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# A NOTE ON TIME-DEPENDENT ADDITIVE FUNCTIONALS

ADRIEN BARRASSO AND FRANCESCO RUSSO

Abstract. This note develops shortly the theory of non-homogeneous additive functionals and is a useful support for the analysis of time-dependent Markov processes and related topics. It is a significant tool for the analysis of Markovian BSDEs in law. In particular we extend to a non-homogeneous setup some results concerning the quadratic variation and the angular bracket of Martingale Additive Functionals (in short MAF) associated to a homogeneous Markov processes.

## 1. Introduction

The notion of Additive Functional of a general Markov process is due to E.B Dynkin and has been studied since the early '60s by the Russian, French and American schools of probability, see for example [4, 8, 16]. A mature version of the homogeneous theory may be found for example in [7], Chapter XV. In that context, given an element $x$ in some state space $E$, $\mathbb{P}^x$ denotes the law of a time-homogeneous Markov process with initial value $x$.

An *additive functional* is a right-continuous process $(A_t)_{t\geq 0}$ defined on a canonical space, adapted to the canonical filtration such that for any $s \leq t$ and $x \in E$, $A_{s+t} = A_s + A_t \circ \theta_s$ $\mathbb{P}^x$-a.s., where $\theta$ is the usual shift operator on the canonical space. If moreover $A$ is under any law $\mathbb{P}^x$ a martingale, then it is called a Martingale Additive Functional (MAF). The quadratic variation and angular bracket of a MAF were shown to be AFs in [7]. We extend this type of results to a more general definition of an AF which is closer to the original notion of Additive Functional associated to a stochastic system introduced by E.B. Dynkin, see [9] for instance.

Our setup is as follows. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with time index $[0,T]$ and state space $E$ being a Polish space. For any $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ corresponds to the probability law (defined on some canonical filtered space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}\right)$) of a Markov process starting from point $x$ at time $s$. On $(\Omega, \mathcal{F})$, we define a *non-homogeneous Additive Functional* (shortened by AF) as a real-valued random-field $A := (A^t_u)_{0\leq t\leq u\leq T}$ verifying the two following conditions.

1. For any $0 \leq t \leq u \leq T$, $A^t_u$ is $\mathcal{F}_{t,u}$-measurable;
2. for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $(\mathcal{F}^{s,x}_t)_{t\in[0,T]}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s.

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$\mathcal{F}_{t,u}$ denotes the $\sigma$-field generated by the canonical process between time $t$ and $u$, and $\mathcal{F}_t^{s,x}$ is obtained by adding the $\mathbb{P}^{s,x}$ negligible sets to $\mathcal{F}_t$. $A^{s,x}$ will be called the *cadlag version of* $A$ *under* $\mathbb{P}^{s,x}$. If for any $(s,x)$, $A^{s,x}$ is a $(\mathbb{P}^{s,x},(\mathcal{F}_t)_{t\in[0,T]})$-square integrable martingale then $A$ will be called a square integrable Martingale Additive Functional (in short, square integrable MAF).

The main contributions of the paper are essentially the following. In Section 3, we recall the definition and prove some basic results concerning canonical Markov classes. In Section 4, we start by defining an AF in Definition 4.1. In Proposition 4.4, we show that if $(M_u^t)_{0\leq t\leq u\leq T}$ is a square integrable MAF, then there exists an AF $([M]_u^t)_{0\leq t\leq u\leq T}$ which for any $(s,x)\in[0,T]\times E$, has $[M^{s,x}]$ as cadlag version under $\mathbb{P}^{s,x}$. Corollary 4.11 states that given two square integrable MAFs $(M_u^t)_{0\leq t\leq u\leq T}$, $(N_u^t)_{0\leq t\leq u\leq T}$, there exists an AF, denoted by $(\langle M,N\rangle_u^t)_{0\leq t\leq u\leq T}$, which has $\langle M^{s,x},N^{s,x}\rangle$ as cadlag version under $\mathbb{P}^{s,x}$. Finally, we prove in Proposition 4.17 that if $M$ or $N$ is such that for $\mathbb{P}^{s,x}$, its cadlag version under $\mathbb{P}^{s,x}$, its angular bracket is absolutely continuous with respect to some continuous non-decreasing function $V$, then there exists a Borel function $v$ such that for any $(s,x)$, $\langle M^{s,x},N^{s,x}\rangle=\int_s^{\cdot\vee s}v(r,X_r)dV_r$.

The present note constitutes a support for the authors, in the analysis of deterministic problems related to Markovian type backward stochastic differential equations where the forward process is given in law, see e.g. [2]. Indeed, when the forward process of the BSDE does not define a stochastic flow (typically if it is not the strong solution of an SDE but only a weak solution), we cannot exploit the mentioned flow property to show that the solution of the BSDE is a function of the forward process, as it is usually done, see Remark 5.35 (ii) in [17] for instance.

## 2. Preliminaries

The present section is devoted to fix some basic notions, notations and vocabulary. A topological space $E$ will always be considered as a measurable space with its Borel $\sigma$-field which shall be denoted $\mathcal{B}(E)$ and if $S$ is another topological space equipped with its Borel $\sigma$-field, $\mathcal{B}(E,S)$ (resp. $\mathcal{B}_b(E,S)$, resp. $\mathcal{C}(E,S)$, resp. $\mathcal{C}_b(E,S)$) will denote the set of Borel (resp. bounded Borel, reps. continuous, resp. bounded continuous) functions from $E$ to $S$. Let $T\in\mathbb{R}_+^*$, $d\in\mathbb{N}^*$, then $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ will denote the space of bounded continuous real valued functions on $[0,T]\times\mathbb{R}^d$ which are differentiable in the first variable, twice differentiable in the second with bounded continuous partial derivatives.

Let $(\Omega,\mathcal{F})$, $(E,\mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega,\mathcal{F})$ to $(E,\mathcal{E})$ shall often be called a *random variable* (with values in $E$), or in short r.v. If $\mathbb{T}$ is some set, an indexed set of r.v. with values in $E$, $(X_t)_{t\in\mathbb{T}}$ will be called a *random field* (indexed by $\mathbb{T}$ with values in $E$). In particular, if $\mathbb{T}$ is an interval included in $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a *stochastic process* (indexed by $\mathbb{T}$ with values in $E$). Given a stochastic process, if the mapping

$$\begin{array}{rcl}(t,\omega) & \longmapsto & X_t(\omega)\\ (\mathbb{T}\times\Omega,\mathcal{B}(\mathbb{T})\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{E})\end{array}$$

is measurable, then the process $(X_t)_{t\in\mathbb{T}}$ will be called a *measurable process* (indexed by $\mathbb{T}$ with values in $E$).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. For any $p \geq 1$, $L^p := L^p(\mathbb{R})$ will denote the set of real valued random variables with finite $p$-th moment. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be *modifications (or versions) of each other* if for every $t \in \mathbb{T}$, $\mathbb{P}(X_t = Y_t) = 1$. If the probability space is equipped with a right-continuous filtration, then $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ will be called *stochastic basis* and will be said to *fulfill the usual conditions* if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets.

Concerning spaces of real valued stochastic processes on the above mentioned stochastic basis, $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1, \infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements $M$ such that $\sup\limits_{t\in\mathbb{T}} |M_t| \in L^p$ and in this set we identify indistinguishable elements. $\mathcal{H}^p$ is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathbb{E}[|\sup\limits_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}^p_0$ will denote the Banach subspace of $\mathcal{H}^p$ whose elements start at zero.

A crucial role in the present note, as well as in classical stochastic analysis is played by localization via stopping times. If $\mathbb{T} = [0, T]$ for some $T \in \mathbb{R}^*_+$, a stopping time will be intended as a random variable with values in $[0, T] \cup \{+\infty\}$ such that for any $t \in [0, T]$, $\{\tau \leq t\} \in \mathcal{F}_t$. We define a *localizing sequence of stopping times* as an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$. Let $Y$ be a process and $\tau$ a stopping time, we denote $Y^\tau$ the process $t \mapsto Y_{t\wedge\tau}$ which we call *stopped process*. If $\mathcal{C}$ is a set of processes, we define its *localized class* $\mathcal{C}_{loc}$ as the set of processes $Y$ such that there exists a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every $n$, the stopped process $Y^{\tau_n}$ belongs to $\mathcal{C}$.

We say some words about the concept of bracket related to two processes: the square bracket and the angular bracket. They coincide if at least one of the two processes is continuous. For any $M, N \in \mathcal{M}$ $[M, N]$ denotes the *covariation* of $M, N$. If $M = N$, we write $[M] := [M, N]$. $[M]$ is called *quadratic variation* of $M$. If $M, N \in \mathcal{H}^2_{loc}$, $\langle M, N\rangle$ (or simply $\langle M\rangle$ if $M = N$) will denote their (predictable) *angular bracket*. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M, N)_{\mathcal{H}^2} := \mathbb{E}[M_T N_T] = \mathbb{E}[\langle M, N\rangle_T]$ which makes it a Hilbert space. Two elements $M, N$ of $\mathcal{H}^2_{0,loc}$ will be said to be *strongly orthogonal* if $\langle M, N\rangle = 0$.

If $A$ is an adapted process with bounded variation then $Var(A)$ (resp. $Pos(A)$, $Neg(A)$) will denote its total variation (resp. positive variation, negative variation), see Proposition 3.1, chap. 1 in [15]. In particular for almost all $\omega \in \Omega$, $t \mapsto Var_t(A(\omega))$ is the total variation function of the function $t \mapsto A_t(\omega)$.

For more details concerning these notions, one may consult [18] or [15] for example.

## 3. Markov Classes

We recall here some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [7], concerning non-homogeneous Markov classes, our reference was Chapter VI of [10].

**3.1. Definition and basic results.** The first definition refers to the canonical space that one can find in [14], see paragraph 12.63.

*Notation* 3.1. In the whole section $E$ will be a fixed Polish space (a separable completely metrizable topological space), and $\mathcal{B}(E)$ its Borel $\sigma$-field. $E$ will be called the *state space*.

We consider $T \in \mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to $E$ right-continuous with left limits and continuous at time $T$ (e.g. cadlag). For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on $\Omega$ the $\sigma$-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

On the measurable space $(\Omega, \mathcal{F})$, we introduce the *canonical process*

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}) & \longrightarrow & (E, \mathcal{B}(E)), \end{array} \tag{3.1}$$

and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.

$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ will be called the *canonical space* (associated to $T$ and $E$). For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r | r \in [t, u + \frac{1}{n}])$.

*Remark* 3.2. All the results of the present paper remain valid if $\Omega$ is the space of continuous functions from $[0,T]$ to $E$, and if the time index is equal to $\mathbb{R}_+$.

We recall that since $E$ is Polish, then $\mathbb{D}(E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in Chapter 3 of [11]), and for which the Borel $\sigma$-field is $\mathcal{F}$ (see Proposition 7.1 in Chapter 3 of [11]). This in particular implies that $\mathcal{F}$ is separable, as the Borel $\sigma$-field of a separable metric space.

*Remark* 3.3. The above $\sigma$-fields fulfill the properties below.

(1) For any $0 \leq t \leq u < T$, $\mathcal{F}_{t,u} = \mathcal{F}_u \cap \mathcal{F}_{t,T}$;
(2) for any $t \geq 0$, $\mathcal{F}_t \vee \mathcal{F}_{t,T} = \mathcal{F}$;
(3) for any $(s,x) \in [0,T] \times E$, the two first items remain true when considering the $\mathbb{P}^{s,x}$-closures of all the $\sigma$-fields;
(4) for any $t \geq 0$, $\Pi := \{F = F_t \cap F_T^t | (F_t, F_T^t) \in \mathcal{F}_t \times \mathcal{F}_{t,T}\}$ is a $\pi$-system generating $\mathcal{F}$, i.e. it is stable with respect to the intersection.

**Definition 3.4.** The function

$$P : \begin{array}{rcl} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2 \times E \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

will be called *transition kernel* if, for any $s,t$ in $[0,T]$, $x \in E$, $A \in \mathcal{B}(E)$, it verifies the following.

(1) $P_{s,t}(\cdot, A)$ is Borel,
(2) $P_{s,t}(x, \cdot)$ is a probability measure on $(E, \mathcal{B}(E))$,
(3) if $t \leq s$ then $P_{s,t}(x,A) = \mathbb{1}_A(x)$,
(4) if $s < t$, for any $u > t$, $\int_E P_{s,t}(x,dy) P_{t,u}(y,A) = P_{s,u}(x,A)$.

The latter statement is the well-known *Chapman-Kolmogorov equation*.

**Definition 3.5.** A transition kernel $P$ is said to be *measurable in time* if for every $t \in [0,T]$ and $A \in \mathcal{B}(E)$, $(s,x) \longmapsto P_{s,t}(x,A)$ is Borel.

*Remark* 3.6. Let $P$ be a transition kernel which is measurable in time, let $\phi \in \mathcal{B}(E,\mathbb{R})$ and $t \in [0,T]$. Assume that for any $(s,x) \in [0,T] \times E$, the integral $\int |\phi|(y)P_{s,t}(x,dy)$ exists. Then the mapping $(s,x) \mapsto \int \phi(y)P_{s,t}(x,dy)$ is Borel. This can be easily shown by approximating $\phi$ by simple functions and using the definition.

**Definition 3.7.** A *canonical Markov class* associated to a transition kernel $P$ is a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega,\mathcal{F})$ and verifying for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x,A), \tag{3.2}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = P_{t,u}(X_t,A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{3.3}$$

The statement below comes Formula 1.7 in Chapter 6 of [10].

**Proposition 3.8.** *For any $(s,x) \in [0,T] \times E$, $t \geq s$ and $F \in \mathcal{F}_{t,T}$ yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \quad \mathbb{P}^{s,x}\, a.s. \tag{3.4}$$

Property (3.4) is often called *Markov property*. We recall here the concept of homogeneous canonical Markov classes and its links with Markov classes.

*Notation* 3.9. A mapping

$$\tilde{P}: \begin{array}{rcl} E \times [0,T] \times \mathcal{B}(E) & \longrightarrow & [0,1] \\ (t,x,A) & \longmapsto & \tilde{P}_t(x,A), \end{array} \tag{3.5}$$

will be called a *homogeneous transition kernel* if

$$P : (s,t,x,A) \longmapsto \tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$$

is a transition kernel in the sense of Definition 3.4. This in particular implies $\tilde{P} = P_{0,\cdot}(\cdot,\cdot)$.

A set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the canonical space associated to $T$ and $E$ (see Notation 3.1) will be called a *homogeneous canonical Markov class* associated to a homogeneous transition kernel $\tilde{P}$ if

$$\begin{cases} \forall t \in [0,T] \quad \forall A \in \mathcal{B}(E) \quad , \mathbb{P}^x(X_t \in A) = \tilde{P}_t(x,A) \\ \forall 0 \leq t \leq u \leq T \quad , \mathbb{P}^x(X_u \in A|\mathcal{F}_t) = \tilde{P}_{u-t}(X_t,A) \quad \mathbb{P}^{s,x}\text{a.s.} \end{cases} \tag{3.6}$$

Given a homogeneous canonical Markov class $(\mathbb{P}^x)_{x\in E}$ associated to a homogeneous transition kernel $\tilde{P}$, one can always consider the canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to the transition kernel

$$P : (s,x,t,A) \longmapsto \tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}.$$

In particular, for any $x \in E$, we have $\mathbb{P}^{0,x} = \mathbb{P}^x$.

For the rest of this section, we are given a canonical Markov class

$$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$$

whose transition kernel is measurable in time. Proposition A.10 in [3] states the following.

**Proposition 3.10.** *For any event $F \in \mathcal{F}$, $(s,x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel. For any random variable $Z$, if the function $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty,\infty]$), then it is Borel.*

**Definition 3.11.** For any $(s,x) \in [0,T] \times E$ we will consider the $(s,x)$-*completion* $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ of the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$ , by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}^{s,x}_t$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$, for every $t \in [0,T]$.

We remark that, for any $(s,x) \in [0,T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions, see 1.4 in [15] Chapter I.

We recall the following simple consequence of Remark 32 in [5] Chapter II.

**Proposition 3.12.** *Let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$, $\mathbb{P}$ a probability on $(\Omega, \mathcal{F})$ and $\mathcal{G}^{\mathbb{P}}$ the $\mathbb{P}$-closure of $\mathcal{G}$. Let $Z^{\mathbb{P}}$ be a real $\mathcal{G}^{\mathbb{P}}$-measurable random variable. There exists a $\mathcal{G}$-measurable random variable $Z$ such that $Z = Z^{\mathbb{P}}$ $\mathbb{P}$-a.s.*

From this we can deduce the following.

**Proposition 3.13.** *Let $(s,x) \in [0,T] \times E$ be fixed, $Z$ be a random variable and $t \in [s,T]$. Then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}^{s,x}_t]$ $\mathbb{P}^{s,x}$ a.s.*

*Proof.* $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable and therefore $\mathcal{F}^{s,x}_t$-measurable. Moreover, let $G^{s,x} \in \mathcal{F}^{s,x}_t$, by Remark 32 in [5] Chapter II, there exists $G \in \mathcal{F}_t$ such that $\mathbb{P}^{s,x}(G \cup G^{s,x}) = \mathbb{P}^{s,x}(G \backslash G^{s,x})$ implying $\mathbb{1}_G = \mathbb{1}_{G^{s,x}}$ $\mathbb{P}^{s,x}$ a.s. So

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}}\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_G Z\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}} Z\right],
\end{aligned}$$

where the second equality occurs because of the definition of $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$. □

In particular, under the probability measure $\mathbb{P}^{s,x}$, $(\mathcal{F}_t)_{t\in[0,T]}$-martingales and $(\mathcal{F}^{s,x}_t)_{t\in[0,T]}$-martingales coincide.

We now show that in our setup, a canonical Markov class verifies the *Blumenthal 0-1 law* in the following sense.

**Proposition 3.14.** *Let $(s,x) \in [0,T] \times E$ and $F \in \mathcal{F}_{s,s}$. Then $\mathbb{P}^{s,x}(F)$ is equal to $1$ or to $0$; In other words, $\mathcal{F}_{s,s}$ is $\mathbb{P}^{s,x}$-trivial.*

*Proof.* Let $F \in \mathcal{F}_{s,s}$ as introduced in Notation 3.1. Since by Remark 3.3, $\mathcal{F}_{s,s} = \mathcal{F}_s \cap \mathcal{F}_{s,T}$, then $F$ belongs to $\mathcal{F}_s$ so by conditioning we get

$$\begin{aligned}
\mathbb{E}^{s,x}[\mathbb{1}_F] &= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{1}_F] \\
&= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_s]] \\
&= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{E}^{s,X_s}[\mathbb{1}_F]],
\end{aligned}$$

where the latter equality comes from (3.4) because $F \in \mathcal{F}_{s,T}$. But $X_s = x$, $\mathbb{P}^{s,x}$ a.s., so

$$\begin{aligned}\mathbb{E}^{s,x}[\mathbb{1}_F] &= \mathbb{E}^{s,x}[\mathbb{1}_F \mathbb{E}^{s,x}[\mathbb{1}_F]] \\ &= \mathbb{E}^{s,x}[\mathbb{1}_F]^2.\end{aligned}$$

□

**3.2. Examples of canonical Markov classes.** We will list here some well-known examples of canonical Markov classes and some more recent ones.

- Let $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$. We are given $b \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$, $\alpha \in \mathcal{C}_b(\mathbb{R}_+ \times \mathbb{R}^d, S^*_+(\mathbb{R}^d))$ (where $S^*_+(\mathbb{R}^d)$ is the space of symmetric strictly positive definite matrices of size $d$) and $K$ a Lévy kernel (this means that for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a $\sigma$-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\sup\limits_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel) such that for any $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{y}{1+\|y\|^2} K(t,x,dy)$ is bounded continuous.

  Let $a$ denote the operator defined on some $\phi \in \mathcal{C}^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ by

$$\partial_t \phi + \frac{1}{2} Tr(\alpha \nabla^2 \phi) + (b, \nabla\phi) + \int \left( \phi(\cdot, \cdot + y) - \phi - \frac{(y, \nabla\phi)}{1 + \|y\|^2} \right) K(\cdot, \cdot, dy) \quad (3.7)$$

  In [20] (see Theorem 4.3 and the penultimate sentence of its proof), the following is shown. For every $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (see Definition 3.1) such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a local martingale for every $\phi \in \mathcal{C}^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\mathbb{P}^{s,x}(X_s = x) = 1$. Moreover $(\mathbb{P}^{s,x})_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d}$ defines a canonical Markov class and its transition kernel is measurable in time.
- The case $K = 0$ was studied extensively in the celebrated book [21] in which it is also shown that if $b$, $\alpha$ are bounded and continuous in the second variable, then there exists a canonical Markov class with transition kernel measurable in time $(\mathbb{P}^{s,x})_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d}$ such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a local martingale for any $\phi \in \mathcal{C}^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.
- In [19], a canonical Markov class whose transition kernel is the weak fundamental solution of a parabolic PDE in divergence form is exhibited.
- In [13], diffusions on manifolds are studied and shown to define canonical Markov classes.
- Solutions of PDEs with distributional drift are exhibited in [12] and shown to define canonical Markov classes.

Some of previous examples were only studied as homogeneous Markov processes but can easily be shown to fall in the non-homogeneous setup of the present paper as it was illustrated in [3].

## 4. Martingale Additive Functionals

We now introduce the notion of non-homogeneous Additive Functional that we use in the paper. This looks to be a good compromise between the notion of Additive Functional associated to a stochastic system introduced by E.B. Dynkin (see for example [9]) and the more popular notion of homogeneous Additive Functional studied extensively, for instance by C. Dellacherie and P.A. Meyer in [7] Chapter XV. This section consists in extending some essential results stated in [7] Chapter XV to our setup.

Our framework is still the canonical space introduced at Notation 3.1. In particular $X$ is the canonical process.

**Definition 4.1.** We denote $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. On $(\Omega, \mathcal{F})$, we define a *non-homogeneous Additive Functional* (shortened AF) as a random-field $A := (A^t_u)_{(t,u)\in\Delta}$ indexed by $\Delta$ with values in $\mathbb{R}$, verifying the two following conditions.

(1) For any $(t,u) \in \Delta$, $A^t_u$ is $\mathcal{F}_{t,u}$-measurable;
(2) for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s.

$A^{s,x}$ will be called the *cadlag version of $A$ under $\mathbb{P}^{s,x}$*.

An AF will be called a *non-homogeneous square integrable Martingale Additive Functional* (shortened square integrable MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a square integrable martingale. More generally an AF will be said to verify a certain property (being non-negative, increasing, of bounded variation, square integrable, having $L^1$-terminal value) if under any $\mathbb{P}^{s,x}$ its cadlag version verifies it.

Finally, given an increasing AF $A$ and an increasing function $V$, $A$ will be said to be *absolutely continuous with respect to $V$* if for any $(s,x) \in [0,T] \times E$, $dA^{s,x} \ll dV$ in the sense of stochastic measures.

*Remark* 4.2. Let $(\mathbb{P}^x)_{x\in E}$ be a homogeneous canonical Markov class (see Notation 3.9). We recall that in the classical literature (see Definition 3 of [7] for instance), an adapted right-continuous process $A$ on the canonical space is called an Additive Functional if for all $0 \leq t \leq u \leq T$ and $x \in E$

$$A_u = A_t + A_{u-t} \circ \theta_t \quad \mathbb{P}^x \text{ a.s.}, \tag{4.1}$$

where $\theta_t : \omega \mapsto \omega\left((t + \cdot) \wedge T\right)$ denotes the shift operator at time $t$.

Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be the canonical Markov class related to $(\mathbb{P}^x)_{x\in E}$ in the sense of Notation 3.9. If for every $0 \leq t \leq u \leq T$, Equation (4.1) holds for all $\omega$, then the random field $(t,u) \longmapsto A_u - A_t$ is a non-homogeneous Additive Functional in the sense of Definition 4.1.

**Example 4.3.** Let $\phi \in \mathcal{C}([0,T] \times E, \mathbb{R})$, $\psi \in \mathcal{B}_b([0,T] \times E, \mathbb{R})$ and $V : [0,T] \longmapsto \mathbb{R}$ be right-continuous and non-decreasing function. Then the random field $A$ given by

$$A^t_u := \phi(u, X_u) - \phi(t, X_t) - \int_t^u \psi(r, X_r) dV_r, \tag{4.2}$$

defines a non-homogeneous Additive Functional. Its cadlag version under $\mathbb{P}^{s,x}$ may be given by

$$A^{s,x} = \phi(\cdot \vee s, X_{\cdot\vee s}) - \phi(s,x) - \int_s^{\cdot\vee s} \psi(r, X_r) dV_r. \tag{4.3}$$

We now adopt the setup of the first item of Section 3.2. We consider some $\phi \in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)$, then the random field $M$ given by

$$M_u^t := \phi(u, X_u) - \phi(t, X_t) - \int_t^u a(\phi)(r, X_r)dr, \tag{4.4}$$

defines a square integrable MAF with cadlag version under $\mathbb{P}^{s,x}$ given by

$$M^{s,x} = \phi(\cdot \vee s, X_{\cdot \vee s}) - \phi(s, x) - \int_s^{\cdot \vee s} a(\phi)(r, X_r)dr. \tag{4.5}$$

In this section for a given MAF $(M_u^t)_{(t,u)\in\Delta}$ we will be able to exhibit two AF, denoted respectively by $([M]_u^t)_{(t,u)\in\Delta}$ and $(\langle M\rangle_u^t)_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Moreover we will show that the Radon-Nikodym derivative of the mentioned angular bracket of a MAF with respect to our reference function $V$ is a time-dependent function of the underlying process.

**Proposition 4.4.** *Let $(M_u^t)_{(t,u)\in\Delta}$ be a square integrable MAF, and for any $(s,x) \in [0,T] \times E$, $[M^{s,x}]$ be the quadratic variation of its cadlag version $M^{s,x}$ under $\mathbb{P}^{s,x}$. Then there exists an AF which we will call $([M]_u^t)_{(t,u)\in\Delta}$ and which, for any $(s,x) \in [0,T] \times E$, has $[M^{s,x}]$ as cadlag version under $\mathbb{P}^{s,x}$.*

*Proof.* We adapt Theorem 16 Chapter XV in [7] to a non homogeneous set-up but the reader must keep in mind that our definition of Additive Functional is different from the one related to the homogeneous case.

For the whole proof $t < u$ will be fixed. We consider a sequence of subdivisions of $[t,u]$: $t = t_1^k < t_2^k < \cdots < t_k^k = u$ such that $\min_{i<k} (t_{i+1}^k - t_i^k) \underset{k\to\infty}{\longrightarrow} 0$. Let $(s,x) \in [0,t] \times E$ with corresponding probability $\mathbb{P}^{s,x}$. For any $k$, we have $\sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k}\right)^2 = \sum_{i<k} (M_{t_{i+1}^k}^{s,x} - M_{t_i^k}^{s,x})^2$ $\mathbb{P}^{s,x}$ a.s., so by definition of quadratic variation we know that

$$\sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} [M^{s,x}]_u - [M^{s,x}]_t. \tag{4.6}$$

In the sequel we will construct an $\mathcal{F}_{t,u}$-measurable random variable $[M]_u^t$ such that for any $(s,x) \in [0,t] \times E$, $\sum_{i\leq k} \left(M_{t_{i+1}^k}^{t_i^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} [M]_u^t$. In that case $[M]_u^t$ will then be $\mathbb{P}^{s,x}$ a.s. equal to $[M^{s,x}]_u - [M^{s,x}]_t$.

Let $x \in E$. Since $M$ is a MAF, for any $k$, $\sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k}\right)^2$ is $\mathcal{F}_{t,u}$-measurable and therefore $\mathcal{F}_{t,u}^{t,x}$-measurable. Since $\mathcal{F}_{t,u}^{t,x}$ is complete, the limit in probability of this sequence, $[M^{t,x}]_u - [M^{t,x}]_t$, is still $\mathcal{F}_{t,u}^{t,x}$-measurable. By Proposition 3.12, there is an $\mathcal{F}_{t,u}$-measurable variable which depends on $(t,x)$, that we call $a_t(x,\omega)$ such that

$$a_t(x,\omega) = [M^{t,x}]_u - [M^{t,x}]_t, \mathbb{P}^{t,x} \text{ a.s.} \tag{4.7}$$

We will show below that there is a jointly measurable version of $(x,\omega) \mapsto a_t(x,\omega)$. For every integer $n \geq 0$, we set $a_t^n(x,\omega) := n \wedge a_t(x,\omega)$ which is in particular limit in probability of $n\wedge \sum_{i\leq k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2$ under $\mathbb{P}^{t,x}$. For any integers $k,n$ and any $x\in E$, we define the finite positive measures $\mathbb{Q}^{k,n,x}$, $\mathbb{Q}^{n,x}$ and $\mathbb{Q}^x$ on $(\Omega,\mathcal{F}_{t,u})$ by

(1) $\mathbb{Q}^{k,n,x}(F):=\mathbb{E}^{t,x}\left[\mathbb{1}_F\left(n\wedge\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2\right)\right]$;
(2) $\mathbb{Q}^{n,x}(F):=\mathbb{E}^{t,x}[\mathbb{1}_F\,(a_t^n(x,\omega))]$;
(3) $\mathbb{Q}^{x}(F):=\mathbb{E}^{t,x}[\mathbb{1}_F\,(a_t(x,\omega))]$.

When $k$ and $n$ are fixed, for any fixed $F$, by Proposition 3.10, the mapping $x\longmapsto \mathbb{E}^{t,x}\left[F\left(n\wedge\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2\right)\right]$ is Borel.

Then $n\wedge\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{t,x}}{\longrightarrow}} a_t^n(x,\omega)$, and this sequence is uniformly bounded by the constant $n$, so the convergence takes place in $L^1$, therefore $x\longmapsto\mathbb{Q}^{n,x}(F)$ is also Borel as the pointwise limit in k of the functions $x\longmapsto\mathbb{Q}^{k,n,x}(F)$. Similarly, $a_t^n(x,\omega)\underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} a_t(x,\omega)$ and is non-decreasing, so by monotone convergence theorem, being a pointwise limit in $n$ of the functions $x\longmapsto\mathbb{Q}^{n,x}(F)$, the function $x\longmapsto\mathbb{Q}^{x}(F)$ is Borel. We recall that $\mathcal{F}$ is separable. The just two mentioned properties and the fact that, for any $x$, we also have (by item 3. above) $\mathbb{Q}^x\ll\mathbb{P}^{t,x}$, allows to show (see Theorem 58 Chapter V in [6]) the existence of a jointly measurable (for $\mathcal{B}(E)\otimes\mathcal{F}_{t,u}$) version of $(x,\omega)\mapsto a_t(x,\omega)$, that we recall to be densities of $\mathbb{Q}^x$ with respect to $\mathbb{P}^{t,x}$. That version will still be denoted by the same symbol.

We can now set $[M]^t_u(\omega)=a_t(X_t(\omega),\omega)$, which is a correctly defined $\mathcal{F}_{t,u}$-measurable random variable. For any $x$, since $\mathbb{P}^{t,x}(X_t=x)=1$, we have the equalities

$$[M]^t_u=a_t(x,\cdot)=[M^{t,x}]_u-[M^{t,x}]_t\ \mathbb{P}^{t,x}\text{a.s.}\tag{4.8}$$

We will moreover prove that

$$[M]^t_u=[M^{s,x}]_u-[M^{s,x}]_t\quad\mathbb{P}^{s,x}\text{ a.s.},\tag{4.9}$$

holds for every $(s,x)\in[0,t]\times E$, and not just in the case $s=t$ that we have just established in (4.8).

Let us fix $s<t$ and $x\in E$. We show that under any $\mathbb{P}^{s,x}$, $[M]^t_u$ is the limit in probability of $\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2$. Indeed, let $\epsilon>0$: the event

$$\left\{\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right\}$$

belongs to $\mathcal{F}_{t,T}$ so by conditioning and using the Markov property (3.4) we have

$$\begin{aligned}&\mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)\\ =\ &\mathbb{E}^{s,x}\left[\mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\middle|\mathcal{F}_t\right)\right]\\ =\ &\mathbb{E}^{s,x}\left[\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)\right].\end{aligned}$$

For any fixed $y$, by (4.6) and (4.8), $\mathbb{P}^{t,y}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$ tends to zero when $k$ goes to infinity, for every realization $\omega$, it yields that

$$\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$$

tends to zero when $k$ goes to infinity. Since this sequence is dominated by the constant 1, that convergence still holds under the expectation with respect to the probability the probability $\mathbb{P}^{s,x}$, thanks to the dominated convergence theorem.

So we have built an $\mathcal{F}_{t,u}$-measurable variable $[M]^t_u$ such that under any $\mathbb{P}^{s,x}$ with $s\leq t$, $[M^{s,x}]_u-[M^{s,x}]_t=[M]^t_u$ a.s. and this concludes the proof. □

We will now extend the result about quadratic variation to the angular bracket of MAFs. The next result can be seen as an extension of Theorem 15 Chapter XV in [7] to a non-homogeneous context.

**Proposition 4.5.** *Let $(B^t_u)_{(t,u)\in\Delta}$ be an increasing AF with $L^1$-terminal value, for any $(s,x)\in[0,T]\times E$, let $B^{s,x}$ be its cadlag version under $\mathbb{P}^{s,x}$ and let $A^{s,x}$ be the predictable dual projection of $B^{s,x}$ in $(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x})$. Then there exists an increasing AF with $L^1$ terminal value $(A^t_u)_{(t,u)\in\Delta}$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $A$ is $A^{s,x}$.*

*Proof.* The first half of the demonstration will consist in showing that

$$\forall(s,x)\in[0,t]\times E,\ (A^{s,x}_u-A^{s,x}_t)\text{ is }\mathcal{F}^{s,x}_{t,u}-\text{measurable.}\qquad(4.10)$$

We start by recalling a property of the predictable dual projection which we will have to extend slightly. Let us fix $(s,x)$ and the corresponding stochastic basis $(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x})$. For any $F\in\mathcal{F}^{s,x}$, let $N^{s,x,F}$ be the cadlag version of the martingale, $r\longmapsto\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r]$. Then for any $0\leq t\leq u\leq T$, the predictable projection of the process $r\mapsto\mathbb{1}_F\mathbb{1}_{[t,u[}(r)$ is $r\mapsto N^{s,x,F}_{r^-}\mathbb{1}_{[t,u[}(r)$, see the proof of Theorem 43 Chapter VI in [6]. Therefore by definition of the dual predictable projection (see Definition 73 Chapter VI in [6]) we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u-A^{s,x}_t)\right]=\mathbb{E}^{s,x}\left[\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r\right],\qquad(4.11)$$

for any $F\in\mathcal{F}^{s,x}$.

We will now prove some technical lemmas which in a sense extend this property, and will permit us to operate with a good common version of the random variable $\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r$ not depending on $(s,x)$.

For the rest of the proof, $0\leq t<u\leq T$ will be fixed.

*Notation* 4.6. Let $F\in\mathcal{F}_{t,T}$. We denote for any $r\in[t,T],\omega\in\Omega$, $N^F_r(\omega):=\mathbb{P}^{t,X_t(\omega)}(F)$.

It is clear that $N^F$ previously introduced is an $(\mathcal{F}_{t,r})_{r\in[t,T]}$-adapted process which does not depend on $(s,x)$, which takes values in $[0,1]$ for all $r,\omega$ and by Proposition 3.8, for any $(s,x)\in[0,t]\times E$, $N^{s,x,F}$ is, on $[t,T]$, a $\mathbb{P}^{s,x}$-version of $N^F$.

**Lemma 4.7.** *Let $F \in \mathcal{F}_{t,T}$. There exists an $\mathcal{F}_{t,u}$-measurable random variable which we will denote $\int_t^u N^F_{r^-} dB_r$ such that for any $(s,x) \in [0,t] \times E$, $\int_t^u N^F_{r^-} dB_r = \int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ $\mathbb{P}^{s,x}$ a.s.*

*Remark* 4.8. By definition, the process $N^F$ introduced in Notation 4.6 and the r.v. $\int_t^u N^F_{r^-} dB_r$ will not depend on any $(s,x)$.

*Proof.* In some sense we wish to integrate $r \mapsto N^F_{r^-}$ against $B^t$ for fixed $\omega$. However first we do not know a priori if the paths $r \mapsto N^F_r$ and $r \mapsto B^t_r$ are measurable, second $r \mapsto N^F_r$ may not have a left limit and $B^t$ may be not of bounded variation. So it is not clear if $\int_t^u N^F_{r^-} dB^t_r$ makes sense for any $\omega$. Moreover under a certain $\mathbb{P}^{s,x}$, $N^{F,s,x}$ and $B^{s,x}_\cdot - B^{s,x}_t$ are only versions of $N^F$ and $B^t$ and not indistinguishable to them. Even if we could compute the aforementioned integral, it would not be clear if $\int_t^u N^F_{r^-} dB^t_r = \int_t^u N^{s,x,F}_{r^-} dB^{s,x}_r$ $\mathbb{P}^{s,x}$ a.s.

We start by some considerations about $B$, setting $W_{tu} := \{\omega : \sup\limits_{r \in [t,u] \cap \mathbb{Q}} B^t_r < \infty\}$ which is $\mathcal{F}_{t,u}$-measurable, and for $r \in [t,u]$

$$\bar{B}^t_r(\omega) := \begin{cases} \sup\limits_{\substack{t \le v < r \\ v \in \mathbb{Q}}} B^t_v(\omega) \text{ if } \omega \in W_{tu} \\ 0 \text{ otherwise.} \end{cases}$$

$\bar{B}^t$ is an increasing, finite (for all $\omega$) process. In general, it is neither a measurable nor an adapted process; however for any $r \in [t,u]$, $\bar{B}^t_r$ is still $\mathcal{F}_{t,u}$-measurable. Since it is increasing, it has right and left limits at each point for every $\omega$, so we can define the process $\tilde{B}^t$ indexed on $[t,u]$ below:

$$\tilde{B}^t_r := \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \bar{B}^t_v, r \in [t,u], \tag{4.12}$$

when $u \in ]t,T[$ and $\tilde{B}^t_T := B^t_T$ if $u = T$. Therefore $\tilde{B}^t$ is an increasing, cadlag process. It is constituted by $\mathcal{F}_{t,u}$-measurable random variables, and by Theorem 15 Chapter IV of [5], $\tilde{B}^t$ is a also a measurable process (indexed by $[t,u]$).

We can show that $\tilde{B}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from $B^{s,x}_\cdot - B^{s,x}_t$ for any $(s,x) \in [0,t] \times E$. Indeed, let $(s,x)$ be fixed. Since $B^{s,x}_\cdot - B^{s,x}_t$ is a version of $B^t$ and $\mathbb{Q}$ being countable, there exists a $\mathbb{P}^{s,x}$-null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$ and $r \in \mathbb{Q} \cap [t,u]$, $B^{s,x}_r(\omega) - B^{s,x}_t(\omega) = B^t_r(\omega)$. Therefore for any $\omega \in \mathcal{N}^c$ and $r \in [t,u]$,

$$\begin{aligned} \tilde{B}^t_r(\omega) &= \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \sup_{\substack{t \le w < v \\ w \in \mathbb{Q}}} B^t_w(\omega) = \lim_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} \sup_{\substack{t \le w < v \\ w \in \mathbb{Q}}} B^{s,x}(\omega)_w - B^{s,x}(\omega)_t \\ &= B^{s,x}(\omega)_r - B^{s,x}(\omega)_t, \end{aligned}$$

where the latter equality comes from the fact that $B^{s,x}(\omega)$ is cadlag and increasing. So we have constructed an increasing finite cadlag (for all $\omega$) process and so the path $r \mapsto \tilde{B}^t(\omega)$ is a Lebesgue integrator on $[t,u]$ for each $\omega$.

We fix now $F \in \mathcal{F}_{t,T}$ and we discuss some issues related to $N^F$ . Since it is positive, we can start defining the process $\bar{N}$, for index values $r \in [t,T[$ by $\bar{N}^F_r := \liminf\limits_{\substack{v \downarrow r \\ v \in \mathbb{Q}}} N^F_v$, and setting $\bar{N}^F_T := N^F_T$. This process is (by similar arguments as for $\tilde{B}^t$ defined in (4.12)), $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ for all $(s,x)\in[0,t]\times E$. For any $r\in[t,T]$, $N^F_r$ (see Notation 4.6) is $\mathcal{F}_{t,r}$-measurable, so $\bar{N}^F_r$ will also be $\mathcal{F}_{t,r}$-measurable for any $r\in[t,T]$ by right-continuity of $\mathcal{F}_{t,\cdot}$ (see Notation 3.1). However, $\bar{N}^F$ is not necessarily cadlag for every $\omega$, and also not necessarily a measurable process. We subsequently define

$$W'_{tu} := \{\omega\in\Omega|\text{there is a cadlag function } f \text{ such that } \bar{N}^F(\omega) = f \text{ on } [t,u]\cap\mathbb{Q}\}.$$

By Theorem 18 b) in Chapter IV of [5], $W'_{tu}$ is $\mathcal{F}_{t,u}$-measurable so we can define on $[t,u]$ $\tilde{N}^F_r := \bar{N}^F_r \mathbb{1}_{W'_{tu}}$. $\tilde{N}^F$ is no longer $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, however, it is now cadlag for all $\omega$ and therefore a measurable process by Theorem 15 Chapter IV of [5]. The r.v. $\tilde{N}^F_r$ are still $\mathcal{F}_{t,u}$-measurable , and $\tilde{N}^F$ is still $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ on $[t,u]$ for any $(s,x)\in[0,t]\times E$.

Finally we can define $\int_t^u N^F_{r^-}dB_r := \int_t^u \tilde{N}^F_{r^-}d\tilde{B}^t_r$ which is $\mathbb{P}^{s,x}$ a.s. equal to $\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r$ for any $(s,x)\in[0,t]\times E$. Moreover, since $\tilde{N}^F$ and $\tilde{B}$ are both measurable with respect to $\mathcal{B}([t,u])\otimes\mathcal{F}_{t,u}$, then $\int_t^u N^F_{r^-}dB_r$ is $\mathcal{F}_{t,u}$-measurable. □

The lemma below is a conditional version of the property (4.11).

**Lemma 4.9.** *For any $(s,x)\in[0,t]\times E$ and $F\in\mathcal{F}^{s,x}_{t,T}$ we have $\mathbb{P}^{s,x}$-a.s.*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\middle|\mathcal{F}_t\right].$$

*Proof.* Let $s,x,F$ be fixed. By definition of conditional expectation, we need to show that for any $G\in\mathcal{F}_t$ we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\middle|\mathcal{F}_t\right]\right] \text{ a.s.}$$

For $r\in[t,u]$ we have $\mathbb{E}^{s,x}[\mathbb{1}_{F\cap G}|\mathcal{F}_r] = \mathbb{1}_G\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r]$ a.s. therefore the cadlag versions of those processes are indistinguishable on $[t,u]$ and the random variables $\int_t^u N^{G\cap F}_{r^-}dB_r$ and $\mathbb{1}_G\int_t^u N^F_{r^-}dB_r$ as defined in Lemma 4.7 are a.s. equal. So by the non conditional property of dual predictable projection (4.11) we have

$$\begin{array}{rcl}\mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)\right] &=& \mathbb{E}^{s,x}\left[\int_t^u N^{G\cap F}_{r^-}dB_r\right]\\ &=& \mathbb{E}^{s,x}\left[\mathbb{1}_G\int_t^u N^F_{r^-}dB_r\right]\\ &=& \mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r|\mathcal{F}_t\right]\right],\end{array}$$

which concludes the proof. □

**Lemma 4.10.** *For any $(s,x)\in[0,t]\times E$ and $F\in\mathcal{F}_{t,T}$ we have $\mathbb{P}^{s,x}$-a.s.,*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)|X_t\right].$$

*Proof.* By Lemma 4.9 we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u - A^{s,x}_t)|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\middle|\mathcal{F}_t\right].$$

By Lemma 4.7, $\int_t^u N^F_{r^-}dB_r$ is $\mathcal{F}_{t,T}$ measurable so the Markov property (3.4) implies

$$\mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\middle|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N^F_{r^-}dB_r\middle|X_t\right],$$

therefore $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_t\right]$ is a.s. equal to a $\sigma(X_t)$-measurable r.v and so is a.s. equal to $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x}-A_t^{s,x})|X_t\right]$. □

We are now able to prove (4.10) which is the first important issue of the proof of Proposition 4.5, which states that By definition, a predictable dual projection is adapted so we already know that $(A_u^{s,x}-A_t^{s,x})$ is $\mathcal{F}_u^{s,x}$-measurable, therefore by Remark 3.3, it is enough to show that it is also $\mathcal{F}_{t,T}^{s,x}$-measurable. So we are going to show that

$$A_u^{s,x}-A_t^{s,x}=\mathbb{E}^{s,x}\left[A_u^{s,x}-A_t^{s,x}|\mathcal{F}_{t,T}\right]\ \mathbb{P}^{s,x}\text{ a.s.} \tag{4.13}$$

For this we will show that

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x}-A_t^{s,x})\right]=\mathbb{E}^{s,x}\left[\mathbb{1}_F\mathbb{E}^{s,x}\left[A_u^{s,x}-A_t^{s,x}|\mathcal{F}_{t,T}\right]\right], \tag{4.14}$$

for any $F\in\mathcal{F}$. We will prove (4.14) for $F\in\mathcal{F}$ event of the form $F=F_t\cap F_{t,T}$ with $F_t\in\mathcal{F}_t$ and $F_{t,T}\in\mathcal{F}_{t,T}$. By item 4. of Remark 3.3, such events form a $\pi$-system $\Pi$ which generates $\mathcal{F}$. Consequently, by the monotone class theorem, (4.14) will remain true for any $F\in\mathcal{F}$ and even in $\mathcal{F}^{s,x}$ since $\mathbb{P}^{s,x}$-null set will not impact the equality. This will imply (4.13) so that $A_u^{s,x}-A_t^{s,x}$ is $\mathcal{F}_{t,T}^{s,x}$-measurable. At this point, as we have anticipated, we prove (4.14) for a fixed $F=F_t\cap F_{t,T}\in\Pi$. By Lemma 4.10 we have

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x}-A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_t\right]\right]\\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|X_t\right]\right]\\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]|X_t\right]\right],
\end{aligned}$$

where the latter equality holds since $\sigma(X_t)\subset\mathcal{F}_{t,T}$. Now observe that since $\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]$ is $\mathcal{F}_{t,T}$-measurable, the Markov property (3.4) allows us to substitute the conditional $\sigma$-field $\sigma(X_t)$ with $\mathcal{F}_t$ and obtain

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x}-A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]|\mathcal{F}_t\right]\right]\\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]\right]\\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{1}_{F_{t,T}}\mathbb{E}^{s,x}\left[(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]\right]\\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_F\mathbb{E}^{s,x}\left[(A_u^{s,x}-A_t^{s,x})|\mathcal{F}_{t,T}\right]\right].
\end{aligned}$$

This concludes the proof of (4.14), therefore (4.13) holds so that $A_u^{s,x}-A_t^{s,x}$ is $\mathcal{F}_{t,u}^{s,x}$-measurable and so (4.10) is established. This concludes the first part of the proof of Proposition 4.5.

We pass to the second part of the proof of Proposition 4.5 where we will show that for given $0<t<u$ there is an $\mathcal{F}_{t,u}$-measurable r.v. $A_u^t$ such that for every $(s,x)\in[0,t]\times E$, $(A_u^{s,x}-A_t^{s,x})=A_u^t\quad\mathbb{P}^{s,x}$ a.s.

Similarly to what we did with the quadratic variation in Proposition 4.4, we start by noticing that for any $x\in E$, since $(A_u^{t,x}-A_t^{t,x})$ is $\mathcal{F}_{t,u}^{t,x}$-measurable, there exists by Proposition 3.12 an $\mathcal{F}_{t,u}$-measurable r.v. $a(x,\omega)$ such that

$$a(x,\omega)=A_u^{t,x}-A_t^{t,x}\quad\mathbb{P}^{t,x}\text{ a.s.} \tag{4.15}$$

As in the proof of Proposition 4.4, we will show the existence of a jointly measurable version of $(x,\omega)\mapsto a(x,\omega)$. For every $x\in E$ we define on $\mathcal{F}_{t,u}$ the positive measure

$$\mathbb{Q}^x:F\longmapsto\mathbb{E}^{t,x}\left[\mathbb{1}_F(A_u^{t,x}-A_t^{t,x})\right]=\mathbb{E}^{t,x}\left[\mathbb{1}_Fa(x,\omega)\right]. \tag{4.16}$$

By Lemma 4.7, and (4.11), for every $F \in \mathcal{F}_{t,u}$ we have

$$\mathbb{Q}^x(F) = \mathbb{E}^{t,x}\left[\int_t^u N_{r^-}^F dB_r\right], \tag{4.17}$$

and we recall that $\int_t^u N_{r^-}^F dB_r$ does not depend on $x$. So by Proposition 3.10 $x \longmapsto \mathbb{Q}^x(F)$ is Borel for any $F$. Moreover, for any $x$, $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$. Again by Theorem 58 Chapter V in [6], there exists a version $(x,\omega) \mapsto a(x,\omega)$ measurable for $\mathcal{B}(E) \otimes \mathcal{F}_{t,u}$ of the related Radon-Nikodym densities.

We can now set $A_u^t(\omega) := a(X_t(\omega),\omega)$ which is then an $\mathcal{F}_{t,u}$-measurable r.v. Since $\mathbb{P}^{t,x}(X_t = x) = 1$ and (4.15) hold, we have

$$A_u^t = a(X_t,\cdot) = a(x,\cdot) = A_u^{t,x} - A_t^{t,x} \quad \mathbb{P}^{t,x} \text{ a.s.} \tag{4.18}$$

We now fix $s < t$ and $x \in E$ and we want to show that we still have $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$ a.s. So, as above, we consider $F \in \mathcal{F}_{t,u}$ and, thanks to (4.11) we compute

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\int_t^u N_{r^-}^F dB_r\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\mathbb{1}_F A_u^t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t\right].
\end{aligned} \tag{4.19}$$

Indeed, concerning the fourth equality we recall that, by (4.16), (4.17) and (4.18), we have $\mathbb{E}^{t,x}\left[\int_t^u N_{r^-}^F dB_r\right] = \mathbb{E}^{t,x}\left[\mathbb{1}_F A_u^t\right]$ for all $x$, so this equality becomes an equality whatever random variable we plug into $x$. The third and fifth equalities come from the Markov property (3.4) since $\int_t^u N_{r^-}^F dB_r$ and $A_u^t$ are $\mathcal{F}_{t,T}$-measurable. Then, adding $\mathbb{P}^{s,x}$-null sets does not change the validity of (4.19), so we have for any $F \in \mathcal{F}_{t,u}^{s,x}$ that $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t\right]$.

Finally, since we had shown in the first half of the proof that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,u}^{s,x}$-measurable, and since $A_u^t$ also has, by construction, the same measurability property, we can conclude that $A_u^{s,x} - A_t^{s,x} = A_u^t$ $\mathbb{P}^{s,x}$ a.s.

Since this holds for every $t \leq u$ and $(s,x) \in [0,t] \times E$, $(A_u^t)_{(t,u)\in\Delta}$ is the desired AF, which ends the proof of Proposition 4.5. □

**Corollary 4.11.** *Let $M$, $M'$ be two square integrable MAFs, let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of $M$ (respectively $M'$) under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF with $L^1$ terminal condition denoted $\langle M, M'\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, M'\rangle$ is $\langle M^{s,x}, M'^{s,x}\rangle$. If $M = M'$ the AF $\langle M, M'\rangle$ will be denoted $\langle M\rangle$ and is increasing.*

*Proof.* If $M = M'$, the corollary comes from the combination of Propositions 4.4 and 4.5, and the fact that the angular bracket of a square integrable martingale is the dual predictable projection of its quadratic variation. Otherwise, it is clear that $M + M'$ and $M - M'$ are square integrable MAFs, so we can consider the increasing MAFs $\langle M - M'\rangle$ and $\langle M + M'\rangle$. We introduce the AF

$$\langle M, M'\rangle = \frac{1}{4}(\langle M + M'\rangle - \langle M - M'\rangle),$$

which by polarization has cadlag version $\langle M^{s,x}, M'^{s,x}\rangle$ under $\mathbb{P}^{s,x}$. $\langle M, M'\rangle$ is therefore a bounded variation AF with $L^1$ terminal condition. □

We are now going to study the Radon-Nikodym derivative of an increasing continuous AF with respect to some measure. The next result can be seen as an extension of Theorem 13 Chapter XV in [7] in a non-homogeneous setup. We will need the following lemma.

**Lemma 4.12.** *Let $(E,\mathcal{E})$ be a measurable space, let $\mathcal{I}$ be a sub-interval of $\mathbb{R}_+$ and let $f : E\times\mathcal{I} \longrightarrow \mathbb{R}$ be a mapping such that for all $t\in\mathcal{I}$, $x\mapsto f(x,t)$ is measurable with respect to $\mathcal{E}$ and for all $x\in E$, $t\mapsto f(x,t)$ is right-continuous, then $f$ is measurable with respect to $\mathcal{E}\otimes\mathcal{B}(\mathcal{I})$.*

*Proof.* On $(E,\mathcal{E})$ we introduce the filtration $(\mathcal{E}_t)_{t\in\mathcal{I}}$ where $\mathcal{E}_t=\mathcal{E}$ for all $t$. In the filtered space $(E,\mathcal{E},(\mathcal{E}_t)_{t\in\mathcal{I}})$, $f$ defines a right-continuous adapted process and is therefore progressively measurable (see Theorem 15 in [5] Chapter IV for instance), and in particular it is measurable. This means that $f$ is measurable with respect to $\mathcal{E}\otimes\mathcal{B}(\mathcal{I})$. □

**Proposition 4.13.** *Let $A$ be a positive, non-decreasing AF absolutely continuous with respect to some continuous non-decreasing function $V$, and for every $(s,x)\in[0,T[\times E$ let $A^{s,x}$ be the cadlag version of $A$ under $\mathbb{P}^{s,x}$. There exists a Borel function $h\in\mathcal{B}([0,T]\times E,\mathbb{R})$ such that for every $(s,x)\in[0,T]\times E$, $A^{s,x}=\int_s^{\cdot\vee s}h(r,X_r)dV_r$, in the sense of indistinguishability.*

*Proof.* We set
$$C^t_u = A^t_u + (V_u - V_t) + (u-t), \tag{4.20}$$
which is an AF with cadlag versions
$$C^{s,x}_t = A^{s,x}_t + V_t + t, \tag{4.21}$$
and we start by showing the statement for $A$ and $C$ instead of $A$ and $V$. We introduce the intermediary function $C$ so that for any $u>t$ that $\frac{A^{s,x}_u - A^{s,x}_t}{C^{s,x}_u - C^{s,x}_t}\in[0,1]$; that property will be used extensively in connections with the application of dominated convergence theorem.

Since $A^{s,x}$ is non-decreasing for any $(s,x)\in[0,T]\times E$, $A$ can be taken positive (in the sense that $A^t_u(\omega)\geq 0$ for any $(t,u)\in\Delta$ and $\omega\in\Omega$) by considering $A^+$ (defined by $(A^+)^t_u(\omega) := A^t_u(\omega)^+$) instead of $A$.

For $t\in[0,T[$ we set
$$\begin{aligned} K_t &:= \liminf_{n\to\infty} \frac{A^t_{t+\frac{1}{n}}}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} \\ &= \lim_{n\to\infty}\inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}. \end{aligned} \tag{4.22}$$

By positivity, this liminf always exists and belongs to $[0,1]$ since the sequence belongs to $[0,1]$. For every $(s,x)\in[0,T]\times E$, since for all $t\geq s$ and $n\geq 0$, $A^t_{t+\frac{1}{n}} = A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $K^{s,x}$ defined by $K^{s,x}_t := \liminf\limits_{n\to\infty} \frac{A^{s,x}_{t+\frac{1}{n}}-A^{s,x}_t}{C^{s,x}_{t+\frac{1}{n}}-C^{s,x}_t}$ is a $\mathbb{P}^{s,x}$-version of $K$, for $t\in[s,T[$. By Lebesgue Differentiation theorem (see Theorem 12 Chapter XV in [7] for a version of the theorem with a general atomless measure), for any $(s,x)$, for $\mathbb{P}^{s,x}$-almost all $\omega$, since $dC^{s,x}(\omega)$ is absolutely continuous with respect to $dA^{s,x}(\omega)$, $K^{s,x}(\omega)$ is a density of $dA^{s,x}(\omega)$ with respect to $dC^{s,x}(\omega)$.

We now show that there exists a Borel function $k$ in $\mathcal{B}([0,T[\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$, $k(t,X_t)$ is on $[s,T[$ a version of $K$ (and therefore of $K^{s,x}$). For every $t\in[0,T[$, $K_t$ is measurable with respect to $\bigcap\limits_{n\geq 0}\mathcal{F}_{t,t+\frac{1}{n}} = \mathcal{F}_{t,t}$ by construction, taking into account Notation 3.1. So for any $(t,x)\in[0,T]\times E$, by Proposition 3.14, there exists a constant which we denote $k(t,x)$ such that

$$K_t = k(t,x), \quad \mathbb{P}^{t,x}\text{a.s.} \tag{4.23}$$

For any integers $(n,m)$, we define

$$k^{n,m} : (t,x) \mapsto \mathbb{E}^{t,x}\left[\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

and for any $n$

$$k_n : (t,x) \mapsto \mathbb{E}^{t,x}\left[\inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right], \tag{4.24}$$

We start showing that

$$\tilde{k}^{n,m} : \begin{array}{rcl} (s,x,t) & \longmapsto & \mathbb{E}^{s,x}\left[\min\limits_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]\mathbb{1}_{s\leq t}, \\ {[0,T]\times E\times[0,T[} & \longrightarrow & [0,1], \end{array} \tag{4.25}$$

is jointly Borel. In order to do so, we will show that at fixed $t$, $\tilde{k}^{n,m}(\cdot,\cdot,t)$ is Borel, at fixed $(s,x)$, $\tilde{k}^{n,m}(s,x,\cdot)$ is right-continuous and we will conclude on the joint measurability thanks to Lemma 4.12.

If we fix $t\in[0,T[$, then by Proposition 3.10

$$(s,x) \longmapsto \mathbb{E}^{s,x}\left[\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]$$

is a Borel map. Since $(s,x)\mapsto \mathbb{1}_{[t,T]}(s)$ is obviously Borel, considering the product of the two previous maps, $\tilde{k}^{n,m}(\cdot,\cdot,t)$ is Borel. We now fix some $(s,x)$ and show that $\tilde{k}^{n,m}(s,x,\cdot)$ is right-continuous. Since that function is equal to zero on $[0,s[$, showing its continuity on $[s,T[$ will be sufficient. We remark that $A^{s,x}$ is continuous $\mathbb{P}^{s,x}$ a.s. $V$ is continuous, and the minimum of a finite number of continuous functions remains continuous. Let $t_q \underset{q\to\infty}{\longrightarrow} t$ be a converging sequence in $[s,T[$. Then $\min\limits_{n\leq p\leq m}\frac{A^{s,x}_{t_q+\frac{1}{p}}-A^{s,x}_{t_q}}{A^{s,x}_{t_q+\frac{1}{p}}-A^{s,x}_{t_q}+\frac{1}{p}+(V_{t_q+\frac{1}{p}}-V_{t_q})}$ tends a.s. to $\min\limits_{n\leq p\leq m}\frac{A^{s,x}_{t+\frac{1}{p}}-A^{s,x}_{t}}{A^{s,x}_{t+\frac{1}{p}}-A^{s,x}_{t}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_{t})}$, when $q$ tends to infinity. Since for any $s\leq t\leq u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $\frac{A^{t_q}_{t_q+\frac{1}{p}}}{A^{t_q}_{t_q+\frac{1}{p}}+\frac{1}{p}+(V_{t_q+\frac{1}{p}}-V_{t_q})}$ tends a.s. to $\frac{A^{t}_{t+\frac{1}{p}}}{A^{t}_{t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_{t})}$. All those terms belonging to $[0,1]$, by dominated convergence theorem, the mentioned convergence also holds under the expectation, hence the announced continuity related to $\tilde{k}^{n,m}$ is established and as anticipated, $\tilde{k}^{n,m}$ is jointly measurable in all its variables.

Since $k^{n,m}(t,y)=\tilde{k}^{n,m}(t,t,y)$, by composition we can deduce that for any $n,m$, $k^{n,m}$ is Borel. By the dominated convergence theorem, $k^{n,m}$ tends pointwise to $k^n$ (which was defined in (4.24), when $m$ goes to infinity so $k^n$ are also Borel for every $n$. Finally, keeping in mind (4.22) nd (4.23) we have $\mathbb{P}^{t,x}$ a.s.

$$k(t,x)=K_t=\lim_{n\to\infty}\inf_{p\geq n}\frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}.$$

Taking the expectation and again by the dominated convergence theorem, $k^n$ (defined in (4.24)) tends pointwise to $k$ when $n$ goes to infinity so $k$ is Borel.

We now show that, for any $(s,x)\in[0,T]\times E$, $k(\cdot,X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of $K$ on $[s,T[$. Since $\mathbb{P}^{t,x}(X_t=x)=1$, we know that for any $t\in[0,T]$, $x\in E$, we have $K_t=k(t,x)=k(t,X_t)$ $\mathbb{P}^{t,x}$-a.s., and we prove below that for any $t\in[0,T]$, $(s,x)\in[0,t]\times E$, we have $K_t=k(t,X_t)$ $\mathbb{P}^{s,x}$-a.s.

Let $t\in[0,T]$ be fixed. Since $A$ is an AF, for any $n$, $\frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$ is $\mathcal{F}_{t,t+\frac{1}{n}}$-measurable. So the event

$$\left\{\liminf_{n\to\infty}\frac{A^t_{t+\frac{1}{n}}}{A^t_{t+\frac{1}{n}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}=k(t,X_t)\right\}$$

belongs to $\mathcal{F}_{t,T}$ and by Markov property (3.4), for any $(s,x)\in[0,t]\times E$, we get

$$\begin{aligned}\mathbb{P}^{s,x}(K_t=k(t,X_t)) &= \mathbb{E}^{s,x}[\mathbb{P}^{s,x}(K_t=k(t,X_t)|\mathcal{F}_t)]\\ &= \mathbb{E}^{s,x}[\mathbb{P}^{t,X_t}(K_t=k(t,X_t))]\\ &= 1.\end{aligned}$$

For any $(s,x)$, the process $k(\cdot,X_\cdot)$ is therefore on $[s,T[$ a $\mathbb{P}^{s,x}$-modification of $K$ and therefore of $K^{s,x}$. However it is not yet clear if provides another density of $dA^{s,x}$ with respect to $dC^{s,x}$, which was defined at (4.21).

Considering that $(t,u,\omega)\mapsto V_u-V_t$ also defines a positive non-decreasing AF absolutely continuous with respect to $C$, defined in (4.20), we proceed similarly as at the beginning of the proof, replacing the AF $A$ with $V$.

Let the process $K'$ be defined by

$$K'_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)},$$

and for any$(s,x)$, let $K'^{s,x}$ be defined on $[s,T[$ by

$$K'^{s,x}_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)}.$$

Then, for any $(s,x)$, $K'^{s,x}$ on $[s,T[$ is a $\mathbb{P}^{s,x}$-version of $K'$, and it constitutes a density of $dV(\omega)$ with respect to $dC^{s,x}(\omega)$ on $[s,T[$, for almost all $\omega$. One shows then the existence of a Borel function $k'$ such that for any $(s,x)$, $k'(\cdot,X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of $K'$ and a modification of $K'^{s,x}$ on $[s,T[$. So for any $(s,x)$, under $\mathbb{P}^{s,x}$, we can write

$$\left\{\begin{array}{rcl} A^{s,x} & = & \int_s^{\cdot\vee s} K^{s,x}_r dC^{s,x}_r \\ V_{\cdot\vee s} - V_s & = & \int_s^{\cdot\vee s} K'^{s,x}_r dC^{s,x}_r \end{array}\right.$$

Now since $dA^{s,x} \ll dV$, for a fixed $\omega$, the set $\{r\in[s,T] | K'^{s,x}_r(\omega)=0\}$ is negligible with respect to $dV$ so also for $dA^{s,x}(\omega)$ and therefore we can write

$$\begin{array}{rcl} A^{s,x} & = & \int_s^{\cdot\vee s} K^{s,x}_r dC^{s,x}_r \\ & = & \int_s^{\cdot\vee s} \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r\neq 0\}} K'^{s,x}_r dC^{s,x}_r \\ & & + \int_s^{\cdot\vee s} \mathbb{1}_{\{K'^{s,x}_r = 0\}} dA^{s,x}_r \\ & = & \int_s^{\cdot\vee s} \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r\neq 0\}} dV_r, \end{array}$$

where we use the convention that for any two functions $\phi,\psi$ then $\frac{\phi}{\psi}\mathbb{1}_{\psi\neq 0}$ is defined by by

$$\frac{\phi}{\psi}\mathbb{1}_{\{\psi\neq 0\}}(x) = \left\{\begin{array}{l} \frac{\phi(x)}{\psi(x)} \text{ if } \psi(x)\neq 0 \\ 0 \text{ if } \psi(x) = 0. \end{array}\right.$$

We set now $h := \frac{k}{k'}\mathbb{1}_{\{k'_r\neq 0\}}$ which is Borel, and clearly for any $(s,x)$, $h(t,X_t)$ is a $\mathbb{P}^{s,x}$-version of $H^{s,x} := \frac{K^{s,x}}{K'^{s,x}}\mathbb{1}_{\{K'^{s,x}\neq 0\}}$ on $[s,T[$. So by Lemma 5.12 in [2], $H^{s,x}_t = h(t,X_t)$ $dV\otimes d\mathbb{P}^{s,x}$ a.e. and finally we have shown that under any $\mathbb{P}^{s,x}$, $A^{s,x} = \int_s^{\cdot\vee s} h(r,X_r)dV_r$ on $[0,T[$. Without change of notations we extend $h$ to $[0,T]\times E$ by zero for $t=T$. Since $A^{s,x}$ is continuous $\mathbb{P}^{s,x}$-a.s. previous equality extends to $T$. □

**Proposition 4.14.** *Let $(A^t_u)_{(t,u)\in\Delta}$ be an AF with bounded variation and taking $L^1$ values. Then there exists an increasing AF which we denote $(Pos(A)^t_u)_{(t,u)\in\Delta}$ (resp. $(Neg(A)^t_u)_{(t,u)\in\Delta}$ ) and which, for any $(s,x)\in[0,T]\times E$, has $Pos(A^{s,x})$ (resp. $Neg(A^{s,x})$) as cadlag version under $\mathbb{P}^{s,x}$.*

*Proof.* By definition of the total variation of a bounded variation function, the following holds. For every $(s,x)\in[0,T]\times E$, $s\le t\le u\le T$ for $\mathbb{P}^{s,x}$ almost all $\omega\in\Omega$, and any sequence of subdivisions of $[t,u]$: $t=t^k_1 < t^k_2 < \cdots < t^k_k = u$ such that $\min_{i<k}(t^k_{i+1}-t^k_i) \underset{k\to\infty}{\longrightarrow} 0$ we have

$$\sum_{i<k} |A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega)| \underset{k\to\infty}{\longrightarrow} Var(A^{s,x})_u(\omega) - Var(A^{s,x})_t(\omega), \tag{4.26}$$

taking into account the considerations of the end of Section 2. By Proposition 3.3 in [15] Chapter I, we have $Pos(A^{s,x}) = \frac{1}{2}(Var(A^{s,x}) + A^{s,x})$ and $Neg(A^{s,x}) = \frac{1}{2}(Var(A^{s,x}) - A^{s,x})$. Moreover, for any $x \in \mathbb{R}$ we know that $x^+ = \frac{1}{2}(|x| + x)$ and $x^- = \frac{1}{2}(|x| - x)$, so we also have

$$\left\{ \begin{array}{lcl} \sum\limits_{i<k} (A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega))^+ & \underset{k\to\infty}{\longrightarrow} & Pos(A^{s,x})_u(\omega) - Pos(A^{s,x})_t(\omega) \\ \sum\limits_{i<k} (A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega))^- & \underset{k\to\infty}{\longrightarrow} & Neg(A^{s,x})_u(\omega) - Neg(A^{s,x})_t(\omega), \end{array} \right. \tag{4.27}$$

for $\mathbb{P}^{s,x}$ almost all $\omega$. Since the convergence a.s. implies the convergence in probability, for every $(s,x) \in [0,T] \times E$, $s \le t \le u$ and any sequence of subdivisions of $[t,u]$: $t = t^k_1 < t^k_2 < \cdots < t^k_k = u$ such that $\min\limits_{i<k} (t^k_{i+1} - t^k_i) \underset{k\to\infty}{\longrightarrow} 0$, we have

$$\left\{ \begin{array}{lcl} \sum\limits_{i<k} \left( A^{t^k_i}_{t^k_{i+1}} \right)^+ & \overset{\mathbb{P}^{s,x}}{\underset{k\to\infty}{\longrightarrow}} & Pos(A^{s,x})_u - Pos(A^{s,x})_t \\ \sum\limits_{i<k} \left( A^{t^k_i}_{t^k_{i+1}} \right)^- & \overset{\mathbb{P}^{s,x}}{\underset{k\to\infty}{\longrightarrow}} & Neg(A^{s,x})_u - Neg(A^{s,x})_t. \end{array} \right. \tag{4.28}$$

The proof can now be performed according to the same arguments as in the proof of Proposition 4.4, replacing $M$ with $A$, the quadratic increments with the positive (resp. negative) increments, and the quadratic variation with the positive (resp. negative) variation of an adapted process. □

We recall a definition and a result from [2]. We assume for now that we are given a fixed stochastic basis fulfilling the usual conditions, and a non-decreasing function $V$.

*Notation* 4.15. We denote $\mathcal{H}^{2,V} := \{M \in \mathcal{H}^2_0 | d\langle M\rangle \ll dV\}$ and $\mathcal{H}^{2,\perp V} := \{M \in \mathcal{H}^2_0 | d\langle M\rangle \perp dV\}$.

Proposition 3.6 in [2] states the following.

**Proposition 4.16.** *$\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sub-Hilbert spaces of $\mathcal{H}^2_0$ and $\mathcal{H}^2_0 = \mathcal{H}^{2,V} \oplus^{\perp} \mathcal{H}^{2,\perp V}$. Moreover, any element of $\mathcal{H}^{2,V}_{loc}$ is strongly orthogonal to any element of $\mathcal{H}^{2,\perp V}_{loc}$.*

For any $M \in \mathcal{H}^2_0$, we denote by $M^V$ its projection on $\mathcal{H}^{2,V}$.

We can now finally establish the main result of the present note.

**Proposition 4.17.** *Let $V$ be a continuous non-decreasing function. Let $M, N$ be two square integrable MAFs, and assume that the AF $\langle N\rangle$ is absolutely continuous with respect to $V$. There exists a function $v \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x)$, $\langle M^{s,x}, N^{s,x}\rangle = \int_s^{\cdot\vee s} v(r, X_r) dV_r$.*

*Proof.* By Corollary 4.11, there exists a bounded variation AF with $L^1$ values denoted $\langle M, N\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, N\rangle$ is $\langle M^{s,x}, N^{s,x}\rangle$. By Proposition 4.14, there exists an increasing AF with $L^1$ values denoted $Pos(\langle M, N\rangle)$ (resp. $Neg(\langle M, N\rangle)$) such that under any $\mathbb{P}^{s,x}$, the cadlag version of $Pos(\langle M, N\rangle)$ (resp. $Neg(\langle M, N\rangle)$) is $Pos(\langle M^{s,x}, N^{s,x}\rangle)$ (resp. $Neg(\langle M^{s,x}, N^{s,x}\rangle)$). We fix some $(s,x)$ and the associated probability $\mathbb{P}^{s,x}$. Since

$\langle N \rangle$ is absolutely continuous with respect to $V$, comparing Definition 4.1 and Notation 4.15 we have $N^{s,x} \in \mathcal{H}^{2,V}$. Therefore by Proposition 4.16 we have

$$\begin{array}{rcl} \langle M^{s,x}, N^{s,x} \rangle & = & \langle (M^{s,x})^V, N^{s,x} \rangle \\ & = & \frac{1}{4}\langle (M^{s,x})^V + N^{s,x} \rangle - \frac{1}{4}\langle (M^{s,x})^V - N^{s,x} \rangle. \end{array} \tag{4.29}$$

Since both processes $\frac{1}{4}\langle (M^{s,x})^V + N^{s,x} \rangle$, $\frac{1}{4}\langle (M^{s,x})^V - N^{s,x} \rangle$ are increasing and starting at zero, we have $Pos(\langle M^{s,x}, N^{s,x} \rangle) = \frac{1}{4}\langle (M^{s,x})^V + N^{s,x} \rangle$ and

$$Neg(\langle M^{s,x}, N^{s,x} \rangle) = \frac{1}{4}\langle (M^{s,x})^V - N^{s,x} \rangle.$$

Now since $(M^{s,x})^V + N^{s,x}$ and $(M^{s,x})^V - N^{s,x}$ belong to $\mathcal{H}^{2,V}$, we have shown that $dPos(\langle M^{s,x}, N^{s,x} \rangle) \ll dV$ and $dNeg(\langle M^{s,x}, N^{s,x} \rangle) \ll dV$ in the sense of stochastic measures.

Since this holds for all $(s,x)$ Proposition 4.13, insures the existence of two functions $v_+, v_-$ in $\mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x)$, $Pos(\langle M^{s,x}, N^{s,x} \rangle) = \int_s^{\cdot \vee s} v_+(r, X_r) dV_r$ and $Neg(\langle M^{s,x}, N^{s,x} \rangle) = \int_s^{\cdot \vee s} v_-(r, X_r) dV_r$. The conclusion now follows setting $v = v_+ - v_-$. □

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