ESSENTIAL SETS FOR RANDOM OPERATORS
CONSTRUCTED FROM AN ARRATIA FLOW

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Abstract. In this paper we consider a strong random operator $T_t$ which describes a shift of functions from $L_2(\mathbb{R})$ along an Arratia flow. We find a compact set in $L_2(\mathbb{R})$ that doesn’t disappear under $T_t$, and estimate its Kolmogorov widths.

1. Introduction: Arratia Flow and Random Operators

In this paper we consider random operators in $L_2(\mathbb{R})$ which describe shifts of functions along an Arratia flow [1]. Let us recall the definition.

Definition 1.1 ([1]). A family of random processes $\{x(u, s), u \in \mathbb{R}, s \geq 0\}$ is called an Arratia flow if

1) for each $u \in \mathbb{R}$ $x(u, \cdot)$ is a Wiener process with respect to the joint filtration such that $x(u, 0) = u$;
2) for any $u_1 \leq u_2$ and $t \geq 0$

$$x(u_1, t) \leq x(u_2, t) \ a.s.$$  

3) the joint characteristics are

$$d < x(u_1, \cdot), x(u_2, \cdot) > (t) = \mathbb{I}_{x(u_1, t) = x(u_2, t)} dt.$$  

In the informal language, Arratia flow is a family of Wiener processes started from each point of $\mathbb{R}$, which move independently up to the meeting, coalesce, and move together. It was proved in [4, 8] that for any $a, b \in \mathbb{R}$ and $t > 0$ the set $x([a; b], t)$ is finite $a.s.$ Since Arratia flow has a right-continuous modification [3], $x(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ is a step function for any time $t > 0$. Hence, for any $a, b \in \mathbb{R}$ and $t > 0$ with probability one there exists a random point $y \in \mathbb{R}$ for which

$$\lambda\{u \in [a; b]: x(u, t) = y\} > 0, \quad (1.1)$$

where $\lambda$ is Lebesgue measure on $\mathbb{R}$. Since $x(\cdot, t)$ is a right-continuous step function, for a fixed countable set $A$

$$\mathbb{P}\{x(\mathbb{R}, t) \cap A \neq \emptyset\} \geq \mathbb{P}\{x(Q, t) \cap A \neq \emptyset\} \leq \sum_{u \in Q} \mathbb{P}\{x(u, t) \in A\} = 0. \quad (1.2)$$

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Since for any \(a < b\) the difference \(\frac{x(b) - x(a)}{\sqrt{2}}\) is a Wiener processes until the collision happens, and \(\frac{x(b,0) - x(a,0)}{\sqrt{2}} = \frac{b-a}{\sqrt{2}}\), one can find the distribution of the time of coalescence \(\tau_{a,b} = \inf\{s \geq 0| x(a, s) = x(b, s)\}\) of the processes \(x(a, \cdot), x(b, \cdot)\), i.e. for any \(t \geq 0\)

\[
P\{\tau_{a,b} \leq t\} = P\{x(a, t) = x(b, t)\} = \sqrt{\frac{2}{\pi}} \int_{\frac{b-a}{\sqrt{2}}}^{\infty} e^{-\frac{v^2}{2}} dv.
\]  

(1.3)

Let us notice that for a fixed time \(t > 0\) and an Arratia flow \(X = \{x(u, s), u \in \mathbb{R}, s \in [0; t]\}\) there exists an Arratia flow \(Y = \{y(u, r), u \in \mathbb{R}, r \in [0; t]\}\) such that trajectories of \(X\) and \(Y\) don't cross [1, 7]. \(Y\) is called a conjugated (or dual) Arratia flow. It was proved in [10] the following change of variable formula for an Arratia flow.

**Theorem 1.2 ([10]).** For any time \(t > 0\) and nonnegative measurable function \(h : \mathbb{R} \to \mathbb{R}\) such that \(\int_{\mathbb{R}} h(u)du < \infty\)

\[
\int_{\mathbb{R}} h(x(u, t))du = \int_{\mathbb{R}} h(u)dy(u, t) \quad a.s.,
\]  

(1.4)

where the last integral is in sense of Lebesgue-Stieltjes.

In this paper we consider random operators \(T_t, \ t > 0\), in \(L_2(\mathbb{R})\) which are defined as follows

\[
(T_t f)(u) = f(x(u, t)),
\]

where \(f \in L_2(\mathbb{R})\) and \(u \in \mathbb{R}\). It was proved in [5] that \(T_t\) is a strong random operator [11] in \(L_2(\mathbb{R})\), but, as it was shown in [10], is not a bounded one. Really, for the point \(y\) from (1.1) one can introduce a sequence of the intervals \(A_i = [r_i; p_i]\) such that \(y \in A_i\) for any \(i \geq 1\) and \(p_i - r_i \to 0, i \to \infty\). Thus for any \(i \geq 1\)

\[
\|T_T I_{A_i}\|_{L_2(\mathbb{R})}^2 \geq \lambda\{u \in [a; b] : x(u, t) = y\} > 0,
\]

which can’t be true if \(T_t\) was a bounded random operator. Hence, the image of a compact set under \(T_t\) may not be a random compact set. Moreover, as it was mentioned in [9], the image of a compact set under strong random operator may not exist. However, in [10] it was presented a family of compact sets in \(L_2(\mathbb{R})\) whose images under \(T_t\) exist and are random compact sets. In this paper we consider a compact set of this type, and investigate the change of its Kolmogorov widths [12] under \(T_t\).

2. \(T_t\)-essential Functions

If the support of the function \(f \in L_2(\mathbb{R})\) is bounded, \(supp f \subset [a; b]\), then \(T_t f\) equals to 0 with positive probability. Really, by (1.4), one can check that
\[
P \left\{ \int_{-\infty}^{\infty} f^2(x(u, t))du = 0 \right\} \geq P \{ x(\mathbb{R}, t) \cap [a; b] = \emptyset \}
\]
\[
= P \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a,b]}(x(u, t))du = 0 \right\}
\]
\[
= P \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a,b]}(u)dy(u, t) = 0 \right\},
\]
where \( \{y(u, s), u \in \mathbb{R}, s \in [0; t]\} \) is a conjugated Arratia flow. Since, by (1.3),
\[
P \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a,b]}(u)dy(u, t) = 0 \right\} = P \{ y(b, t) = y(a, t) \} > 0,
\]
then \( P \{ \|T_t f\|_{L_2(\mathbb{R})} > 0 \} > 0 \). This leads to the following definition.

**Definition 2.1.** For a fixed \( t > 0 \) a function \( f \in L_2(\mathbb{R}) \) is said to be a \( T_t \)-essential if
\[
P \{ \|T_t f\|_{L_2(\mathbb{R})} > 0 \} = 1.
\]

**Example 2.2.** Let \( f \in L_2(\mathbb{R}) \) be an analytic function which doesn’t equal totally to zero. Denote the set of its zeroes \( Z_f = \{ u \in \mathbb{R} \mid f(u) = 0 \} \). Then, by (1.2), \( P \{ x(\mathbb{R}, t) \cap Z_f = \emptyset \} = 1 \), so \( f \) is a \( T_t \)-essential for any \( t > 0 \).

Let us notice that if \( t_1 \neq t_2 \) then \( T_{t_1} \)-essential function may not be a \( T_{t_2} \)-essential. To introduce a \( T_{t_1} \)-essential that is not \( T_{t_2} \)-essential function let us consider an increasing sequence \( \{u_k\}_{k=0}^{\infty} \) such that \( u_0 = 0, u_1 = 1 \) and for any \( n \in \mathbb{N} \)
\[
u_{2n+1} - u_{2n} = \frac{1}{2^n}, \quad u_{2n} = u_{2n-1} + 2n(\ln 2)^2.
\]

**Theorem 2.3.** The function \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]} \) is a \( T_{t_1} \)-essential, and is not a \( T_{t_2} \)-essential.

**Proof.** To prove that \( f \) is not a \( T_{t_2} \) essential we show that \( P \{ \|T_{t_2} f\|_{L_2(\mathbb{R})} > 0 \} < 1 \). Since \([u_{2k}; u_{2k+1}] \cap [u_{2j}; u_{2j+1}] = \emptyset\) for any \( k \neq j \) then, by (1.4),
\[
P \{ \|T_{t_2} f\|_{L_2(\mathbb{R})} > 0 \} = P \left\{ \int_{-\infty}^{\infty} \left( \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}(x(u, 2)) \right)^2 du > 0 \right\}
\]
\[
= P \left\{ \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}(x(u, 2))du > 0 \right\}
\]
expression and complete the proof

From the fact that

Let

Proof.

Thus by (1.3),

Consequently, the function \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]} \) is not a \( T_2 \)-essential. To prove that \( f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n};u_{2n+1}]} \) is a \( T_1 \)-essential one can show the following estimation.

Lemma 2.4. Let \( \{ w(u_n, \cdot) \}_{n=0}^{\infty} \) be a family of independent Wiener processes on \([0;1]\) such that \( w(u_n,0) = u_n \). Then for any \( n \in \mathbb{N} \)

\[
\mathbb{P} \left\{ \max_{s \in [0;1]} \max_{j=0,2n-1} w(u_j, s) \geq \min_{s \in [0;1]} w(u_{2n}, s) \right\} < \frac{1}{2n^2 \sqrt{\pi } \ln 2} \]

Proof. Let \( w_1, w_2 \) be an independent Wiener processes on \([0;1]\) started from point 0, i.e. \( w_1(0) = w_2(0) = 0 \). It can be noticed that

\[
\mathbb{P} \left\{ \max_{s \in [0;1]} \max_{j=0,2n-1} w(u_j, s) \geq \min_{s \in [0;1]} w(u_{2n}, s) \right\} = \mathbb{P} \left\{ \exists j = 0,2n-1: \max_{s \in [0;1]} w(u_j, s) - \min_{s \in [0;1]} w(u_{2n}, s) \geq 0 \right\}
\]

\[
\leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w(u_j, s) - \min_{s \in [0;1]} w(u_{2n}, s) \geq 0 \right\}
\]

\[
\leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w_1(s) - \min_{s \in [0;1]} w_2(s) \geq u_{2n} - u_j \right\}.
\]

From the fact that \( \{ u_n \}_{n=0}^{\infty} \) is an increasing sequence we can estimate the last expression and complete the proof

\[
\sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0;1]} w_1(s) - \min_{s \in [0;1]} w_2(s) \geq u_{2n} - u_j \right\}
\]

\[
\leq \frac{1}{\sqrt{\pi }} \sum_{j=0}^{2n-1} \frac{1}{u_{2n} - u_j} e^{-\frac{(u_{2n} - u_j)^2}{4}}
\]

\[
\leq \frac{2n - 1}{\sqrt{\pi } (u_{2n} - u_{2n-1})} e^{-\frac{(u_{2n} - u_{2n-1})^2}{4}}
\]

\[
\leq \frac{1}{2n^2 \sqrt{\pi } \ln 2}.
\]
Let us prove that the function $f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}, u_{2n+1})}$ is a $T_1$-essential. Using the reasoning from the first part of the proof it can be checked that for the considered function $f$ the following equality holds

$$
\mathbb{P}\{ \|T_1 f\|_{L_2(\mathbb{R})} > 0 \} = \mathbb{P}\left\{ \sum_{n=0}^{\infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) > 0 \right\}.
$$

Let us prove that

$$
\mathbb{P}\left\{ \limsup_{n \to \infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) \geq 1 \right\} = 1. \quad (2.1)
$$

Build a new processes $\{\tilde{y}(u_t, \cdot)\}_{n=0}^{\infty}$ such that $\{\tilde{y}(u_t, \cdot)\}_{n=0}^{\infty}$ and $\{y(u_t, \cdot)\}_{n=0}^{\infty}$ have the same distributions in $C([0; 1])$ by $\tau [f, g] := \inf \{ t \mid f(t) = g(t) \}$. Put $\tilde{y}(u_0, \cdot) := w(u_0, \cdot)$. Then for any $n \in \mathbb{N}$, $s \in [0; 1]$ one can define

$$
\tilde{y}(u_n, s) = w(u_n, s) \mathbb{I}\{ s < \tau[w(u_{n-1}, \cdot), \tilde{y}(u_{n-1}, \cdot)] \} + \tilde{y}(u_{n-1}, s) \mathbb{I}\{ s \geq \tau[w(u_{n-1}, \cdot), \tilde{y}(u_{n-1}, \cdot)] \}.
$$

According to constructions of stochastic processes $\{\tilde{y}(u_t, \cdot)\}_{n=0}^{\infty}$

$$
\mathbb{P}\{ \exists N \in \mathbb{N} : \forall n \geq N \quad \tilde{y}(u_{2n}, t) = w(u_{2n}, t),
\tilde{y}(u_{2n+1}, t) = w(u_{2n+1}, t) \mathbb{I}\{ t < \tau[w(u_{2n}, \cdot), w(u_{2n+1}, \cdot)] \} + w(u_{2n}, t) \mathbb{I}\{ t \geq \tau[w(u_{2n}, \cdot), w(u_{2n+1}, \cdot)] \} = 1. \quad (2.2)
$$

Thus

$$
\mathbb{P}\{ \exists N \in \mathbb{N} : \forall n \geq N \quad \tilde{y}(u_{2n+1}, t) - \tilde{y}(u_{2n}, t) = w(u_{2n+1}, t) - w(u_{2n}, t) \} = 1.
$$

For the considered sequence $\{u_n\}_{n=0}^{\infty}$ and any $n \in \mathbb{N}$ the following inequality holds

$$
\mathbb{P}\{ w(u_{2n+1}, t) - w(u_{2n}, t) \geq 1 \} = \int_{1}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{(v-1)^2}{4}} dv \geq \frac{1}{\sqrt{4\pi}} \int_{1}^{\infty} e^{-\frac{v^2}{4}} dv.
$$

Therefore, by the Borel-Cantelli lemma and (2.2),

$$
\mathbb{P}\{ \limsup_{n \to \infty} (\tilde{y}(u_{2n+1}, t) - \tilde{y}(u_{2n}, t)) \geq 1 \} = 1.
$$

Using the observation from Example 2.2 one can introduce a family of $T_1$-essential functions for all $t > 0$.

For any $\varepsilon > 0$ let us consider an integral operator $K_\varepsilon$ in $L_2(\mathbb{R})$ with the kernel

$$
k_\varepsilon(v_1, v_2) = \int_{\mathbb{R}} p_\varepsilon(u - v_1) p_\varepsilon(u - v_2) dy(u, t), \quad (2.3)
$$
where \( v_1, v_2 \in \mathbb{R} \), and \( p_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-u^2/2\varepsilon} \). By the change of variables formula for an Arratia flow [10],

\[
(K_\varepsilon f, f) = \int_{\mathbb{R}} (f * p_\varepsilon)^2(x(u, t))\,du.
\]

**Lemma 2.5.** For any \( \varepsilon > 0 \) and nonzero function \( f \in L_2(\mathbb{R}) \)

\[
P\{ (K_\varepsilon f, f) \neq 0 \} = 1.
\]

**Proof.** According to (2.1) it is sufficient to note that \( f * p_\varepsilon \) is an analytic function. Consequently, for any \( t > 0 \) the following relations are true

\[
P\{ (K_1 f, f) > 0 \} = P\{ \Vert T_1(f * p_1) \Vert_{L_2(\mathbb{R})} > 0 \} = P\{ x(\mathbb{R}, t) \cap Z_{f * p_1} = \emptyset \} = 1.
\]

\[\square\]

According to the last theorem and (2.4), for any \( \varepsilon > 0 \) and nonzero \( f \in L_2(\mathbb{R}) \) the function \( f * p_\varepsilon \) is a \( T_1 \)-essential for each \( t > 0 \).

### 3. Change of Compact Sets under a Strong Random Operator

Generated by an Arratia Flow

As it was noticed in the introduction any function with bounded support isn’t a \( T_1 \)-essential. Consequently, if \( K \subseteq L_2(\mathbb{R}) \) is a compact set of functions with uniformly bounded supports such that \( T_t(K) \) is well-defined, then the image \( T_t(K) \) equals to \( \{0\} \) with positive probability. It was shown in [10] that \( T_t \) may also change the geometry of \( K \) even in the case of a compact set \( K \) for which \( T_t(K) \neq \{0\} \) a.s. For example, the image \( T_t(K) \) of a convergent sequence and its limiting point may not have limiting points. In this section we build a compact set \( K \) for which \( T_t(K) \neq \{0\} \) a.s. and investigate the change of its Kolmogorov-widths in \( L_2(\mathbb{R}) \) under random operator \( T_t \).

**Definition 3.1 ([12]).** The **Kolmogorov n-width** of a set \( C \subseteq H \) in a Hilbert space \( H \) is given by

\[
d_n(C) = \inf_{\dim L \leq n} \sup_{f \in C} \inf_{g \in L} \|f - g\|_H,
\]

where \( L \) is a subspace of \( H \).

We consider the following compact set in \( L_2(\mathbb{R}) \)

\[
K = \left\{ f \in W_2^1(\mathbb{R}) \mid \int_{\mathbb{R}} f^2(u)(1 + |u|^3)\,du + \int_{\mathbb{R}} (f'(u))^2(1 + |u|)^7\,du \leq 1 \right\}.
\]

Estimations on its Kolmogorov-widths in \( L_2(\mathbb{R}) \) are presented in the next lemma.

**Lemma 3.2.** There exist positive constants \( C_1, C_2 \) such that for any \( n \in \mathbb{N} \)

\[
\frac{C_1}{n} \leq d_n(K) \leq \frac{C_2}{n^{1/2}}.
\]
Proof. Let \( n \in \mathbb{N} \) be fixed. To estimate \( d_n(K) \) from above one can consider the partition \( \{ u_k \}_{k=0}^{n-1} \) of \([-n^{\frac{1}{2}}; n^{\frac{1}{2}}]\) into \( n \) segments \( \{ [u_k; u_{k+1}], k = 0, n-1 \} \) with equal lengths. Let us show that for the \( n \)-dimensional subspace \( L_n = LS\{ \Pi_{[u_k; u_{k+1}]}, k = 0, n-1 \} \)

\[
\sup_{f \in K} \inf_{g \in L_n} \| f - g \|_{L_2(\mathbb{R})} \leq \frac{C_2}{n^{\frac{1}{2}}}. \]

If \( f \in K \) then \( \int_{|u|>c} f^2(u)(1+|u|)^3 du \leq 1 \). Thus for any \( C > 0 \)

\[
\int_{|u|>c} f^2(u) du \leq \frac{1}{(1+C)^3} \int_{|u|>c} f^2(u)(1+|u|)^3 du \leq \frac{1}{C^3}. \]

So, for the function \( g_f = \sum_{k=0}^{n-1} f(u_k) \Pi_{[u_k; u_{k+1}]} \in L_n \) the following estimation is true

\[
\| f - g_f \|_{L_2(\mathbb{R})}^2 \leq \frac{1}{n^{\frac{1}{2}}} + \int_{|u| \leq n^{\frac{3}{2}}} (f(u) - g_f(u))^2 du. \]

By the Cauchy inequality, for \( f \in K \) and \( u \in [u_k; u_{k+1}] \)

\[
\left( \int_{u_k}^{u} f'(v) dv \right)^2 \leq \int_{u_k}^{u} \frac{dv}{(1+|v|)^2} \leq u - u_k.
\]

Consequently,

\[
\int_{|u| \leq n^{\frac{3}{2}}} (f(u) - g_f(u))^2 du = \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left( \int_{u_k}^{u} f'(v) dv \right)^2 du
\]

\[
\leq \frac{1}{2} \sum_{k=0}^{n-1} (u_{k+1} - u_k)^2 = \frac{2}{n^{\frac{3}{2}}},
\]

and the upper estimation for \( d_n(K) \) holds with the constant \( C_2 = 3\frac{2}{n} \).

To get a lower estimation for \( d_n(K) \) we use the theorem about \( n \)-width of \( (n+1) \)-dimensional ball [12]. Let \( \{ u_k \}_{k=0}^{2(n+1)} \) be a partition of \([0; 1]\) into \( 2(n+1) \) segments \( \{ [u_k; u_{k+1}], k = 0, 2n+1 \} \) with equal lengths. Consider \( (n+1) \)-dimensional space \( L_{n+1} = LS\{ f_k, k = 0, n \} \), where the functions \( f_k, k = 0, n \), are defined as follows

\[
f_k = \begin{cases} 
0, & u \notin [u_{2k}; u_{2k+1}], \\
1, & u \in [u_{2k} + \frac{1}{6(n+1)}; u_{2k+1}], \\
6(n+1)(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{1}{6(n+1)}], \\
-6(n+1)(u - u_{2k+1}), & u \in [u_{2k} + \frac{2}{6(n+1)}; u_{2k+1}].
\end{cases}
\]

We show that if \( c = \frac{2\gamma^2}{(n+1)^2} \) then the ball \( B_{n+1} = \{ f \in L_{n+1} \| f \|_{L_2(\mathbb{R})} \leq \frac{1}{\sqrt{cn}} \} \) is a subset of \( K \). Since \( \| f_k \|_{L_2(\mathbb{R})}^2 = \frac{5}{18(n+1)}, k = 0, n \), then for any \( f \in B_{n+1} \)

such that \( f = \sum_{k=0}^{n} c_k f_k \) the following relation holds \( \sum_{k=0}^{n} c_k^2 \leq \frac{36}{5cn} \). Thus according
to (3.2),
\[
\int_{\mathbb{R}} f^2(u)(1 + |u|)^3 du + \int_{\mathbb{R}} (f'(u))^2 (1 + |u|)^7 du \\
\leq 2^3 \|f\|_{L^2(\mathbb{R})}^2 + 2^7 \cdot \sum_{k=0}^{n} c_k^2 \left( \int_{u_{2k+1}}^{u_{2k}+1} (6(n+1))^2 du \right) \\
\leq \frac{2^3}{c_n^2} + 2^{10} \cdot 3n \cdot \frac{36}{5cn} \leq \frac{1}{c} \cdot \frac{2^3(5 + 2^9 \cdot 3^3)}{5} = 1.
\]

Consequently, \( B_{n+1} \subset K \) and \( d_n(K) \geq d_n(B_{n+1}) \). Due to the theorem about \( n \)-width of \((n+1)\)-dimensional ball, \( d_n(B_{n+1}) = \frac{1}{\sqrt{n}} \) [12]. So the lower estimation for \( d_n(K) \) holds with \( C_1 := \sqrt{c} \).

\[\Box\]

To show that estimations from above for the Kolmogorov-widths of the considered compact set \( K \) don’t change under \( T_t \) one may use the same idea as in Lemma 2.

**Theorem 3.3.** There exists \( \Omega \) of probability one such that for any \( \omega \in \Omega \) and \( n \in \mathbb{N} \)
\[
d_n(T_t^\omega(K)) \leq \frac{C(\omega)}{n^{\overline{m}}}, \tag{3.3}
\]
where the constant \( C(\omega) > 0 \) doesn’t depend on \( n \).

**Proof.** For a fixed \( n \in \mathbb{N} \) let us consider a partition \( \{u_k\}_{k=0}^{n} \) of \([-n^{\frac{1}{m}}; n^{\frac{1}{m}}]} \) into \( n \) segments with equal lengths. To prove (3.3) it’s sufficient to show the following inequality for the linear space \( L_n^\omega = LS\{ T_t^\omega \Pi_{[u_k; u_{k+1}]}, \ k = 0, n-1 \} \) with dimension at most \( n \)
\[
\sup_{h_1 \in T_t^\omega(K)} \inf_{h_2 \in L_n^\omega} \|h_1 - h_2\|_{L^2(\mathbb{R})} \leq \frac{C(\omega)}{n^{\overline{m}}}.
\]

According to the change of variable formula for an Arratia flow, one can check the equality for any \( f \in K \)
\[
\left\| T_t^\omega f - T_t^\omega \left( \sum_{k=0}^{n-1} f(u_k)\Pi_{[u_k; u_{k+1}]} \right) \right\|_{L^2(\mathbb{R})}^2 = \int_{|u| > n^{\frac{1}{m}}} f^2(u)dy(u, t, \omega) \\
+ \int_{|u| \leq n^{\frac{1}{m}}} \left( f(u) - \sum_{k=0}^{n-1} f(u_k)\Pi_{[u_k; u_{k+1}]}(u) \right)^2 dy(u, t, \omega).
\]
To estimate from above the last integral let us notice that
\[
\int_{|u| \leq n^\frac{1}{5}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k, u_{k+1}]}(u)\right)^2 dy(u, t, \omega)
\leq \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^u |f'(v)| dv \right)^2 dy(u, t, \omega).
\]

Due to (3.1), for any \( f \in K \) and \( u \in [u_k; u_{k+1}] \)
\[
\left(\int_{u_k}^u |f'(v)| dv \right)^2 \leq \int_{u_k}^u \frac{dv}{(1 + |v|)^{\frac{7}{2}}} \leq u_{k+1} - u_k.
\]

Thus
\[
\sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^u |f'(v)| dv \right)^2 dy(u, t, \omega) \leq \sum_{k=0}^{n-1} (u_{k+1} - u_k) \int_{u_k}^{u_{k+1}} dy(u, t, \omega)
= \frac{2}{n^\frac{7}{2}} (y(n^\frac{1}{2}, t, \omega) - y(-n^\frac{1}{2}, t, \omega)).
\]

For an Arratia flow \( \{y(u, s), u \in \mathbb{R}, s \in [0; t]\} \) the following relation is true [2]
\[
\lim_{|u| \to \infty} \frac{|y(u, t)|}{|u|} = 1 \text{ a.s.}
\]

Consequently, for any \( \omega \in \bar{\Omega} = \{\omega' \in \Omega \mid \lim_{|u| \to \infty} \frac{|y(u, t, \omega')|}{|u|} = 1\} \) the estimation holds
\[
\int_{|u| \leq n^\frac{1}{5}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k, u_{k+1}]}(u)\right)^2 dy(u, t, \omega) \leq \frac{4c(\omega)}{n^\frac{7}{2}} \tag{3.4}
\]
with the constant
\[
c(\omega) = \sup_{|u| \geq 1} \frac{|y(u, t, \omega)|}{|u|}. \tag{3.5}
\]

Let us prove that for any \( \omega \in \bar{\Omega} \) there exists a constant \( \bar{c}(\omega) \) such that
\[
\int_{|u| > n^\frac{1}{5}} f^2(u) dy(u, t, \omega) \leq \frac{\bar{c}(\omega)}{n^\frac{7}{2}}.
\]

It can be noticed that
\[
\int_{|u| > n^\frac{1}{5}} f^2(u) dy(u, t, \omega) \leq \frac{1}{n^\frac{7}{2}} \int_{|u| > n^\frac{1}{5}} f^2(u)(1 + |u|)^3 dy(u, t).
\]
Denote by \( \{\theta_j\}_{j=1}^{\infty} \) a sequence of jump points of the function \( y(\cdot, t) \) on \( \mathbb{R}_+ \). Thus
one may show
\[
\int_{u>n^{\frac{1}{2}}} f^2(u)(1 + u)^3 dy(u,t) = \sum_{\theta_i \geq n^{\frac{1}{2}}} f^2(\theta_i)(1 + \theta_i)^3 \Delta y(\theta_i, t)
= \sum_{k=1}^{\infty} \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i)(1 + \theta_i)^3 \Delta y(\theta_i, t)
\leq \sum_{k=1}^{\infty} (2 + k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t).
\]

According to the Cauchy inequality and (3.1), for any \( u \in \mathbb{R}_+ \) the following relations hold
\[
f^2(u) \leq \int_{u}^{\infty} (f'(v))^2 (1 + v)^7 dv \cdot \int_{u}^{\infty} \frac{dv}{(1 + v)^7} \leq \frac{1}{6u^6}.
\]
Consequently, due to (3.5), the inequalities are true
\[
\sum_{k=1}^{\infty} (2 + k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t)
\leq \sum_{k=1}^{\infty} (2 + k)^3 \frac{1}{6k^6} (y(k + 1, t) - y(k, t))
\leq \frac{16c}{3} \sum_{k=1}^{\infty} \frac{1}{k^2}.
\]
Hence, for any \( \omega \in \tilde{\Omega} \) there exists the constant \( C_1(\omega) = \frac{16c(\omega)}{3} \) such that
\[
\int_{u>n^{\frac{1}{2}}} f^2(u) dy(u,t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{1}{2}}},
\]
Similarly, it can be proved that \( \int_{u<-n^{\frac{1}{2}}} f^2(u) dy(u,t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{1}{2}}} \). According to this and (3.4), for any \( \omega \in \tilde{\Omega} \) an upper estimation for \( d_n(T_t^\omega(K)) \) is true. \( \square \)

The functions from Lemma 2 that were used to build the \((n + 1)\)-dimensional subspace are not \( T_t \)-essential for any \( t > 0 \). Thus the image of this subspace under the random operator \( T_t \) may be equal to \( \{0\} \) with positive probability. So, one can ask about the existence of a finite-dimensional subspace such that for any \( t > 0 \) its image under \( T_t \) is a linear subspace with the same dimension.

4. A Subspace Preserving the Dimension under a Random Operator Generated by an Arratia Flow

In this section for any \( t > 0 \) and \( n \in \mathbb{N} \) we present a family \( \{g_k, k = 0, n\} \) of linearly independent \( T_t \)-essential functions such that their images under \( T_t \) are linearly independent. Such a family generates a subspace which preserves the
dimension under a random operator generated by an Arratia flow. It can be used to get a lower estimation of $d_n(T_i(K))$.

Let us fix any $n \in \mathbb{N}$, and build a family of $(n+1)$ linearly independent functions in the following way. Let $\{u_k\}_{k=0}^{2(n+1)}$ be a partition of $[0; n^{-2}]$ into $2(n+1)$ segments with equal lengths. For any $k = 0, n$ define $f_k$ by

$$f_k = \begin{cases} 0, & u \notin [u_{2k}; u_{2k+1}], \\ 1, & u \in [u_{2k} + \frac{n^{-2}}{6(n+1)}; u_{2k} + \frac{2n^{-2}}{6(n+1)}], \\ \frac{6(n+1)}{n^{-2}}(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{n^{-2}}{6(n+1)}], \\ -\frac{6(n+1)}{n^{-2}}(u - u_{2k+1}), & u \in [u_{2k} + \frac{2n^{-2}}{6(n+1)}; u_{2k+1}]. \end{cases} \quad (4.1)$$

Lemma 4.1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the functions $\{f_k*p_c, k = 0, n\}$ are linearly independent.

Proof. Since the considered functions $\{f_k, k = 0, n\}$ are linearly independent, its Gram determinant doesn’t equal to 0, i.e. $G(f_0, \ldots, f_n) \neq 0$. For each $k = 0, n$

$$f_k*p_c \rightarrow f_k, \ \varepsilon \rightarrow 0.$$ 

Hence, due to the continuity of the Gram determinant, one may notice that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$

$$G(f_0*p_c, \ldots, f_n*p_c) \neq 0,$$

and the desired result is proved. \hfill \square

Theorem 4.2. There exists a set $\Omega_0$ of probability one such that for any $\omega \in \Omega_0$ the functions $T_i^c(f_0*p_c), \ldots, T_i^c(f_n*p_c)$ are linearly independent.

Proof. Denote by $K_c$ the integral operator in $L_2(\mathbb{R})$ with the kernel $k_c$. To prove the statement of the theorem it’s enough to show that on some $\Omega_0$ of probability one the following inequality holds ($K_c f, f) > 0$, for any nonzero $f \in LS\{f_0, \ldots, f_n\}$. Due to (1.4)

$$(K_c f, f) = \sum_{\theta} (f*p_c)^2(\theta) \Delta y(\theta, t), \quad (4.2)$$

where $\theta$ is a point of jump of the function $y(\cdot, t)$.

It was proved in [6] that there exists $\Omega_0$ of probability one such that for any $\omega \in \Omega_0$ a linear span of the functions $\{p_c(\cdot - \theta(\omega))|_{[0; 1]}\}_{\theta(\omega)}$ is dense in $L_2([0; 1])$. Thus on the set $\Omega_0$ for any $f \in LS\{f_0, \ldots, f_n\} \subset L_2([0; 1])$ one can find a random point $\theta_f$ such that $(f(\cdot), p_c(\cdot - \theta_f)) \neq 0$. Since $y(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing, $\Delta y(\theta, t) > 0$ for any jump-point $\theta$. Consequently, on the set $\Omega_0$

$$\sum_{\theta} (f*p_c)^2(\theta) \Delta y(\theta, t) = \sum_{\theta} (f(\cdot), p_c(\cdot - \theta))^2 \Delta y(\theta, t) \geq (f(\cdot), p_c(\cdot - \theta))_t^2 \Delta y(\theta_f, t) > 0,$$

which proves the theorem. \hfill \square
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