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ESSENTIAL SETS FOR RANDOM OPERATORS CONSTRUCTED FROM AN ARRATIA FLOW

A. A. DOROGOVTSSEV AND IA. A. KORENOVSKA

ABSTRACT. In this paper we consider a strong random operator T_t which describes a shift of functions from $L_2(\mathbb{R})$ along an Arratia flow. We find a compact set in $L_2(\mathbb{R})$ that doesn't disappear under T_t , and estimate its Kolmogorov widths.

1. Introduction: Arratia Flow and Random Operators

In this paper we consider random operators in $L_2(\mathbb{R})$ which describe shifts of functions along an Arratia flow [1]. Let us recall the definition.

Definition 1.1 ([1]). A family of random processes $\{x(u, s), u \in \mathbb{R}, s \geq 0\}$ is called an *Arratia flow* if

- 1) for each $u \in \mathbb{R}$ $x(u, \cdot)$ is a Wiener process with respect to the joint filtration such that $x(u, 0) = u$;
- 2) for any $u_1 \leq u_2$ and $t \geq 0$

$$x(u_1, t) \leq x(u_2, t) \text{ a.s.}$$

- 3) the joint characteristics are

$$d \langle x(u_1, \cdot), x(u_2, \cdot) \rangle (t) = \mathbb{I}_{\{x(u_1, t) = x(u_2, t)\}} dt.$$

In the informal language, Arratia flow is a family of Wiener processes started from each point of \mathbb{R} , which move independently up to the meeting, coalesce, and move together. It was proved in [4, 8] that for any $a, b \in \mathbb{R}$ and $t > 0$ the set $x([a; b], t)$ is finite a.s. Since Arratia flow has a right-continuous modification [3], $x(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ is a step function for any time $t > 0$. Hence, for any $a, b \in \mathbb{R}$ and $t > 0$ with probability one there exists a random point $y \in \mathbb{R}$ for which

$$\lambda\{u \in [a; b] : x(u, t) = y\} > 0, \quad (1.1)$$

where λ is Lebesgue measure on \mathbb{R} . Since $x(\cdot, t)$ is a right-continuous step function, for a fixed countable set A

$$\begin{aligned} \mathbb{P}\{x(\mathbb{R}, t) \cap A \neq \emptyset\} &= \mathbb{P}\{x(\mathbb{Q}, t) \cap A \neq \emptyset\} \\ &\leq \sum_{u \in \mathbb{Q}} \mathbb{P}\{x(u, t) \in A\} = 0. \end{aligned} \quad (1.2)$$

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Since for any $a < b$ the difference $\frac{x(b,\cdot)-x(a,\cdot)}{\sqrt{2}}$ is a Wiener processes until the collision happens, and $\frac{x(b,0)-x(a,0)}{\sqrt{2}} = \frac{b-a}{\sqrt{2}}$, one can find the distribution of the time of coalescence $\tau_{a,b} = \inf\{s \geq 0 \mid x(a,s) = x(b,s)\}$ of the processes $x(a,\cdot), x(b,\cdot)$, i.e. for any $t \geq 0$

$$\begin{aligned} \mathbb{P}\{\tau_{a,b} \leq t\} &= \mathbb{P}\{x(a,t) = x(b,t)\} \\ &= \sqrt{\frac{2}{\pi}} \int_{\frac{b-a}{\sqrt{2t}}}^{\infty} e^{-\frac{v^2}{2}} dv. \end{aligned} \quad (1.3)$$

Let us notice that for a fixed time $t > 0$ and an Arratia flow $X = \{x(u,s), u \in \mathbb{R}, s \in [0;t]\}$ there exists an Arratia flow $Y = \{y(u,r), u \in \mathbb{R}, r \in [0;t]\}$ such that trajectories of X and $\tilde{Y} = \{y(u,t-r), u \in \mathbb{R}, r \in [0;t]\}$ don't cross [1, 7]. Y is called a conjugated (or dual) Arratia flow. It was proved in [10] the following change of variable formula for an Arratia flow.

Theorem 1.2 ([10]). *For any time $t > 0$ and nonnegative measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}} h(u)du < \infty$*

$$\int_{\mathbb{R}} h(x(u,t))du = \int_{\mathbb{R}} h(u)dy(u,t) \quad a.s., \quad (1.4)$$

where the last integral is in sense of Lebesgue-Stieltjes.

In this paper we consider random operators T_t , $t > 0$, in $L_2(\mathbb{R})$ which are defined as follows

$$(T_t f)(u) = f(x(u,t)),$$

where $f \in L_2(\mathbb{R})$ and $u \in \mathbb{R}$. It was proved in [5] that T_t is a strong random operator [11] in $L_2(\mathbb{R})$, but, as it was shown in [10], is not a bounded one. Really, for the point y from (1.1) one can introduce a sequence of the intervals $A_i = [r_i; p_i]$ such that $y \in A_i$ for any $i \geq 1$ and $p_i - r_i \rightarrow 0, i \rightarrow \infty$. Thus for any $i \geq 1$

$$\|T_t \mathbb{1}_{A_i}\|_{L_2(\mathbb{R})}^2 \geq \lambda\{u \in [a; b] : x(u,t) = y\} > 0,$$

which can't be true if T_t was a bounded random operator. Hence, the image of a compact set under T_t may not be a random compact set. Moreover, as it was mentioned in [9], the image of a compact set under strong random operator may not exist. However, in [10] it was presented a family of compact sets in $L_2(\mathbb{R})$ whose images under T_t exist and are random compact sets. In this paper we consider a compact set of this type, and investigate the change of its Kolmogorov widths [12] under T_t .

2. T_t -essential Functions

If the support of the function $f \in L_2(\mathbb{R})$ is bounded, $\text{supp} f \subset [a; b]$, then $T_t f$ equals to 0 with positive probability. Really, by (1.4), one can check that

$$\begin{aligned} \mathbb{P} \left\{ \int_{-\infty}^{\infty} f^2(x(u, t)) du = 0 \right\} &\geq \mathbb{P} \{ x(\mathbb{R}, t) \cap [a; b] = \emptyset \} \\ &= \mathbb{P} \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a; b]}(x(u, t)) du = 0 \right\} \\ &= \mathbb{P} \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a; b]}(u) dy(u, t) = 0 \right\}, \end{aligned}$$

where $\{y(u, s), u \in \mathbb{R}, s \in [0; t]\}$ is a conjugated Arratia flow. Since, by (1.3),

$$\mathbb{P} \left\{ \int_{-\infty}^{\infty} \mathbb{I}_{[a; b]}(u) dy(u, t) = 0 \right\} = \mathbb{P} \{ y(b, t) = y(a, t) \} > 0,$$

then $\mathbb{P} \{ \|T_t f\|_{L_2(\mathbb{R})} = 0 \} > 0$. This leads to the following definition.

Definition 2.1. For a fixed $t > 0$ a function $f \in L_2(\mathbb{R})$ is said to be a T_t -essential if

$$\mathbb{P} \{ \|T_t f\|_{L_2(\mathbb{R})} > 0 \} = 1.$$

Example 2.2. Let $f \in L_2(\mathbb{R})$ be an analytic function which doesn't equal totally to zero. Denote the set of its zeroes $Z_f = \{u \in \mathbb{R} \mid f(u) = 0\}$. Then, by (1.2), $\mathbb{P} \{x(\mathbb{R}, t) \cap Z_f = \emptyset\} = 1$, so f is a T_t -essential for any $t > 0$.

Let us notice that if $t_1 \neq t_2$ then T_{t_1} -essential function may not be a T_{t_2} -essential. To introduce a T_1 -essential that is not T_2 -essential function let us consider an increasing sequence $\{u_k\}_{k=0}^{\infty}$ such that $u_0 = 0, u_1 = 1$ and for any $n \in \mathbb{N}$

$$u_{2n+1} - u_{2n} = \frac{1}{2^n}, \quad u_{2n} = u_{2n-1} + 2n(\ln 2)^{\frac{1}{2}}.$$

Theorem 2.3. The function $f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}$ is a T_1 -essential, and is not a T_2 -essential.

Proof. To prove that f is not a T_2 essential we show that $\mathbb{P} \{ \|T_2 f\|_{L_2(\mathbb{R})} > 0 \} < 1$. Since $[u_{2k}; u_{2k+1}] \cap [u_{2j}; u_{2j+1}] = \emptyset$ for any $k \neq j$ then, by (1.4),

$$\begin{aligned} \mathbb{P} \{ \|T_2 f\|_{L_2(\mathbb{R})}^2 > 0 \} &= \mathbb{P} \left\{ \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}(x(u, 2)) \right)^2 du > 0 \right\} \\ &= \mathbb{P} \left\{ \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}(x(u, 2)) du > 0 \right\} \end{aligned}$$

$$\begin{aligned}
&= \mathbb{P} \left\{ \sum_{n=0}^{\infty} (y(u_{2n+1}, 2) - y(u_{2n}, 2)) > 0 \right\} \\
&= \mathbb{P} \{ \exists n \geq 0 : y(u_{2n+1}, 2) \neq y(u_{2n}, 2) \} \\
&\leq \sum_{n=0}^{\infty} P \{ y(u_{2n+1}, 2) \neq y(u_{2n}, 2) \}.
\end{aligned}$$

Thus by (1.3),

$$\sum_{n=0}^{\infty} \mathbb{P} \{ y(u_{2n+1}, 2) \neq y(u_{2n}, 2) \} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{4\pi}} \int_{-\frac{1}{2^{n+1}}}^{\frac{1}{2^{n+1}}} e^{-\frac{v^2}{4}} dv \leq \frac{1}{\sqrt{\pi}} < 1.$$

Consequently, the function $f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}$ is not a T_2 -essential. To prove that $f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}$ is a T_1 -essential one can show the following estimation.

Lemma 2.4. *Let $\{w(u_n, \cdot)\}_{n=0}^{\infty}$ be a family of independent Wiener processes on $[0; 1]$ such that $w(u_n, 0) = u_n$. Then for any $n \in \mathbb{N}$*

$$\mathbb{P} \left\{ \max_{s \in [0; 1]} \max_{j=0, 2n-1} w(u_j, s) \geq \min_{s \in [0; 1]} w(u_{2n}, s) \right\} < \frac{1}{2^{n^2} \sqrt{\pi} \ln 2}.$$

Proof. Let w_1, w_2 be an independent Wiener processes on $[0; 1]$ started from point 0, i.e. $w_1(0) = w_2(0) = 0$. It can be noticed that

$$\begin{aligned}
&\mathbb{P} \left\{ \max_{s \in [0; 1]} \max_{j=0, 2n-1} w(u_j, s) \geq \min_{s \in [0; 1]} w(u_{2n}, s) \right\} \\
&= \mathbb{P} \left\{ \exists j = \overline{0, 2n-1} : \max_{s \in [0; 1]} w(u_j, s) - \min_{s \in [0; 1]} w(u_{2n}, s) \geq 0 \right\} \\
&\leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0; 1]} w(u_j, s) - \min_{s \in [0; 1]} w(u_{2n}, s) \geq 0 \right\} \\
&\leq \sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0; 1]} w_1(s) - \min_{s \in [0; 1]} w_2(s) \geq u_{2n} - u_j \right\}.
\end{aligned}$$

From the fact that $\{u_n\}_{n=0}^{\infty}$ is an increasing sequence we can estimate the last expression and complete the proof

$$\begin{aligned}
&\sum_{j=0}^{2n-1} \mathbb{P} \left\{ \max_{s \in [0; 1]} w_1(s) - \min_{s \in [0; 1]} w_2(s) \geq u_{2n} - u_j \right\} \\
&\leq \frac{1}{\sqrt{\pi}} \sum_{j=0}^{2n-1} \frac{1}{u_{2n} - u_j} e^{-\frac{(u_{2n} - u_j)^2}{4}} \\
&\leq \frac{2n-1}{\sqrt{\pi}(u_{2n} - u_{2n-1})} e^{-\frac{(u_{2n} - u_{2n-1})^2}{4}} \\
&\leq \frac{1}{2^{n^2} \sqrt{\pi} \ln 2}.
\end{aligned}$$

□

Let us prove that the function $f = \sum_{n=0}^{\infty} \mathbb{I}_{[u_{2n}; u_{2n+1}]}$ is a T_1 -essential. Using the reasoning from the first part of the proof it can be checked that for the considered function f the following equality holds

$$P\{ \|T_1 f\|_{L_2(\mathbb{R})} > 0 \} = P\left\{ \sum_{n=0}^{\infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) > 0 \right\}.$$

Let us prove that

$$P\left\{ \limsup_{n \rightarrow \infty} (y(u_{2n+1}, 1) - y(u_{2n}, 1)) \geq 1 \right\} = 1. \tag{2.1}$$

Build a new processes $\{\tilde{y}(u_n, \cdot)\}_{n=0}^{\infty}$ such that $\{\tilde{y}(u_n, \cdot)\}_{n=0}^{\infty}$ and $\{y(u_n, \cdot)\}_{n=0}^{\infty}$ have the same distributions in $\mathcal{C}([0; 1])^{\infty}$ in the following way [4]. Let $\{w(u_n, \cdot)\}_{n=0}^{\infty}$ be a given family of Wiener processes on $[0; 1]$, $w(u_n, 0) = u_n$. Let us denote collision time of $f, g \in \mathcal{C}([0; 1])$ by $\tau[f, g] := \inf\{t \mid f(t) = g(t)\}$. Put $\tilde{y}(u_0, \cdot) := w(u_0, \cdot)$. Then for any $n \in \mathbb{N}$, $s \in [0; 1]$ one can define

$$\begin{aligned} \tilde{y}(u_n, s) &= w(u_n, s) \mathbb{I}\{s < \tau[w(u_n, \cdot), \tilde{y}(u_{n-1}, \cdot)]\} \\ &\quad + \tilde{y}(u_{n-1}, s) \mathbb{I}\{s \geq \tau[w(u_n, \cdot), \tilde{y}(u_{n-1}, \cdot)]\}. \end{aligned}$$

According to constructions of stochastic processes $\{\tilde{y}(u_n, \cdot)\}_{n=0}^{\infty}$

$$\begin{aligned} P\{ \exists N \in \mathbb{N} : \forall n \geq N \quad &\tilde{y}(u_{2n}, t) = w(u_{2n}, t), \\ &\tilde{y}(u_{2n+1}, t) = w(u_{2n+1}, t) \mathbb{I}\{t < \tau[w(u_{2n}, \cdot), w(u_{2n+1}, \cdot)]\} \\ &+ w(u_{2n}, t) \mathbb{I}\{t \geq \tau[w(u_{2n}, \cdot), w(u_{2n+1}, \cdot)]\} \} = 1. \end{aligned} \tag{2.2}$$

Thus

$$P\{ \exists N \in \mathbb{N} : \forall n \geq N \quad \tilde{y}(u_{2n+1}, t) - \tilde{y}(u_{2n}, t) = w(u_{2n+1}, t) - w(u_{2n}, t) \} = 1.$$

For the considered sequence $\{u_n\}_{n=0}^{\infty}$ and any $n \in \mathbb{N}$ the following inequality holds

$$P\{ w(u_{2n+1}, t) - w(u_{2n}, t) \geq 1 \} = \int_1^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{(v-\frac{1}{2k})^2}{4}} dv \geq \frac{1}{\sqrt{4\pi}} \int_1^{\infty} e^{-\frac{v^2}{4}} dv.$$

Therefore, by the Borel-Cantelli lemma and (2.2),

$$P\{ \limsup_{n \rightarrow \infty} (\tilde{y}(u_{2n+1}, t) - \tilde{y}(u_{2n}, t)) \geq 1 \} = 1.$$

□

Using the observation from Example 2.2 one can introduce a family of T_t -essential functions for all $t > 0$.

For any $\varepsilon > 0$ let us consider an integral operator K_ε in $L_2(\mathbb{R})$ with the kernel

$$k_\varepsilon(v_1, v_2) = \int_{\mathbb{R}} p_\varepsilon(u - v_1) p_\varepsilon(u - v_2) dy(u, t), \tag{2.3}$$

where $v_1, v_2 \in \mathbb{R}$, and $p_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{u^2}{2\varepsilon}}$. By the change of variables formula for an Arratia flow [10],

$$(K_\varepsilon f, f) = \int_{\mathbb{R}} (f * p_\varepsilon)^2(x(u, t)) du. \tag{2.4}$$

Lemma 2.5. *For any $\varepsilon > 0$ and nonzero function $f \in L_2(\mathbb{R})$*

$$P \{ (K_\varepsilon f, f) \neq 0 \} = 1.$$

Proof. According to (2.1) it is sufficient to note that $f * p_\varepsilon$ is an analytic function. Consequently, for any $t > 0$ the following relations are true

$$\begin{aligned} P \{ (K_1 f, f) > 0 \} &= P \{ \|T_t(f * p_1)\|_{L_2(\mathbb{R})} > 0 \} \\ &= P \{ x(\mathbb{R}, t) \cap Z_{f * p_1} = \emptyset \} = 1. \end{aligned}$$

□

According to the last theorem and (2.4), for any $\varepsilon > 0$ and nonzero $f \in L_2(\mathbb{R})$ the function $f * p_\varepsilon$ is a T_t -essential for each $t > 0$.

3. Change of Compact Sets under a Strong Random Operator Generated by an Arratia Flow

As it was noticed in the introduction any function with bounded support isn't a T_t -essential. Consequently, if $K \subseteq L_2(\mathbb{R})$ is a compact set of functions with uniformly bounded supports such that $T_t(K)$ is well-defined, then the image $T_t(K)$ equals to $\{0\}$ with positive probability. It was shown in [10] that T_t may also change the geometry of K even in the case of a compact set K for which $T_t(K) \neq \{0\}$ a.s. For example, the image $T_t(K)$ of a convergent sequence and its limiting point may not have limiting points. In this section we build a compact set K for which $T_t(K) \neq \{0\}$ a.s. and investigate the change of its Kolmogorov-widths in $L_2(\mathbb{R})$ under random operator T_t .

Definition 3.1 ([12]). The *Kolmogorov n -width* of a set $C \subseteq H$ in a Hilbert space H is given by

$$d_n(C) = \inf_{\dim L \leq n} \sup_{f \in C} \inf_{g \in L} \|f - g\|_H,$$

where L is a subspace of H .

We consider the following compact set in $L_2(\mathbb{R})$

$$K = \left\{ f \in W_2^1(\mathbb{R}) \mid \int_{\mathbb{R}} f^2(u)(1 + |u|)^3 du + \int_{\mathbb{R}} (f'(u))^2 (1 + |u|)^7 du \leq 1 \right\}. \tag{3.1}$$

Estimations on its Kolmogorov-widths in $L_2(\mathbb{R})$ are presented in the next lemma.

Lemma 3.2. *There exist positive constants C_1, C_2 such that for any $n \in \mathbb{N}$*

$$\frac{C_1}{n} \leq d_n(K) \leq \frac{C_2}{n^{\frac{3}{10}}}.$$

Proof. Let $n \in \mathbb{N}$ be fixed. To estimate $d_n(K)$ from above one can consider the partition $\{u_k\}_{k=0}^n$ of $[-n^{\frac{1}{5}}; n^{\frac{1}{5}}]$ into n segments $\{[u_k; u_{k+1}], k = 0, n-1\}$ with equal lengths. Let us show that for the n -dimensional subspace $L_n = \overline{LS\{\Pi_{[u_k; u_{k+1}]}, k = 0, n-1\}}$

$$\sup_{f \in K} \inf_{g \in L_n} \|f - g\|_{L_2(\mathbb{R})} \leq \frac{C_2}{n^{\frac{3}{10}}}.$$

If $f \in K$ then $\int_{\mathbb{R}} f^2(u)(1 + |u|)^3 du \leq 1$. Thus for any $C > 0$

$$\int_{|u| > c} f^2(u) du \leq \frac{1}{(1+C)^3} \int_{|u| > c} f^2(u)(1 + |u|)^3 du \leq \frac{1}{C^3}.$$

So, for the function $g_f = \sum_{k=0}^{n-1} f(u_k) \Pi_{[u_k; u_{k+1}]} \in L_n$ the following estimation is true

$$\|f - g_f\|_{L_2(\mathbb{R})}^2 \leq \frac{1}{n^{\frac{3}{5}}} + \int_{|u| \leq n^{\frac{1}{5}}} (f(u) - g_f(u))^2 du.$$

By the Cauchy inequality, for $f \in K$ and $u \in [u_k; u_{k+1}]$

$$\left(\int_{u_k}^u f'(v) dv \right)^2 \leq \int_{u_k}^u \frac{dv}{(1+|v|)^7} \leq u - u_k.$$

Consequently,

$$\begin{aligned} \int_{|u| \leq n^{\frac{1}{5}}} (f(u) - g_f(u))^2 du &= \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^u f'(v) dv \right)^2 du \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} (u_{k+1} - u_k)^2 = \frac{2}{n^{\frac{3}{5}}}, \end{aligned}$$

and the upper estimation for $d_n(K)$ holds with the constant $C_2 = 3^{\frac{1}{2}}$.

To get a lower estimation for $d_n(K)$ we use the theorem about n -width of $(n+1)$ -dimensional ball [12]. Let $\{u_k\}_{k=0}^{2(n+1)}$ be a partition of $[0; 1]$ into $2(n+1)$ segments $\{[u_k; u_{k+1}], k = 0, 2n+1\}$ with equal lengths. Consider $(n+1)$ -dimensional space $L_{n+1} = \overline{LS\{f_k, k = 0, n\}}$, where the functions $f_k, k = 0, n$, are defined as follows

$$f_k = \begin{cases} 0, & u \notin [u_{2k}; u_{2k+1}], \\ 1, & u \in [u_{2k} + \frac{1}{6(n+1)}; u_{2k} + \frac{2}{6(n+1)}], \\ 6(n+1)(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{1}{6(n+1)}], \\ -6(n+1)(u - u_{2k+1}), & u \in [u_{2k} + \frac{2}{6(n+1)}; u_{2k+1}]. \end{cases} \quad (3.2)$$

We show that if $c = \frac{2^3(5+2^9 \cdot 3^3)}{5}$ then the ball $B_{n+1} = \{f \in L_{n+1} \mid \|f\|_{L_2(\mathbb{R})} \leq \frac{1}{\sqrt{cn}}\}$ is a subset of K . Since $\|f_k\|_{L_2(\mathbb{R})}^2 = \frac{5}{18(n+1)}, k = 0, n$, then for any $f \in B_{n+1}$ such that $f = \sum_{k=0}^n c_k f_k$ the following relation holds $\sum_{k=0}^n c_k^2 \leq \frac{36}{5cn}$. Thus according

to (3.2),

$$\begin{aligned} & \int_{\mathbb{R}} f^2(u)(1+|u|)^3 du + \int_{\mathbb{R}} (f'(u))^2 (1+|u|)^7 du \\ & \leq 2^3 \|f\|_{L_2(\mathbb{R})}^2 + 2^7 \cdot \sum_{k=0}^n c_k^2 \left(\int_{u_{2k}}^{u_{2k+\frac{1}{6(n+1)}}} (6(n+1))^2 du + \int_{u_{2k+\frac{2}{6(n+1)}}}^{u_{2k+1}} (6(n+1))^2 du \right) \\ & \leq \frac{2^3}{cn^2} + 2^{10} \cdot 3n \frac{36}{5cn} \leq \frac{1}{c} \cdot \frac{2^3(5+2^9 \cdot 3^3)}{5} = 1. \end{aligned}$$

Consequently, $B_{n+1} \subset K$ and $d_n(K) \geq d_n(B_{n+1})$. Due to the theorem about n -width of $(n+1)$ -dimensional ball, $d_n(B_{n+1}) = \frac{1}{\sqrt{cn}}$ [12]. So the lower estimation for $d_n(K)$ holds with $C_1 := \sqrt{c}$. \square

To show that estimations from above for the Kolmogorov-widths of the considered compact set K don't change under T_t one may use the same idea as in Lemma 2.

Theorem 3.3. *There exists $\tilde{\Omega}$ of probability one such that for any $\omega \in \tilde{\Omega}$ and $n \in \mathbb{N}$*

$$d_n(T_t^\omega(K)) \leq \frac{C(\omega)}{n^{\frac{3}{10}}}, \tag{3.3}$$

where the constant $C(\omega) > 0$ doesn't depend on n .

Proof. For a fixed $n \in \mathbb{N}$ let us consider a partition $\{u_k\}_{k=0}^n$ of $[-n^{\frac{1}{5}}; n^{\frac{1}{5}}]$ into n segments with equal lengths. To prove (3.3) it's sufficient to show the following inequality for the linear space $L_n^\omega = LS\{T_t^\omega \mathbb{I}_{[u_k; u_{k+1}]}, k = \overline{0, n-1}\}$ with dimension at most n

$$\sup_{h_1 \in T_t^\omega(K)} \inf_{h_2 \in L_n^\omega} \|h_1 - h_2\|_{L_2(\mathbb{R})} \leq \frac{C(\omega)}{n^{\frac{3}{10}}}.$$

According to the change of variable formula for an Arratia flow, one can check the equality for any $f \in K$

$$\begin{aligned} & \left\| T_t^\omega f - T_t^\omega \left(\sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k; u_{k+1}]} \right) \right\|_{L_2(\mathbb{R})}^2 = \int_{|u| > n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \\ & + \int_{|u| \leq n^{\frac{1}{5}}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k; u_{k+1}]}(u) \right)^2 dy(u, t, \omega). \end{aligned}$$

To estimate from above the last integral let us notice that

$$\begin{aligned} & \int_{|u| \leq n^{\frac{1}{5}}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k; u_{k+1}]}(u) \right)^2 dy(u, t, \omega) \\ & \leq \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^u |f'(v)| dv \right)^2 dy(u, t, \omega). \end{aligned}$$

Due to (3.1), for any $f \in K$ and $u \in [u_k; u_{k+1}]$

$$\left(\int_{u_k}^u |f'(v)| dv \right)^2 \leq \int_{u_k}^u \frac{dv}{(1+|v|)^7} \leq u_{k+1} - u_k.$$

Thus

$$\begin{aligned} \sum_{k=0}^{n-1} \int_{u_k}^{u_{k+1}} \left(\int_{u_k}^u |f'(v)| dv \right)^2 dy(u, t, \omega) & \leq \sum_{k=0}^{n-1} (u_{k+1} - u_k) \int_{u_k}^{u_{k+1}} dy(u, t, \omega) \\ & = \frac{2}{n^{\frac{4}{5}}} (y(n^{\frac{1}{5}}, t, \omega) - y(-n^{\frac{1}{5}}, t, \omega)). \end{aligned}$$

For an Arratia flow $\{y(u, s), u \in \mathbb{R}, s \in [0; t]\}$ the following relation is true [2]

$$\lim_{|u| \rightarrow \infty} \frac{|y(u, t)|}{|u|} = 1 \quad \text{a.s.}$$

Consequently, for any $\omega \in \tilde{\Omega} = \{\omega' \in \Omega \mid \lim_{|u| \rightarrow \infty} \frac{|y(u, t, \omega')|}{|u|} = 1\}$ the estimation holds

$$\int_{|u| \leq n^{\frac{1}{5}}} \left(f(u) - \sum_{k=0}^{n-1} f(u_k) \mathbb{I}_{[u_k; u_{k+1}]}(u) \right)^2 dy(u, t, \omega) \leq \frac{4c(\omega)}{n^{\frac{3}{5}}} \quad (3.4)$$

with the constant

$$c(\omega) = \sup_{|u| \geq 1} \frac{|y(u, t, \omega)|}{|u|}. \quad (3.5)$$

Let us prove that for any $\omega \in \tilde{\Omega}$ there exists a constant $\tilde{c}(\omega)$ such that

$$\int_{|u| > n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{\tilde{c}(\omega)}{n^{\frac{3}{5}}}.$$

It can be noticed that $\int_{|u| > n^{\frac{1}{5}}} f^2(u) dy(u, t) \leq \frac{1}{n^{\frac{3}{5}}} \int_{|u| > n^{\frac{1}{5}}} f^2(u) (1 + |u|)^3 dy(u, t)$. Denote by $\{\theta_j\}_{j=1}^{\infty}$ a sequence of jump points of the function $y(\cdot, t)$ on \mathbb{R}_+ . Thus

one may show

$$\begin{aligned} \int_{u > n^{\frac{1}{5}}} f^2(u)(1+u)^3 dy(u, t) &= \sum_{\theta_i \geq n^{\frac{1}{5}}} f^2(\theta_i)(1+\theta_i)^3 \Delta y(\theta_i, t) \\ &= \sum_{k=1}^{\infty} \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i)(1+\theta_i)^3 \Delta y(\theta_i, t) \\ &\leq \sum_{k=1}^{\infty} (2+k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t). \end{aligned}$$

According to the Cauchy inequality and (3.1), for any $u \in \mathbb{R}_+$ the following relations hold

$$f^2(u) \leq \int_u^{\infty} (f'(v))^2 (1+v)^7 dv \cdot \int_u^{\infty} \frac{dv}{(1+v)^7} \leq \frac{1}{6u^6}.$$

Consequently, due to (3.5), the inequalities are true

$$\begin{aligned} &\sum_{k=1}^{\infty} (2+k)^3 \sum_{\{i: \theta_i \in [k; k+1)\}} f^2(\theta_i) \Delta y(\theta_i, t) \\ &\leq \sum_{k=1}^{\infty} (2+k)^3 \frac{1}{6k^6} (y(k+1, t) - y(k, t)) \\ &\leq \frac{16c}{3} \sum_{k=1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

Hence, for any $\omega \in \tilde{\Omega}$ there exists the constant $C_1(\omega) = \frac{16c(\omega)}{3}$ such that

$$\int_{u > n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}.$$

Similarly, it can be proved that $\int_{u < -n^{\frac{1}{5}}} f^2(u) dy(u, t, \omega) \leq \frac{C_1(\omega)}{n^{\frac{3}{5}}}$. According to this

and (3.4), for any $\omega \in \tilde{\Omega}$ an upper estimation for $d_n(T_t^\omega(K))$ is true. \square

The functions from Lemma 2 that were used to build the $(n+1)$ -dimensional subspace are not T_t -essential for any $t > 0$. Thus the image of this subspace under the random operator T_t may be equal to $\{0\}$ with positive probability. So, one can ask about the existence of a finite-dimensional subspace such that for any $t > 0$ its image under T_t is a linear subspace with the same dimension.

4. A Subspace Preserving the Dimension under a Random Operator Generated by an Arratia Flow

In this section for any $t > 0$ and $n \in \mathbb{N}$ we present a family $\{g_k, k = \overline{0, n}\}$ of linearly independent T_t -essential functions such that their images under T_t are linearly independent. Such a family generates a subspace which preserves the

dimension under a random operator generated by an Arratia flow. It can be used to get a lower estimation of $d_n(T_t(K))$.

Let us fix any $n \in \mathbb{N}$, and build a family of $(n+1)$ linearly independent functions in the following way. Let $\{u_k\}_{k=0}^{2(n+1)}$ be a partition of $[0; n^{-2}]$ into $2(n+1)$ segments with equal lengths. For any $k = \overline{0, n}$ define f_k by

$$f_k = \begin{cases} 0, & u \notin [u_{2k}; u_{2k+1}], \\ 1, & u \in [u_{2k} + \frac{n^{-2}}{6(n+1)}; u_{2k} + \frac{2n^{-2}}{6(n+1)}], \\ \frac{6(n+1)}{n^{-2}}(u - u_{2k}), & u \in [u_{2k}; u_{2k} + \frac{n^{-2}}{6(n+1)}], \\ -\frac{6(n+1)}{n^{-2}}(u - u_{2k+1}), & u \in [u_{2k} + \frac{2n^{-2}}{6(n+1)}; u_{2k+1}]. \end{cases} \quad (4.1)$$

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the functions $\{f_k * p_\varepsilon, k = \overline{0, n}\}$ are linearly independent.*

Proof. Since the considered functions $\{f_k, k = \overline{0, n}\}$ are linearly independent, its Gram determinant doesn't equal to 0, i.e. $G(f_0, \dots, f_n) \neq 0$. For each $k = \overline{0, n}$

$$f_k * p_\varepsilon \rightarrow f_k, \quad \varepsilon \rightarrow 0.$$

Hence, due to the continuity of the Gram determinant, one may notice that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$

$$G(f_0 * p_\varepsilon, \dots, f_n * p_\varepsilon) \neq 0,$$

and the desired result is proved. □

Theorem 4.2. *There exists a set Ω_0 of probability one such that for any $\omega \in \Omega_0$ the functions $T_t^\omega(f_0 * p_\varepsilon), \dots, T_t^\omega(f_n * p_\varepsilon)$ are linearly independent.*

Proof. Denote by K_ε the integral operator in $L_2(\mathbb{R})$ with the kernel k_ε . To prove the statement of the theorem it's enough to show that on some Ω_0 of probability one the following inequality holds $(K_\varepsilon f, f) > 0$, for any nonzero $f \in LS\{f_0, \dots, f_n\}$. Due to (1.4)

$$(K_\varepsilon f, f) = \sum_{\theta} (f * p_\varepsilon)^2(\theta) \Delta y(\theta, t), \quad (4.2)$$

where θ is a point of jump of the function $y(\cdot, t)$.

It was proved in [6] that there exists Ω_0 of probability one such that for any $\omega \in \Omega_0$ a linear span of the functions $\{p_\varepsilon(\cdot - \theta(\omega))|_{[0;1]}\}_{\theta(\omega)}$ is dense in $L_2([0; 1])$. Thus on the set Ω_0 for any $f \in LS\{f_0, \dots, f_n\} \subset L_2([0; 1])$ one can find a random point θ_f such that $(f(\cdot), p_\varepsilon(\cdot - \theta_f)) \neq 0$. Since $y(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing, $\Delta y(\theta, t) > 0$ for any jump-point θ . Consequently, on the set Ω_0

$$\begin{aligned} \sum_{\theta} (f * p_\varepsilon)^2(\theta) \Delta y(\theta, t) &= \sum_{\theta} (f(\cdot), p_\varepsilon(\cdot - \theta))^2 \Delta y(\theta, t) \\ &\geq (f(\cdot), p_\varepsilon(\cdot - \theta_f))^2 \Delta y(\theta_f, t) > 0, \end{aligned}$$

which proves the theorem. □

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