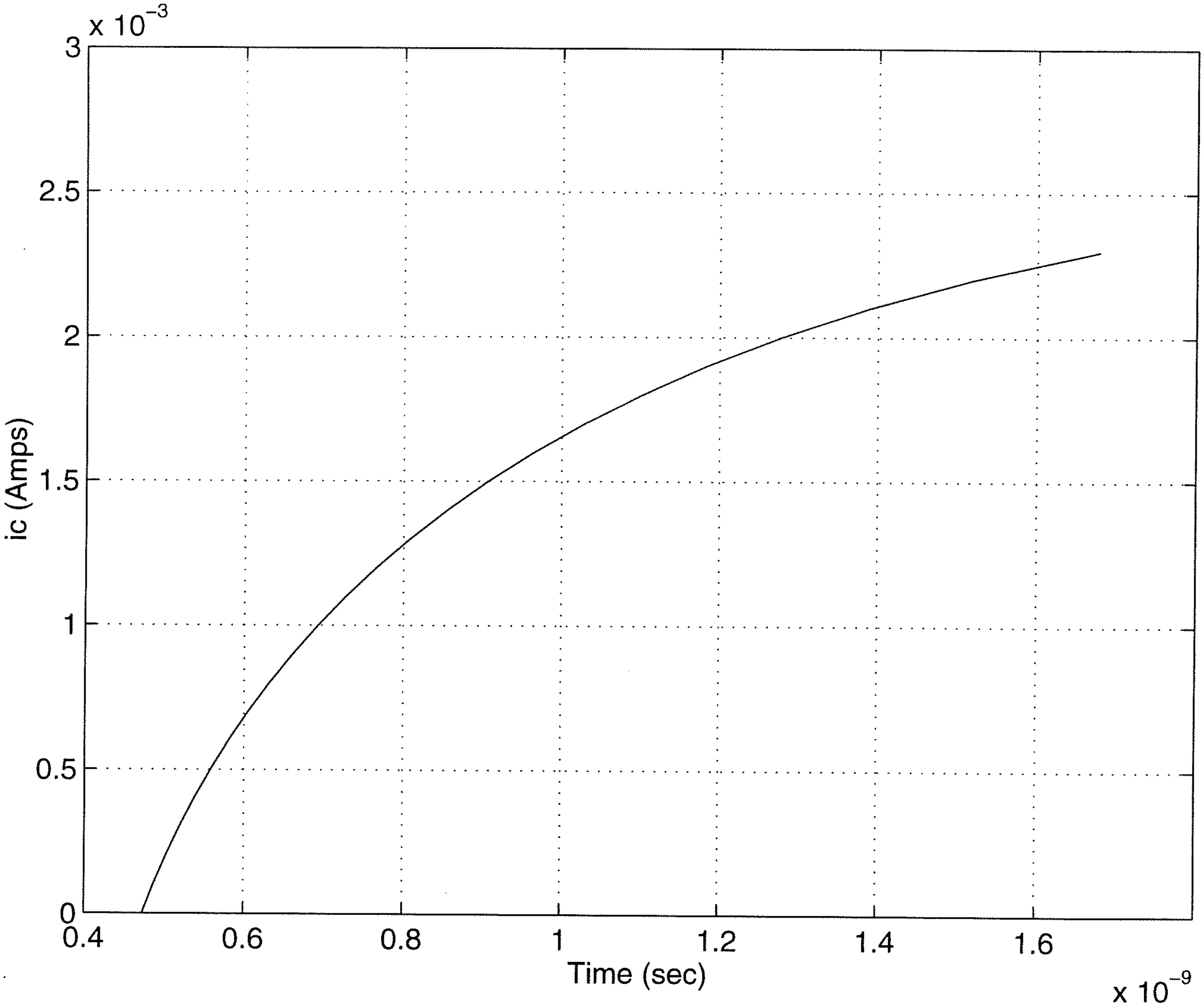

Figure 13: Graph of i_c V.s. t for Linear τ, W = $5.0\mu m$ and $C_L = 1$ pF

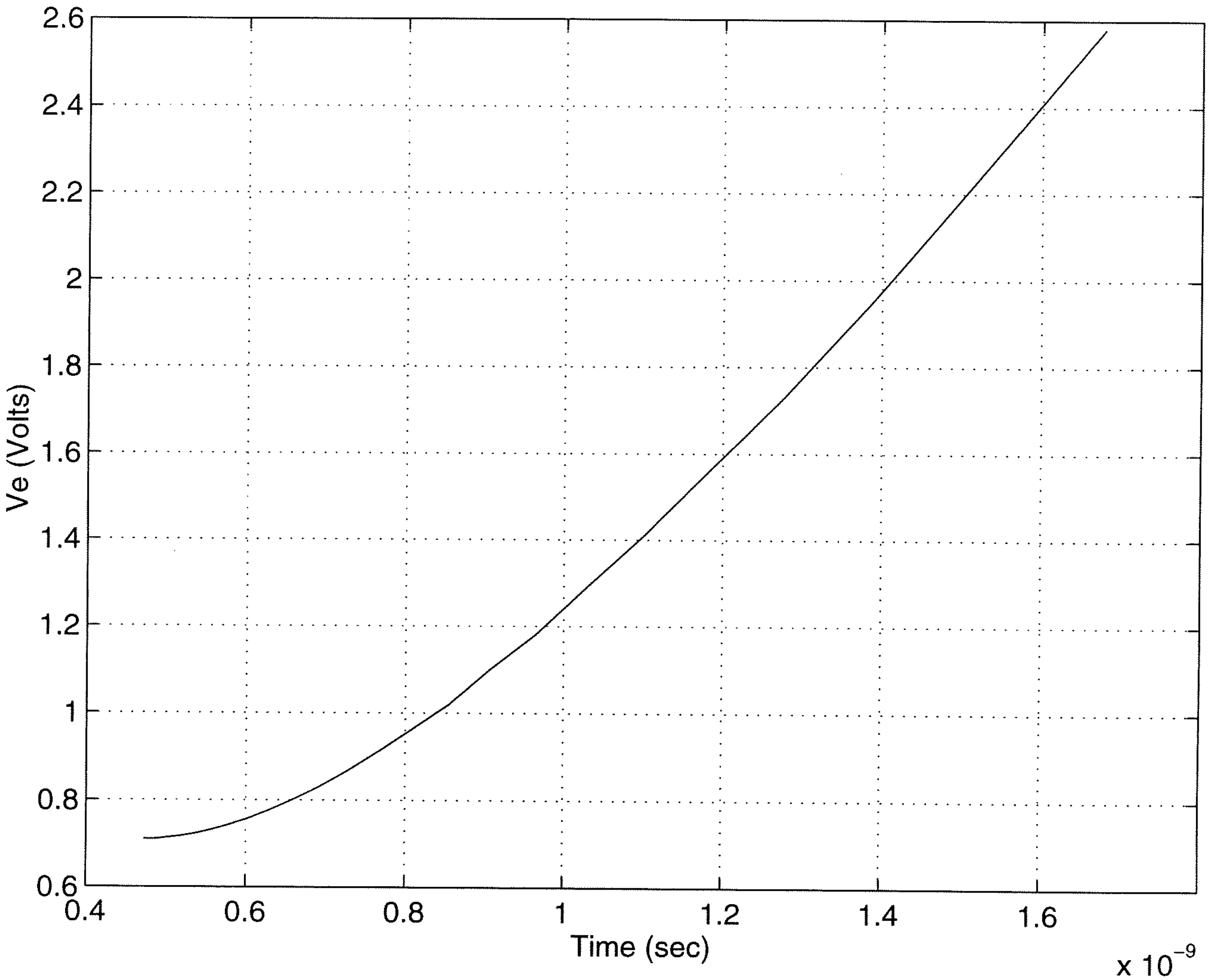

Figure 14: Graph of V_e V.s. t for Linear τ, W $= 5.0\mu m$ and $C_L = 1$ pF

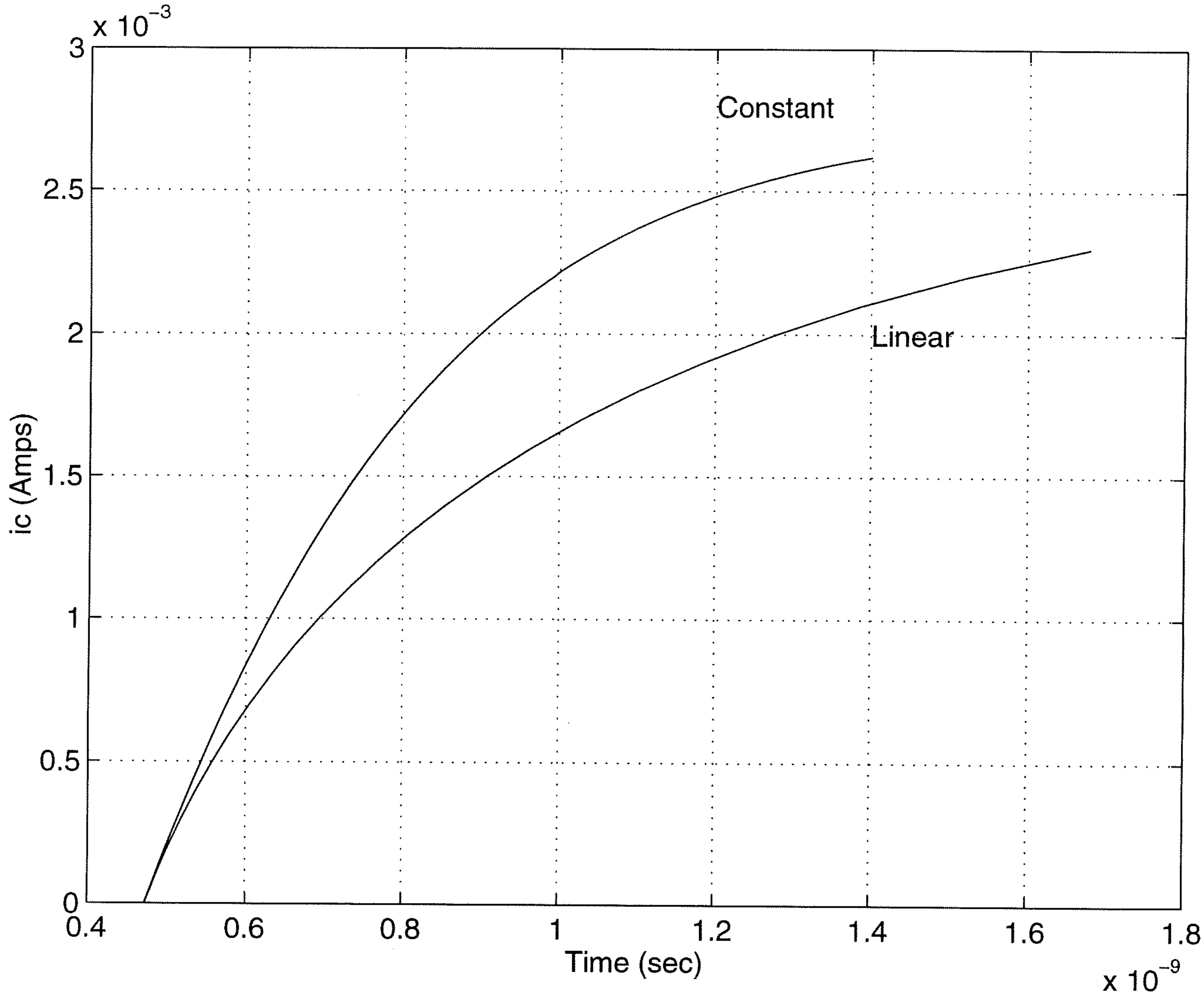

Figure 15: Comparison Graph of i_c V.s. t for Constant and Linear τ, W = $5.0\mu m$ and $C_L = 1$ pF

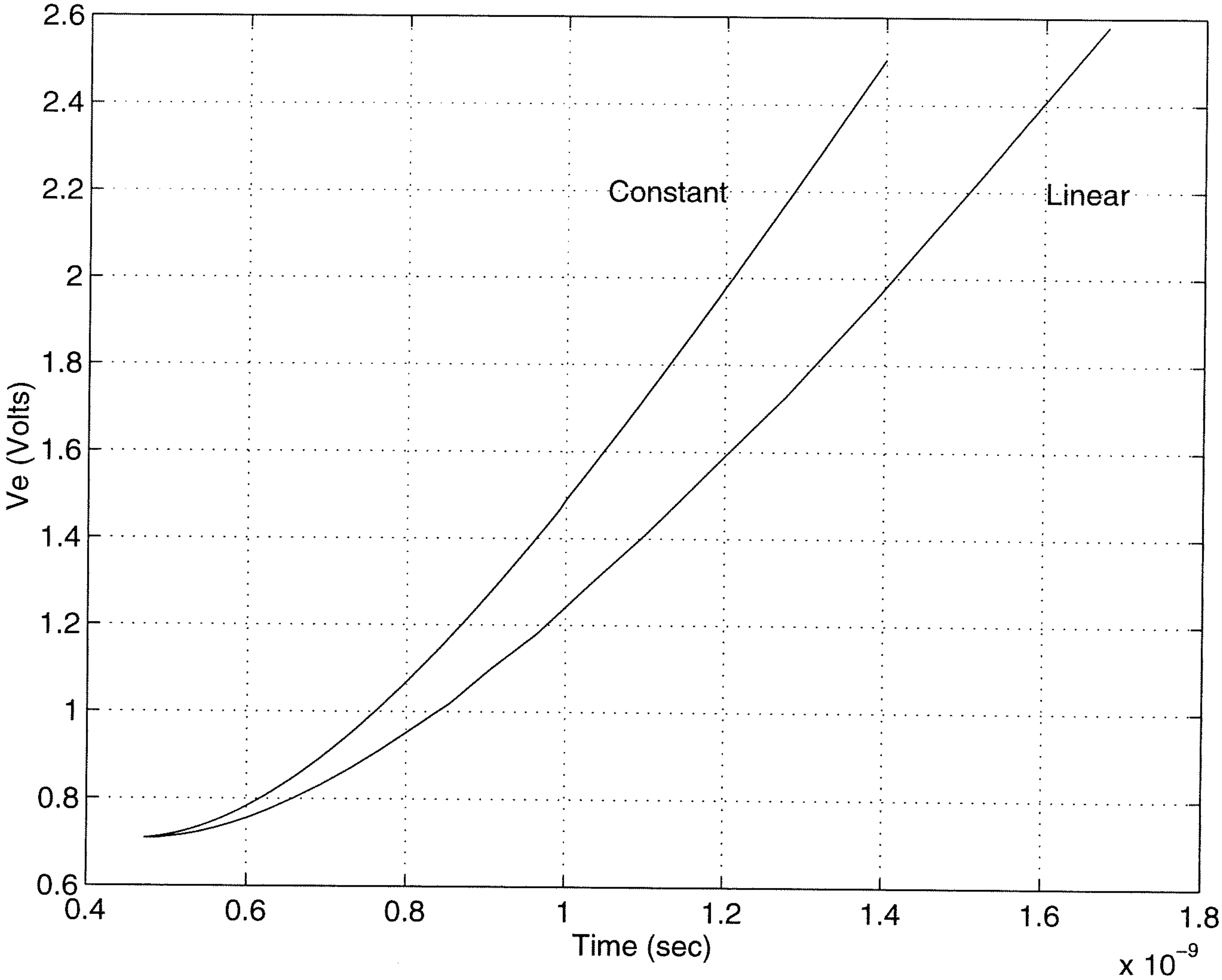

Figure 16: Comparison Graph of V_e V.s. t for Constant and Linear τ, W = $5.0\mu m$ and $C_L = 1$ pF

4.3 Generalized τ

We assume $\tau = \tau_f \mathcal{F}(i_c, V_{bc})$. This equations accounts for the fact that τ varies with both the collector current and the base-collector voltage. The expression above for τ is the same as the one used in SPICE,

$$\tau = \tau_f \left(1 + X_{tf} e^{\frac{V_{bc}}{1.44 V_{tf}}} \left(\frac{i_c}{i_c + I_{tf}}\right)^2\right)$$

By substituting this τ into equation (8) we obtain

$$\tau_f \frac{d(\mathcal{F} i_c)}{dt} + i_c \left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{i_c^2}{\beta I_k} = I_{mos} \frac{C_L}{C_b + C_L}$$

This equation cannot be solved. The equation can be simplified using $i_c = I_{cP}$ (peak value). At this instant in time $\frac{di_c}{dt} = 0$. The equation can therefore be rewritten as

$$\tau_f I_{cP} \frac{\partial \mathcal{F}}{\partial V_{bc}} \frac{\partial V_e}{\partial t} + I_{cP} \left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{I_{cP}^2}{\beta I_k} = I_{mos} \frac{C_L}{C_b + C_L}$$

By substituting (7) into the above equation we obtain:

$$\tau_f I_{cP} \frac{d\mathcal{F}}{dV_{bc}} \left(\frac{I_{mos} + I_{cP}}{C_b + C_L}\right) + I_{cP} \left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{I_{cP}^2}{\beta I_k} = I_{mos} \frac{C_L}{C_b + C_L} \tag{34}$$

$$\frac{\partial \mathcal{F}}{\partial V_{bc}} = \frac{\partial \left(1 + X_{tf} e^{\frac{V_{bc}}{1.44 V_{tf}}} \left(\frac{I_{cP}}{I_{cP} + I_{tf}}\right)^2\right)}{\partial V_{bc}}$$

$$\frac{\partial \mathcal{F}}{\partial V_{bc}} = \frac{X_{tf}}{1.44 V_{tf}} e^{\frac{V_{bc}}{1.44 V_{tf}}} \left(\frac{I_{cP}}{I_{cP} + I_{tf}}\right)^2 \tag{35}$$

By substituting (35) into (34) the resulting equation is:

$$\tau_f I_{cP} \left(\frac{X_{tf}}{1.44 V_{tf}} e^{\frac{V_{bc}}{1.44 V_{tf}}} \left(\frac{I_{cP}}{I_{cP} + I_{tf}}\right)^2\right) \left(\frac{I_{mos} + I_{cP}}{C_b + C_L}\right) + I_{cP} \left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right)$$
$$+ \quad \frac{I_{cP}^2}{\beta I_k} = I_{mos} \frac{C_L}{C_b + C_L}$$

$$\tau_f \left(\frac{X_{tf}}{1.44 V_{tf}} e^{\frac{V_{bc}}{1.44 V_{tf}}} \frac{I_{cP}^3}{(I_{cP} + I_{tf})^2} \left(\frac{I_{mos} + I_{cP}}{C_b + C_L}\right)\right) +$$
$$I_{cP} \left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) + \frac{I_{cP}^2}{\beta I_k} - I_{mos} \frac{C_L}{C_b + C_L} = 0 \tag{36}$$

Then if we let

$$
\begin{aligned}
f(I_{cP}) &= \frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \left(\frac{I_{cP}^3}{(I_{cP}+I_{tf})^2} \right) \left(\frac{I_{cP}+I_{mos}}{C_b+C_L} \right) \\
&\quad + \frac{I_{cP}^2}{\beta I_k} + I_{cP}\left(\frac{1}{\beta} + \frac{C_b}{C_b+C_L}\right) - I_{mos}\frac{C_L}{C_b+C_L} = 0
\end{aligned}
$$

$$
\begin{aligned}
f(I_{cP}) &= \frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \left(\frac{I_{cP}^4 + I_{cP}^3 I_{mos}}{C_b+C_L} \right) + \frac{(I_{cP}+I_{tf})^2 I_{cP}^2}{\beta I_k} \\
&\quad + (I_{cP}+I_{tf})^2 I_{cP}\left(\frac{1}{\beta} + \frac{C_b}{C_B+C_L}\right) - (I_{cP}+I_{tf})^2 \frac{I_{mos}C_L}{C_b+C_L} = 0
\end{aligned}
$$

$$
\begin{aligned}
f(I_{cP}) &= \frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{I_{cP}^4}{C_b+C_L} + \frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{I_{cP}^3 I_{mos}}{C_b+C_L} + \frac{I_{cP}^4}{\beta I_k} \\
&\quad + \frac{2I_{cP}^3 I_{tf}}{\beta I_k} + \frac{I_{cP}^2 I_{tf}^2}{\beta I_k} + I_{cP}^3\left(\frac{1}{\beta} + \frac{C_b}{C_B+C_L}\right) + 2I_{cP}^2 I_{tf}\left(\frac{1}{\beta} + \frac{C_b}{C_B+C_L}\right) \\
&\quad + I_{cP} I_{tf}^2\left(\frac{1}{\beta} + \frac{C_b}{C_B+C_L}\right) - \frac{I_{cP}^2 I_{mos} C_L}{C_b+C_L} - \frac{2I_{cP} I_{tf} I_{mos} C_L}{C_b+C_L} - \frac{I_{tf}^2 I_{mos} C_L}{C_b+C_L} = 0
\end{aligned}
$$

$$
\begin{aligned}
f(I_{cp}) = \quad & I_{cP}^4 \left(\frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{1}{C_B+C_L} + \frac{1}{\beta I_k} \right) \\
+ \quad & I_{cP}^3 \left(\frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{I_{mos}}{C_B+C_L} + \frac{2I_{tf}}{\beta I_k} + \frac{1}{\beta} + \frac{C_b}{C_b+C_L} \right) \\
+ \quad & I_{cP}^2 \left(\frac{I_{tf}^2}{\beta I_k} + 2I_{tf}\left(\frac{1}{\beta} + \frac{C_b}{C_b+C_L}\right) - \frac{I_{mos}C_L}{C_b+C_L} \right) \\
+ \quad & I_{cP} \left(I_{tf}^2 \left(\frac{1}{\beta} + \frac{C_b}{C_b+C_L}\right) - \frac{2I_{tf} I_{mos} C_L}{C_b+C_L} \right) \\
- \quad & \frac{I_{tf}^2 I_{mos} C_L}{C_b+C_L} = 0
\end{aligned}
$$

Then if we let:

$$
\begin{aligned}
X_1 &= \left(\frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{1}{C_B+C_L} + \frac{1}{\beta I_k} \right) \\
X_2 &= \left(\frac{\tau_f X_{tf}}{1.44V_{tf}} e^{\frac{V_{bc}}{1.44V_{tf}}} \frac{I_{mos}}{C_B+C_L} + \frac{2I_{tf}}{\beta I_k} + \frac{1}{\beta} + \frac{C_b}{C_b+C_L} \right)
\end{aligned}
$$

$$
\begin{aligned}
X_3 &= \left(\frac{I_{tf}^2}{\beta I_k} + 2I_{tf}\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) - \frac{I_{mos}C_L}{C_b + C_L}\right) \\
X_4 &= \left(I_{tf}^2\left(\frac{1}{\beta} + \frac{C_b}{C_b + C_L}\right) - \frac{2I_{tf}I_{mos}C_L}{C_b + C_L}\right) \\
X_5 &= -\frac{I_{tf}^2 I_{mos}C_L}{C_b + C_L} = 0
\end{aligned}
$$

the equation becomes:

$$
f(I_{cP}) = X_1 I_{cP}^4 + X_2 I_{cP}^3 + X_3 I_{cP}^2 + X_4 I_{cP} + X_5 = 0 \tag{37}
$$

4.4 Newton-Raphson Iteration

From the Newton-Rhaphson method [6] we know that for any function $F(x_n)$, the root of that function can be approximated ever more closely by iterating the formula

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

Then if we use the seed value of $I_{cP} = \lambda_1$ in equation (37) as suggested in [1], and apply the Newton-Rhaphson method we obtain

$$\begin{aligned} f(I_{cP} = \lambda_1) &= X_1\lambda_1^4 + X_2\lambda_1^3 + X_3\lambda_1^2 + X_4\lambda_1 + X_5 \\ f'(I_{cP} = \lambda_1) &= 4X_1\lambda_1^3 + 3X_2\lambda_1^2 + 2X_3\lambda_1 + X_4 \end{aligned}$$

$$\begin{aligned} I_{cP2} &= \lambda_1 - \frac{f(\lambda_1)}{f'(\lambda_1)} \\ &= \lambda_1 - \frac{X_1\lambda_1^4 + X_2\lambda_1^3 + X_3\lambda_1^2 + X_4\lambda_1 + X_5}{4X_1\lambda_1^3 + 3X_2\lambda_1^2 + 2X_3\lambda_1 + X_4} \\ &= \frac{4X_1\lambda_1^4 + 3X_2\lambda_1^3 + 2X_3\lambda_1^2 + X_4\lambda_1 - (X_1\lambda_1^4 + X_2\lambda_1^3 + X_3\lambda_1^2 + X_4\lambda_1 + X_5)}{4X_1\lambda_1^3 + 3X_2\lambda_1^2 + 2X_3\lambda_1 + X_4} \\ &= \frac{3\lambda_1^4 X_1 + 2\lambda_1^3 X_2 + \lambda_1^2 X_3 - X_5}{4X_1\lambda_1^3 + 3X_2\lambda_1^2 + 2X_3\lambda_1 + X_4} \end{aligned}$$

So from this it can be seen that

$$I_{cP(n+1)} = \frac{3I_{cPn}^4 X_1 + 2I_{cPn}^3 X_2 + I_{cPn}^2 X_3 - X_5}{4X_1 I_{cPn}^3 + 3X_2 I_{cPn}^2 + 2X_3 I_{cPn} + X_4} \tag{38}$$

4.5 Piecewise Delay Expression

When τ is in the generalized form, a piecewise delay of the expression for i_c can be divided into three non overlapping times

1) Junction Delay

This is the time between t_0 and t_1. It is the time from when the input voltage falls from V_{oh} until the base-emitter voltage (V_{be}) is equal to the diode voltage (V_{diode}) and hence Q_1 turns on. The delay expression is the same as in (2)

2) Diffusion Delay

This is the time between t_1 and t_2. t_2 is defined as the time required to provide the forward stored charge for the BJT. It is the time required for the collector current to reach its peak value (I_{cP}) at its initial rate of increase. So if we assume that at t_1, $i_c = 0$ and at t_2, $i_c = I_{cP}$ then we can find the slope of the line by using the equation,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence,

$$\frac{i_c - 0}{t - t_1} = \frac{I_{cP} - 0}{t_2 - t_1}$$

$$i_c = \frac{I_{cP}}{t_2 - t_1}(t - t_1)$$

$$i_c = \frac{I_{cP}}{t_2 - t_1}t - \frac{I_{cP}}{t_2 - t_1}t_1$$

So the slope of the the line is $\frac{I_{cP}}{t_2 - t_1}$. If we assume that between t_1 and t_2 all the MOS current (I_{mos}) is used to charge the base and that the instantaneous Q_1 transit time is a constant, we can then say

$$\frac{I_{mos}}{\tau_f} = \frac{I_{cP}}{t_2 - t_1}$$

$$t_2 - t_1 = \frac{\tau_f I_{cP}}{I_{mos}} \tag{39}$$

3) Load Delay

This is the time between t_2 and t_d. t_d is the time when the output voltage reached half the supply voltage ($V_e = \frac{V_{dd}}{2}$). Using I_{cP} derived from the Newton-Raphson method and equation (7) we obtain:

$$\begin{aligned}
\frac{dV_e}{dt} &= \frac{I_{mos} + I_{cP}}{C_b + C_L} \\
dV_e &= \frac{I_{mos} + I_{cP}}{C_b + C_L} dt \\
\int_{V_e(t_1)}^{V_e t_d} dV_e &= \frac{I_{mos} + I_{cP}}{C_b + C_L} \int_{t_2}^{t_d} dt \\
\Big[V_e \Big]_{V_e(t_1)}^{V_e t_d} &= \frac{I_{mos} + I_{cP}}{C_b + C_L} \Big[t \Big]_{t_2}^{t_d} \\
V_e(t_d) - V_e(t_1) &= \frac{I_{mos} + I_{cP}}{C_b + C_L} (t_d - t_2) \\
t_d - t_2 &= \frac{(C_b + C_L)(V_e(t_d) - V_e(t_1))}{I_{mos} + I_{cP}}
\end{aligned}$$

Then since $V_e(t_d) = \frac{V_{dd}}{2}$

$$t_d - t_2 = \frac{(C_b + C_L)(\frac{V_{dd}}{2} - V_e(t_1))}{I_{mos} + I_{cP}} \tag{40}$$

Hence,

$$t_d = t_2 + \frac{(C_b + C_L)(\frac{V_{dd}}{2} - V_e(t_1))}{I_{mos} + I_{cP}}$$

Since

$$t_2 = \frac{\tau_f I_{cP}}{I_{mos}} + t_1$$

The eqaution for the piecewise delay can be written as:

$$t_d = (t_1 - t_0) + \frac{\tau_f I_{cP}}{I_{mos}} + \frac{(C_b + C_L)(\frac{V_{dd}}{2} - V_e(t_1))}{I_{mos} + I_{cP}} \tag{41}$$

Equation (41) was used in the program to find the time delay for the BiCMOS transistor for varying MOS gate width and load capacitance. The results are shown on the Fig. 15 and 16. From Fig. 15 it can be seen that the time delay t_d decreases with $0 \leq W < 4\mu$m and increases with $W > 4\mu$m. I can also be observed that the optimum MOS gate width in terms of time delay is approximately $4\mu m$. From Fig. 16 it can be seen that the time delay increases with increasing load capacitance. Hence the optimum load capacitance in terms of time delay is the smallest capacitance possible.

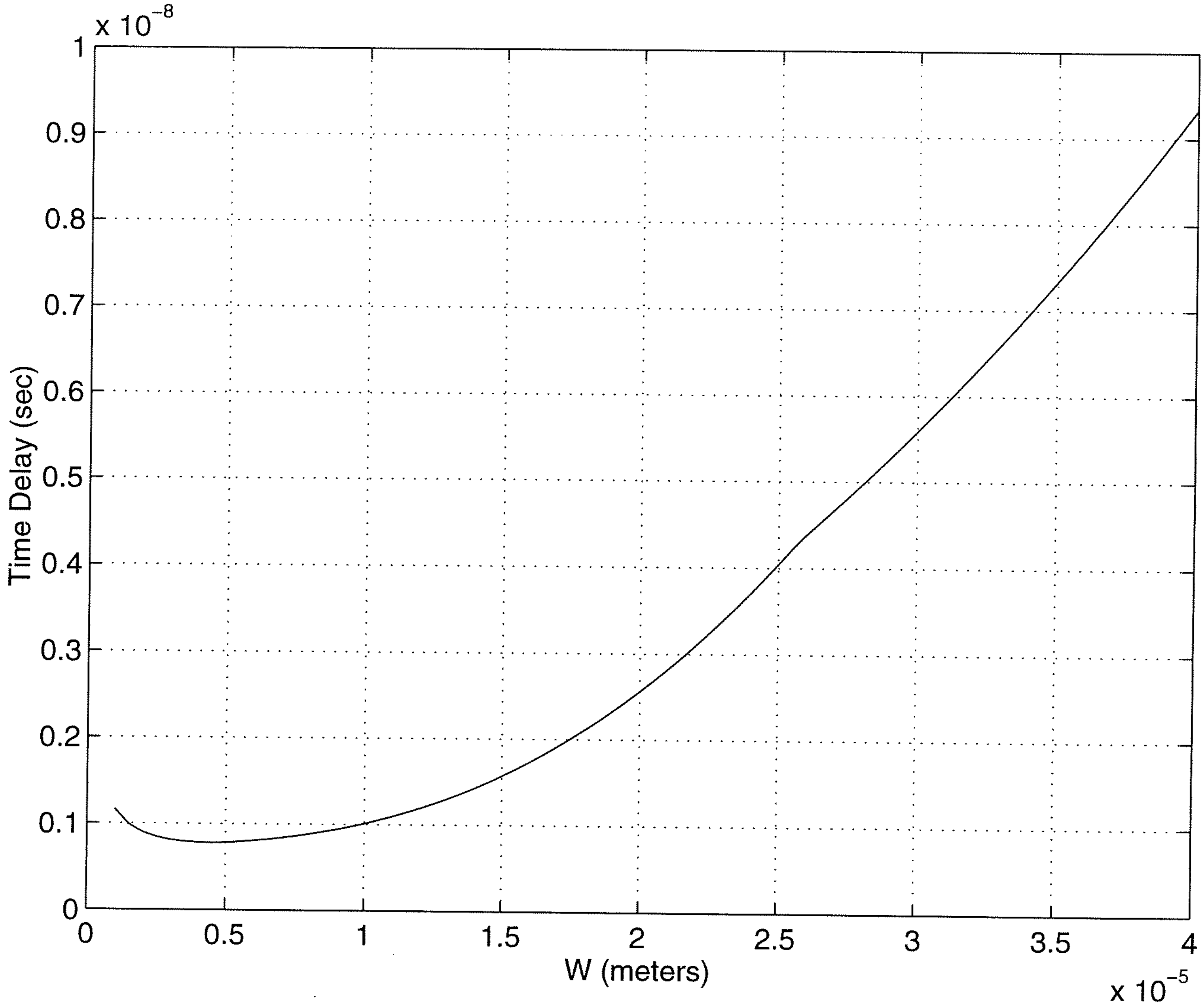

Figure 17: Graph of t_d V.s. W for Generalized τ, $C_L = 1$ pF

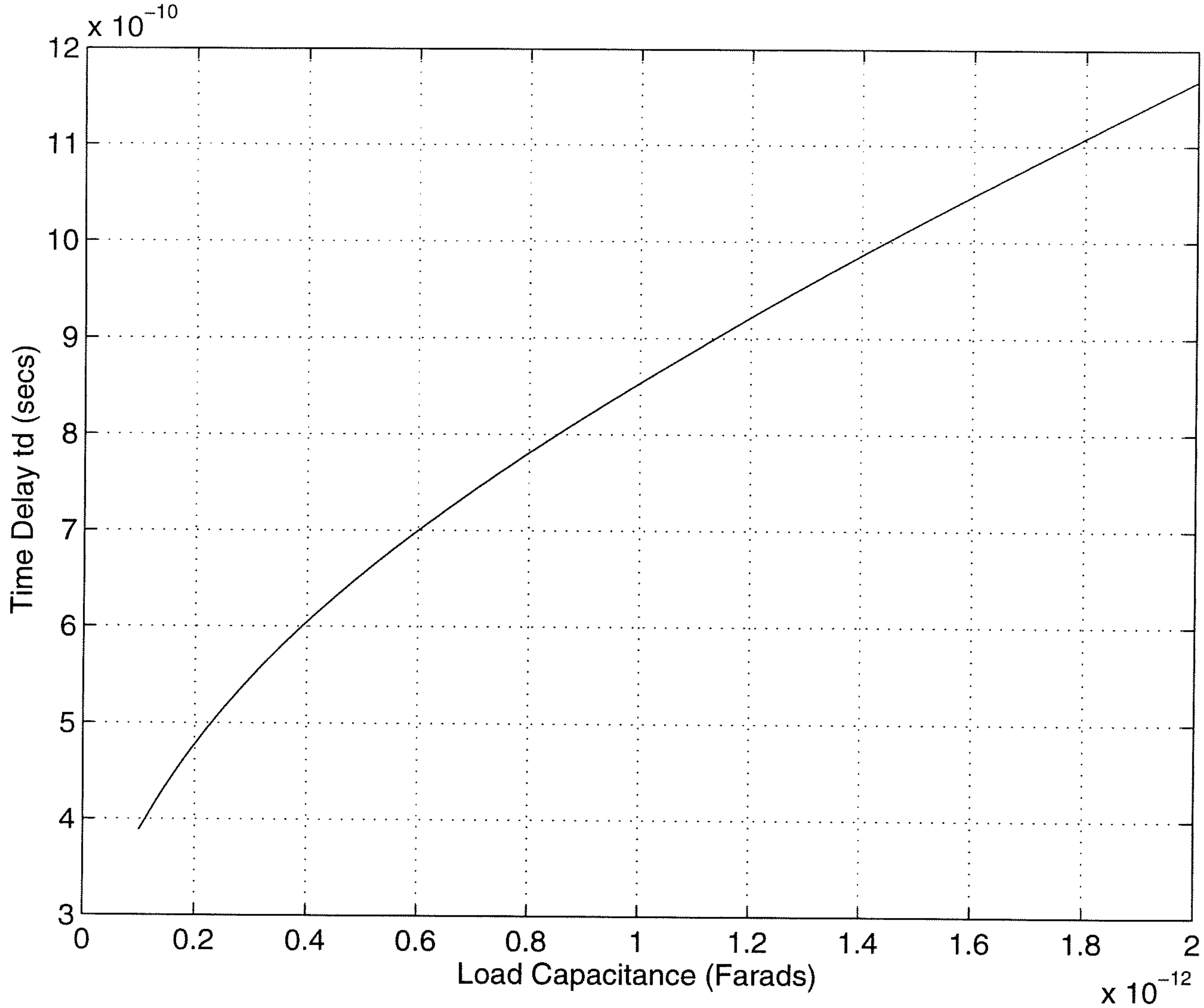

Figure 18: Graph of t_d V.s. C_L for Generalized τ, W = $4.0\mu m$

5 Discrepancies Between the Presented Analysis and [1]

5.1 Discrepancies in the Constant τ analysis

a) According to the authors of [1], for constant τ

$$i_c = \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 - \lambda_2}{2} \tanh\left(\frac{\tau + c}{\tau'}\right)$$

where $\tau' = \frac{2\tau_f \beta I_k}{(\lambda_1 - \lambda_2)}$ and $c = \left(\frac{\tau'}{2}\right) \ln\left(\frac{-\lambda_1}{\lambda_2}\right) - t_1$.

This result is not the same as the one obtained in the presented analysis. Although both i_c and τ' are the same for both, for the presented analysis $c = \left(\frac{\tau'}{2}\right) \ln\left(\frac{-\lambda_2}{\lambda_1}\right) - t_1$.

5.2 Discrepancies in the Linear τ analysis

a) In their analysis for Linear τ the authors of [1] first assume that,

$$\tau = \tau_f \left(1 + \frac{Bi_c}{I_k}\right)$$

But when substituting this τ into the differential equation for i_c ([1] page 1458 equation (9))the i_c equation is rearranged as:

$$-\tau_f \beta I_k \left(1 + \frac{2Bi_c}{I_k}\right) \frac{di_c}{dt} = (i_c - \lambda_1)(i_c - \lambda_2)$$

From this it can be seen that a different formula has been used to describe τ

$$\tau = \tau_f \left(1 + \frac{2Bi_c}{I_k}\right)$$

Hence in [1] ,

$$t = t_1 + \frac{\tau_f \beta I - k}{\lambda_1 - \lambda_2} \left[\left(1 + \frac{2\lambda_1 B}{I_k}\right) \ln\left(1 - \frac{i_c}{\lambda_1}\right) + \left(1 + \frac{2\lambda_2 B}{I_k}\right) ln\left(1 - \frac{i_c}{\lambda_2}\right)\right]$$

For the analysis in this work the original form of τ was used throughout hence the equivalent equation (33) does not have the $2B$ factor and instead has just a B.

b) For the output voltage waveform, the result in [1] is:

$$\begin{aligned} V_e &= V_e(t_1) + \frac{\tau_c \beta I_k}{(C_b + C_L)(\lambda_1 - \lambda_2)} \left(\left(1 + \frac{2\lambda_1 B}{I_k}\right) \left(-i_c - (I_{mos} + \lambda_1) \ln\left(1 - \frac{i_c}{\lambda_1}\right)\right) \right. \\ &\left. + \left(1 + \frac{2\lambda_2 B}{I_k}\right) \left(i_c + (I_{mos} + \lambda_1) \ln\left(1 - \frac{i_c}{\lambda_1}\right)\right) \right) \end{aligned}$$

It is obvious from this that again the $2B$ factor has been used.

6 Conclusion

The transient analysis of the BiCMOS circuit has been carried out using three increasingly accurate models for the base pushout effect. The differential equations for i_c and V_e have been solved for constant and linear τ. Also the peak current and time delay have been obtained from the generalized τ analysis. Finally a program was developed to use the equation obtained and output data points that could then be analyzed using MATLAB.

From the study of the t_1 V.s. C_L curve it was observed that t_1 increases with load $C_L > 0.2$pF and that a minimum t_1 can be obtained with a load capacitance of approximately 0.2pF. From the study of the t_1 V.s. W curve it was observed that t_1 increases with $W > 4\mu$m and that a t_1 can be obtained with a gate width of approximately 4μm. From the study of the i_c V.s. t curves for the constant and linear τ analysis it was observed that i_c is larger at all times for the constant τ simulation. From the study of the the V_e V.s. t curves for constant and linear τ analysis it was observed that for constant τ the slope was larger and t_d was smaller. Also from the analysis of the t_d V.s. C_L curve using generalized τ it was observed that increasing the load capacitance increases the time delay. Finally from the t_d V.s. W for generalized τ analysis it was observed the optimum gate width is approximately $4\mu m$.

Appendix

A Table of symbols

Symbol	Description
β	BJT current gain.
B	Linear Proportionality constant in τ dependence on i_c.
C_b	Sum of C_{bc} and C_p.
C_{bc}, C_{be}	Base-collector, base-emitter capacitance of Q_1.
C_{cs}	Collector-substrate capacitance of Q_1.
C_d	Diffusion capacitance of Q_1.
C_{ib}	Couplling capacitance between input and B_1.
C_{jp1}, C_{jn3}	Total drain junction capacitance of M_1, M_3, etc.
C_L	Total load capacitance at output($C_{load} + C_{pe}$).
C_{ovn}, C_{ovp}	NMOS, PMOS gate-drain overlap capacitance per unit width.
C_{ox}	MOS gate oxide capacitance per unit area.
C_p	Parasitic capacitance at Q_1 base.
C_{pe}	Parasitic capacitance at Q_1 emitter output node.
$Cjc0, C_{je0}, C_{js0}$	Base collector, base-emitter, collector-substrate zero bias depletion capacitance.
E_{critP}	Critical field velocity saturation in M_1
E_{xtraP}	Mobility reduction factor for a field along the channel of M_1
ϵ_0	Dielectric constant of a vacuum.
ϵ_{ox}	Dielectric constant of SiO_2.
i_b, i_c	Transient base, collector current in Q_1.
I_{cP}	Peak value of Q_1 transient collector current.
I_k	Knee current at Q_1.
$I_m[\,]$	MOS drain current function.
I_{mos1}	Average M_1 drain current from t_0 to t_1.
I_{mos}	Average M_1 drain current from t_1 to t_d.
I_{tf}	SPICE parameter controlling the τ dependence on current.
I_s	Saturation current for Q_1.
L_e	Emitter length of Q_1.
m_c, m_e, m_s	Base-collector, base-emitter, substrate junction grading factor.
μ_{0p}	Low-field M_1 mobility.
μ_p	M_1 mobility with field dependence included.
ϕ_c, ϕ_e, ϕ_s	Base-collector, base-emitter, substrate built in potential.

Symbol	Description
R_c, R_b	BJT collector, base resistance.
t_0	Time instant when input falls.
t_1	Time when V_{be} reaches V_{diode}.
t_2	Time when tangen to i_c at t_1 reaches I_{cP}.
t_d	Time when output reaches $\frac{V_{dd}}{2}$.
τ	Instantaneous Q_1 transit time, function of i_c, V_{bc}.
τ_f	Low current transit time of Q_1 without base pushout.
τ_F	Forward transit time for Q_1.
τ_R	Reverse transit time for Q_1.
V_b, V_e	Q_1 base, emitter voltage.
V_{bc}, V_{be}	Q_1 base-collector, base-emitter voltage.
V_{dd}	Supply voltage.
V_{diode}	Q_1 turn-on voltage.
V_{dpSAT}	M_1 drain saturation voltage.
V_{ih}, V_{il}	Input high, low voltage.
V_{normP}	Mobility reduction factor for a field perpendicular to the channel.
V_{ol}	Initial output low voltage.
V_{sd}, V_{sg}	M_1 source-drain, source-gate bias(positive quantities).
V_{tf}	SPICE parameter controlling τ dependence on V_{bc}.
V_{tp}	M_1 threshold voltage.
V_T	Q_1 thermal voltage.
X_{tf}	SPICE parameter controlling τ dependence on i_c, V_{bc}.
W_1, W_2, etc	Gate width of MOSFET M_1, M_2, etc.

B Parasitic Capacitances

The parasitic capacitance at the base of B_1 and Q_1 comes from the gate-drain overlap and drain junction capacitance of M_1 and M_3 and the gate capacitance of M_4. Thus

$$C_p = C_{ovp}W_1 + C_{ovn}W_3 + C_{jp1} + C_{jn3} + 2C_{ovn}W_4 \tag{1}$$

The parasitic capacitance at the emitter E_1 and Q_1 is dure to the gate-drain overlap and drain-juncton capacitance of M_2, the collector-base and collector-substrate capacitance of Q_2.

$$C_{pe} = C_{ovn}W_2 + C_{jn2} + C_{bc2} + C_{cs2} \tag{2}$$

From [5] it can be seen that

$$C_{bc} = \frac{\tau_R I_s}{V_T} e^{\frac{V_{bc}}{V_T}} + \frac{C_{jc0}}{\left[1 - \left(\frac{V_{bc}}{\phi_c}\right)\right]^{m_c}} \tag{3}$$

and

$$C_{cs} = \frac{C_{js0}}{\left[1 - \left(\frac{V_{cs}}{\phi_s}\right)\right]^{m_s}} \tag{4}$$

The junction capacitance in the above equations have implicit voltage dependencies. The bias at which the capacitance is evaluated is the mean voltage value in the relevant time period.

C MOSFET Current

The drain current of M_1 accounts for mobility reduction due to the vertical and transverse electric fields (parameters V_{normP} and E_{traP}) and saturation voltage reduction due to the velocity saturation (parameter E_{critP}). I_m is expressed as a function of the source-gate (V_{sg}) and source-drain voltage (V_{sd}).

$$\begin{aligned}
I_m[V_{sg}, V_{sd}] &= \mu_p C_{ox} \frac{W}{L} \left(V_{sg} - V_{tp} - V_{sd-eff}\right) V_{sd-eff} \\
\mu_p &= \mu_{0p} \frac{1}{1 + \left(\frac{V_{sg} - V_{tp}}{V_{normP}}\right)} \cdot \frac{1}{1 + \frac{V_{dpSAT}}{E_{xtraP} L}} \\
V_{sd-eff} &= \min[V_{sd}, V_{dpSAT}] \\
V_{dpSAT} &= (V_{sg} - V_{tp}) + E_{critP} L - \sqrt{(V_{sg} - V_{tp})^2 + (E_{critP} L)^2}
\end{aligned} \tag{5}$$

D V_b and V_e at t_0 and t_1

When there is an abrupt fall in the input, the capacitance ($C_{ib} = C_{ovp}W_1 + C_{ovn}W_3 + 0.5C_{ox}W_3L$) between input and the base (B_1) causes B_1 to fall. The net capacitance at B_1 is $C_b = C_p + C_L$ which is parallel with the series combination of C_{be} and C_L. Where C_{be} is defined in [5] as:

$$C_{be} = \frac{\tau_F I_s}{V_T} e^{\frac{V_{be}}{V_T}} + \frac{C_{je0}}{\left[1 - \left(\frac{V_{be}}{\phi_e}\right)\right]^{m_e}} \tag{6}$$

Therefore,

$$V_b(t_0) = (V_{il} - V_{ih}) \frac{C_{ib}}{C_b + \frac{C_{be}C_L}{C_{be}+C_L}} \tag{7}$$

As a result of the "bump" at B, the output gets bumped to

$$V_e(t_0) = V_{ol} + V_b(t_0) \frac{C_{be}}{C_{be} + C_L} \tag{8}$$

When the input transition is not abrupt, $V_b(t_0)$and $V_e(t_0)$ are simply the initial values of 0 and V_{ol}. The change in V_e from t_0 to t_1 is realted to the change in V_be during the same period by the ratio of the impedance at the output to that across the base-emitter juntion. At the instant $t = t_1$, the voltage across the base-emitter capacitor is V_{diode}. The voltage output is therefore

$$V_e(t_1) = V_e(t_0) + (V_{diode} - (V_B(t_0) - V_e(t_0))) \frac{C_{be}}{C_L} \tag{9}$$

$$V_b(t_1) = V_e(t_1) + V_{diode} \tag{10}$$

E The Program

Table of constants used in program

Symbol	**Constant name**	**Value**
L	L	5.0×10^{-6}
W	W	5.0×10^{-6}
C_{ox}	COX	1.40538×10^{-7}
μ_{0p}	UOP	250.0
V_{normP}	VNORMP	5.0×10^{4}
$1/E_{xtraP}$	IEXTRAP	0
E_{critP}	ECRITP	1.0×10^{6}
V_{il}	VIL	1.5
V_{ih}	VIH	2.88
V_{ol}	VOL	0.7
V_{diode}	VDIODE	0.7
C_{be}	CBE	16.196×10^{-15}
C_{bc}	CBC	7.92×10^{-15}
C_{cs}	CCS	80.0×10^{-15}
C_{ovn}, C_{ovp}	COV	350.0×10^{-12}
$C_{je0}, C_{jc0}, C_{js0}$	CJ	0.33×10^{-3}
V_{tp}	VTP	-0.8
I_{tf}	ITF	40.0×10^{-3}
V_{tf}	VTF	3.0
X_{tf}	XTF	100.0
C_L	CL	1.0×10^{-12}
τ_f	TAUF	20.0×10^{-12}
β	BETA	100.0
B	B	1.5
I_k	IK	2.0×10^{-3}
t_0	T0	0
V_{bc}	VBC	-4.3
V_{dd}	VDD	5.0

V_{il} and V_{ih} were both obtained using SPICE simulation. The SPICE .out file used to determine V_{il} and V_{ih} is included below.

```
**** 04/09/97 20:10:45 ******* Win32s Evaluation PSpice (April 1995) *****

 * INVERTER OUTPUT


 ****     CIRCUIT DESCRIPTION


******************************************************************************




M1 3 1 2 2 MOD1 L=10U W=20U
M2 3 1 0 0 MOD2 L=10U W=20U
VDD 2 0 5
VIN 1 0  PWL(0 0 0.5US 0 1US 5 )

.MODEL MOD1 PMOS VTO=-0.8 KP=32.87E-6 GAMMA=1.36 PHI=0.6
+ LAMBDA=1.605E-2 CGSO=350E-12 CGDO=350E-12 RSH=25 CJ=0.33E-3
+ MJSW=0.5 TOX=2.50E-8 NSUB=5.0E16 NSS=1.0E10
+ TPG=-1 XJ=0.2E-4 LD=0.15E-4 U0=250

.MODEL MOD2 NMOS VTO=0.8 KP=32.87E-6 GAMMA=1.36 PHI=0.6
+ LAMBDA=1.605E-2 CGSO=350E-12 CGDO=350E-12 RSH=25 CJ=0.33E-3
+ MJSW=0.5 TOX=2.50E-8 NSUB=1.0E16 NSS=1.0E10
+ TPG=-1 XJ=0.2E-4 LD=0.15E-4 U0=500



.probe
.TRAN 0.0001US 1US
.END
```

```
**** 04/09/97 20:10:45 ******* Win32s Evaluation PSpice (April 1995) *****

 * INVERTER OUTPUT

 ****     MOSFET MODEL PARAMETERS

******************************************************************************

               MOD1             MOD2
               PMOS             NMOS
       LEVEL    1                1
         TPG   -1               -1
           L  100.000000E-06   100.000000E-06
           W  100.000000E-06   100.000000E-06
          LD   15.000000E-06    15.000000E-06
         VTO   -.8               .8
          KP   32.870000E-06    32.870000E-06
       GAMMA    1.36             1.36
         PHI     .6               .6
      LAMBDA     .01605           .01605
         RSH   25               25
          IS   10.000000E-15    10.000000E-15
        PBSW     .8               .8
          CJ  330.000000E-06   330.000000E-06
        MJSW     .5               .5
        CGSO  350.000000E-12   350.000000E-12
        CGDO  350.000000E-12   350.000000E-12
        NSUB   50.000000E+15    10.000000E+15
         NSS   10.000000E+09    10.000000E+09
         TOX   25.000000E-09    25.000000E-09
          XJ   20.000000E-06    20.000000E-06
         VFB    0                0
          K1    0                0
          K2    0                0
          U0  250              500
        TEMP    0                0
         VDD    0                0
       XPART    0                0
```

```
**** 04/09/97 20:10:45 ******* Win32s Evaluation PSpice (April 1995) *****

 * INVERTER OUTPUT

 ****     INITIAL TRANSIENT SOLUTION       TEMPERATURE =   27.000 DEG C

****************************************************************************

 NODE   VOLTAGE     NODE   VOLTAGE     NODE   VOLTAGE     NODE   VOLTAGE

(    1)    0.0000  (    2)    5.0000  (    3)    5.0000


    VOLTAGE SOURCE CURRENTS
    NAME          CURRENT

    VDD          -5.010E-12
    VIN           0.000E+00

    TOTAL POWER DISSIPATION   2.51E-11  WATTS


          JOB CONCLUDED

          TOTAL JOB TIME            7.52
```

```
C PROGRAM:  delay_models.f
C AUTHOR:   Beshara Elmufdi
C DATE:     4/9/97

      REAL*8 W, L, COX, CBE, CBC, CCS, COV, CJ, CL
      REAL*8 VDIODE, VIL, VIH, VOL, VTP, VTF, VBC, VDD
      REAL*8 VNORMP, ITF, IEXTRAP, IK, ECRITP, XTF, TO
      REAL*8 BETA, B, TAUF, UOP

      REAL*8 CP, CPE, CIB, CB, VBTOB, VBT1B, VETOB
      REAL*8 VET1B, IMB1, IMB2, IMOS1, T1, IMOS
      REAL*8 L1, L2, C, TPR



      L = 5.0E-6
      W = 5.0E-6
      COX = 1.40538E-7
      UOP = 250.0
      VNORMP = 5.0E4
      IEXTRAP = 0
      ECRITP = 1.0E6
      VIL = 1.5
      VIH = 2.88
      VOL = 0.7
      VDIODE = 0.7
      CBE = 16.196E-15
      CBC = 7.92E-15
      CCS = 80.0E-15
      COV = 350.0E-12
      CJ  = 0.33E-3
      VTP = -0.8
      ITF = 40.0E-3
      VTF = 3.0
      XTF = 100.0
      CL = 1.0E-12
      TAUF = 20.0E-12
```

```
      BETA = 100.0
      B = 1.5
      IK = 2.0E-3
      T0 = 0
      VBC = -4.3
      VDD = 5.0

      CALL TODATA(W, L, CL, COV, CJ, CBE,
     +  CBC, CCS, COX, VIH, VIL, VOL, VDIODE,
     +  VDD, VTP, VNORMP, ECRITP, IEXTRAP,
     +  UOP, T0, CP, CPE, CIB, CB, VBT0B,
     +  VBT1B, VET0B, VET1B, IMOS1, T1)

      CALL IM(VDD - VIL, VDD - VBT1B, COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB1)
      CALL IM(VDD - VIL, VDD - (VDD/2.0 + VDIODE), COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB2)

      IMOS = 0.5*(IMB1 + IMB2)

      CALL LDA(IK, BETA, CB, CL, IMOS,L1, L2)

      CALL TPRIME(L1, L2, TAUF, BETA, IK, TPR)

      CALL CFACT(L1, L2, TPR, T1, C)
      CALL CONSTICVE(L1, L2, T1, TAUF, BETA, IK,
     + IMOS, C, TPR,VET1B, CB, CL, VDD)

      CALL LINTVE(L1, L2, T1, BETA, TAUF, B,
     +  IK, VET1B, CB, CL, VDD)

      CALL TMDLYW( COV, CJ, CBC, CBE, CCS, COX,
     +  CL, VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP,
     +  ECRITP, IEXTRAP, UOP, T0, XTF, VTF, ITF, VBC,
     +  BETA, IK, TAUF)

      CALL TMDLYCL( COV, CJ, CBC, CBE, CCS, COX,
     +  VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP,
     +  ECRITP, IEXTRAP, UOP, T0, XTF, VTF, ITF, VBC,
```

```
     +  BETA, IK, TAUF)

      CALL TMDWCL( COV, CJ, CBC, CBE, CCS, COX,
     +  VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP,
     +  ECRITP, IEXTRAP, UOP, T0, XTF, VTF, ITF, VBC,
     +  BETA, IK, TAUF)

      STOP
      END


C     **********************************************************************
C     **********************************************************************
C     ******************                            **************************
C     ******************     FUNCTIONS              **************************
C     ******************                            **************************
C     **********************************************************************
C     **********************************************************************


C     **********************************************************************
C     Subroutine to calculate all inital capacintances, voltages and
C     currents needed to calculate t1.
C     **********************************************************************

      SUBROUTINE T0DATA(W, L, CL, COV, CJ, CBE,
     +  CBC, CCS, COX, VIH, VIL, VOL, VDIODE,
     +  VDD, VTP, VNORMP, ECRITP, IEXTRAP,
     +  UOP, T0, CP, CPE, CIB, CB, VBT0B,
     +  VBT1B, VET0B, VET1B, IMOS1, T1)


      REAL*8  W, L, CL, COV, CJ, CBE, CBC, CCS, COX
      REAL*8  VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP
      REAL*8  ECRITP, IEXTRAP, UOP, T0

      REAL*8 CP, CPE, CIB, CB, VBT0B, VBT1B, VET0B, VET1B
      REAL*8 IMB1, IMB2, IMOS1, T1
```

```
      CALL PARACAPS(W, L, COV, CJ, CBC, CCS,
     +   COX, CP, CPE, CIB, CB)

      CALL VBCONST(VIL, VIH, VOL, VDIODE, CIB, CB, CBE, CL,
     + VBT0B, VBT1B, VET0B, VET1B)

      CALL IM(VDD - VIL, VDD - VBT1B, COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB1)

      CALL IM(VDD - VIL, VDD - VBT1B, COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB2)

      IMOS1 = 0.5*(IMB1 + IMB2)

      CALL IM(VDD - VIL, VDD - (VDD/2.0 + VDIODE), COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB2)

      CALL T1CONST(CB, CL, CBE, T0, IMOS1, VBT0B, VBT1B, T1)

      RETURN
      END


C     ************************************************************************
C     Subroutine to find the parasitic capacitances for the inverter due to
C     the total drain capacitance, the gate-drain overlap capacitance and
C     the gate oxide capacitance of M1 and M3, and the base-collector and
C     collector-substrate capacitances of Q1
C     ************************************************************************

      SUBROUTINE PARACAPS(W, L, COV, CJ, CBC, CCS,
     +                    COX, CP, CPE, CIB, CB)

      REAL*8  W, L, COV, CJ, CBC, CCS, COX
      REAL*8  CP, CPE, CIB, CB

      CP = COV*W + COV*W + CJ*W*L + CJ*W*L + 2.0*COV*W
      CPE = COV*W + CJ*W*L + CBC + CCS
      CIB = COV*W + COV*W + 0.5*COX*W*L
```

```
      CB = CP + CBC

      RETURN
      END


C     *******************************************************************
C     Subroutine to find the Q1 base and emitter voltages at times t0 and
C     t1.
C     *******************************************************************

      SUBROUTINE VBCONST(VIL, VIH, VOL, VDIODE, CIB, CB, CBE, CL,
     + VBT0B, VBT1B, VET0B, VET1B)

      REAL*8 VIL, VIH, VOL, VDIODE, CIB, CB, CBE, CL
      REAL*8 VBT0B, VBT1B, VET0B, VET1B

      VBT0B = (VIL - VIH)*(CIB/(CB + (CBE*CL/(CBE + CL))))
      VET0B = VOL + VBT0B*(CBE/(CBE + CL))
      VET1B = VET0B + (VDIODE - (VBT0B - VET0B))*(CBE/CL)
      VBT1B = VET1B + VDIODE

      RETURN
      END



C     *******************************************************************
C     Subroutine to calculate the MOS drain current.
C     *******************************************************************

      SUBROUTINE IM(VSG, VSD, COX, W, L, VTP, UOP,
     + ECRITP, IEXTRAP, VNORMP, IMB)

      REAL*8 VSD, VSG, COX, W, L, VTP, UOP
      REAL*8 ECRITP, IEXTRAP, VNORMP
      REAL*8 UP, VSDEFF, VDPSAT
      REAL*8 IMB
```

```
      VDPSAT = (VSG - VTP) + ECRITP*L - ((VSG - VTP)**2. +
     + (ECRITP*L)**2.)**0.5

      IF(VSD .LT. VDPSAT) THEN
       VSDEFF = VSD
      ELSE
        VSDEFF = VDPSAT
      ENDIF
      UP = UOP*(1./(1. + ((VSG - VTP)/VNORMP)))*
     + (1./(1. + (VDPSAT*IEXTRAP/L)))

      IMB = UP*COX*(W/L)*(VSG - VTP - VSDEFF)*VSDEFF

      RETURN
      END


C     *******************************************************************
C     Subroutine to calculate the time when VBE reaches VDIODE (T1)
C     *******************************************************************

      SUBROUTINE T1CONST(CB, CL, CBE, T0, IMOS1, VBT0, VBT1, T1CONSTB)

      REAL*8 CB, CL, CBE, T0, IMOS1, VBT1, VBT0
      REAL*8 T1CONSTB

      T1CONSTB = T0 + (CB + (CBE*CL/(CBE + CL)))*((VBT1 - VBT0)/IMOS1)
      RETURN
      END



C     *******************************************************************
C     Subroutine to calculate the lambda factor2, lambda1 and lambda2,
C     for the ic differential equation.
C     *******************************************************************
```

```
      SUBROUTINE LDA(IK, BETA, CB, CL, IMOS, LDA1B, LDA2B)

      REAL*8 LDA1B,LDA2B
      REAL*8 IK, BETA, CB, CL, IMOS
      REAL*8 D1, D2, D3

      D1 = (1. + ((BETA * CB)/(CB + CL)))*(IK)
      D2 = (4.*IMOS*BETA*IK*CL)/(CB + CL)
      D3 = D1**2. + D2
      D3 = SQRT(D3)
      D1 = -D1
      LDA1B = 0.5*(D1 + D3)
      LDA2B = 0.5*(D1-D3)
      RETURN
      END


C     *****************************************************************
C     Subroutine to calculate the tau prime factor for the ic
C     differential equation.
C     *****************************************************************
      SUBROUTINE TPRIME(L1, L2, TAUF, BETA, IK,TPRIMEB)

       REAL*8 TPRIMEB
       REAL*8 TAUF, BETA, IK, L1, L2

       TPRIMEB = (2.*TAUF*BETA*IK)/(L1 - L2)
       RETURN
       END


C      ****************************************************************
C      Subroutine to calculate the c factor for the ic differential
C      eqation.
C      ****************************************************************

       SUBROUTINE CFACT(L1, L2, TPR, T1,CFACTB)
```

```
      REAL*8 CFACTB
      REAL*8 TPR, L1, L2, T1, C1

      C1 = TPR/2.
      C1 = C1*LOG(-L2/L1)
      CFACTB = C1 - T1
      RETURN
      END




C     **************************************************************
C     Subroutine to calculate various values of ic and Ve for
C     constant tau. The output is written to a matlab script file
C     **************************************************************
      SUBROUTINE CONSTICVE(L1, L2, T1, TAUF,
     + BETA, IK, IMOS, C, TPR,VET1B, CB,
     + CL, VDD)

      REAL*8 L1, L2, T1, TAUF, BETA, IK
      REAL*8 IMOS, C, TPR,VET1B, CB, CL, VDD

      REAL*8 T, ICCONSTB, ICC2B, VECONSTB


      OPEN(100, FILE='const_ic.m')
      OPEN(101, FILE='const_ve.m')

      WRITE(100,*) '% const_ic.m'
      WRITE(100,*) '% DATE FOR IC USING CONSTANT TAU ANALYSIS'

      WRITE(100,*) 'Data = ['


      WRITE(101,*) '% const_ve.m'
      WRITE(101,*) '% DATE FOR VE USING CONSTANT TAU ANALYSIS'
```

```
      WRITE(101,*) 'Data = ['

14    FORMAT(E11.3, ',', 3X, E15.6)
      DO 12 T = T1, 1.E-8, 1.0E-11
          CALL ICC2(L1, L2, T, T1, TAUF, BETA, IK, ICC2B)
          CALL ICCONST(L1, L2, T, C, TPR,ICCONSTB)
          CALL VECONST(VET1B, CB, CL, L1, L2, TPR, C, T, T1,
     +                 IMOS, TAUF, BETA, IK, VECONSTB)
          WRITE(100,14) T, ICCONSTB
          WRITE(101,14) T, VECONSTB
          IF( VECONSTB .GT. (VDD/2.0)) GOTO 13
12    CONTINUE
13    CONTINUE

      WRITE(100,*) ']'
      WRITE(100,*) ' '
      WRITE(100,*) 't = Data(:,1)'
      WRITE(100,*) 'ic = Data(:,2)'
      WRITE(100,*) ' '

      WRITE(101,*) ']'
      WRITE(101,*) ' '
      WRITE(101,*) 't = Data(:,1)'
      WRITE(101,*) 've = Data(:,2)'
      WRITE(101,*) ' '

      CLOSE(101)
      CLOSE(100)

      RETURN
      END



C     ***********************************************************
C     Subroutine to calculate the collector current for Q1 (ic) for
C     constant tau using the tau prime and c factors previously
```

```
C       computed.
C       ***************************************************************

        SUBROUTINE ICCONST(L1, L2, T, C, TPR, ICCONSTB)

        REAL*8 ICCONSTB
        REAL*8 L1, L2, T, C, TPR

        ICCONSTB = ((L1 - L2)/2.)*TANH((T + C)/TPR)
        ICCONSTB = ((L1 + L2)/2.) + ICCONSTB
        RETURN
        END


C       ***************************************************************
C       Subroutine to calculate the collector current for Q1 (ic) for
C       constant tau without using the tau prime and c factors
C       previously computed.
C       ***************************************************************

        SUBROUTINE ICC2(L1, L2, T, T1, TAUF, BETA, IK, ICC2B)

        REAL*8 ICC2B
        REAL*8 L1, L2, T, T1, TAUF, BETA, IK
        REAL*8 X

        X = ((T - T1)*(L1 - L2))/(TAUF*BETA*IK)
        ICC2B = (L1*L2)*(1 - EXP(-X))/(L2 - L1*EXP(-X))
        RETURN
        END


C       ***************************************************************
C       Subroutine to calculate the Q1 emitter voltage (Ve) for
C       constant tau.
C       ***************************************************************
```

```
      SUBROUTINE VECONST(VET1B, CB, CL, L1, L2, TAUPR, C, T, T1,
     +                   IMOS, TAUF, BETA, IK, VECONSTB)

      REAL*8 VECONSTB
      REAL*8 VET1B, CB, CL, T, T1, IMOS, L1, L2, TAUPR, C
      REAL*8 TAUF, BETA, IK
      REAl*8 D1, D2, D3

      D1 = (IMOS + (L1 + L2)/2)*(T - T1)
      D2 = TAUF*BETA*IK
      D3 = LOG((COSH( (T+C)/TAUPR ))/(COSH( (T1 + C)/TAUPR )))

      VECONSTB = VET1B + (1/(CB + CL))*( D1 + D2*D3)

      RETURN
      END


C     ****************************************************************
C     Subroutine to calculate various values of t and Ve for
C     linear tau. The output is written to a matlab script file
C     ****************************************************************

      SUBROUTINE LINTVE(L1, L2, T1, BETA, TAUF, B,
     +  IK, VET1B, CB, CL, VDD)


      REAL*8  L1, L2, T1, BETA, TAUF, B
      REAL*8  IK, VET1B, CB, CL, VDD

      REAL*8  IC, TLINB, VELINB


      OPEN(200, FILE='lin_ic.m')
      OPEN(201, FILE='lin_ve.m')

      WRITE(200,*) '% lin_ic.m'
```

```
      WRITE(200,*) '% DATA FOR IC USING LINEAR TAU ANALYSIS'

      WRITE(200,*) 'Data = ['


      WRITE(201,*) '% const_ve.m'
      WRITE(201,*) '% DATA FOR VE USING LINEAR TAU ANALYSIS'

      WRITE(201,*) 'Data = ['

4     FORMAT(E14.5, ',', 3X, E12.3)
6     FORMAT(E14.5, ',', 3X, E15.6)


      DO 3 IC = 0, 1, 1.E-4
            CALL TLIN(L1, L2, T1, BETA, TAUF, B, IK, IC, TLINB)
            CALL VELIN(L1, L2, VET1B, BETA, IK, B, IC, T1, TAUF,
     +      CB, CL, VELINB)
            WRITE (200,4)  TLINB, IC
            WRITE(201,4)   TLINB, VELINB
            IF((VDD/2.0) - VELINB) 5, 3, 3
3     CONTINUE
5     CONTINUE

      WRITE(200,*) ']'
      WRITE(200,*) ' '
      WRITE(200,*) 't = Data(:,1)'
      WRITE(200,*) 'ic = Data(:,2)'
      WRITE(200,*) ' '

      WRITE(201,*) ']'
      WRITE(201,*) ' '
      WRITE(201,*) 't = Data(:,1)'
      WRITE(201,*) 've = Data(:,2)'
      WRITE(201,*) ' '

      CLOSE(201)
      CLOSE(200)
```

```
      RETURN
      END


C     ************************************************************
C     Subroutine to calculate the time (t) at which the Q1 collector
C     current (ic) equals a particular value, using a linear tau
C     analysis.
C     ************************************************************

      SUBROUTINE TLIN(L1, L2, T1, BETA, TAUF, B, IK, IC, TLINB)

      REAL*8 TLINB
      REAL*8 T1, BETA, TAUF, L1, L2, B, IK, IC
      REAL*8 D1, D2, D3

      D1 = (1. + (B*L1/IK))*LOG(1 - IC/L1)
      D2 = (1. + (B*L2/IK))*LOG(1 - IC/L2)
      D3 = (TAUF*BETA*IK)/(L1 - L2)
      TLINB = T1 - D3*(D1 - D2)
      RETURN
      END


C     ************************************************************
C     Subroutine to calculate the Q1 emitter voltage (Ve) for
C     linear tau.
C     ************************************************************

      SUBROUTINE VELIN(L1, L2, VET1B, BETA, IK, B, IC, T1, TAUF,
     + CB, CL, VELINB)

      REAL*8 VELINB
      REAL*8 VET1B, BETA, IK, L1, L2, B, IC, T1, TAUF, CB, CL
      REAL*8 D1, D2, D3

      D1 = (1. + (L2*B/IK))*(IC + (L2 + IMOS)*LOG(1 - (IC/L2)) )
```

```
      D2 = (1. + (L1*B/IK))*(-IC - (L1 + IMOS)*LOG(1 - (IC/L1)) )
      D3 = (TAUF*BETA*IK)/((L1 - L2)*(CB + CL))

      VELINB = VET1B + D3*(D2 + D1)
      RETURN
      END


C     ************************************************************
C     Subroutine to calculate the peak Q1 collector current (Icp)
C     using a generalized tau and the Newton_Raphson Iteration
C     technique.
C     ************************************************************

      SUBROUTINE NEWTON(SEED, TAUF, XTF, VTF, ITF, VBC, CB, CL, BETA,
     +                  IK, IMOS, NEWTONB)

      REAL*8 NEWTONB
      REAL*8 SEED, TAUF, XTF, VTF, ITF, VBC, CB, CL, BETA
      REAL*8 IK, IMOS
      REAL*8 X1, X2, X3, X4, X5, ICPOLD, ICPNEW, N

      X1 = ((TAUF*XTF* EXP( VBC/(1.44*VTF) )) / (1.44*VTF)) *
     +     (1.0/(CB+CL))

      X2 = X1*IMOS + (2.0*ITF)/(BETA*IK) + (1.0/BETA) + (CB/(CB+CL))

      X1 = X1 + (1.0/(BETA*IK))

      X3 = ((ITF**2.0)/(BETA*IK)) + (2.0*ITF*( (1.0/BETA) +
     +     (CB/(CB+CL)))) - (IMOS*CL)/(CB+CL)

      X4 = (ITF**2.0)*((1.0/BETA) + (CB/(CB+CL))) - (2.0*ITF*IMOS*CL)/
     +     (CB+CL)
      X5 = -((ITF**2.0)*IMOS*CL)/(CB+CL)

      ICPOLD = SEED
      DO 10 N = 1, 100
```

```
        FX = F(X1, X2, X3, X4, X5, ICPOLD)
        FPRIMEX = FPRIME(X1, X2, X3, X4, ICPOLD)
        ICPNEW = ICPOLD - (FX/FPRIMEX)
        IF(ABS(ICPOLD - ICPNEW) .LT. 1E-12) GOTO 20
        ICPOLD = ICPNEW

 10   CONTINUE
 20   CONTINUE

      NEWTONB = ICPNEW

      RETURN
      END



C     ************************************************************
C     Function to calculate the value of a linear function of degree
C     4 with values X1, X2, X3, X4 and X5 and variable ICP
C     ************************************************************
      FUNCTION F(X1, X2, X3, X4, X5, ICP)

      REAL*8 X1, X2, X3, X4, X5, ICP

      F = X1*ICP**4 + X2*ICP**3 + X3*ICP**2 + X4*ICP + X5
      RETURN
      END



C     ************************************************************
C     Function to calculate the derivative of a linear function of
C     degree 4 with values X1, X2, X3 and X4 and variable ICP
C     ************************************************************

      FUNCTION FPRIME(X1, X2, X3, X4, ICP)
```

```
      REAL*8 X1, X2, X3, X4, ICP

      FPRIME = 4*X1*ICP**3 + 3*X2*ICP**2 + 2*X3*ICP + X4
      RETURN
      END




C     ************************************************************
C     Subroutine to calculate the time (td) for the Q1 emitter
C     voltage (Ve) reaches Vdd/2, using the previously calculated
C     Q1 collector current for generalized tau (Icp) .
C     ************************************************************

      SUBROUTINE OPTDELAY(T1, T0, TAUF, ICP, IMOS, CB,
     +                    CL, VDD, VET1B, TD)

      REAL*8 TD
      REAL*8 T1, T0, TAUF, ICP, IMOS, CB, CL, VDD, VET1B


      TD = T1 - T0 + (TAUF*ICP)/(IMOS) +
     +  (CB+CL)*(VDD/2 - VET1B)/(IMOS + ICP)

      RETURN
      END



C     ************************************************************
C     Subroutine to calculate the Q1 collector current for
C     generalized tau (Icp) and the time (td) needed for the Q1
C     emitter voltage (Ve) to reach Vdd/2, for various MOS gate
C     widths (W). The output is written to a matlab script file.
C     ************************************************************
```

```
      SUBROUTINE TMDLYW( COV, CJ, CBC, CBE, CCS, COX,
     +  CL, VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP,
     +  ECRITP, IEXTRAP, UOP, TO, XTF, VTF, ITF, VBC,
     +  BETA, IK, TAUF)

      REAL*8  COV, CJ, CBC, CCS, COX
      REAL*8  CL, VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP
      REAL*8  ECRITP, IEXTRAP, UOP, TO, XTF, VTF, ITF, VBC
      REAL*8  BETA, IK, TAUF

      REAL*8 CP, CBE, CIB, CB, VBTOB, VBT1B, VETOB, VET1B
      REAL*8 IMB1, IMB2, IMOS1, IMOS, T1, L1, L2
      REAL*8 NEWTONB, TD, W, L

      OPEN(300, FILE='gen_w.m')
      OPEN(350, FILE='t1_w.m')

      WRITE(300,*) '% gen_w.m'
      WRITE(300,*) '% DATA FOR TD USING GENRALIZED TAU ANALYSIS'
      WRITE(300,*) '% AND VARYING W'

      WRITE(300,*) 'Data = ['

      WRITE(350,*) '% t1_w.m'
      WRITE(350,*) '% DATA FOR T1 USING GENRALIZED TAU ANALYSIS'
      WRITE(350,*) '% AND VARYING W'

      WRITE(350,*) 'Data = ['

40    FORMAT(E11.3, ',', 3X, E15.6)

c      L = 2.0E-6

      DO 50 W = 1.0E-6, 40.0E-6, 5.0E-7

      L = 0.5*W

      CALL TODATA(W, L, CL, COV, CJ, CBE,
     +  CBC, CCS, COX, VIH, VIL, VOL, VDIODE,
```

```
     +  VDD, VTP, VNORMP, ECRITP, IEXTRAP,
     +  UOP, TO, CP, CPE, CIB, CB, VBTOB,
     +  VBT1B, VETOB, VET1B, IMOS1, T1)

      CALL IM(VDD - VIL, VDD - VBT1B, COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB1)

      CALL IM(VDD - VIL, VDD - (VDD/2.0 + VDIODE), COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB2)

      IMOS = 0.5*(IMB1 + IMB2)

      CALL LDA(IK, BETA, CB, CL, IMOS, L1, L2)

      CALL NEWTON(L1, TAUF, XTF, VTF, ITF, VBC, CB, CL, BETA,
     +            IK, IMOS, NEWTONB)

      CALL OPTDELAY(T1, TO, TAUF, NEWTONB, IMOS, CB,
     +                    CL, VDD, VET1B, TD)

      WRITE(300,40) W, TD
      WRITE(350,40) W, T1

50    CONTINUE

      WRITE(300,*) ']'
      WRITE(300,*) ' '
      WRITE(300,*) 'W = Data(:,1)'
      WRITE(300,*) 'td = Data(:,2)'
      WRITE(300,*) ' '

      CLOSE(300)


      WRITE(350,*) ']'
      WRITE(350,*) ' '
      WRITE(350,*) 'W = Data(:,1)'
      WRITE(350,*) 't1 = Data(:,2)'
      WRITE(350,*) ' '
```

```
      CLOSE(350)

      RETURN
      END



C     ************************************************************
C     Subroutine to calculate the Q1 collector current for
C     generalized tau (Icp) and the time (td) needed for the Q1
C     emitter voltage (Ve) to reach Vdd/2, for various load
C     capacitances (CL). The output is written to a matlab script
C     file.
C     ************************************************************

      SUBROUTINE TMDLYCL( COV, CJ, CBC, CBE, CCS, COX,
     +  VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP,
     +  ECRITP, IEXTRAP, UOP, T0, XTF, VTF, ITF, VBC,
     +  BETA, IK, TAUF)

      REAL*8  COV, CJ, CBC, CCS, COX
      REAL*8  VIH, VIL, VOL, VDIODE, VDD, VTP, VNORMP
      REAL*8  ECRITP, IEXTRAP, UOP, T0, XTF, VTF, ITF, VBC
      REAL*8  BETA, IK, TAUF

      REAL*8 CP, CBE, CIB, CB, VBT0B, VBT1B, VET0B, VET1B
      REAL*8 IMB1, IMB2, IMOS1, IMOS, T1, L1, L2
      REAL*8 NEWTONB, TD, W, L, CL



      OPEN(400, FILE='t1_cl.m')

      WRITE(400,*) '% gen_cl.m'
      WRITE(400,*) '% DATA FOR TD USING GENRALIZED TAU ANALYSIS'
      WRITE(400,*) '% AND VARYING CL'
```

```
      WRITE(400,*) 'Data = ['

      OPEN(500, FILE='t1_cl.m')

      WRITE(500,*) '% t1_cl.m'
      WRITE(500,*) '% DATA FOR T1 USING GENRALIZED TAU ANALYSIS'
      WRITE(500,*) '% AND VARYING CL'

      WRITE(500,*) 'Data = ['

60    FORMAT(E11.3, ',', 3X, E15.6)

      W = 4.0E-6
      L = 2.0E-6

      DO 70 CL = 1.0E-13, 2.0E-12, 1.0E-14

      CALL TODATA(W, L, CL, COV, CJ, CBE,
     +  CBC, CCS, COX, VIH, VIL, VOL, VDIODE,
     +  VDD, VTP, VNORMP, ECRITP, IEXTRAP,
     +  UOP, T0, CP, CPE, CIB, CB, VBT0B,
     +  VBT1B, VET0B, VET1B, IMOS1, T1)


      CALL IM(VDD - VIL, VDD - VBT1B, COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB1)

      CALL IM(VDD - VIL, VDD - (VDD/2.0 + VDIODE), COX, W,
     + L, VTP, UOP, ECRITP, IEXTRAP, VNORMP, IMB2)

      IMOS = 0.5*(IMB1 + IMB2)

      CALL LDA(IK, BETA, CB, CL, IMOS, L1, L2)


      CALL NEWTON(L1, TAUF, XTF, VTF, ITF, VBC, CB, CL, BETA,
     +            IK, IMOS, NEWTONB)

      CALL OPTDELAY(T1, T0, TAUF, NEWTONB, IMOS, CB,
```

```
     +                       CL, VDD, VET1B, TD)

      WRITE(400,60) CL, TD
      WRITE(500,60) CL, T1

70    CONTINUE

      WRITE(400,*) ']'
      WRITE(400,*) ' '
      WRITE(400,*) 'cl = Data(:,1)'
      WRITE(400,*) 'td = Data(:,2)'
      WRITE(400,*) ' '

      CLOSE(400)

      WRITE(500,*) ']'
      WRITE(500,*) ' '
      WRITE(500,*) 'cl = Data(:,1)'
      WRITE(500,*) 't1 = Data(:,2)'
      WRITE(500,*) ' '

      CLOSE(500)


      RETURN
      END
```

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