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Wavelets, Coorbit Theory, and Projective Representations

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Abstract

Banach spaces of functions, or more generally, of distributions are one of the main topics in analysis. In this thesis, we present an abstract framework for construction of invariant Banach function spaces from projective group representations. Coorbit theory gives a unified method to construct invariant Banach function spaces via representations of Lie groups. This theory was introduced by Feichtinger and Gröchenig in [23, 24, 25, 26] and then extended in [9]. We generalize this concept by constructing coorbit spaces using projective representation which is first studied by O. Christensen in [10]. This allows us to describe wider classes of function spaces as coorbits, in order to construct frames and atomic decompositions for these spaces. As in the general coorbit theory, we construct atomic decompositions and Banach frames for coorbit spaces under certain smoothness conditions. By this modification, we can discretize the Bergman spaces $A^p_\alpha(\mathbb{B}^n)$ via the family of projective representations $\{\rho_s\}$ of the group $\mathrm{SU}(n, 1)$, for any real parameter $s > n$.

Chapter 1
Introduction

Bergman spaces on the unit disk $\mathbb{D}$ are among the interesting function spaces which are studied by many mathematicians. In 1950 S. Bergman introduced these spaces in his book [3]. For $1 \leq p < \infty$, Bergman space is the space of all holomorphic functions that belong to $L^p(\mathbb{D})$. One of the problems that arises is to give a discrete description of these spaces. The same question can be asked for Bergman spaces on the unit ball in $\mathbb{C}^n$, or more generally, on a bounded symmetric domain. Recently, J. Christensen, Gröchenig, and Ólafsson, used the coorbit theory to describe Bergman spaces on the unit ball as coorbits of L^p-spaces in their paper [6]. They constructed these coorbits via the representation (projective representation)

$$\pi_s(x)f(z) = (-(z,b)+\bar{d})^{-s} f(x^{-1} \cdot z) \tag{1.1}$$

of the group $\mathrm{SU}(n,1)$. Here $x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$, where A is an $n \times n$ matrix, and b, c are vectors in $\mathbb{C}^n$, and $d \in \mathbb{C}$. This function is a multi-valued function for non-integer values of $s > n$, due to the term $(-(z,b)+\bar{d})^{-s}$. To make this function into a single valued function, we have two approaches. The first is done in [6], when they defined this function on a simply connected subgroup of $\mathrm{SU}(n,1)$. This subgroup is diffeomorphic to $\mathrm{SU}(n,1)/K$, where K is the maximal compact subgroup of $\mathrm{SU}(n,1)$. The second approach is to consider this function as a projective representation of $\mathrm{SU}(n,1)$.

In the same paper [6], the authors constructed frames and atomic decompositions for Bergman spaces via a representation of a finite covering group of $\mathrm{SU}(n,1)$ under the restriction that the parameter $s > n$ is rational.

In this thesis, we obtain frames and atomic decompositions for Bergman spaces via the full group $\mathrm{SU}(n,1)$ by using the second approach. To do this, we modify the coorbit theory so that we construct a coorbit space by using the projective representation.

In 1980's H. Feichtinger and K. Gröchenig developed the theory of coorbits [23, 24, 25, 26], which become a powerful tool for discretizing wide classes of function spaces. Assume that $(\pi, \mathcal{H})$ is an integrable representation with respect to the measure $w\,dx$ on a Lie group G, where w is a submultiplicative weight function and dx is left invariant Haar measure on G. Fix u from the space of analyzing vectors

$$\mathcal{A}_w := \left\{u \in \mathcal{H} \mid (\pi(\cdot)u, u) \in L^1_w(G)\right\}$$

and define the Banach space

$$\mathcal{H}^1_w := \left\{v \in \mathcal{H} \mid (\pi(\cdot)u, v) \in L^1_w(G)\right\}.$$

If we denote the conjugate dual space of $\mathcal{H}^1_w$ by $(\mathcal{H}^1_w)^*$, then the coorbit space of a left invariant Banach space B is

$$CoB := \left\{\phi \in (\mathcal{H}^1_w)^* \mid \langle \phi, \pi(\cdot)u\rangle \in B\right\},$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing of $\mathcal{H}^1_w$ and $(\mathcal{H}^1_w)^*$. The coorbit space CoB is a π-invariant Banach space, which is isometrically isomorphic to a reproducing kernel Banach subspace of B. Feichtinger and Gröchenig showed that, if $\{x_i\}$ is a well spread set of G, then $\{\pi(x_i)u\}$ forms a frame and an atomic decomposition for CoB under certain assumptions on u, see [24].

J. Christensen and Ólafsson generalized the definition of coorbit spaces in [9] to remove the integrability and irreducibility restrictions. Let π be a representation on a Fréchet space $\mathcal{S}$, where $\mathcal{S}$ is continuously embedded and weakly dense in its conjugate dual $\mathcal{S}^*$. If we fix a cyclic vector $u \in \mathcal{S}$ and define the wavelet transform $W_u(\phi)(x) = \langle \phi, \pi(x)u \rangle$, then under the assumption in [9], the coorbit space of a left invariant Banach function space B

$$\mathrm{Co}_\pi^u B := \{\phi \in \mathcal{S}^* \mid W_u(\phi) \in B\}$$

is a π^*-invariant Banach space, which is isometrically isomorphic to the reproducing kernel Banach subspace

$$B_u := \{f \in B \mid f * W_u(u) = f\}.$$

In [10], O. Christensen extended the definition of the Feichtinger-Gröchenig coorbit space to be applied for an irreducible unitary projective representation ρ on a Hilbert space $\mathcal{H}$, under the restriction of the integrability of ρ. He proved that the same results are true as in Feichtinger-Gröchenig theory.

In this thesis, we give a general definition of the coorbit space that arises from the projective representation to describe wider classes of function spaces as coorbits. In particular, we apply the coorbit theory to Bergman spaces when the projective representation is not integrable. Then, we construct frames and atomic decompositions for these spaces.

This thesis will be organized as follows: Chapter 2 is devoted to give a background for the main tools of coorbit theory. We define some basic concepts like a Lie group and its left invariant Haar measure, strongly continuous representation, continuous wavelet transform, and left/right invariant Banach function spaces; moreover, we define a reproducing kernel Banach space. We conclude this chapter with some results related to the square integrable representations. In Chapter 3, the coorbit theory, which is founded by Feichtinger and Gröchenig, is studied in detail. First, we give a quick survey of Feichtinger-Gröchenig theory without proofs. Then, we describe the general coorbit theory of J. Christensen and Ólafsson. Next, we summarize the main results about discretization of coorbit spaces. It is shown that the family $\{\pi(x_i)u\}$ forms a frame and an atomic decomposition for the coorbit space $\mathrm{Co}_\pi^u B$. In Chapter 4, we study a concrete example of a dual pairing coorbit, Bergman spaces on the unit disk, which is studied in [6, 8, 9]. In Chapter 5, we formulate our main result about the existence of the twisted convolutive coorbits, and the existence of its atomic decompositions. First, we give background about a projective representation of a Lie group, and we introduce the twisted convolution. Then, we define the twisted coorbit space via a projective representation and study the connection between the regular coorbits and the twisted coorbits. Finally, we find conditions that ensure that the family $\{\rho(x_i)u\}$ forms a frame and an atomic decomposition for the twisted coorbit. Then, last chapter is devoted to describe Bergman spaces, $A_s^p(\mathbb{B}^n)$, on the unit ball as twisted coorbits of L^p spaces via a projective representation on the group $\mathrm{SU}(n,1)$ for all values of the parameter $s > n$. We conclude Chapter 5 with a construction of atomic decompositions for $A_s^p(\mathbb{B}^n)$.

Chapter 2
Banach Function Spaces and Wavelets

2.1 Preliminaries

A Lie group [1] is a group endowed with the structure of a smooth manifold such that the group multiplication and the group inversion are smooth. Typical examples of Lie groups are the classical linear matrix groups, like the orthogonal group $\mathrm{O}(n)$, the unitary group $\mathrm{U}(n)$, and the special linear group $\mathrm{SL}(n, \mathbb{R})$. A Fréchet space is a locally convex, complete, Hausdorff topological vector space with topology induced by a countable family of semi-norms. Fréchet spaces are a generalization of Banach spaces, for more details see [40].

Through this thesis we assume that G is a Lie group which is $\sigma-$compact and $\mathcal{S}$ is a Fréchet space. We assume that the conjugate linear dual $\mathcal{S}^*$ of $\mathcal{S}$ equipped with the weak*-topology. Moreover, $\mathcal{S}$ is continuously embedded and weakly dense in $\mathcal{S}^*$. We will use $\langle \cdot, \cdot \rangle$ to denote the dual pairing of the spaces $\mathcal{S}$ and $\mathcal{S}^*$. As usual we denote the group of all bounded automorphisms on $\mathcal{S}$ with bounded inverses by $\mathrm{GL}(\mathcal{S})$, and the subgroup of all unitary such automorphisms by $\mathcal{U}(\mathcal{S})$.

2.2 Continuous Representation

Let V be a complete locally convex Hausdorff topological vector space. A strongly continuous representation [2] of G on the space V is a continuous group homomorphism $\pi : G \to \mathrm{GL}(V)$ in the sense that the map $x \mapsto \pi(x)v$ is continuous for all $v \in V$. That is, $\pi : G \to \mathrm{GL}(V)$ is a continuous group homomorphism when the group $\mathrm{GL}(V)$

[1] We only study finite dimensional Lie groups.
[2] In this thesis we sometimes write continuous representation or even representation instead of strongly continuous representation.

is endowed with the strong operator topology. If $V = \mathcal{S}$, then the strong continuity condition and the continuity of the action $(x, v) \mapsto \pi(x)v$ from $G \times \mathcal{S}$ into $\mathcal{S}$ are equivalent (see [46]). Let $\mathcal{H}$ be a Hilbert space, if $\pi(x)$ is a unitary representation on $\mathcal{H}$ for all $x \in G$, then the representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is called unitary representation. A subspace W of $\mathcal{S}$ is called an invariant subspace if $\pi(x)W \subset W$ for all $x \in G$. A representation $(\pi, \mathcal{S})$ is called irreducible if the only invariant closed subspaces of $\mathcal{S}$ are 0 and $\mathcal{S}$ itself. A nonzero vector $u \in \mathcal{S}$ is called π-cyclic if the span of the set $\{\pi(x)u \mid x \in G\}$ is a dense subset of $\mathcal{S}$. This is equivalent to the following condition:

$$\text{If } \langle \lambda, \pi(x)u \rangle = 0 \text{ for all } x \in G, \text{ then } \lambda = 0.$$

It is not hard to see that a representation is irreducible if and only if every nonzero vector is cyclic (see [13]).

2.2.1 Contragradient Representation

Let π be a strongly continuous representation of G on the Fréchet space $\mathcal{S}$. We define the contragradient representation (or the dual representation, see [27]) π^* on $\mathcal{S}^*$ by

$$\langle \pi^*(x)\phi, v \rangle := \langle \phi, \pi(x^{-1})v \rangle \tag{2.1}$$

for all $v \in \mathcal{S}$ and all $\phi \in \mathcal{S}^*$. This relation defines an actual representation of G on the space $\mathcal{S}^*$ as we show in the following lemma.

Lemma 2.1. *Let π be a representation of G on the Fréchet space $\mathcal{S}$ and let S^* be the conjugate dual of $\mathcal{S}$ equipped with the weak*-topology. The mapping π^* which is defined in 2.1 is a strongly continuous representation of G on the space $\mathcal{S}^*$.*

Proof. First, we show that π^* is a homomorphism. Fix $\phi \in \mathcal{S}^*$, for any $v \in \mathcal{S}$ we have

$$\begin{aligned}\langle \pi^*(xy)\phi, v \rangle =& \langle \phi, \pi((xy)^{-1})v \rangle \\ =& \langle \phi, \pi(y^{-1})\pi(x^{-1})v \rangle\end{aligned}$$

$$=\langle \pi^*(y)\phi, \pi(x^{-1})v\rangle$$
$$=\langle \pi^*(x)\pi^*(y)\phi, v\rangle.$$

Thus, $\pi^*(xy) = \pi^*(x)\pi^*(y)$. Now, we show that $x \mapsto \pi^*(x)\phi$ is continuous from G into $\mathcal{S}^*$ for all $\phi \in \mathcal{S}^*$. Let $\{x_\alpha\}$ be a net in G such that $x_\alpha \to x$ in G, then $\pi(x_\alpha^{-1})v \to \pi(x_\alpha^{-1})v$ in $\mathcal{S}$, because the inversion map and $x \to \pi(x)v$ are continuous. The continuity of the dual paring implies that $\langle \phi, \pi(x_\alpha^{-1})v\rangle \to \langle \phi, \pi(x^{-1})v\rangle$, or equivalently $\langle \pi^*(x_\alpha)\phi, v\rangle \to \langle \pi^*(x)\phi, v\rangle$. Thus, $\pi^*(x_\alpha)\phi \to \pi^*(x)\phi$ weakly. Therefore the mapping $x \mapsto \pi^*(x)\phi$ is continuous. □

2.3 Haar Measure on a Lie Group

In this section, we study a special kind of Borel measures on a (Hausdorff) locally compact topological group. This measure is invariant under left translation of Borel sets. For instance, Lebesgue measure on $\mathbb{R}^n$ is a left translation invariant Borel measure. However, we are looking for a generalization of Lebesgue measure. For a reference we encourage the reader to see, for example, [2, 16, 21].

Let X be a locally compact topological space. Recall that the Borel σ-algebra $\mathcal{B}$ on X is the σ-algebra that is generated by the topology of X, i.e., the set of all open subsets of X. A Boral measure μ is a positive measure defined on the Borel algebra $\mathcal{B}$. Moreover, μ is called regular if

1. μ is finite on every compact subset of X,
2. $\mu(B) = \inf\{\mu(U) : U \text{ is open and } B \subset U\}$ for $B \in \mathcal{B}$,
3. $\mu(B) = \sup\{\mu(K) : K \text{ is compact and } K \subset B\}$ for $B \in \mathcal{B}$ with a finite measure.

Let G be a locally compact group, a Borel measure μ_G on G is called left invariant if $\mu_G(aB) = \mu_G(B)$ for all $B \in \mathcal{B}$ and $a \in G$. This condition can be written equivalently

$$\int_G f(ax)\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x)$$

for any $f \in \mathcal{C}_c(G)$ and $a \in G$. We will use $d\mu(x)$ or dx instead of $d\mu_G(x)$ if there is no confusion.

A left invariant regular Borel measure is called left Haar measure. In the same way we define the right Haar measure. The following theorem states the existence and the uniqueness of Haar measure.

Theorem 2.2. *Let G be a locally compact group. There exists a nonzero left Haar measure on G which is unique up to a positive constant.*

One can see that if μ_G is the left Haar measure on G, then $\mu_G(U) > 0$ for any non-empty open subset of G, and if G is compact then $\mu_G(G)$ is finite. In this case, we normalize the measure such that $\int_G d\mu_G(x) = 1$.

If μ is a left Haar measure on G, then for any $a \in G$, we define a new left Haar measure $\mu_a(B) = \mu(Ba)$ for $B \in \mathcal{B}$. By the previous theorem, there is a positive constant, $\Delta(a)$, which depends on a such that

$$\int_G f(x)\, d\mu_a(x) = \Delta(a) \int_G f(x)\, d\mu(x).$$

Hence, we define a function $\Delta : G \mapsto \mathbb{R}_+$, which is called the modular function. A group is called unimodular if the modular function identically equals to 1. In the following theorem we summarize some properties of the modular function.

Theorem 2.3.

1. *The modular function is a continuous group homomorphism from G into $\mathbb{R}_+$, the multiplicative group of positive real numbers.*

2. *If G is abelian or compact then G is unimodular.*

3. Let $a \in G$ and $f \in L^1(G)$. Then

$$\int_G f(xa)\, d\mu(x) = \Delta(a^{-1}) \int_G f(x)\, d\mu(x).$$

4. If $f \in L^1(G)$, then

$$\int_G f(x^{-1})\Delta(x^{-1})\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

If G acts continuously on a manifold $\mathbb{M}$, then a Borel measure μ on $\mathbb{M}$ is called G-invariant if

$$\int_{\mathbb{M}} f(g \cdot x)\, d\mu(x) = \int_{\mathbb{M}} f(x)\, d\mu(x)$$

for all $f \in \mathcal{C}_c(\mathbb{M})$. If K is a closed subgroup of G, then G/K is manifold, and G acts on G/K by $g \cdot (aK) = (ga)K$. If we denote the natural projection from G onto G/K by π, then for any f on G/K the function $f \circ \pi$ is on G. We conclude this section with the following proposition about the existence of G-invariant measure on the quotient space G/K when K is compact. There is a more general statement when K is closed, for example see [20].

Proposition 2.4. *Let K be a compact subgroup of G and let μ_G be the left Haar measure on G. Then, there exists a G-invariant measure $\mu_{G/K}$ on G/K. Moreover, we can normalize $\mu_{G/K}$ so that*

$$\int_{G/K} f(x)\, d\mu_{G/K}(x) = \int_G f \circ \pi(x)\, d\mu_G(x)$$

where $\pi : G \to G/K$ is the natural projection $\pi(x) = xK$.

2.4 Banach Function Spaces

In this section, we describe an important class of function spaces and give some definitions that will be used to construct Banach representations.

2.4.1 Banach Function Spaces and BF-Spaces

Let $(\mathbb{M}, \mu)$ be a measure space. A Banach function space is a vector space B of equivalence classes of measurable functions on $\mathbb{M}$ for which there exists a mapping $f \mapsto \|f\|_B$ such that

1. the value $\|f\|_B$ is non-negative and $\|f\|_B = 0$ if and only if $f = 0$ μ-almost everywhere,
2. for any scalar λ, we have $\|\lambda f\|_B = |\lambda| \; \|f\|_B$,
3. the triangle inequality $\|f + g\|_B \leq \|f\|_B + \|g\|_B$ holds for all f and g in B,
4. the space B is complete in the topology defined by $\|.\|_B$.

In this thesis, we assume that the space $\mathbb{M}$ is a σ-finite measure space with measure μ. We say that a Banach function space B is solid if for measurable functions f and g on $\mathbb{M}$ for which $|g(x)| \leq |f(x)|$ for almost all $x \in \mathbb{M}$ and $f \in B$, then $g \in B$ with $\|g\|_B \leq \|f\|_B$.

A well known family of solid Banach function spaces are the L^p-spaces. Let μ_n be the Lebesgue measure on $\mathbb{R}^n$. For $1 \leq p < \infty$, we define

$$L^p(\mathbb{R}^n) := \Big\{ f : \mathbb{R}^n \to \mathbb{C} \mid \|f\|_p := \Big(\int_{\mathbb{R}^n} |f(x)|^p d\mu_n(x) \Big)^{1/p} < \infty \Big\}.$$

Other examples of solid Banach function spaces are the family of weighted L^p-spaces. A weight w is a continuous function $w : \mathbb{M} \mapsto \mathbb{R}_+$. We define the Banach function space $L^p_w(\mathbb{M})$ for $1 \leq p < \infty$, as

$$L^p_w(\mathbb{M}) := \Big\{ f : \mathbb{M} \to \mathbb{C} \mid \|f\|_{L^p_w} := \Big(\int_{\mathbb{M}} |f(x)|^p \, w(x) \, d\mu(x) \Big)^{1/p} < \infty \Big\}.$$

Notice that $f \in L^p(\mathbb{M})$ if and only if $|f|^p w \in L^1(\mathbb{M})$. Also w-weighted L^p-spaces are nothing but L^p spaces with a new measure $dv = w \, d\mu$ which can be explained by the

Radon-Nikodym derivative

$$w = \frac{dv}{d\mu}.$$

If $\mathbb{M} = G$, where G a Lie group (or more generally Hausdorff locally compact topological group), we define the left and right translation operators on B by

$$L_a f(x) := f(a^{-1}x)$$

and

$$R_a f(x) := f(xa)$$

for $f \in B$. We say that B is left invariant if $L_a f \in B$ for all $f \in B$ and $f \mapsto L_a f$ is continuous. Similarly, we define right invariant Banach function spaces on G. We say that the right translation is continuous if $x \mapsto R_x f$ is continuous for all $f \in B$, i.e., $x \mapsto R_x$ is a continuous representation of G on B. In this thesis we deal with specific function spaces which are called BF-spaces.

Definition 2.5. *A Banach function space B on G is called a BF-space if*

1. *Its topological dual space B^* is a Banach function space on G, and the dual paring is given by integral.*

2. *The convergence in the space B implies convergences (locally) in Haar measure on G.*

In the following example we show that the family of Banach function spaces $L^p_w(G)$, with submultiplicative weight, are left invariant with continuous left translation. More specific, it is a typical example of left-invariant BF-spaces.

Example 2.6. *Let $w : G \mapsto \mathbb{R}_+$ be a weight function. It is called submultiplicative if $w(xy) \leq w(x)w(y)$ for all $x, y \in G$. We always assume that $w(x) \geq 1$. Assume that $d\mu_G$ is the left invariant Haar measure on G. We claim that the w-weighted L^p-space*

$$L^p_w(G) := \left\{ f : \|f\|_{L^p_w} := \Big(\int_G |f(x)|^p \, w(x) d\mu_G(x) \Big)^{1/p} < \infty \right\}$$

is a solid left invariant Banach function space in which the left translation is continuous. Indeed, solidity comes from the fact that $L^p_w(G) = L^p(G, w d\mu_G)$. For $f \in L^p_w(G)$ and $a \in G$, we have

$$\begin{aligned}
\|L_a f\|^p_{L^p_w} &= \int_G |L_a f(x)|^p w(x) d\mu_G(x) \\
&= \int_G |f(a^{-1}x)|^p w(x)\, d\mu_G(x) \\
&= \int_G |f(x)|^p w(ax)\, d\mu_G(x) \\
&\leq \int_G |f(x)|^p w(a) w(x)\, d\mu_G(x) \\
&= w(a) \int_G |f(x)|^p w(x)\, d\mu_G(x) \\
&= w(a) \|f\|^p_{L^p_w} < \infty.
\end{aligned}$$

It follows that $L_a f \in L^p_w(G)$, and $f \mapsto L_a f$ is continuous. So, the space $L^p_w(G)$ is left invariant. Finally, the continuity of the left translation follows from the fact that the space of compactly supported continuous functions is dense in $L^p_w(G)$, and the continuity on a compact set implies uniform continuity on that set.

2.4.2 Convolution on BF-Spaces

Let $dx := d\mu_G(x)$ be the left Haar measure on G. For two functions f and g on the group G, we define the convolution by

$$f * g(x) := \int_G f(y) g(y^{-1}x)\, dx \quad \text{for all } x \in G$$

whenever the integral is defined. In particular, the convolution is well defined under the conditions of the following lemma. Through this thesis we define $g^\vee(x) := g(x^{-1})$ for a function g on the group G.

Lemma 2.7. *Let B be a left-invariant BF-space on G. Fix a function g on G. If the mappings $x \mapsto F(x)g^\vee(x)$ are in $L^1(G)$, for all $F \in B$, then the convolution*

$$F * g(x) := \int_G F(y) g(y^{-1}x)\, dy$$

is well defined for all $x \in G$. Moreover, if the mapping

$$F \mapsto F * g(1) = \int_G F(y) g(y^{-1})\, dy$$

is continuous on B, then the mapping $F \mapsto F * g(x)$ is continuous for all $x \in G$.

Proof. For $F \in B$, we have

$$\begin{aligned}\int_G \left|F(y)g(y^{-1}x)\right| \, dy &= \int_G \left|F(y)g\left((x^{-1}y)^{-1}\right)\right| \, dy \\ &= \int_G \left|F(xy)g(y^{-1})\right| \, dy \\ &= \int_G \left|L_{x^{-1}}F(y)g(y^{-1})\right| \, dy.\end{aligned}$$

The last integral is finite because $L_{x^{-1}}F$ is again in B. Thus, the convolution is well defined.

Now, assume that the mapping $F \mapsto F * g(1)$ is continuous. The same calculations as above show that, for any $x \in G$, one has

$$F * g(x) = L_{x^{-1}}F * g(1).$$

The continuity of $F \mapsto L_x F$ and the continuity of $F \mapsto F * g(1)$ show that the mapping $F \mapsto F * g(x)$ is continuous for all $x \in G$, that is,

$$\begin{aligned}|F * g(x)| =& |L_{x^{-1}}F * g(1)| \\ \leq& C_1 \left\|L_{x^{-1}}F\right\|_B \\ \leq& C \left\|F\right\|_B .\end{aligned}$$

□

2.4.3 Reproducing Kernel Banach Space

Reproducing Kernel Banach spaces have an important role in sampling theory. For example, if we can reconstruct a function f by

$$f(t) = \int_{\mathbb{R}} f(w) k(t, w)\, dw$$

for some kernel k, then under certain assumptions one can choose a sample $\{t_i\}_{i\in I}$ of $\mathbb{R}$ such that

$$Tf := \sum_i f(t_i)k(\cdot, t_i).$$

is invertible and

$$f(t) = \sum_i f(t_i)\, T^{-1}k(\cdot, t_i)(t).$$

For more details see [33].

A Hilbert space $\mathcal{H}$ of functions on a set X is called a reproducing kernel Hilbert space (RKHS) if the evaluation map $E_x : \mathcal{H} \to \mathbb{C}$, $E_x(f) = f(x)$, is continuous for all $x \in X$. Every RKHS has a unique reproducing kernel $k : X \times X \to \mathbb{C}$ such that $f(x) = (k(x, \cdot), f)_{\mathcal{H}}$. This result follows from Riesz representation theorem; indeed if the evaluation map E_x is continuous, then there is a unique element k_x in $\mathcal{H}$ such that $(f, k_x)_{\mathcal{H}} = f(x)$ for all $x \in X$. In particular, $(k_y, k_x)_{\mathcal{H}} = k_x(y)$, and the function $k(x, y) := k_x(y)$ is the desired kernel(see [1]). In Banach function spaces, the situations are different, because we no longer can apply the Riesz representation theorem. Mathematicians intended to generalize this concept for Banach spaces and they came up with different definitions that are compatible with their areas of experience. For example see [48, 49]. We will use the following definition which generalizes the RKHS definition.

Definition 2.8. *Let X be a set, and let $K : X \times X \to \mathbb{C}$ be a function. Assume that B and B' are Banach function spaces on X. We say that (B, B') is a pair of reproducing kernel Banach spaces (RKBS) with reproducing kernel K on X if*

1. *The evaluation maps $E_x : B \to \mathbb{C}$ and $E'_x : B' \to \mathbb{C}$ are continuous for all $x \in X$.*

2. *$K_x := K(x, \cdot) \in B$ and $K^x := K(\cdot, x) \in B'$ for all $x \in X$.*

3. *There exists a bilinear form $\beta(\cdot,\cdot) : B \times B' \to \mathbb{C}$ such that $\beta(f, K^x) = f(x)$ and $\beta(K_x, g) = g(x)$, for all $x \in X$, $f \in B$, and $g \in B'$.*

Notice that if $B = \mathcal{H}$ is a Hilbert space, then we can choose $B' = \mathcal{H}$. If the evaluation map is continuous, then for any $x \in X$ there is a unique reproducing kernel k such that $k(x,\cdot)$ and $k(\cdot,x)$ are in $\mathcal{H}$. Define the sesquilinear form β to be the inner product of $\mathcal{H}$, then $\beta(f, k(x,\cdot)) = f(x)$ and $\beta(k(\cdot,x), f) = f(x)$ for all $x \in X$ and all $f \in \mathcal{H}$. That is, every RKHS is a RKBS.

The following definition is useful when B' is topological dual of B. We will use this definition through out this thesis.

Definition 2.9. *Let B and its topological dual B^* are both Banach function spaces on a set X, and let $k : X \times X \to \mathbb{C}$ be a function on $X \times X$. The space B is called a reproducing kernel Banach space with a reproducing kernel k if*

1. *the evaluation maps E_x and E_x^*, on B and B^* respectively, are continuous for all $x \in X$.*
2. *$k_x := k(x,\cdot) \in B$ and $k^x := k(\cdot,x) \in B^*$ for all $x \in X$.*
3. *$\langle f, k_x \rangle = f(x)$ and $\langle k^x, g \rangle = g(x)$ for all $x \in X$.*

2.5 Continuous Voice Transform

In this section, we define the concept of voice transform and we summarize some properties and results that maybe found in [41, 31, 35, 47].

Definition 2.10. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. For a fixed vector $u \in \mathcal{H}$ we define the voice transform to be the linear mapping $V_u : \mathcal{H} \to \mathcal{C}_b(G)$ given by*

$$V_u(v)(x) := (v, \pi(x)u)_{\mathcal{H}}$$

for $v \in \mathcal{H}$ and $x \in G$.

Remark 2.11. *Some authors call the voice transform the wavelet transform, because the voice transform is a generalization of the wavelet transform, which comes from the quasi-regular representation of the group $ax + b$.*

Note that the voice transform is a bounded operator. Indeed,

$$|V_u(v)(x)| \leq \|v\|\|u\| \text{ for all } x \in G.$$

Hence,

$$\|V_u(v)\|_{sup} \leq \|v\|\|u\|.$$

The following lemma gives a characterization of cyclic vectors of a representation.

Lemma 2.12. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. For a fixed nonzero vector $u \in \mathcal{H}$ the voice transform V_u is injective if and only if u is π-cyclic.*

Proof. Suppose that V_u is injective. If $(v, \pi(x)u) = 0$ for all $x \in G$, then $V_u(v) = 0$, and hence $v = 0$. Which proves that u is π-cyclic.

Conversely, suppose that u is π-cyclic. If $V_u(v) = 0$, then

$$(v, \pi(x)u) = V_u(v)(x) = 0$$

for all $x \in G$, which implies that $v = 0$, and hence V_u is injective. □

The following corollary describe the connection between the injectivity of the voice transform and the irreducibility of the representation. In fact, it gives a new characterization of the concept of irreducible unitary representation (see [31, 41]).

Corollary 2.13. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. The voice transform V_u is injective for all nonzero vectors $u \in \mathcal{H}$ if and only if π is irreducible.*

Proof. By Lemma 2.12, W_u is injective for all nonzero $u \in \mathcal{H}$ if and only if u is π-cyclic for all nonzero $u \in \mathcal{H}$ if and only if $\mathcal{H}$ is irreducible. □

One of the important tools in the coorbit theory (which is the main topic in this thesis) on reproducing kernel Banach spaces is the following reproducing formula for the voice transform

$$V_u(v) * V_u(u) = V_u(v)$$

for all $v \in \mathcal{H}$. Unfortunately, this formula is not hold in general. However, square-integrable representations satisfy the reproducing formula, which is a consequence of a result due to Duflo and Moore in [17].

Definition 2.14. *An irreducible unitary representation $(\pi, \mathcal{H})$ is called square-integrable if there is a nonzero vector $u \in \mathcal{H}$ such that $V_u(u) \in L^2(G)$, i.e.*

$$\int_G |(u, \pi(x)u)_{\mathcal{H}}|^2 \, dx < \infty,$$

such vector u is called π-admissible.

Remark 2.15. *If we drop the irreduciblity condition in the definition of the admissibility, we say that a nonzero vector u is admissible if $V_u(v) \in L^2(G)$ for all $v \in \mathcal{H}$. However, in an irreducible representation these two definitions are equivalent as we will see in the corollary of the following theorem.*

The following theorem is one of the important ingredients of the convolutive coorbit theory (see [17]). A self-adjoint operator A on a Hilbert space is called positive if $(Av, v) \geq 0$ for all vectors v in its domain.

Theorem 2.16. (Doflo − Moore)*Let $(\pi, \mathcal{H})$ be a square-integrable representation of G.Then*

1. *there exists a positive self-adjoint operator A_π which is defined on a dense subset D of $\mathcal{H}$ such that $u \in \mathcal{H}$ is admissible if and only if $u \in D$. Moreover, the orthogonality relation holds*

$$\int_G (v_1, \pi(x)u_1)\,(\pi(x)u_2, v_2)\,dx = (A_\pi u_2, A_\pi u_1)\,(v_1, v_2)$$

for all $u_1, u_2 \in D$ and $v_1, v_2 \in \mathcal{H}$.

2. *In addition, if G is a unimodular, then $D = \mathcal{H}$ and $A_\pi = c_\pi Id_{\mathcal{H}}$. Thus, all vectors of $\mathcal{H}$ are admissible and*

$$\int_G (v_1, \pi(x)u_1)\,(\pi(x)u_2, v_2)\,dx = c_\pi^2 (u_2, u_1)\,(v_1, v_2)$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$. The constant c_π is called the formal dimension of π.

Now we have the following corollary:

Corollary 2.17. *Let u be an admissible vector in a square-integrable representation $(\pi, \mathcal{H})$. Then, for any $v \in \mathcal{H}$, we have $V_u(v) \in L^2(G)$. In this case the voice transform $V_u : \mathcal{H} \to L^2(G)$ is a bounded mapping.*

Proof. By Theorem 2.16, u is in the domain of A_π and

$$\int_G |(v, \pi(x)u)|^2\,dx = \int_G (v, \pi(x)u)\,(\pi(x)u, v)\,dx = \|A_\pi u\|^2\,\|v\|^2$$

and the result is obtained.

□

2.6 General Voice Transform

In the previous section, we defined the voice (wavelet) transform related to a unitary representation of a group G on a Hilbert space $\mathcal{H}$. In this section, we generalize this concept for representations on Fréchet spaces.

As usual, we assume that $\mathcal{S}$ is a Fréchet space and we assume that $\mathcal{S}^*$ is the conjugate dual space of $\mathcal{S}$. If $(\pi, \mathcal{S})$ is a strongly continuous representation of G, then for a vector $u \in \mathcal{S}$ the function $x \mapsto \langle \lambda, \pi(x)u \rangle$ is in the space $\mathcal{C}(G)$ for any $\lambda \in \mathcal{S}^*$ and so, the linear mapping $\mathcal{S}^* \ni \lambda \mapsto \langle \lambda, \pi(\cdot)u \rangle \in \mathcal{C}(G)$ is well defined. Let π^* be the dual representation of π as defined in 2.1. Then, the voice transform is defined as follows.

Definition 2.18. *For a vector $u \in \mathcal{S}$, we define the voice (wavelet) transform V_u : $\mathcal{S}^* \mapsto \mathcal{C}(G)$ by $V_u(\phi)(x) := \langle \pi^*(x^{-1})\phi, u \rangle$. The values $\langle \pi^*(x^{-1})\phi, u \rangle$ are called the wavelet coefficients.*

This definition is an extension of the Hilbert space case. If $\mathcal{S} = \mathcal{H}$, then $\mathcal{S}^* = \mathcal{H}$ and we have

$$(\pi^*(x^{-1})v, u) = (v, \pi(x)u) = V_u(v)(x)$$

in the usual sense. As we have seen in the Hilbert space case, the voice transform V_u is a bounded mapping. Moreover, if u is a nonzero π-cyclic vector, then the voice transform is injective as we state in the following theorem.

Theorem 2.19. *Let $(\mathcal{S}^*, \pi^*)$ be the dual representation of $(\mathcal{S}, \pi)$ and let u be a nonzero vector in the Fréchet space $\mathcal{S}$. The Wavelet transform $V_u : \mathcal{S}^* \to \mathcal{C}(G)$ is injective if and only if u is π-cyclic.*

Proof. The proof is the same as the case of Hilbert space, the only difference is replacing the inner product by the dual pairing. □

2.6.1 Wavelets on Compact Groups

Let G be a compact group, and let $(\pi, \mathcal{H})$ be any irreducible unitary representation of G. Then, any nonzero vector $u \in \mathcal{H}$ is admissible. Indeed, compact groups have finite Haar measure, and also the function $x \mapsto |(u, \pi(x)u)|^2$ is continuous. These facts together demonstrate our claim. As a consequence, the wavelet transform V_u : $\mathcal{H} \to L^2(G)$ is injective.

2.6.2 Non-square Integrable Representation

This example shows the existence of a non-square integrable irreducible unitary representation. Let G be the additive real group $\mathbb{R}$ and let $\mathcal{H} = \mathbb{C}$. It is clear that $\pi_\sigma(x) := e^{i\sigma x}$ defines a unitary representation of G on the space $\mathcal{H}$, where σ any nonzero real number. For any nonzero $z \in \mathcal{H}$ we have

$$\begin{aligned}\int_{\mathbb{R}} |(z, \pi(x)z)|^2\, dx &= \int_{\mathbb{R}} |(z, e^{i\sigma x}z)|^2\, dx \\ &= \int_{\mathbb{R}} |z\bar{z}e^{-i\sigma x}|^2\, dx \\ &= |z|^4 \int_{\mathbb{R}} dx = \infty.\end{aligned}$$

Therefore, π is not a square-integrable representation.

Chapter 3

Convolutive Coorbit Theory and Discretization

3.1 Convolutive Coorbit Theory

In analysis, one of the most important topics is to study features of function spaces. In 1980's H.G Feichtinger and K. Gröchenig introduced the theory of coorbit spaces when they were constructing frames and atomic decompositions of some function spaces on Lie groups [24, 25]. In the Feichtinger-Gröchenig the theory, coorbit space is a Banach space constructed by starting with an irreducible, unitary, integrable representation of a locally compact group on a Hilbert space.

Analysts have become more interested in studying coorbit spaces, and they have generalized coorbit spaces to represent wider classes of function spaces, such as Besov spaces and Bergman spaces, when the integrability or irreduciblity are no longer valid on their representations. See for example [7, 8, 9, 14, 15, 12, 24, 25, 36, 37, 39].

3.1.1 Feichtinger-Gr¨ochenig Theory

We now present in short the coorbit spaces that were introduced by H.G Feichtinger and K. Gröchenig and we give some results about these spaces. To describe the Feichtingerand Gröchenig coorbit spaces, we go back to Theorem 2.16, in which we can construct a coorbit space that is isomorphic to a reproducing kernel Banach space. If π is square integrable and u is any admissible vector, one can normalize u such that $\|Au\| = 1$, and obtains the reproducing formula

$$W_u(v) * W_u(u) = W_u(v)$$

for all $v \in \mathcal{H}$. The construction of the Feichtinger-Gröchenig coorbit spaces can be summarized as follows (see [24, 25]):

1. Let $\mathcal{H}$ be a Hilbert space. Fix a weight function w on G as in example 2.6; that is, w is a submultiplicative with $w(x) \geq 1$. Moreover, we assume that $w(x) = \Delta(x^{-1})w(x^{-1})$, where Δ is the modular function of the Haar measure on G. Let $(\pi, \mathcal{H})$ be an irreducible, unitary, w-integrable, continuous representation on the group G (we can assume that G is a locally compact Hausdorff group).

2. Define the space of analyzing vectors to be

$$\mathcal{A}_w := \left\{ u \in \mathcal{H} \mid (\pi(\cdot)u, u)_{\mathcal{H}} \in L^1_w(G) \right\}.$$

This space is not trivial because the representation is w-integrable.

3. Fix a nonzero analyzing vector $u \in \mathcal{A}_w$ and define the wavelet transform on $\mathcal{H}$ by $W_u(v)(x) = (v, \pi(x)u)$. Define the space

$$\mathcal{H}^1_w := \left\{ v \in \mathcal{H} \mid W_u(v) \in L^1_w(G) \right\}$$

with norm $\|v\|_{\mathcal{H}^1_w} := \|W_u(v)\|_{L^1_w}$. Then consider the conjugate dual space $(\mathcal{H}^1_w)^*$ of the space $\mathcal{H}^1_w$ endowed with the weak*-topology. We will write $\langle \cdot, \cdot \rangle$ for the dual pairing if there is no confusion.

4. It was proved that (see [24]) the continuous embedding $\mathcal{H}^1_w \hookrightarrow \mathcal{H} \hookrightarrow (\mathcal{H}^1_w)^*$ holds. Moreover, $\mathcal{H}^1_w$ is dense in $\mathcal{H}$ with the latter is weakly dense in $(\mathcal{H}^1_w)^*$. This allows us to extend the wavelet transform onto $(\mathcal{H}^1_w)^*$ by

$$W_u(v^*)(x) = \langle v^*, \pi(x)u \rangle \quad \text{for all} \quad v^* \in \left(\mathcal{H}^1_w\right)^*.$$

5. Let B be a solid Banach function space on G such that

   (i) The space B is continuously embedded in the space $L^1_{loc}(G)$; that is, for any compact subset K of G there exists a constant C_K such that

$$\int_K |f(x)| \, dx \leq C_K \|f\|_B \quad \text{for all } f \in B,$$

(ii) the relation $B * L^1_w(G) \subset B$ holds with

$$\|f * g\|_B \leq \|f\|_B \|g\|_{L^1_w}.$$

6. The coorbit space of the Banach space B is

$$\mathrm{Co}B := \left\{ v \in \left(\mathcal{H}^1_w\right)^* \mid W_u(v) \in B \right\}$$

with norm $\|v\|_{\mathrm{Co}B} := \|W_u(v)\|_B$.

By the integrability of the representation, the representation is square integrable because

$$|\langle u, \pi(x)u\rangle|^2 \leq \|u\|^2 \, |\langle u, \pi(x)u\rangle| \, .$$

It follows that, by theorem 2.16, we can normalize u such that $\|Au\| = 1$ and the reproducing formula

$$W_u(v) * W_u(u) = W_u(v)$$

holds for all $v \in \mathcal{H}$. Also, by the assumption in Step (5), the mapping $F \mapsto F * W_u(u)$ is continuous on B. Thus it is a projection on the space $B * W_u(u)$, and hence the space $B * W_u(u)$ can be described as

$$B * W_u(u) := \{F \in B \mid F * W_u(u) = F\} \, .$$

This space is a closed subspace of B and hence is a Banach space.

Let us now summarize the properties of the coorbit space which is constructed by Feichtinger-Gröchenig. For the proof and more details see [24, 25].

Theorem 3.1. *1. The coorbit space* $\mathrm{Co}B$ *is a* π*-invariant Banach space which is continuously embedded into* $(\mathcal{H}^1_w)^*$.

2. The definition of $\mathrm{Co}B$ *is independent of the choice of the analyzing vector* $u \in \mathcal{A}_w$*, i.e., different vectors* $u \in \mathcal{A}_w$ *define the same space with equivalent norms.*

The following theorem is the main result about coorbit spaces.

Theorem 3.2. *1. A function $F \in B$ is of the form $F = W_u(v)$ for some $v \in \mathrm{Co}B$ if and only if $F \in B * W_u(u)$, i.e., F satisfies the reproducing formula $F = F * W_u(u)$.*

2. The Wavelet transform $W_u : \mathrm{Co}B \to B * W_u(u)$ is an isometric isomorphism, and the mapping $B \ni F \mapsto F * W_u(u) \in B$ is a bounded projection from B onto $B * W_u(u)$.

3. The space $B*W_u(u)$ is contained in $L^1_{1/w}(G) \cap \mathcal{C}(G)$. In particular, the evaluation mapping $F \mapsto F(x)$ is continuous.

Thus, the coorbit space $\mathrm{Co}B$ is isomorphic to a reproducing kernel Banach subspace of B with kernel $K(x, y) = W_u(u)(x^{-1}y)$.

3.1.2 Coorbit Theory: Dual Pairing

In this section, we describe a wide class of function spaces as coorbit spaces, in which the Feichtinger-Gröchenig theory fails to apply. Then we study more properties of coorbit spaces.

In the Feichtinger-Gröchenig theory, the representation should be irreducible, unitary, and integrable. Even though this theory described many of interesting examples of function spaces, there are may interesting representations that are not integrable nor irreducible and not even unitary. In [9], J. Christensen and G. Ólafsson generalized the concept of coorbit spaces to describe a wider classes of function spaces. Furthermore, they gave examples of Banach function spaces as coorbit spaces the representation is not integrable (see [7, 8]).

To construct the coorbit space of a left-invariant BF-space on G via a representation on that group, we start by defining the analyzing vector.

Definition 3.3. *Assume $(\pi, \mathcal{S})$ is a representation of G on the Fréchet space $\mathcal{S}$ which is continuously embedded and weakly dense in its conjugate dual $\mathcal{S}^*$. A π-cyclic vector $u \in \mathcal{S}$ is called a π-analyzing vector for $\mathcal{S}$ if the reproducing formula*

$$W_u(\lambda) * W_u(u) = W_u(\lambda)$$

holds for all $\lambda \in \mathcal{S}^*$.

Let $u \in \mathcal{S}$ and define the set

$$B_u := \{f \in B | f * W_u(u) = f\}.$$

It is easy to see that the set B_u is a normed subspace of B (maybe trivial subspace) with norm inherited from B. The following assumptions are made on the Banach space B to define, what we call, the coorbit space of B.

Assumption 3.4. *Let B be a left-invariant BF-space on G. Assume that there exists a nonzero π-analyzing vector $u \in \mathcal{S}$ satisfying the following properties:*

(R1) The mapping $B \ni f \mapsto \int_G f(x)W_u(u)(x^{-1})\,dx \in \mathbb{C}$ is continuous.

(R2) If $f * W_u(u) = f \in B$, then the mapping $\mathcal{S} \ni v \mapsto \int_G f(x)W_v(u)(x^{-1})\,dx \in \mathbb{C}$ is in $\mathcal{S}^*$.

Remark 3.5. *(i) The conditions (R1) says that the convolution $B * W_u(u)$ is defined as we saw in Lemma 2.7, and the continuity condition says that the evaluation map on B_u is continuous as we will see in the following lemma.*

(ii) The condition (R2) says that every function in the spaces B_u corresponds to a distribution in $\mathcal{S}^*$.

Lemma 3.6. *Let B be a BF- space on G. If B and u satisfy (R1), then the space B_u is closed in B and hence a reproducing kernel Banach space with reproducing kernel $k(x, y) = L_yW_u(u)(x)$.*

Proof. We start the proof by showing that the space B_u is a closed subspace of B. Let $\{f_i\}$ be a sequence in the space B_u that converges to $f \in B$ in norm. The convergence in the space B implies the convergence in measure, and hence there exists a subsequence f_{i_k} of f_i that converges to f almost everywhere. Assumption $(R1)$ says that the convolution $f * W_u(u)(x)$ is well defined for all $x \in G$ as we have seen in Lemma 2.7. We claim that $f = f * W_u(u)$ almost everywhere on G. Indeed, by the definition of the space B_u, for almost all $x \in G$ we have

$$f_{i_k}(x) = f_{i_k} * W_u(u)(x).$$

By Lemma 2.7, the function $f \mapsto f * W_u(u)(x)$ is continuous. Passing the limit as $k \mapsto \infty$ we have

$$f(x) = f * W_u(u)(x)$$

for almost all $x \in G$. Therefore the function f is in the space B_u and we proved the closeness part. Now, let us prove that B_u is a reproducing Banach space, i.e., the evaluation map $E_x(f) = f(x)$ is continuous on B_u for $x \in G$. For a function $f \in B_u$, we have $f = f * W_u(u)$ and hence

$$|E_x(f)| = |f(x)| = |f * W_u(u)(x)|.$$

Again, by the continuity of $f \mapsto f * W_u(u)(x)$ as in Lemma 2.7 we have

$$|E_x(f)| \leq C\|f\|_B.$$

Finally, for $f \in B_u$ we have

$$\begin{aligned} f(x) = f * W_u(u)(x) &= \int_G f(y) W_u(u)(y^{-1}x)\, dy \\ &= \int_G f(y) L_y W_u(u)(x)\, dy \\ &= \int_G f(y) k(x, y)\, dy \end{aligned}$$

which proves that k is the reproducing kernel, and the proof is completed. □

The following result shows that the assumptions on the space B in [9] imply the assumptions that we stated, as it is proved in [9, Theorem 2.3].

Lemma 3.7. *Assume $u \in \mathcal{S}$ is a π-cyclic vector that satisfies*

1. *The reproducing formula $W_u(v) * W_u(u) = W_u(v)$ for all $v \in \mathcal{S}$, and*

2. *the mapping $\mathcal{S}^* \ni \lambda \mapsto W_u(\lambda) * W_u(u)(1) \in \mathbb{C}$ is weakly continuous.*

Then u is a π-analyzing vector for $\mathcal{S}$.

Proof. By our assumptions, the space $\mathcal{S}$ is weakly dense in $\mathcal{S}^*$. Therefore, for any $\lambda \in \mathcal{S}^*$ we can choose a net $\{\lambda_i\}$ in $\mathcal{S}$ such that $\lambda_i \to \lambda$ weakly, that is

$$\langle \lambda_i, v \rangle \to \langle \lambda, v \rangle$$

for all $v \in \mathcal{S}$. Using Assumption (1), we have

$$W_u(\lambda_i) * W_u(u) = W_u(\lambda_i)$$

for all λ_i. On the other hand,

$$\begin{aligned} W_u(\lambda_i) * W_u(u)(x) =& \int_G W_u(\lambda_i)(y) W_u(u)(y^{-1}x)\, dy \\ =& \int_G \left\langle \pi^*(y^{-1})(\lambda_i), u \right\rangle \left\langle \pi^*(x^{-1}y)(u), u \right\rangle\, dy \\ =& \int_G \left\langle \pi^*((xy)^{-1})(\lambda_i), u \right\rangle \left\langle \pi^*(y)(u), u \right\rangle\, dy \\ =& \int_G \left\langle \pi^*(x^{-1})(\lambda_i), \pi(y)u \right\rangle \left\langle \pi^*(y)(u), u \right\rangle\, dy \\ =& W_u\left(\pi^*(x^{-1})(\lambda_i)\right) * W_u(u)(1) \end{aligned}$$

and hence

$$W_u(\lambda_i) = W_u\left(\pi^*(x^{-1})(\lambda_i)\right) * W_u(u)(1)$$

As $\lambda_i \to \lambda$ weakly, the left hand side

$$W_u(\lambda_i)(x) \to W_u(\lambda)(x)$$

for all $x \in G$, because $\langle\ \lambda_i, \pi(x)u\rangle \to \langle\ \lambda, \pi(x)u\rangle$ for all $x \in G$. Also, by Assumption (2) the right hand side

$$W_u\left(\pi^*(x^{-1})(\lambda_i)\right) * W_u(u)(1) \to W_u\left(\pi^*(x^{-1})(\lambda)\right) * W_u(u)(1)$$

for all $x \in G$. But $W_u\left(\pi^*(x^{-1})(\lambda)\right) * W_u(u)(1) = W_u(\lambda) * W_u(u)(x)$; therefore,

$$W_u(\lambda) * W_u(u)(x) = W_u(\lambda)(x)$$

for all $\lambda \in \mathcal{S}^*$. $\square$

Lemma 3.8. *Assume B and $u \in \mathcal{S}$ satisfy Assumption 3.4. The set*

$$\Lambda = \{\lambda \in \mathcal{S}^* \mid W_u(\lambda) \in B\}$$

is a Banach space with norm

$$\|\lambda\| = \|W_u(\lambda)\|_B.$$

Proof. First the linearity of W_u ensures that Λ is a linear space contains the zero vector. Let us now show that $\|.\|$ is an actual norm. The linearity of W_u and the fact that $\|.\|_B$ is a norm prove all conditions of the norm, except the non-degeneracy condition. The value $\|\lambda\| = 0$ if and only if $\|W_u(\lambda)\|_B = 0$. The last statement is true if and only if $\lambda = 0$ because of the cyclicity of u. This shows that Λ is a normed space. Now, let us show that Λ is complete. Let $\{\lambda_n\}$ be a Cauchy sequence in the space Λ. For $m, n \in \mathbb{N}$ we have

$$\|W_u(\lambda_n) - W_u(\lambda_m)\|_B = \|W_u(\lambda_n - \lambda_m)\|_B = \|\lambda_n - \lambda_m\|$$

which means that $\{W_u(\lambda_n)\}$ is a Cauchy sequence in the space B, and hence it has a limit, $F \in B$, say. We claim that $\lambda_n \to \lambda$ for some $\lambda \in \Lambda$, i.e., $W_u(\lambda) \in B$. The same argument as in the proof of Lemma 3.6, shows that $F * W_u(u) = F$. On the other hand, define λ by

$$\langle \lambda, v \rangle = F * W_v(u)(1).$$

By Assumption (R2), λ is in the space $\mathcal{S}^*$, and

$$\begin{aligned} W_u(\lambda)(y) &= \langle \lambda, \pi(y)u \rangle \\ &= \int_G F(x) \langle \pi^*(x)u, \pi(y)u \rangle \, dx \\ &= \int_G F(x) \left\langle u, \pi(x^{-1}y)u \right\rangle \, dx \\ &= \int_G F(x) W_u(u)(x^{-1}y) \, dx \\ &= F * W_u(u)(y) \\ &= F(y) \end{aligned}$$

for almost all $y \in G$. So, $W_u(\lambda) \in B$, and hence $\lambda \in \Lambda$. Thus, the space Λ is complete. □

Remark 3.9. *Note that we used all conditions of Assumption 3.4 to show that the space Λ is well defined Banach space. Actually, we used that u is cyclic to show that Λ is indeed a normed space, and we used the reproducing formula and the conditions $(R1), (R2)$ to show that Λ is complete.*

Now, we can introduce the definition of the coorbit space.

Definition 3.10. *Let $(\mathcal{S}, \pi)$ be a representation of G and let B be a left-invariant BF-space on G. Assume that $u \in \mathcal{S}$ is a π-analyzing vector satisfying Assumption 3.4. A coorbit space of B related to the representation π is the Banach space*

$$\mathrm{Co}_\pi^u B := \{\phi \in \mathcal{S}^* \mid W_u(\phi) \in B\}$$

with the norm

$$\|\phi\|_{\mathrm{Co}_\pi^u B} := \|W_u(\phi)\|_B.$$

Sometimes the coorbit space is a trivial space with no interesting structure. Most of the time we will require that $W_u(u) \in B$. Automatically, this implies that $W_u(u) \in B_u$ or equivalently $u \in \mathrm{Co}_\pi^u B$. In this case we get a non-trivial coorbit space.

In the following theorem, it is shown that the coorbit space is isomorphic to a reproducing Banach space.

Theorem 3.11. *Assume that B and u satisfy Assumption 3.4, then*

1. $W_u(v) * W_u(u) = W_u(v)$ *for* $v \in \mathrm{Co}_\pi^u B$.
2. *The space* $\mathrm{Co}_\pi^u B$ *is a* π^**-invariant Banach space.*
3. $W_u : \mathrm{Co}_\pi^u B \to B$ *intertwines* π^* *and the left translation.*
4. *If the left translation is continuous, then* π^* *acts continuously on* $\mathrm{Co}_\pi^u B$.
5. $W_u : \mathrm{Co}_\pi^u B \to B_u$ *is an isometric isomorphism.*

Proof. (1) The reproducing formula is true for all $\phi \in \mathcal{S}^*$, and hence it is true for the space $\mathrm{Co}_\pi^u B$.

(2) We proved that the space $\mathrm{Co}_\pi^u B$ is a Banach space. Let us now prove that $\mathrm{Co}_\pi^u B$ is π^*-invariant. Assume that $\lambda \in \mathrm{Co}_\pi^u B$, for any fixed $y \in G$, we have

$$\begin{aligned} W_u(\pi^*(y)\lambda)(x) &= \langle \pi^*(y)\lambda, \pi(x)u \rangle \\ &= \left\langle \lambda, \pi(y^{-1}x)u \right\rangle \\ &= W_u(\lambda)(y^{-1}x) \\ &= L_y W_u(\lambda)(x). \end{aligned}$$

Thus, we have

$$W_u(\pi^*(y)\lambda) = L_y W_u(\lambda). \tag{3.1}$$

The fact that $\lambda \in \mathrm{Co}_\pi^u B$ and B is left invariant yield that $L_y W_u(u)$ is in B, and hence $W_u(\pi^*(y)\lambda)$ is in $\mathrm{Co}_\pi^u B$. This proves that the space is π^*-invariant.

(3) In 3.1, we found that $W_u(\pi^*(y)\lambda) = L_y W_u(\lambda)$, which means that $W_u(u)$ intertwines π^* and the left translation.

(4) By our assumption, the function $G \ni x \mapsto L_x F \in B$ is continuous. So, for any $\epsilon > 0$, there exists a neighborhood U of the identity such that

$$\|L_x F - F\|_B < \epsilon$$

for all $x \in U$. Using 3.1, we have

$$\begin{aligned}\|\pi^*(x)\lambda - \lambda\|_{\mathrm{Co}_\pi^u B} =& \|W_u(\pi^*(x)\lambda - \lambda)\|_B \\ =& \|L_x W_u(\lambda) - W_u(\lambda)\|_B\end{aligned}$$

for $x \in G$ and $\lambda \in \mathrm{Co}_\pi^u B$. But the definition of $\mathrm{Co}_\pi^u B$ ensures that $W_u(\lambda)$ is in B. Thus

$$\|\pi^*(x)\lambda - \lambda\|_{\mathrm{Co}_\pi^u B} < \epsilon$$

for all $x \in U$. It follows that $x \mapsto \pi^*(x)(\lambda)$ is continuous at the identity. For any $y \in G$, one can write

$$\begin{aligned}\|\pi^*(x)\lambda - \pi^*(y)\lambda\|_{\mathrm{Co}_\pi^u B} =& \|\pi^*(y)\pi^*(y^{-1}x)\lambda - \pi^*(y)\lambda\|_{\mathrm{Co}_\pi^u B} \\ =& \|\pi^*(y)\left(\pi^*(y^{-1}x)\lambda - \lambda\right)\|_{\mathrm{Co}_\pi^u B} \\ =& \|W_u\left(\pi^*(y)\left(\pi^*(y^{-1}x)\lambda - \lambda\right)\right)\|_B \\ =& \|L_y W_u\left(\pi^*(y^{-1}x)\lambda - \lambda\right)\|_B \\ \leq& C\|W_u\left(\pi^*(y^{-1}x)\lambda - \lambda\right)\|_B \\ =& C\|\pi^*(y^{-1}x)\lambda - \lambda\|_{\mathrm{Co}_\pi^u B}.\end{aligned}$$

As $y \to x$, we have $y^{-1}x \to 1$. Thus, the term $\|\pi^*(x)\lambda - \pi^*(y)\lambda\|_{\mathrm{Co}_\pi^u B}$ can be made as small as we please, and our assertion is proved.

(5) Let us show that $W_u\left(\mathrm{Co}_\pi^u B\right) = B_u$. If $\lambda \in \mathrm{Co}_\pi^u B$, then $W_u(\lambda) \in B$. By Part (1) we have $W_u(\lambda) * W_u(u) = W_u(\lambda)$, hence $W_u(\lambda) \in B_u$. On the other hand, if $F \in B_u$, then $F * W_u(u) = F$ and the assumption (R2) ensures that λ, which is defined by

$$\langle \lambda, v \rangle = \int_G F(x) \left\langle \pi^*(x)u, v \right\rangle \, dx,$$

is in the space $\mathcal{S}^*$. Same calculations as in Part(2) show that $W_u(\lambda) = F = F * W_u(u)$, therefore $F \in W_u\left(\mathrm{Co}_\pi^u B\right)$ and this shows that $W_u : \mathrm{Co}_\pi^u B \to B_u$ is surjective. From the definition of the norm of $\mathrm{Co}_\pi^u B$ we conclude that W_u is an isometry.

□

In practice, the previous assumptions can be weakened in order cover a wider classes of function spaces. Moreover, these modified assumptions are easy to deal with.

Assumption 3.12. *Let B be a left-invariant BF-space on G, and let π be a representation of G on a Fréchet spaces $\mathcal{S}$ which is continuously embedded and weakly dense in its conjugate dual $\mathcal{S}^*$. Assume that there exists a π-analyzing vector $u \in \mathcal{S}$ such that*

(R1/2) The mapping

$$B \times \mathcal{S} \ni (f, v) \mapsto f * W_v(u)(1) = \int_G f(x) W_v(u)(x^{-1}) \, dx \in \mathbb{C}$$

is continuous.

Remark 3.13. *Note that if*

$$B = L_w^p(G) = \{f : G \to \mathbb{C} \mid \|f\|_{L_w^p} := \left(\int |f(x)|^p \, w(x) \, dx\right)^{1/p} < \infty\}$$

then the continuity condition will be a duality condition, i.e., $\mathcal{S} \ni v \mapsto W_v(u)^\vee \in L^q_{w^{-q/p}}(G)$ is continuous, where $W_v(u)^\vee(x) = W_v(u)(x^{-1})$ and $\frac{1}{p} + \frac{1}{q} = 1$. In this case *we have*

$$
\begin{aligned}
\left|\int_G f(x)W_v(u)(x^{-1})\,dx\right| &= \left|\int_G f(x)(w(x))^{1/p}W_v(u)(x^{-1})(w(x))^{-1/p}\,dx\right| \\
&\le \left(\int_G |f(x)|^p\, w(x)\,dx\right)^{1/p} \times \\
&\quad \left(\int_G |W_v(u)(x^{-1})|^q\,(w(x))^{-q/p}\,dx\right)^{1/q} \\
&= \|f\|_{L^p_w}\|W_v(u)^\vee\|_{L^q_{w^{-q/p}}} \\
&\le C\|f\|_{L^p_w}\|v\|_\alpha
\end{aligned}
$$

for some semi-norm $\|.\|_\alpha$ on $\mathcal{S}$.

Under these assumptions the coorbit space is well defined and the results of Theorem 3.11 are still true.

Theorem 3.14. *Assume that B and $u \in \mathcal{S}$ satisfy Assumption 3.12. Then the coorbit space $\mathrm{Co}_\pi^u B$ is well defined and the results of Theorem 3.11 are still true.*

Proof. We only have to show that $(R1/2)$ implies both $(R1)$ and $(R2)$, which is obvious. So, the assumptions of Theorem 3.11 are satisfied and our assertion is true. □

We end this section with the following theorem which shows how the coorbit space depends on the analyzing vector.

Theorem 3.15. *Assume that u_1 and u_2 are π-analyzing vectors for $\mathcal{S}$ which satisfy Assumption 3.4, and the following properties hold for $i, j \in \{1, 2\}$:*

1. *there are nonzero constants $C_{i,j}$ such that $W_{u_i}(\lambda) * W_{u_j}(u_i) = C_{i,j}W_{u_j}(\lambda)$ for all $\lambda \in \mathcal{S}^*$,*
2. *the mapping $B_{u_i} \ni f \mapsto f * W_{u_j}(u_i) \in B$ is continuous.*

Then $Co_\pi^{u_1} B = Co_\pi^{u_2} B$ with equivalent norms.

Proof. Let $\lambda \in Co_\pi^{u_1} B$. Then $W_{u_1}(\lambda) \in B$. By Assumption (1),

$$W_{u_2}(\lambda) = C_{1,2}^{-1} W_{u_1}(\lambda) * W_{u_2}(u_1).$$

By Assumption (2), we conclude that $W_{u_2}(\lambda) \in B$ and $\|W_{u_2}(\lambda)\|_B \leq A\|W_{u_1}\|_B$. Similarly, by interchanging u_1 and u_2 we have the same for $\lambda \in Co_\pi^{u_2} B$, and the proof is completed. □

3.2 Sampling Theory: Convolutive Coorbits

3.2.1 Sampling on Hilbert Spaces

In sampling theory, we are interested in reconstructing a continuous signal f from a discrete set of values $\{f(x_i)\}$. One of the important tools in this field is the frame theory. This theory generalize the definition of the orthonormal-basis of a Hilbert space. Let $\mathcal{H}$ be a separable Hilbert space a sequence $\{e_i\}_{i\in\mathbb{N}}$ of vectors in $\mathcal{H}$ form a basis if $\operatorname{span}\{e_i \mid i \in \mathbb{N}\}$ is dense in $\mathcal{H}$ and $(e_i, e_j)_\mathcal{H} = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function. It follows that, for $\{(f, e_i)_\mathcal{H}\}$, the signal f can be reconstructed uniquely by $f = \sum_i (f, e_i)_\mathcal{H}\, e_i$. However, we are looking to reconstruct f from the sample $\{f(x_i)\}$. If we assume that $\mathcal{H}$ is a reproducing Hilbert space, then one can replace (f, e_i) by the evaluation map at x_i which gives the require reconstruction. Frame theory is established by R. Duffin and A. Schaeffer in 1952 [18]. A sequence $\{f_i\}$ of elements of $\mathcal{H}$ is called a Hilbert frame if there are positive constants A and B such that

$$A\|f\|^2 \leq \sum_i |(f, f_i)|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. The numbers A and B are called frame bounds. If $A = B$, then the frame is called a tight frame (see [11] for more details). Note that the condition $A\|f\|^2 \leq \sum_i |(f, f_i)|^2$ implies that the frame $\{f_i\}$ is complete, i.e., the closure of $\operatorname{span}\{f_i : i \in \mathbb{N}\}$ equals to $\mathcal{H}$. The condition $\sum_i |(f, f_i)|^2 \leq B\|f\|^2$ ensures that the operator $T : \mathcal{H} \to \ell^2(\mathbb{N})$, $T(f) = \{(f, f_i)\}$ is bounded. If we denote the adjoint operator of T by T^*, then the frame operator is defined to be $S : \mathcal{H} \to \mathcal{H}$, $S := T^*T$. One can see that $S(f) = \sum_i (f, f_i) f_i$ and S is a bounded, invertible, self adjoint, and positive operator (see, for example, [11]). Therefore, we can reconstruct f by $f = \sum_i (f, f_i) S^{-1} f_i$.

3.2.2 Sequence Spaces and Banach Frames

In this section, we have a background and some results about atomic decompositions and frames of coorbit spaces constructed by convolution which is defined by a given representation on G. A Banach space of sequences $\{x_i\}_{i\in I}$ is called a Banach sequence space with index I. For any solid BF-space B, we can associate a sequence Banach space B^d which is first introduced in [24]. For example, a natural Banach sequence space that corresponds to the $L^p(\mathbb{R})$ spaces is the sequence space $l^p(\mathbb{Z})$ space. We need the following definition in order to introduce the associated Banach sequence space of B.

Definition 3.16. *For a relatively compact neighborhood U of the identity, the family $\{x_i\}_{i\in I}$ of elements in G is called U-well spread in G if*

i. $G \subset \bigcup_{i\in I} x_i U$, *and*

ii. if there exists an real integer N such that

$$\sup_i \#\{j : x_i U \cap x_j U \neq \phi\} \leq N.$$

Now, let us define the sequence space B^d that associated to the space B.

Definition 3.17. *Let B be a solid Banach space. Assume that the family $\{x_i\}_{i\in I}$ is U-well spread in G. The associated sequence space B^d is the space*

$$B^d = \big\{\{\lambda_i\}_{i\in I} | \sum_{i\in I} |\lambda_i| 1_{x_i U} \in B\big\}$$

equipped with the norm

$$\|\{\lambda_i\}_{i\in I}\|_{B^d} := \left\| \sum_i |\lambda_i| 1_{x_i U} \right\|_B .$$

The space B^d is a solid Banach space in the sense that for $\{\lambda_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ with $\{\lambda_i\}_{i\in I} \in B^d$ and $|\eta_i| \leq |\lambda_i|$ for all $i \in I$, then $\{\eta_i\}_{i\in I} \in B$ and $\|\{\eta_i\}_{i\in I}\|_{B^d} \leq \|\{\lambda_i\}_{i\in I}\|_{B^d}$. A typical example is the space $B = L^p(G)$ with the corresponding sequence space $B^d = l^p(I)$. More properties of B^d can be found in [24].

Now, we introduce the definition of atomic decompositions and Banach frames which were first introduced in [28].

Definition 3.18. *Let B be a Banach space, and let B^* its dual space. If there is an associated Banach sequence space B^d with index set I, such that for $\lambda_i \in B^*$ and $\phi_i \in B$, we have*

i. $\{\lambda_i(f)\}_{i\in I} \in B^d$ *for all* $f \in B$,

ii. the norms $\|\lambda_i(f)\|_{B^d}$ *and* $\|f\|_B$ *are equivalent, that is, there exist* $A, B > 0$ *such that*

$$A\|f\|_B \leq \|\lambda_i(f)\|_{B^d} \leq B\|f\|_B,$$

iii. f *can be written as* $f = \sum_{i\in I} \lambda_i(f)\phi_i$.

Then $\{(\lambda_i, \phi_i)\}_{i\in I}$ is an atomic decomposition of B with respect to B^d.

More generally, a Banach frame for a Banach space can be defined as the following:

Definition 3.19. *Let B be a Banach space, and let B^* its dual space. If there is an associated Banach sequence space B^d with index set I, such that for $\lambda_i \in B^*$ we have*

i. $\{\lambda_i(f)\}_{i\in I} \in B^d$ *for all* $f \in B$, *ii.* *the norms* $\|\lambda_i(f)\|_{B^d}$ *and* $\|f\|_B$ *are equivalent, that is, there exist* $A, B > 0$ *such that*

$$A\|f\|_B \leq \|\lambda_i(f)\|_{B^d} \leq B\|f\|_B,$$

iii. *there is a bounded reconstruction operator* $T : B^d \to B$ *such that*

$$T(\{\lambda_i(f)\}_{i\in I}) = f.$$

Then $\{\lambda_i\}_{i\in I}$ *is a Banach frame for* B *with respect to* B^d*. The constants* A *and* B *are called frame bounds. The frame called a tight frame if* $A = B$*.*

The following concept will be used in our theory of this chapter and later on.

Definition 3.20. *Let* U *be a relatively compact neighborhood of the identity. A family* $\{\psi_i\}_{i\in I}$ *of non-negative functions on* G *is called a bounded uniform partition of unity subordinate to* U *(or* U-BUPU*), if there is a* U*-well spread family* $\{x_i\}_{i\in I}$ *in* G *such that* $\operatorname{supp}\psi_i \subseteq x_iU$ *and* $\sum\limits_{i\in I}\psi_i(x) = 1$ *for all* $x \in G$*. Note that the sum is finite for a given* $x \in G$*.*

The following example is a concrete example of a U-BUPU family, which is used to prove some results in the upcoming chapters.

Example 3.21. *Consider the one dimensional torus* $\mathbb{T}$*, for a fixed integer* $N \in \mathbb{N}$ *define*

$$V := \left\{e^{i\theta} | -\frac{\pi}{N} \leq \theta \ < \frac{\pi}{N}\right\}$$

where $i = \sqrt{-1}$*. We will construct a* V*-well spread family* $\{t_j\}_{j=1}^N$ *as follows, define*

$$t_k := e^{\frac{2\pi}{N}(k-1)i} \quad \text{for all } k = 1, ..., N.$$

Then

$$t_kV = \left\{e^{\theta i} \mid \frac{-\pi + 2\pi(k-1)}{N} \leq \theta \ < \frac{\pi + 2\pi(k-1)}{N}\right\}$$

hence $t_jV \cap t_{j+1}V = \phi$. *It follows that*

$$\sup_k \#\{j : t_kV \cap t_jV \neq \emptyset\} = 0.$$

On the other hand, it is clear that

$$\mathbb{T} = \bigcup_{j=1}^{N} t_jV.$$

Hence $\{t_j\}_{j=1}^N$ *is a* V*-well spread set. Therefore, the family* $\{\eta_j\}_{j=1}^n$*, where*

$$\eta_j := 1_{t_jV}$$

is a V-BUPU.

3.2.3 Discretization: Feichtinger-Gr¨ochenig Theory

In this section, we summarize the mechanism of the sampling theory on the coorbit spaces that were constructed by Feichtinger-Gröchenig theory. For more details and proofs see [24] and [28]. In this theory, we assume that

1. w is a submultiplicative weight function on G such that $w(x) \geq 1$ and $w(x) = \Delta(x^{-1})w(x^{-1})$.

2. $(\pi, \mathcal{H})$ is an irreducible, unitary, w-integrable representation.

3. The space B is a solid left-invariant Banach function space which is continuously embedded in $L^1_{loc}(G)$ and the relation $B * L^1_w(G) \subset B$ holds with

$$\|f * g\|_B \leq \|f\|_B \|g\|_{L^1_w}.$$

The main ingredient to construct a Banach frame and an atomic decomposition is the oscillation function which is given in the following definition.

Definition 3.22. *Let U be a relatively compact neighborhood of the identity, and let F be a function on G. The U-left oscillation of F is*

$$M_U^l F(x) := \sup_{u \in U} |F(ux) - F(x)|$$

and the U-right oscillation of F is

$$M_U^r F(x) := \sup_{u \in U} |F(xu^{-1}) - F(x)|.$$

Discretization of coorbit spaces in this theory can be done if we have more restrictions on the analyzing vector $u \in \mathcal{A}_w$ which can be chosen from the better vectors. The better vectors construction depends on the Wiener spaces, which we introduce in summary as the following:

Let Q be a compact neighborhood of the identity, the control function of a function F on G is defined by

$$KF(x) := \sup_{y \in xQ} |F(y)|.$$

Let B be a BF-space, we define the space

$$W(B) := \{F \in B \mid KF \in B\}$$

with norm

$$\|F\|_{W(B)} := \|F\|_B.$$

This space is independent of the choice of the compact subset Q. Now, we define the set of better vectors (or basic atoms) as

$$\mathcal{B}_w := \left\{u \in \mathcal{H} \mid W_u(u) \in W\left(L_w^1(G)\right)\right\}.$$

This set of better vectors is a subset of $\mathcal{A}_w$ and it is still dense in $\mathcal{H}_w^1$ by irreducibility. In the following theorem, we can see the advantages of the better vectors on the discretization. Here we introduce a discretization operators which discretize the identity operator on the space $B * W_u(u)$.

Theorem 3.23. *Let $\{x_i\}_{i\in I}$ be a U-well spread family, and let $\{\psi_i\}_{i\in I}$ be any U-BUPU family with* $\mathrm{supp}\psi_i \subset x_iU$. *If $u \in \mathcal{B}_w$ and $M_U^l W_u(u) \in L_w^1(G)$, then the following operators are well define bounded operators from $B * W_u(u)$ into itself:*

(1) $T_U f := \sum_i f(x_i)\psi_i * W_u(u)$.

(2) $S_U f := \sum_i c_i f(x_i)\psi_i * L_{x_i}W_u(u)$, *where* $c_i = \int \psi_i$.

(3) $R_U f := \sum_i c_i(f)\psi_i * L_{x_i}W_u(u)$, *where* $c_i(f) = \int f(x)\psi_i(x)\,dx$.

Here, the sum is pointwise limit of partial sums, and if the compactly supported continuous functions are dense in B, then the sum is in norm. Moreover, these operators converge to the identity operator on $B * W_u(u)$ as $\mathrm{Int}(U) \to \{1\}$, *and the convergence is in the operator norm.*

The convergence of the operators in the previous theorem is in the following sense: Denote the interior of U by $\mathrm{Int}(U)$ and the identity operator on $B * W_u(u)$ by $\mathrm{id}_{B*W_u(u)}$. For any family $\{\{\psi_i^\alpha\}_{i\in I}, U_\alpha\}_\alpha$ of U_α's-BUPU we define the partial ordering by inclusion on U_α's. Then

$$\|T_{U_\alpha} - \mathrm{id}_{B*W_u(u)}\|_{op} \to 0$$

as $\mathrm{Int}(U_\alpha) \to \{1\}$.

One can choose U_α small enough such that these operators are invertible. For example, if we choose U_α such that $\|M_{U_\alpha}^l W_u(u)\|_B < 1$, then T_{U_α} is invertible because $\|T_{U_\alpha} - Id_{B*W_u(u)}\|_{op} < \|M_{U_\alpha}^l W_u(u)\|_B$ as proved in [28]. Now, we state the main results about the coorbit frames and decompositions.

Theorem 3.24. *Under the assumptions of this section, let $u \in \mathcal{B}_w$, and choose U small enough such that the discretization operators of the identity are invertible. If $M_U^l W_u(u) \in L^1(G)$, then* *i. (Banach frame by T_U and S_U) The family $\{\pi(x_i)u\}$ is a frame of $\mathrm{Co}_\pi^u B$ with respect to the sequence space B^d, with reconstruction operators given by*

(a) (the T_U operator)

$$v \mapsto W_u^{-1} T_U^{-1} \left(\sum_i (v, \pi(x_i)u)_{\mathcal{H}} \psi_i * W_u(u) \right),$$

(b) (the S_U operator)

$$v \mapsto W_u^{-1} S_U^{-1} \left(\sum_i c_i (v, \pi(x_i)u)_{\mathcal{H}} \psi_i * W_u(u) \right)$$

where $c_i = \int \psi_i$.

ii. (Atomic decomposition by S_U and R_U)

(a) Let

$$\lambda_i(v) := \left(S_U^{-1} W_u(v) \right)(x_i).$$

Then $\{(\lambda_i, \pi(x_i)u)\}$ is an atomic decomposition for $\mathrm{Co}_\pi^u B$ with respect to the sequence space B^d.

(b) Let

$$\lambda_i(v) := \int_G \left(R_U^{-1} W_u(v) \right)(x) \psi_i(x)\, dx.$$

Then $\{(\lambda_i, \pi(x_i)u)\}$ is an atomic decomposition for $\mathrm{Co}_\pi^u B$ with respect to the sequence space B^d.

Any vector $v \in \mathrm{Co}_\pi^u B$ can be reconstructed by $v = \sum_i \lambda_i(v)\pi(x_i)u$ with convergence in weak-topology. If the compactly supported continuous functions are dense in B, then the convergence is in norm.*

3.2.4 Discretization: Convolutive Coorbits via Weakly Smooth Vectors.

As we have seen in the previous section, the integrability condition is assumed in the discretization of coorbit spaces in the Feichtinger-Gröchenig theory. In this section, we are going to replace this condition by a smoothness condition which will enable us to cover more function spaces that the Feichtinger-Gröchenig theory. We summarize the results from [5].

Let G be a Lie Group, and let $\mathfrak{g}$ be its Lie algebra of dimension n. Fix a basis $\{E_1, E_2, ..., E_n\}$ for $\mathfrak{g}$. For a function $f \in B$, we say that f is left differentiable in the direction of $X \in \mathfrak{g}$, if

$$L(X)f(x) := \frac{d}{ds}\bigg|_{s=0} L_{e^{sX}} f(x) = \frac{d}{ds}\bigg|_{s=0} f(e^{-sX}x)$$

exists for all $x \in G$.

We say that f is right differentiable in the direction of $X \in \mathfrak{g}$ if

$$R(X)f(x) := \frac{d}{ds}\bigg|_{s=0} R_{e^{sX}} f(x) = \frac{d}{ds}\bigg|_{s=0} f(xe^{sX})$$

exists for all $x \in G$. A function f is left differentiable if it is differentiable in the direction of X for all $X \in \mathfrak{g}$, and the same for right differentiablity.

Let $N \in \mathbb{N}$. Then for any multi-index $\alpha = (\alpha(1), \alpha(2), ..., \alpha(N)) \in \{1, 2, ..., n\}^N$, we say that α is of order N and we write $|\alpha| = N$. We define

$$R^\alpha f := R(E_{\alpha(N)})R(E_{\alpha(N-1)})...R(E_{\alpha(1)})f$$

and

$$L^\alpha f := L(E_{\alpha(N)})L(E_{\alpha(N-1)})...L(E_{\alpha(1)})f$$

whenever the derivatives exist. We use the convention that $E_0 = \mathrm{id}_{\mathfrak{g}}$ for $|\alpha| = 0$.

Finally, a function f on G is left differentiable of order N if $L^\alpha f$ exists for all α with $|\alpha| = N$. Similarly, we define the right differentiablity of a function of order N. To discretize the coorbit space we need the following concept of smoothness.

Definition 3.25. *Let $(\pi, \mathcal{S})$ be a representation of G on a Fréchetspace $\mathcal{S}$ which is continuously embedded and weakly dense in its dual $\mathcal{S}^*$. Let $E1, E2, ..., E3$ be a basis for the Lie algebra $\mathfrak{g}$ of G.*

(1) A vector $u \in \mathcal{S}$ is called π-weakly differentiable in the direction of $X \in \mathfrak{g}$, the Lie algebra of G, if there is a vector, denoted by $\pi(X)u$, in $\mathcal{S}$ such that

$$\langle \lambda, \pi(X)u \rangle = \frac{d}{ds}\bigg|_{s=0} \left\langle \lambda, \pi(e^{sX})u \right\rangle$$

for all $\lambda \in \mathcal{S}^*$. A vector $u \in \mathcal{S}$ is called a π-weakly differentiable of order 1 if for $\alpha \in \{1, 2, ..., n\}$ there is a vector $\pi(E_\alpha)u \in \mathcal{S}$ such that

$$\langle \lambda, \pi(E_\alpha)u \rangle = \frac{d}{ds}\bigg|_{s=0} \langle \lambda, \pi(e^{sE_\alpha})u \rangle$$

for all $\lambda \in \mathcal{S}^*$.

(2) A vector $u \in \mathcal{S}$ is called a π-weakly differentiable of order 2 if for all $\alpha \in \{0, 1, ..., n\} \times \{0, 1, ..., n\}$ there is a vector $\pi(E_\alpha)u \in \mathcal{S}$ such that

$$\langle \lambda, \pi(E_\alpha)u \rangle = \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \langle \lambda, \pi(e^{sE_{\alpha(2)}})\pi(e^{tE_{\alpha(1)}})u \rangle$$

for all $\lambda \in \mathcal{S}^*$. Inductively we define π-weakly differentiable vector of order N

(3) A distribution $\lambda \in \mathcal{S}^*$ is called π^*-weakly differentiable in the direction of $X \in \mathfrak{g}$ if there is a distribution, denoted by $\pi^*(X)\lambda$, in $\mathcal{S}^*$ such that

$$\langle \pi^*(X)\lambda, v \rangle = \frac{d}{ds}\bigg|_{s=0} \left\langle \pi^*(e^{sX})\lambda, v \right\rangle$$

for all $v \in \mathcal{S}$. A vector $u \in \mathcal{S}$ is called a π-weakly differentiable of order 1 if for $\alpha \in \{1, 2, ..., n\}$ there is a distribution $\pi^*(E_\alpha)u \in \mathcal{S}^*$ such that

$$\langle \pi^*(E_\alpha)\lambda, v \rangle = \frac{d}{ds}\bigg|_{s=0} \langle \pi^*(e^{sE_\alpha})\lambda, v \rangle$$

for all $v \in \mathcal{S}$. Similarly we define π^*-weak differentiablity of order N.

Note that if $\pi(E_{\alpha(N)}), \pi(E_{\alpha(N-1)}), ...\pi(E_{\alpha(1)})$ exist, then

$$\pi(E_\alpha)u = \pi(E_{\alpha(N)})\pi(E_{\alpha(N-1)})...\pi(E_{\alpha(1)})u$$

and the same for $\pi^*(E_\alpha)$.

For the rest of this chapter, we assume that B is a BF-space on G satisfying the following assumptions:

Assumption 3.26. *Assume B is a solid bi-invariant BF-space of G such that*

I. The operators $f \mapsto L_a f$ and $f \mapsto R_a f$ are uniformly bounded on compact subsets of G, in the sense that for any compact subset U of G, there is a constant C_U such that

$$\sup_{x\in U} \|L_x f\|_B \leq C_U \|f\|_B$$

and

$$\sup_{x\in U} \|R_x f\|_B \leq C_U \|f\|_B,$$

II. the right translation $x \mapsto R_x f$ is continuous for all $f \in B$.

Furthermore, we define U_ϵ as follows

$$U_\epsilon := \{\exp(t_1E_1)\exp(t_2E_2)...\exp(t_nE_n) : -\epsilon \leq t_j \leq \epsilon, 1 \leq j \leq n := \dim(G)\}$$

Lemma 3.27. *If $u \in \mathcal{S}$ is π-weakly differentiable up to order N, then the function $x \mapsto W_u(\lambda)(x)$ is right differentiable up to order N for all $\lambda \in \mathcal{S}^*$ and $R^\alpha W_u(\lambda)(x) = W_{\pi(E_\alpha)u}(\lambda)$ for any multi-index α. Similarly, if $\lambda \in \mathcal{S}^*$ is π^*-weakly differentiable, then the function $x \mapsto W_v(\lambda)(x)$ is left differentiable for all $v \in \mathcal{S}$ and $L^\alpha W_u(\lambda)(x) = (-1)^{|\alpha|} W_u(\pi^*(E_\alpha)\lambda)$ for any multi-index $\alpha \in \{1, 2, ..., \dim(G)\}^N$, $N \in \mathbb{N}$.*

Proof. Assume that $u \in \mathcal{S}$ is a π-weakly differentiable vector. For any $X \in \mathfrak{g}$ and any $\lambda \in \mathcal{S}^*$ we have

$$\begin{aligned} R(X)W_u(\lambda)(x) &= \frac{d}{ds}\bigg|_{s=0} W_u(\lambda)(xe^{sX}) \\ &= \frac{d}{ds}\bigg|_{s=0} \left\langle \lambda, \pi(xe^{sX})u \right\rangle \\ &= \frac{d}{ds}\bigg|_{s=0} \left\langle \pi^*(x^{-1})\lambda, \pi(e^{sX})u \right\rangle \\ &= \left\langle \pi^*(x^{-1})\lambda, \pi(X)u \right\rangle \\ &= W_{\pi(X)u}(\lambda)(x). \end{aligned}$$

By induction, we have $R^\alpha W_u(\lambda)(x) = W_{\pi(E_\alpha)u}(\lambda)(x)$.

Next, we assume that $\lambda \in \mathcal{S}^*$ is π^*-weakly differentiable, then

$$\begin{aligned} L(X)W_u(\lambda)(x) &= \frac{d}{ds}\bigg|_{s=0} W_u(\lambda)(e^{-sX}x) \\ &= \frac{d}{ds}\bigg|_{s=0} \left\langle \lambda, \pi(e^{-sX})\pi(x)u \right\rangle \\ &= \langle \pi^*(-X)\lambda, \pi(x)u \rangle \\ &= - W_u\left(\pi^*\left(X\right)\lambda\right)(x). \end{aligned}$$

Again, the induction gives the result for $L^\alpha W_u(\lambda)$. $\square$

Now, we discretize the reproducing kernel Banach space B_u in order to discretize the coorbit space $\mathrm{Co}_\pi^u B$. Remember that the mapping

$$B \ni f \mapsto \int_G f(y) L_x W_u(u)^\vee(y)\, dy \in \mathbb{C}$$

is continuous for all $x \in G$ (see Lemma 2.7). This allows us to present $L_x W_u(u)^\vee$ as a functional on B with the pairing

$$\langle L_x W_u(u)^\vee, f \rangle = \int_G f(y) L_x W_u(u)^\vee(y)\, dy = f * W_u(u)(x).$$

In particular,

$$\langle L_x W_u(u)^\vee, f\rangle = f(x)$$

for $f \in B_u$.

Theorem 3.28. *Let $(\pi, \mathcal{S})$ be a representation of G, and let B be a BF-space on G satisfying Assumption 3.26. Assume that $u \in \mathcal{S}$ is π-weakly and π^*-weakly differentiable up to order of $\dim G$, and satisfies Assumption 3.12 as well. Assume that $W_u(u) \in B$ and assume that the mappings*

$$f \mapsto f * |W_{\pi(E_\alpha)u}(u)| \qquad \text{and} \qquad f \mapsto f * |W_u(\pi^*(E_\alpha)u)|$$

are continuous on B for all α with $|\alpha| \leq \dim(G)$. Then

1. *We can choose ϵ small enough such that for any U_ϵ-BUPU $\{\psi_i\}$ the following three operators*

$$\begin{aligned} T_1 f &:= \sum_i f(x_i)(\psi_i * W_u(u)) \\ T_2 f &:= \sum_i \lambda_i(f) L_{x_i} W_u(u)\ , \qquad \left(\lambda_i(f) = \int f(x)\psi_i(x)\, dx\right) \\ T_3 f &:= \sum_i c_i f(x_i) L_{x_i} W_u(u)\ , \qquad \left(c_i = \int \psi_i(x)\, dx\right) \end{aligned}$$

   are all invertible on B_u. The convergence of the sums above is pointwise and, if the continuous compactly supported functions are dense in B, then the convergence is also in norm.

2. *(Frame) The family $\{L_{x_i} W_u(u)^\vee\}$ is a frame for the space B_u with respect to the sequence space B^d. That means, the norms $\|f\|_B$ and $\|\{f(x_i)\}\|_{B^d}$ are equivalent, and any $f \in B_u$ can be reconstructed by*

$$f = T_1^{-1} A(\{f(x_i)\})$$

where $A : B^d \to B_u$ is a bounded operator given by

$$A(\{\lambda_i\}) = \sum_i \lambda_i \, \psi_i * W_u(u).$$

3. *(Atomic decomposition) The families $\{\lambda_i \circ T_2^{-1}, L_{x_i} W_u(u)\}$ and $\{c_i L_{x_i} W_u(u) \circ T_3^{-1}, L_{x_i} W_u(u)\}$ are atomic decompositions of B_u with respect to the sequence space B^d. That means, the norms $\|f\|_B$ and $\|\{\lambda_i \circ T_2^{-1}(f)\}\|_{B^d}$ are equivalent, and $f \in B_u$ can be written as $f = \sum_i \lambda_i(T_2^{-1} f) L_{x_i} W_u(u)$. The same for the other atomic decomposition.*

   The convergence of the sums is pointwise, and if the compactly supported continuous functions are dense in B, then the convergence is also in norm.

Proof. Set $\Phi(x) = W_u(u)(x)$, then by Lemma 3.27 we know that Φ is left and right differentiable up to order $\dim G$. Moreover, $R^\alpha W_u(u)(x) = W_{\pi(E_\alpha)u}(u)$ and $L^\alpha W_u(u)(x) = (-1)^{|\alpha|} W_u(\pi^*(E_\alpha)\, u)$. It follows that all assumptions of Theorem 2.6 in [7] are satisfied and the results hold. □

As a consequence of the above theorem and the fact that the spaces B_u and $\mathrm{Co}_\pi^u B$ are isometrically isomorphic, we have the following result about the existence of a frame and an atomic decomposition of the coorbit space of a function space under a smoothness condition on the kernel. For the proof, see [5].

Theorem 3.29. *Let $(\pi, \mathcal{S})$ be a representation of G, and let B be a BF-space on G. Assume that $u \in \mathcal{S}$ is a π-analyzing vector satisfying Assumption 3.12, which is both π-weakly and π^*-weakly differentiable. Furthermore, assume that $W_u(u) \in B$ and the mappings*

$$f \mapsto f * |W_{\pi(E_\alpha)u}(u)| \qquad \text{and} \qquad f \mapsto f * |W_u(\pi^*(E_\alpha)u)|$$

are continuous on B for all α with $|\alpha| \le \dim(G)$.

Then, we can choose ϵ small enough such that for any U_ϵ-well spread set $\{x_i\}$ the family $\{\pi(x_i)u\}$ is a frame for $\mathrm{Co}_\pi^u B$ with respect to the sequence space B^d, and the families $\{\lambda_i \circ T_2^{-1} \circ W_u \, , \pi^*(x_i)u\}$ and $\{c_i T_3^{-1} \circ W_u \, , \pi^*(x_i)u\}$ are atomic decompositions for $\mathrm{Co}_\pi^u B$ with respect to the sequence space B^d. In particular, $\phi \in \mathrm{Co}_\pi^u B$ can be reconstructed by

$$\phi = W_u^{-1} T_1^{-1} \Big(\sum_i W_u(\phi)(x_i) \psi_i * W_u(u) \Big)$$

$$\phi = \sum_i \lambda_i \big(T_2^{-1} W_u(\phi) \big) \pi(x_i)(u)$$

$$\phi = \sum_i c_i \; T_3^{-1} W_u(\phi) \, \pi(x_i)(u)$$

with convergence in $\mathcal{S}^*$. The convergence is in $\mathrm{Co}_\pi^u B$ if $C_c(G)$ is dense in B.

Chapter 4
Bergman Spaces on the Unit Disc

As an important application of coorbit theory, we can construct a frame and an atomic decomposition for spaces that can be described by coorbits. In this chapter we study as a concrete example the Bergman spaces on the unit disc.

4.1 Bergman Spaces

In this section we define Bergman spaces and summarize some results about Bergman spaces which can be found in, for example, [19], [30].

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc. Let dz denote the Lebesgue measure $dxdy$ where $z = x + iy$. For $s > -1$, let

$$d\mu_s(z) := \frac{s+1}{\pi}\left(1 - |z|^2\right)^s dz$$

be the $s-$weighted measure on the disc $\mathbb{D}$. For $1 \leq p < \infty$, define the $s-$ weighted L^p space :

$$L^p_s(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ measurabe } \mid \|f\|_{L^p_s} := \left(\int_{\mathbb{D}} |f(z)|^p \, d\mu_s(z)\right)^{1/p} < \infty \right\}$$

If we denote the space of holomorphic functions on the unit disc $\mathbb{D}$ by $\mathcal{O}(\mathbb{D})$, then we define the Bergman spaces, $A^p_s(\mathbb{D})$, to be

$$A^p_s(\mathbb{D}) := \{f \in L^p_s(\mathbb{D}) \mid f \in \mathcal{O}(\mathbb{D})\}$$

with norm

$$\|f\|_{A^p_s(\mathbb{D})} := \|f\|_{L^p_s(\mathbb{D})} .$$

Now let us summarize some important properties of Bergman spaces. We start by proving that Bergman spaces are Banach spaces. However, we need the following lemma, from [30], to prove our claim.

Lemma 4.1. *Fix $1 \leq p < \infty$, $s > -1$. For every compact subset of $\mathbb{D}$ there exists a positive constant C such that*

$$\sup_{z \in K} |f(z)| \leq C \|f\|_{L_s^p(\mathbb{D})}$$

for all $f \in A_s^p(\mathbb{D})$.

Theorem 4.2. *Assume that $1 \leq p < \infty$, and $s > -1$. Bergman spaces are closed subspaces of $L_s^p(\mathbb{D})$, and hence they are Banach spaces.*

Proof. Let $\{f_n\}$ be a sequence in the space $A_s^p(\mathbb{D})$ such that $f_n \to f$ in the space $L_s^p(\mathbb{D})$. According to Theorem 5.2 from [42], it is enough to show that $f_n \to f$ uniformly on every compact subet of the unit disc $\mathbb{D}$. For any $n, m \in \mathbb{N}$ we have $f_n - f_m \in A_s^p(\mathbb{D})$, therefore, one can apply Lemma 4.1. Thus

$$\sup_{z \in K} |f_n(z) - f_m(z)| \leq C \|f_n - f_m\|_{L_s^p(\mathbb{D})}$$

By the uniform Cauchy criteria, $f_n \to f$ uniformly on K. $\square$

In particular, $A_s^2(\mathbb{D})$ is a Hilbert space with the inner product:

$$(f, g)_s := \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu_s(z).$$

As a consequence of Lemma 4.1, Bergman spaces are reproducing kernel Banach spaces.

Theorem 4.3. *For $1 \leq p < \infty$, and $s > -1$, Bergman spaces are reproducing kernel Banach spaces. In particular, $A_s^2(\mathbb{D})$ is a reproducing kernel Hilbert space.*

Proof. We have to show that the evaluation map $f \mapsto f(z)$ is continuous. Let $z \in \mathbb{D}$, choose K to be a closed disc with center z and radius $r < \min\{|z|, 1-|z|\}$.Then the estimation of $|f(z)|$ given by Lemma 4.1:

$$|f(z)| \leq \sup_{w \in K} |f(w)| \leq C\|f\|_{L^p_s(\mathbb{D})}$$

$\square$

In the following theorem we find the orthogonal projection of the space $A^2_s(\mathbb{D})$ explicitly.

Theorem 4.4. *For $s > -1$, the orthogonal projection $\mathbf{P}_s : L^2_s(\mathbb{D}) \to A^2_s(\mathbb{D})$ onto $A^2_s(\mathbb{D})$ is given by*

$$\mathbf{P}_s f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^{s+2}}\, d\mu_s(w).$$

In particular, for $f \in A^2_s(\mathbb{D})$, we have

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^{s+2}}\, d\mu_s(w).$$

If we define $k(z,w) := (1-\bar{w}z)^{-(s+2)}$, then for any $f \in A^2_s(\mathbb{D})$ we have

$$f(z) = \int_{\mathbb{D}} f(w)k(z,w)\, d\mu_s(w).$$

therefore, the function k is a reproducing kernel.

4.2 The Group of Automorphisms of the Unit Disc

As a natural question one would ask about the natural groups that act on Bergman spaces. For that reason we will study the group of automorphisms on the unit disc. Recall that the group of automorphisms of the unit disc is the group of all biholomorphic functions from $\mathbb{D}$ onto itself. In complex analysis, it is a well known fact that any automorphism of the unit disc is of the form

$$f(z) = e^{i\theta}\frac{z+a}{1+\bar{a}z}$$

for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Note that if $f(z) = e^{i\theta}\frac{z+a}{1+\bar{a}z}$ for some θ, then $\theta_n := \theta + 2n\pi$ gives the same function for all $n \in \mathbb{Z}$. One can also describe the automorphisms group in terms of linear matrix groups. Let $SU(1,1)$ be the group

$$SU(1,1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

It is not hard to see that the mapping

$$f \mapsto \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} e^{i\theta_n/2} & ae^{i\theta_n/2} \\ e^{-i\theta_n/2}\bar{a} & e^{-i\theta_n/2}\bar{a} \end{pmatrix}$$

for some choice of θ_n is a double valued function,

$$\frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} e^{i\theta/2} & ae^{i\theta/2} \\ e^{-i\theta/2}\bar{a} & e^{-i\theta/2}\bar{a} \end{pmatrix} \text{ and } \frac{-1}{\sqrt{1-|a|^2}} \begin{pmatrix} e^{i\theta/2} & ae^{i\theta/2} \\ e^{-i\theta/2}\bar{a} & e^{-i\theta/2}\bar{a} \end{pmatrix}$$

corresponds to the same f. When identifying these two matrices we have an isomorphism between the group of all automorphisms on $\mathbb{D}$ and $SU(1,1)/\{\pm 1\}$.

From the discussion above, the group $SU(1,1)$ is the natural group acting on the disc, and hence on Bergman spaces. The following theorem describes the action of $SU(1,1)$ on the unit disc.

Theorem 4.5. *The group $SU(1,1)$ acts transitively on the unit disc $\mathbb{D}$ by the action*

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z = \frac{az+b}{\bar{b}z+\bar{a}}.$$

More over if we denote the origin in $\mathbb{C}$ by o, then the subgroup

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

is the stabilizer of o.

Proof. Direct calculations show that $g_1 \cdot (g_2 \cdot z) = (g_1 g_2) \cdot z$ for all $g_1, g_2 \in SU(1,1)$. Let us show that $SU(1,1)$ acts transitively on $\mathbb{D}$, for any $z \in \mathbb{D}$ define

$$g_z = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$$

then we have

$$\left| \frac{1}{\sqrt{1-|z|^2}} \right|^2 - \left| \frac{z}{\sqrt{1-|z|^2}} \right|^2 = 1$$

and $g_z \cdot o = z$, which shows that $g_z \in SU(1,1)$, and hence the action is transitive. For the zero stabilizer we have $g \cdot 0 = 0$ if and only if $b = 0$ and hence

$$g = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$$

with $|a| = 1$. $\square$

As a consequence we can describe the unit disc as a homogeneous space.

Corollary 4.6. *The unit disc $\mathbb{D}$ is homeomorphic to $SU(1,1)/K$, where*

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

4.3 Discrete Series Representations of SU(1,1)

Let $g \in SU(1,1), z \in \mathbb{D}$, and define

$$j(g,z) := \frac{d(g \cdot z)}{dz}$$

then

$$\begin{aligned} j(g,z) =& \frac{a(\bar{b}z + \bar{a}) - \bar{b}(az + \bar{a})}{(\bar{b}z + \bar{a})^2} \\ =& \frac{1}{(\bar{b}z + \bar{a})^2} \end{aligned}$$

and the function j satisfies the cocycle condition

$$j(g_1 g_2, z) = j(g_1, g_2 \cdot z) j(g_2, z).$$

The calculations in [43] show that the the measure $d\mu(z) = \frac{1}{\pi}(1 - |z|^2)^{-2} dz$ is an $SU(1,1)$-invariant measure on $\mathbb{D}$. Therefore, for all integer values of $s > -1$, the function

$$\pi_s(g) f(z) := \left(j(g^{-1}, z)\right)^{s/2} f(g^{-1} \cdot z)$$

defines a unitary representation of $SU(1,1)$ in the space $L^2_{s-2}(\mathbb{D})$. This representation is not irreducible, however if we restrict that representation on the Bergman space $\mathcal{H}_s := A^2_{s-2}(\mathbb{D})$, the result is an irreducible representation (see [43]). In term of elements of $SU(1,1)$ one can write the representation by the formula:

$$\pi_s \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} f(z) = (-\bar{b}z + a)^{-s} f\left(\frac{\bar{a}z - b}{-\bar{b}z + a}\right) \tag{4.1}$$

This family of representations, $\{\pi_s\}_{s=0}^{\infty}$ is called the discrete series representations of $SU(1,1)$.

Note that we restrict ourselves to $s = 0, 1, 2, 3, 4, 5, \ldots$, because otherwise the term

$$(-\bar{b}z + a)^{-s}$$

will be undefined as single valued function. We now define a discrete series representations for all $s > -1$. For this reason we will restrict our representations to a simply connected subgroup of $SU(1,1)$. To construct this group we use the Iwasawa decomposition of $SU(1,1) = S \times K$, where

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

and $S \simeq SU(1,1)/K \simeq \mathbb{D}$, which implies that S is simply connected group. Also there is a one to one correspondence between functions on S(or equivalently K-right invariant functions on G) and functions on $\mathbb{D}$, which is given by $f(x) = \widetilde{f}(x \cdot o)$. This correspondence induces an $SU(1,1)$-invariant measure on S given by

$$\int_S f(x)\,dx = \frac{s+1}{\pi} \int_{\mathbb{D}} \widetilde{f}(z) \frac{dz}{(1-|z|^2)^2} \tag{4.2}$$

For $s > -1$, we define the s-weighted L^p space on S as

$$L^p_s(S) = \{f : S \to \mathbb{C} : \int_S |f(x)|^p \,(1-|x\cdot o|^2)^s dx < \infty\}$$

with norm

$$\|f\|_{L^p_s(S)} := \left(\int_S |f(x)|^p \,(1-|x\cdot o|^2)^s dx \right)^{1/p}$$

Using (4.2), we see that

$$\int_S |f(x)|^p \,(1-|x\cdot o|^2)^{(s+2)} dx = \frac{s+1}{\pi} \int_{\mathbb{D}} |\widetilde{f}(z)|^p \,(1-|z|^2)^s dx$$

Therefore we have the isometry

$$\|f\|^p_{L^p_{(s+2)}(S)} = \|\widetilde{f}\|^p_{L^p_s(\mathbb{D})}. \tag{4.3}$$

Since the universal covering of a simply connected group is isomorphic to itself, the restriction of π_s on the group S is a well defined unitary representation for all real values $s > -1$. We will also denote this restriction by π_s.

From now on, we will work with the subgroup S instead of the full group $SU(1,1)$, and the representation π_s is the one that defined on S.

4.4 Wavelets on the Bergman Space ${A_s}^2(\mathbb{D})$

In this section we define the wavelet transform on the Bergman space A^p_{s-2}, to use it later to define coorbit spaces of Bergman spaces. Form now on, through this chapter, we use $\mathcal{H}_s = A^p_{s-2}$, and $u = 1_{\mathbb{D}}$ the characteristic function on the disc $\mathbb{D}$.

The function u is in the space $\mathcal{H}_s = A^2_{(s-2)}(\mathbb{D})$. Indeed, the integral:

$$\int_{\mathbb{D}} 1_{\mathbb{D}}(z)\, d\mu_{s-2}(z) = \frac{s-1}{\pi} \int_{\mathbb{D}} \left(1-|z|^2\right)^{s-2} dz$$

$$
\begin{aligned}
&= \frac{s-1}{\pi} \int_0^{2\pi} \int_0^1 \left(1-r^2\right)^{s-2} r \, dr d\theta \\
&= \frac{s-1}{\pi} \int_0^{2\pi} \int_0^1 \frac{1}{2} u^{s-2} \, du d\theta \\
&= 1
\end{aligned}
$$

is finite, and so $u \in \mathcal{O}(\mathbb{D}) \cap L^2_{s-2}(\mathbb{D})$. It follows that, one can define the wavelet (voice) transform $W_u : \mathcal{H}_s \to \mathcal{C}_b(G)$ by:

$$
W_u^s(v)(x) = (v, \pi_s(x)u)_{\mathcal{H}_s} = (v, \pi_s(x)u)_{(s-2)}
$$

which can be concretely given in following result.

Proposition 4.7. *For $s > 1$, the voice transform W_u^s on the space $\mathcal{H}_s = A^2_{s-2}$ is given by*

$$
W_u^s(v) \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = (\bar{a})^{-s} \, v \left(\frac{b}{\bar{a}} \right).
$$

In particular,

$$
|W_u^s(u)(x)| = |a|^{-s} = (1 - |x \cdot o|^2)^{s/2}.
$$

Proof. The voice transform for a function $v \in A^2_s(\mathbb{D})$ is given by $W_u^s(v)(x) = (v, \pi(x)u)$, for

$$
x = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}
$$

we have

$$
\begin{aligned}
W_u^s(v)(x) &= \frac{s-1}{\pi} \int_{\mathbb{D}} v(z) \, \overline{\pi_s(x)u(z)} \left(1 - |z|^2\right)^{s-2} dz \\
&= \frac{s-1}{\pi} \int_{\mathbb{D}} v(z) \, (-bz + \bar{a})^{-s} \left(1 - |z|^2\right)^{s-2} dz \\
&= \frac{s-1}{\pi} \int_{\mathbb{D}} (\bar{a})^{-s} \, v(z) \left(1 - \frac{b}{\bar{a}} z\right)^{-s} \left(1 - |z|^2\right)^{s-2} dz \\
&= (\bar{a})^{-s} \, v \left(\frac{b}{\bar{\alpha}} \right)
\end{aligned}
$$

where we used Theorem 4.4 in the last step. For the other part, note that $|a|^2-|b|^2=1$ which implies that

$$|a|^{-1}=(1-|b/\bar{a}|^2)^{1/2}=(1-|x\cdot o|^2)^{1/2}.$$

Therefore, $|W_u^s(u)(x)|=|a|^{-s}=(1-|x\cdot o|^2)^{s/2}$. □

In the following lemma we see that for $1<s\leq 2$ the representation π_s is not integrable and hence we can not use the Feichtinger-Gröchenig theory to discretize Bergman spaces for this case.

Lemma 4.8. *The representation $(\pi_s,\mathcal{H}_s)$ of the group S, is square integrable if and only if $s>1$, and it is integrable if and only if $s>2$.*

Proof. Assume $u=1_{\mathbb{D}}$, by Proposition 4.7 we have

$$\begin{aligned}\int_S |(u,\pi_s(x)u)_{\mathcal{H}_s}|^2\,dx &= \int_S (1-|x\cdot o|^2)^2\,dx\\ &= \frac{s-1}{\pi}\int_{\mathbb{D}}(1-|z|^2)^{s-2}\,dz\end{aligned}$$

This integral is finite if and only if $s>2$. A similar argument shows the other part. □

4.5 Bergman Spaces as Coorbits

We have seen in the previous section that $(\pi_s,\mathcal{H}_s)$ is a integrable irreducible unitary representation for $s>2$, which means that we can apply Feichtinger-Gröchenig theory, where as the representation is no longer integrable for $1<s\leq 2$, and hence we will use the construction of the coorbits in the dual pairing.

In this section we are going to write $A_\alpha^p(\mathbb{D})$ as a coorbit space of the Banach space $L^p_{\alpha+2-sp/2}(S)$ by using our Fréchetspace to be the space of smooth vectors $\mathcal{H}_s^\infty$ of the

Hilbert space $\mathcal{H}_s = A^p_{s-2}(\mathbb{D})$. We start by a background about the space of smooth vectors which can be found in [46].

A function $f : G \to \mathcal{H}$ is of class $C^1(G, \mathcal{H})$ if $L(E_j)f$ exists and continuous for all $j = 1, 2, ...n$. For any multi-index α of order N a function f on G is of class $C^N(G, \mathcal{H})$, if for any $|\alpha| = N$, the function $L^\alpha f$ is continuous. Finally, a function f is of class $C^\infty(G, \mathcal{H})$ if it is of class $C^N(G, \mathcal{H})$ for all N. The space $C^\infty(G, \mathcal{H})$ is a Fréchetspace which is topologized by the family of semi-norms:

$$\|f\|_{N,K} := \sup_{x\in K, |\alpha|=N} \|L^\alpha f(x)\|_{\mathcal{H}}$$

for any compact subset K of G.

Definition 4.9. *The space of smooth vectors is defined by*

$$\mathcal{H}^\infty_\pi := \{v \in \mathcal{H}_\pi \mid x \mapsto \pi(x)v \text{ is in} C^\infty(G, \mathcal{H}_\pi)\}$$

with a topology inherited from the space $C^\infty(G, \mathcal{H})$ under the inclusion $v \mapsto F_v$ where $F_v(x) = \pi(x)v$. A representation $\mathcal{H}_\pi$ is called smooth representation if the space of smooth vectors $\mathcal{H}^\infty_\pi$ is dense in $\mathcal{H}_\pi$.

We summarize some properties of the space of the smooth vectors in the following theorem which can be found in [46].

Theorem 4.10. *Let $\mathcal{H}_\pi$ be a square-integrable unitary representation of G. then the following are true:*

1. *The space of smooth vectors $\mathcal{H}^\infty_\pi$ is a Fréchetspace with the family of semi-norms*

$$\|v\|_{N,K} := \sup_{x\in K, |\alpha|=N} \|E_\alpha F_v(x)\|_{\mathcal{H}}.$$

2. *The space $\mathcal{H}^\infty_\pi$ is dense in $\mathcal{H}_\pi$, in particular the representation $\mathcal{H}_\pi$ is smooth.*

3. The space $\mathcal{H}_\pi^\infty$ is π-invariant subspace of $\mathcal{H}_\pi$.

4. The representation $\pi\big|_{\mathcal{H}_\pi^\infty}$ is irreducible.

We will keep denoting the restriction of the representation π on the space $\mathcal{H}^\infty$ by π. If we denote the conjugate dual of $\mathcal{H}^\infty$ by $\mathcal{H}^{-\infty}$, then the dual representation of π will be denoted by π^*, that is

$$\langle \pi^*(x)\lambda, u\rangle = \langle \lambda, \pi(x)u\rangle$$

Go back to Bergman spaces, we start by the following characterization of the smooth vectors and its conjugate dual which can be found in [34].

Lemma 4.11. *The space of smooth vectors $\mathcal{H}_s^\infty$ and its conjugate dual $\mathcal{H}_s^{-\infty}$ are exactly described to be:*

(i) A vector $v \in \mathcal{H}_s^\infty$ if and only if $v = \sum_{k=0}^\infty a_k z^k$ such that for any $m \in \mathbb{N}$ there exists a constant C_m satisfies

$$|a_k|^2 \le C_m(1+k)^{-m}.$$

(ii) A distribution $v \in \mathcal{H}_s^{-\infty}$ if and only if $v = \sum_{k=0}^\infty b_k z^k$ such that there exist both $m \in \mathbb{N}$ and a constant C_m satisfy

$$|b_k|^2 \le C_m(1+k)^{m}.$$

By Theorem 4.10, the space $\mathcal{H}_s^\infty$ is π_s-invariant subspace and it is dense in $\mathcal{H}_s$. We keep denoting the subrepresentation on the space $\mathcal{H}_s^\infty$ by π_s, then π_s is a unitary representation on G.

The following corollaries play an important role in the discretization of Bergman spaces:

Corollary 4.12. *The smooth vectors of the representations $\mathcal{H}_s$ are bounded on the unit disc.*

Proof. Let v be a smooth vector, by Theorem 4.11, we obtain the series representation $v = \sum a_k z^k$ with

$$|a_k|^2 \leq C_m(1+k)^{-m}.$$

We can estimate the sum by

$$\begin{aligned}\sum |a_k| =& \sum (|a_k|^2)^{1/2} \\ \leq& C_4^{1/2} \sum (1+k)^{-2} \\ \leq& C.\end{aligned}$$

Thus, $|v(z)| \leq \sum |a_k| \leq C$ for all $z \in \mathbb{D}$. □

Corollary 4.13. *Assume $u = 1_{\mathbb{D}}$, and assume $v \in \mathcal{H}_s^\infty$. Then, there exists a constant C_v depending continuously on v such that $|W_u(v)(x)| \leq C_v |W_u(u)(x)|$ and $|W_v(u)(x)| \leq C_v |W_u(u)(x)|$ for all $x \in G$.*

Proof. Assume that $v \in \mathcal{H}_s^\infty$. Note that for $x = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, we have

$$\begin{aligned}|W_u(v)(x)| =& \left| (\bar{a})^{-s}\, v\left(\frac{b}{\bar{a}}\right) \right| \\ =& |W_u(u)(x)| |v\left(\frac{b}{\bar{a}}\right)| \\ \leq& C_v |W_u(u)(x)|\end{aligned}$$

where we used Proposition 4.7 and Corollary 4.12. For the other part, we have

$$\begin{aligned}|W_v(u)(x)| &= |(u, \pi_s(x)v)| \ = \ |(v, \pi_s(x^{-1})u)| \\ &= |W_u(v)(x^{-1})| \leq C_v |W_u(u)(x^{-1})|\end{aligned}$$

$$= C_v|W_u(u)(x)|$$

where we used the first part and the fact that $(\pi_s, \mathcal{H}_s)$ is unitary. $\square$

Corollary 4.14. *Assume that $1 \leq p < \infty$, $s > 1$. If v is a smooth vector for $\mathcal{H}_s$, then $W^s_u(v) \in L^p_t(S)$ for $t + ps/2 > 1$.*

Proof. By Corollary 4.13, we have

$$\begin{aligned}
\int_S |W_u(v)(x)|^p(1-|x\cdot o|^2)^t\,dx \leq& C\int_S |W_u(u)(x)|^p(1-|x\cdot o|^2)^t\,dx \\
=& C\int_S (1-|x\cdot o|^2)^{t+ps/2}\,dx \\
=& C\int_{\mathbb{D}} (1-|z|^2)^{t+ps/2-2}\,dz
\end{aligned}$$

this integral is finite if $t + ps/2 - 2 > -1$ or equivalently $t + ps/2 > 1$. $\square$

We claim that the space $Co^u_{\pi_s}L^p_{\alpha+2-sp/2}(S)$ is a non-zero well defined Banach space for $-1 < \alpha < p(s-1) - 1$. More precisely, The space $B = L^p_{\alpha+2-sp/2}(S)$ satisfies Assumption 3.4.

I. The vector $u = 1_{\mathbb{D}}$ is π_s-cyclic vector in $\mathcal{H}^\infty_s$. Which follows from the fact that π_s is irreducible on the full group $SU(1,1)$ and $x \in SU(1,1)$ has the decomposition $x = sk$ where $s \in S$ and $k \in K$. A vector $v \in \mathcal{H}^\infty_s$ can be written as $v = \sum_j c_j\pi_s(x_j)u = \sum_j c_j\pi(s_j)\pi(k_j)u = \sum_j(c_je^{-s\theta_j i})\pi_s(s_j)u$.

II. The vector u is π_s-analyzing vector for $\mathcal{H}^\infty_s$. If $\phi \in \mathcal{H}^{-\infty}_s$, then $\phi = \sum_k a_k\phi_k$ where $\phi_k(z) = z^k$,and hence

$$\begin{aligned}
W_u(\phi)(x) =& \sum_k a_kW_u(\phi_k)(x) \\
=& \sum_k a_k(\bar{a})^{-s}\phi_k(\frac{b}{\bar{a}}) \\
=& (\bar{a})^{-s}\phi(\frac{b}{\bar{a}}).
\end{aligned}$$

Also, simple calculations show that if $x = \begin{pmatrix} a_x & b_x \\ \bar{b}_x & \bar{a}_x \end{pmatrix}$, $y = \begin{pmatrix} a_y & b_y \\ \bar{b}_y & \bar{a}_y \end{pmatrix}$, $w = x \cdot o$, and $z = y \cdot o$, then $\bar{a}_{y^{-1}x} = a_y \bar{a}_x (1 - w\bar{z})$. Now,

$$\begin{aligned} W_u(\phi) * W_u(u)(x) &= \int_S W_u(\phi)(y) W_u(u)(y^{-1}x)\, dy \\ &= \int_S (\bar{a}_y)^{-s} \phi(y \cdot o)(\bar{a}_{y^{-1}x})^{-s} dy \\ &= (\bar{a}_x)^{-s} \int_S (\bar{a}_y)^{-s}(a_y)^{-s}\phi(y\cdot o)(1 - w\bar{z})^{-s} dy \\ &= \frac{s-1}{\pi}(\bar{a}_x)^{-s} \int_{\mathbb{D}} (1-|z|^2)^s \phi(z)(1-w\bar{z})^{-s} \frac{dz}{(1-|z|^2)^2} \\ &= \frac{s-1}{\pi}(\bar{a}_x)^{-s} \int_{\mathbb{D}} \sum_k a_k \phi_k(z) \frac{(1-|z|^2)^{s-2}}{(1-w\bar{z})^s} dy \\ &= \frac{s-1}{\pi}(\bar{a}_x)^{-s} \sum_k a_k \int_{\mathbb{D}} \phi_k(z) \frac{(1-|z|^2)^{s-2}}{(1-w\bar{z})^s} dy \\ &= \frac{s-1}{\pi}(\bar{a}_x)^{-s} \sum_k a_k \phi_k(w) \\ &= \frac{s-1}{\pi} W_u(\phi)(x) \end{aligned}$$

Where we are allowed to interchange the integral and the sum by using Tonelli's Theorem, indeed,

$$\begin{aligned} \int_{\mathbb{D}} \sum_k \left| a_k \phi_k(z) \frac{(1-|z|^2)^{s-2}}{(1-w\bar{z})^s} \right| dy &\le C_1 \int_{\mathbb{D}} \sum_k |a_k| \left| \frac{(1-|z|^2)^{s-2}}{(1-w\bar{z})^s} \right| dy \\ &\le C \int_{\mathbb{D}} 1_{\mathbb{D}}(z) \left| \frac{(1-|z|^2)^{s-2}}{(1-w\bar{z})^s} \right| dy \\ &= C \end{aligned}$$

III. Now we show that the mapping

$$L^p_{\alpha+2-sp/2}(S) \times \mathcal{H}^\infty_s \ni (f, v) \mapsto \int_S f(x) \langle \pi^*_s(x)u, v \rangle \, dx$$

is continuous. As we remarked before this is a duality condition, that is we only have to show that $|W_v(u)^\vee| = |W_u(v)| \in L^q_{-\alpha q/p + sq/2}(G)$. By corollary 4.14, this is true if $-\alpha q/p + sq/2 + sq/2 > 1$ which equivalent to $\alpha < p(s-1) - 1$.

IV. Finally, we show that the coorbit space is non-trivial, by showing that $u \in Co^u_{\mathcal{H}_s^\infty} L^p_{\alpha+2-sp/2}(S)$. According to Corollary 4.14, we have

$$W_u(u) \in L^p_{\alpha+2-sp/2}(S),$$

hence $u \in Co^u_{\mathcal{H}_s^\infty} L^p_{\alpha+2-sp/2}(S)$.

The steps $(I)-(IV)$ prove the following theorem:

Theorem 4.15. *The spaces $Co^u_{\pi_s} L^p_r(G)$ are non-zero well defined π_s-invariant Banach spaces whenever $1 \le p < \infty$ and $-1 < \alpha < p(s-1)-1$.*

We end this section with our main result about Bergman spaces on the unit disc, which will be proved in detail for the general case of the unit ball in Chapter 6, so we will not include the proof here.

Theorem 4.16. *Assume that $1 \le p < \infty$ and $-1 < \alpha < p(s-1)-1$. The space $A^p_\alpha(\mathbb{D})$ is corresponding to the coorbit space $Co^u_{\pi_s} L^p_{\alpha+2-sp/2}(S)$ up to equivalence of norms.*

4.6 Discretization: Bergman Spaces on the Unit Disc

In the previous section we described Bergman spaces as coorbits. In this section we will use the theory of coorbits to construct frames and atomic decompositions for Bergman spaces via the subgroup S. Further, in [6], the authors gave a discretization through a finite covering group of $SU(1,1)$ to include the discrete representation series $(\pi_s, \mathcal{H}_s)$ for rational $s > 1$ and all smooth vectors to be analyzing vectors.

Proposition 4.17. *Assume that $1 \le p < \infty$ and $-1 < \alpha < p(s-1)-1$. The convolution operators $f \mapsto f * |W_u(v)|$ and $f \mapsto f * |W_v(u)|$ are continuous on $L^p_{\alpha+2-sp/2}(S)$ for all $v \in \mathcal{H}_s^\infty$.*

Proof. By Theorem 4.13, it is enough to show that the mapping $f \mapsto f * |W_u(u)|$ is continuous on $L^p_{\alpha+2-sp/2}(S)$. Now, for if $x = \begin{pmatrix} a_x & b_x \\ \bar{b}_x & \bar{a}_x \end{pmatrix}$, $y = \begin{pmatrix} a_y & b_y \\ \bar{b}_y & \bar{a}_y \end{pmatrix}$, $w = x \cdot o$, and $z = y \cdot o$, then $\bar{a}_{y^{-1}x} = a_y \bar{a}_x(1 - w\bar{z})$ and

$$\begin{aligned} f * |W^s_u(u)|(x) =& \int_S f(y)|W^s_u(u)|(y^{-1}x)|\, dy \\ \leq& C_1 \int_S f(y)(1 - |y^{-1}x \cdot o|^2)^{s/2}\, dy \\ =& C \int_{\mathbb{D}} |\widetilde{f}(z)\, \frac{|(1 - |w|^2)^{s/2}\,(1 - |z|^2)^{s/2-2}}{|1 - w\bar{z}|^s}\, dy \end{aligned}$$

According to the Theorem 2.10 in [50], the operator S which is given by

$$Sf(z) = (1 - |w|^2)^{s/2} \int_{\mathbb{B}^n} |f(z)|\, \frac{(1 - |z|^2)^{s/2-2}}{|1 - \langle w, z\rangle\, |^s}\, dz$$

is continuous on $L^p_{\alpha-sp/2}(\mathbb{D})$ whenever $-sp/2 < \alpha - sp/2 + 1 < p\,(s/2 - 1)$ which equivalent to $-1 < \alpha < p(s-1) - 1$. Since $\|f\|_{L^p_{\alpha-sp/2}(\mathbb{D})} = \|\widetilde{f}\|_{L^p_{\alpha+2-sp/2}(S)}$, the operator $F \mapsto f * |W_u(u)|$ is continuous on $L^p_{\alpha+2-sp/2}(S)$. □

Theorem 4.18. *Assume that $1 \leq p < \infty$ and $-1 < \alpha < p(s-1) - 1$. Fix $u = 1_{\mathbb{D}}$. Then we can choose ϵ small enough such that for any U_ϵ-well spread set $\{x_i\}$ there exist a family of functionals $\{\phi_i\}$ on A^p_α such that the family $\{(\phi_i, \pi_s(x_i)u)\}$ forms an atomic decomposition for A^p_α, and the family $\{\pi_s(x_i)u\}$ forms a frame as well. The reconstruction operators are given in Theorem 3.29.*

Proof. We will apply Theorem 3.29. In last section, we proved that all the assumptions of Theorem 3.29 are satisfied and the continuity of the convolution operators are done in the previous proposition. So the existence of a frame and an atomic decomposition is established by Theorem 3.29. □

Chapter 5
Projective Representation, Twisted Convolutive Coorbits, and Discretization.

5.1 Projective Representation

A generalization of group representation is a projective representation. In this section we start by introducing the definition of the continuous projective representation. Then we will construct a representation from given projective representation, and we will introduce the definition of twisted left translation and some useful results. We define $\mathbb{T}$ to be the one dimensional Torus, that is $\mathbb{T} := \{t \in \mathbb{C} \mid \ |t| = 1\}$.

Definition 5.1. *Let $\mathcal{S}$ be a locally convex topological vector space. A continuous projective representation of a Lie group G is a mapping $\rho : G \to GL(\mathcal{S})$ that satisfies the following:*

1. $\rho(1) = \mathrm{id}$.
2. *There is a smooth cocycle $\sigma : G \times G \to \mathbb{T}$, which satisfies the cocycle $\rho(ab) = \sigma(a, b)\rho(a)\rho(b)$.*
3. *For every $v \in \mathcal{S}$ the mapping $a \mapsto \rho(a)v$ is continuous.*

The following are straightforward consequences about the cocycle σ:

1. $\sigma(a, b)\sigma(ab, c) = \sigma(a, bc)\sigma(b, c)$ for all a, b, and c in G,
2. $\sigma(a, 1) = \sigma(1, a) = 1$ for all $a \in G$,
3. $\sigma(a, b)^{-1} = \overline{\sigma(a, b)}$.

We define unitary projective representation, irreducible projective representation, ρ-cyclic, square-integrable projective representation, and ρ-admissible vector in the same way as for representations.

In the following lemma we define the dual projective representation on the conjugate dual of $\mathcal{S}$.

Lemma 5.2. *Let $(\rho, \mathcal{S})$ be a continuous projective representation of G on a Fréchet space $\mathcal{S}$, and let $\mathcal{S}^*$ be the conjugate dual of $\mathcal{S}$ equipped with the weak*-topology. The mapping ρ^*, which is given by*

$$\langle \rho^*(x)\lambda, v\rangle := \langle \lambda, \rho(x)^{-1}v\rangle$$

for all $\lambda \in \mathcal{S}$ and all $v \in \mathcal{S}$, defines a continuous projective representation of G on the space $\mathcal{S}^*$ with the same cocycle as $(\rho, \mathcal{S})$.

Proof. Assume that $\sigma(x, y)$ is a cocycle of $(\rho, \mathcal{S})$. Then we have

$$\begin{aligned}
\langle \rho^*(xy)\lambda, v\rangle =& \langle \lambda, \rho(xy)^{-1}v\rangle \\
=& \langle \lambda, (\sigma(x, y)\rho(x)\rho(y))^{-1}v\rangle \\
=& \langle \lambda, \overline{\sigma(x, y)}\rho(y)^{-1}\rho(x)^{-1}v\rangle \\
=& \langle \sigma(x, y)\lambda, \rho(y)^{-1}\rho(x)^{-1}v\rangle \\
=& \langle \sigma(x, y)\rho^*(x)\rho^*(y)\lambda, v\rangle
\end{aligned}$$

Hence, $\rho^*(xy) = \sigma(x, y)\rho^*(x)\rho^*(y)$. Let us prove the continuity condition. For a net $x_\alpha \to x$ in G we have $x_\alpha^{-1} \to x^{-1}$ and $\sigma(x_\alpha, x_\alpha^{-1}) \to \sigma(x, x^{-1})$, which implies that $\sigma(x_\alpha, x_\alpha^{-1})\rho(x_\alpha^{-1})v \to \sigma(x, x^{-1})\rho(x^{-1})v$ for all $v \in \mathcal{S}$. So $\rho(x_\alpha)^{-1}v \to \rho(x)^{-1}v$. the continuity of the dual pairing implies that $\langle \lambda, \rho(x_\alpha)^{-1}v\rangle \to \langle \lambda, \rho(x)^{-1}v\rangle$ for all $\lambda \in \mathcal{S}^*$. Thus $\rho^*(x_\alpha)v \to \rho^*(x)v$ weakly, i.e., in $\mathcal{S}^*$. □

This projective representation is called the dual projective representation of $(\rho, \mathcal{S})$.

For any projective representation ρ on a given Lie group G we can construct a representation from ρ on a new group related to G which is called the Mackey group of

G (see [10]). This construction will connect the convolutive coorbits and the twisted convolutive coorbits that arise from the projective representation of G.

Definition 5.3. *Let $(\rho, \mathcal{S})$ be a projective representation of G with a cocycle σ. The Mackey group that corresponds to G is the Lie group $G_\sigma := G \times \mathbb{T}$, with multiplication given by*

$$(x,t)(y,z) = (xy, \overline{\sigma(x,y)}\; tz),$$

and equipped by the product manifold structure.

The Mackey group G_σ has left-invariant Haar measure given by $d\mu_{G_\sigma}(x,t) = dx\; dt$, where dx is the left invariant Haar measure of G and dt is the normalized Lebesgue measure on $\mathbb{T}$. If $(\rho, \mathcal{S})$ is a projective representation of G then

$$\pi_\rho(a,z) = z\rho(a)$$

defines a representation of G_σ on the space $\mathcal{S}$.

Let $u \in \mathcal{S}$ be ρ-cyclic vector. We define the projective Wavelet transform (or twisted wavelet transform) $\overset{\#}{W}_u$: $\mathcal{S}^* \to L^2(G)$ by

$$\overset{\#}{W}_u\,(\lambda)(x) := \langle \lambda, \rho(x)u \rangle.$$

Lemma 5.4. *Let $(\rho, \mathcal{S})$ be a projective representation of G and let $(\pi_\rho, \mathcal{S})$ be the corresponding representation of G_σ. Then the following are true:*

1. *The vector $u \in \mathcal{S}$ is ρ-cyclic if and only if u is π_ρ-cyclic.*
2. *The wavelet transform W_u, generated by π_ρ, and the projective wavelet transform $\overset{\#}{W}_u$ are related by*

$$W_u(\lambda)(x,t) = \bar{t}\,\overset{\#}{W}_u\,(\lambda)(x).$$

Proof. (1)The following calculations

$$\mathrm{span}\{\pi(x,z)u \mid (x,z) \in G_\sigma\} = \mathrm{span}\{z\rho(x)u \mid x \in G, z \in \mathbb{T}\}$$

$$=\mathrm{span}\{\rho(x)u \mid x \in G\}$$

show our assertion.

(2) Assume that $(x,t) \in G_\sigma$, then

$$\begin{aligned}
W_u(\lambda)(x,t) =&\langle \lambda, \pi_\rho(x,t)u\rangle \\
=&\langle \lambda, t\rho(x)u\rangle \\
=&\bar{t}\,\langle \lambda, \rho(x)u\rangle \\
=&\bar{t}\ \overset{\#}{W}_u(\lambda)(x).
\end{aligned}$$

□

Now we introduce the twisted left and right invariance operators on a BF-spaces.

Definition 5.5. *Let B be a BF-space on G, and let $(\rho, \mathcal{S})$ be a projective representation on G with a cocycle σ. For a function $f \in B$, we define*

$$L_\#(a)f(x) := \overline{\sigma(a, a^{-1}x)}L_a f(x)$$

and

$$R_\#(a)f(x) := \overline{\sigma(x,a)}R_a f(x),$$

We say that B is twisted left-invariant if $L_\#(a)f \in B$ for all $f \in B$ and $f \mapsto L_\#(a)f$ is continuous for all $a \in G$. Analogously, we define twisted right-invariant spaces.

Example 5.6. *Let $(\rho, \mathcal{S})$ be a continuous projective representation of G and let B be a solid left invariant BF-space on G. If left translation on B is continuous, then $a \mapsto L_\#(a)$ is a continuous projective representation of G on the space B. Indeed, one can use the cocycle property*

$$\sigma(ab, b^{-1}a^{-1}x)\sigma(a,b) = \sigma(a, a^{-1}x)\sigma(b, b^{-1}a^{-1}x)$$

to conclude that $L_\#(ab)f(x) = \sigma(a,b)L_\#(a)L_\#(b)f(x)$. Moreover, solidity and left invariance of B show that $\|L_\#(a)f\| = \|L_a f\|$. Finally, continuity of $a \mapsto L_\#(a)f$

follows from the fact that $a \mapsto L_a f$ and σ are uniformly continuous on compact subsets.

Remark 5.7. *If B is a solid space, then the left invariance of the space B implies twisted-left invariance of the space B.*

5.1.1 Twisted Convolution

As before, $(\rho, \mathcal{S})$ is a projective representation of G with cocycle σ. We define the twisted convolution of functions f and g on G by

$$f\#g(x) := \int_G f(y)L_{\#}(y)g(x)\,dy = \int_G f(y)g(y^{-1}x)\overline{\sigma(y, y^{-1}x)}\,dy$$

whenever the integral exists. In the following theorem we state the relation between the convolution and the twisted convolution.

Theorem 5.8. *Let f and g be measurable functions on G, then the following are equivalent:*

(i) The convolution $f * g(x)$ is defined for $x \in G$.

(ii) The convolution $f * |g|(x)$ is defined for $x \in G$.

(iii) The convolution $|f| * |g|(x)$ is defined for $x \in G$.

(iv) The twisted convolution $f\#g(x)$ is defined for $x \in G$.

(v) The twisted convolution $|f|\#|g|(x)$ is defined for $x \in G$.

Proof. The proof follows from the fact that

$$\begin{aligned}
\left|f(y)g(y^{-1}x)\right| &= \left||f(y)|\left|g(y^{-1}x)\right|\right| \\
&= \left|f(y)g(y^{-1}x)\overline{\sigma(y, y^{-1}x)}\right| \\
&= \left||f(y)|\left|g(y^{-1}x)\right|\overline{\sigma(y, y^{-1}x)}\right|.
\end{aligned}$$

□

As we have seen with the convolution operator, the following lemma gives sufficient conditions to define the twisted convolution operator with $\overset{\#}{W}_u(u)$.

Lemma 5.9. *Let B be a twisted left-invariant Banach function space on G, and fix a vector $u \in \mathcal{S}$. If the mapping $x \to F(x) \overset{\#}{W}_u(u)(x^{-1})$ is in $L^1(G)$ for all $F \in B$, then the twisted convolution*

$$F\# \overset{\#}{W}_u(u)(x) := \int_G F(y) \overset{\#}{W}_u(u)(y^{-1}x)\overline{\sigma(y, y^{-1}x)}\, dy$$

is well defined for all $x \in G$. Moreover, if the mapping $F \mapsto F\# \overset{\#}{W}_u(u)(1)$ is continuous on B, then the mapping $F \mapsto F\# \overset{\#}{W}_u(u)(x)$ is continuous for all $x \in G$.

Proof. For $F \in B$, we have

$$\begin{aligned}
\int_G \left| F(y) \overset{\#}{W}_u(u)(y^{-1}x)\overline{\sigma(y, y^{-1}x)} \right| dy &= \int_G \left| F(y) \overset{\#}{W}_u(u)\left((x^{-1}y)^{-1}\right) \right| dy \\
&= \int_G \left| F(xy) \overset{\#}{W}_u(u)(y^{-1}) \right| dy \\
&= \int_G \left| \overline{\sigma(x^{-1}, xy)} F(xy) \overset{\#}{W}_u(u)(y^{-1}) \right| dy \\
&= \int_G \left| L_\#(x^{-1}) F(y) \overset{\#}{W}_u(u)(y^{-1}) \right| dy.
\end{aligned}$$

The last integral is finite because $L_\#(x^{-1})F$ is again in B. Thus, the function $y \mapsto F(y) \overset{\#}{W}_u(u)(y^{-1}x)\overline{\sigma(y, y^{-1}x)}$ is integrable, and hence, the twisted convolution $F\# \overset{\#}{W}$ (x) is well defined for all $x \in G$.

Now, assume that the mapping $f \mapsto f\# \overset{\#}{W}_u(u)(1)$ is continuous. For any $x \in G$, one has

$$\begin{aligned}
F\# \overset{\#}{W}_u(u)(x) &= \int_G F(y) \overset{\#}{W}_u(u)(y^{-1}x)\overline{\sigma(y, y^{-1}x)}\, dy \\
&= \int_G F(xy) \overset{\#}{W}_u(u)(y^{-1})\overline{\sigma(xy, y^{-1})}\, dy.
\end{aligned}$$

Using the cocycle properties, we have

$$\sigma(x^{-1}, xy)\sigma(y, y^{-1}) = \sigma(x^{-1}, x)\sigma(xy, y^{-1}).$$

It follows that

$$\begin{aligned}
F\# \overset{\#}{W}_u (u)(x) =& \sigma(x^{-1},x)\int_G L_\#(x^{-1})F(y) \overset{\#}{W}_u (u)(y^{-1})\overline{\sigma(y,y^{-1})}\, dy \\
=& \sigma(x^{-1},x)L_\#(x^{-1})F\# \overset{\#}{W}_u (u)(1).
\end{aligned}$$

The continuity of $F \mapsto L_\#(x)F$ and the continuity of $F \mapsto F\# \overset{\#}{W}_u (u)(1)$ show that the mapping $F \mapsto F\# \overset{\#}{W}_u (u)(x)$ is continuous for all $x \in G$. Indeed,

$$\begin{aligned}
\left| F\# \overset{\#}{W}_u (u)(x) \right| =& |L_\#(x^{-1})F\# \overset{\#}{W}_u (u)(1)| \\
\leq & C' \left\| L_\#(x^{-1})F \right\|_B \\
\leq & C \left\| F \right\|_B .
\end{aligned}$$

□

5.2 Twisted Convolutive Coorbit Spaces

Let $u \in \mathcal{S}$ and define the space

$$B_u^\# := \{f \in B | f\# \overset{\#}{W}_u (u) = f\}$$

with norm inherited from B. A ρ-cyclic vector $u \in \mathcal{S}$, is called a ρ-analyzing vector for $\mathcal{S}$ if the reproducing formula

$$\overset{\#}{W}_u (\lambda)\# \overset{\#}{W}_u (u) = \overset{\#}{W}_u (\lambda)$$

holds for all $\lambda \in \mathcal{S}^*$. We state sufficient conditions that make the twisted convolutive coorbit a well defined Banach space.

Assumption 5.10. *Let B be a twisted left-invariant BF-space on G. Assume there exists a nonzero ρ-analyzing vector $u \in \mathcal{S}$ satisfying the following continuity condition: The mapping*

$$(R'1/2) \quad B \times \mathcal{S} \ni (f,v) \mapsto f\# \overset{\#}{W}_v (u)(1) = \int_G f(y) \overset{\#}{W}_v (u)^\vee(y)\overline{\sigma(y,y^{-1})}\, dy \in \mathbb{C}$$

is continuous.

Remark 5.11. 1. *As in the convolutive coorbits, we have weaker assumptions on the twisted convolutive coorbits. However, we will consider the above assumptions, the reason behind that is we only need these assumptions in practice.*

2. *As before, if $B = L^p_w(G)$ then the continuity condition will be a duality requirement, i.e.*
$$\mathcal{S} \ni v \mapsto \overset{\#}{W}_v\,(u)^\vee \in L^q_{w^{-q/p}}(G)$$
is continuous, where $\frac{1}{p} + \frac{1}{q} = 1$.

3. *The above continuity condition ensures that $\overset{\#}{W}_v\,(u)^\vee$ is a continuous linear functional on B, with the paring*
$$\langle \overset{\#}{W}_v\,(u)^\vee, f\rangle = \int_G f(y)\,\overset{\#}{W}_v\,(u)^\vee(y)\overline{\sigma(y,y^{-1})}\,dy.$$

We are now ready to define the twisted convolutive coorbit space.

Definition 5.12. *Let $(\rho, \mathcal{S})$ be a projective representation of G, and let B be a twisted left-invariant BF-space on G. Assume that $u \in \mathcal{S}$ is a ρ-analyzing vector satisfying Assumption 5.10. A twisted convolutive coorbit space of B related to the projective representation ρ is the space*
$$\mathrm{Co}^u_\rho B := \{\lambda \in \mathcal{S}^* |\, \overset{\#}{W}_u\,(\lambda) \in B\}$$
with the norm
$$\|\lambda\|_{\mathrm{Co}^u_\rho B} := \|\,\overset{\#}{W}_u\,(\lambda)\|_B.$$

To connect the theory of twisted convolutive coorbits to the ordinary coorbit theory, we introduce the following function spaces on $G_\sigma = G \times \mathbb{T}$ that corresponds to a function space B on G:
$$\widetilde{B} = \{\widetilde{f} : G \times \mathbb{T} \to \mathbb{C}\ \mid \widetilde{f} \text{ is measuralbe, } f := \int_{\mathbb{T}} |\widetilde{f}(.,t)|\,dt \in B\}$$

with norm $\|\widetilde{f}\|_{\widetilde{B}} := \|f\|_B$, and the space

$$\widehat{B} := \{\widetilde{f} : G \times \mathbb{T} \to \mathbb{C} \ \mid \widetilde{f}(a,t) = \bar{t}f(a), f \in B\}$$

with norm $\|\widetilde{f}\|_{\widehat{B}} := \|f\|_B$. These spaces are Banach function spaces, and the space $\widetilde{B}$ is studied in [26]. It is easy to see that B is isometrically isomorphic to $\widehat{B}$ and the latter is continuously embedded in $\widetilde{B}$. However, if B is a solid Banach space, then $\widetilde{B}$ is solid, but $\widehat{B}$ is not solid in general. For the coorbit theory we use the space $\widehat{B}$ to connect the convolutive coorbits by the twisted convolutive coorbits when the solidity is not needed. However, we will use the space $\widetilde{B}$ when the solidity is needed as in the discretization of the coorbit spaces.

Lemma 5.13. *If $G \times \mathbb{T}$ and $\widehat{B}$ are defined as before, then the following relations hold.*

1. *If the space B is twisted left-invariant, then $\widehat{B}$ is left-invariant.*

2. *For $\widetilde{f} \in \widehat{B}$, we have $\widetilde{f} * W_u(u)(x,z) = \bar{z}\, f \,\#\, \overset{\#}{W}_u\,(u)(x)$.*

3. $\mathrm{Co}^u_\rho B = \mathrm{Co}^u_{\pi_\rho} \widehat{B}$.

4. *The spaces $B^{\#}_u$, $\widehat{B}_u$ are isometrically isomorphic via $\Lambda f(x,t) := \bar{t}\ f(x)$.*

Proof. The first part is done by the following calculations:

$$\begin{aligned} L_{(a,w)}\widetilde{f}(x,z) =& \widetilde{f}(a^{-1}x, \bar{w}z\sigma(a,a^{-1})\overline{\sigma(a^{-1},x)}) \\ =& \widetilde{f}(a^{-1}x, \bar{w}z\sigma(a,a^{-1}x)) \\ =& w\bar{z}\,\overline{\sigma(a,a^{-1}x)}f(a^{-1}x) \\ =& \bar{z}w\, L_a f(x). \end{aligned}$$

Therefore, B is twisted left-invariant if and only if $\widehat{B}$ is left invariant. For the second part, we have

$$\widetilde{f} * W_u(u)(x,z) = \iint \widetilde{f}(y,w)W_u(u)((y,w)^{-1}(x,z))dwdy$$

$$
\begin{aligned}
&= \iint \widetilde{f}(y,w) W_u(u)(y^{-1}x, \bar{w} z \sigma(y,y^{-1})\overline{\sigma(y^{-1},x)}) dw dy \\
&= \bar{z} \int f(y) \overset{\#}{W}_u (u)(y^{-1}x)\overline{\sigma(y,y^{-1})}\sigma(y^{-1},x)\, dy \\
&= \bar{z}\, f \# \overset{\#}{W}_u (u)(x).
\end{aligned}
$$

Next, assume that $\lambda \in \mathcal{S}$. Then, $\lambda \in \mathrm{Co}_\rho^u B$ if and only if $\overset{\#}{W}_u (\lambda) \in B$ if and only if $W_u(\lambda) \in \widehat{B}$ if and only if $\lambda \in \mathrm{Co}_{\pi_\rho}^u \widehat{B}$.

Finally, it is clear that Λ is surjective, and $\|\Lambda f\|_{\widehat{B}} = \|f\|_B$. □

The following theorem is the connection between the coorbit theory that arises from representation theory and the one that arise from projective representation theory.

Theorem 5.14. *If B and u satisfy Assumption 5.10, then $\widehat{B}$ and u satisfy Assumption 3.12.*

Proof. First, by Lemma 5.4 we know that u is π-cyclic, and by Lemma 5.13 the space $\widehat{B}$ is left invariant. Next, denote the wavelet transform related to the representation π_ρ by W_u. Assume that u is a ρ-analyzing vector. We show that u satisfying the reproducing formula $W_u(\lambda) * W_u(u) = W_u(\lambda)$ for all $\lambda \in \mathcal{S}^*$. The same calculations, as in Lemma 5.13 (2), show that

$$
\begin{aligned}
W_u(\lambda) * W_u(u)(x,z) &= \bar{z}\, \overset{\#}{W}_u (\lambda) \# \overset{\#}{W}_u (u)(x) \\
&= \bar{z}\, \overset{\#}{W}_u (\lambda) \\
&= W_u(\lambda)(x,z).
\end{aligned}
$$

Hence $W_u(\lambda) * W_u(u) = W_u(\lambda)$ for all $\lambda \in \mathcal{S}^*$.

Now, let B and u satisfy $(R'1/2)$, and note that

$$\left|\iint \widetilde{f}(x,z) \overset{\#}{W}_v (u)((x,z)^{-1})\, dzdx\right| \leq \left|\iint \bar{z} f(x) \overset{\#}{W}_v (u)(x^{-1}, \bar{z}\sigma(x,x^{-1}))\, dzdx\right|$$
$$\leq \left|\int f(x) W_v(u)((x^{-1})\overline{\sigma(x,x^{-1})}\, dx\right|.$$

It follows that the continuity of $(f,v) \mapsto \int f(x)W_v(u)((x^{-1})\overline{\sigma(x,x^{-1})}\, dx$ on $B \times \mathcal{S}$ implies the continuity of $(\widetilde{f}, v) \mapsto \iint \widetilde{f}(x,z) \overset{\#}{W}_v (u)((x,z)^{-1})\, dzdx$. Therefore $\widehat{B}$ and u satisfy $(R1/2)$. $\square$

The following theorem states that the space $B_u^{\#}$ is a reproducing kernel Banach space.

Theorem 5.15. *Let B be a twisted left-invariant BF-space on G and let $u \in \mathcal{S}$ be a ρ-cyclic vector. If B and u satisfy:*

$$(R'1) \quad \textit{The mapping } \; B \ni f \mapsto f \# \overset{\#}{W}_u (u)(1) \in \mathbb{C} \; \textit{ is continuous,}$$

then $B_u^{\#}$ is closed in B, and hence is a reproducing kernel Banach space with $k(x,y) = L_{\#}(y) \overset{\#}{W}_u (u)(x)$.

Proof. We know that B is isometrically isomorphic to $\widehat{B}$ by the isomorphism $(\Lambda f)(x,t) := \bar{t} f(x)$ and $\Lambda(B_u^{\#}) = \widehat{B}_u$. By Theorem 5.14 (we consider the continuity in the first argument), the space space $\widehat{B}$ and u satisfy $(R1)$. Hence by Lemma 3.6, the space $\widehat{B}_u$ is a closed subspace of $\widehat{B}$. This proves that $B_u^{\#}$ is closed. Also, k is the reproducing kernel, because for $f \in B_u^{\#}$ we have

$$\int_G f(y)k(x,y)dy = \int_G f(y)\overline{\sigma(y,y^{-1}x)} \overset{\#}{W}_u (u)(y^{-1}x)dy$$
$$= f \# \overset{\#}{W}_u (u)(x) = f(x).$$

$\square$

Now we demonstrate our main result about the coorbit space constructed by the twisted convolution.

Theorem 5.16. *Assume that B and u satisfy Assumption 5.10. Then*

1. $\overset{\#}{W}_u(v) \# \overset{\#}{W}_u(u) = \overset{\#}{W}_u(v)$ *for* $v \in \mathrm{Co}^u_\rho B$.

2. *The space* $\mathrm{Co}^u_\rho B$ *is a* ρ^**-invariant Banach space.*

3. $\overset{\#}{W}_u \colon \mathrm{Co}^u_\rho B \to B$ *intertwines* ρ^* *and* $a \mapsto L_\#(a)$.

4. $\overset{\#}{W}_u \colon \mathrm{Co}^u_\rho B \to B^\#_u$ *is an isometric isomorphism.*

Proof. By Theorem 5.14, the space $\widehat{B}$ and u satisfy Assumption 3.12. So we can apply Theorem 3.11 to the space $\widehat{B}$.

(1) For $v \in \mathrm{Co}^u_{\pi_\rho}\widehat{B}$ we have $W_u(\lambda) * W_u(u) = W_u(\lambda)$ for all $\lambda \in \mathcal{S}^*$. Moreover, as we noted in Lemma 5.13 and Lemma 5.4 we have $\mathrm{Co}^u_\rho B = \mathrm{Co}^u_{\pi_\rho}\widehat{B}$ and

$$W_u(v) * W_u(u)(x,z) = \bar{z}\,\overset{\#}{W}_u(v) \# \overset{\#}{W}_u(u)(x)$$

as well as $W_u(v)(x,z) = \bar{z}\,\overset{\#}{W}_u(v)(x)$. Putting all the pieces together, we have

$$\overset{\#}{W}_u(v) \# \overset{\#}{W}_u(u) = \overset{\#}{W}_u(v)$$

for $v \in \mathrm{Co}^u_\rho B$.

(2) We know that the space $\mathrm{Co}^u_{\pi_\rho}\widehat{B} = \mathrm{Co}^u_\rho B$ is π^*_ρ-invariant Banach space. So $W_u(\pi^*_\rho(y,w)\phi) \in \mathrm{Co}^u_{\pi_\rho}\widehat{B}$. On the other hand

$$W_u(\pi^*_\rho(y,w)\phi)(x,z) = \bar{z}w\,\overset{\#}{W}_u(\rho^*(y)\phi)(x),$$

which implies that $\overset{\#}{W}_u(\rho^*(y)\phi) \in B$.

(3) Using the fact that W_u intertwines π^*_ρ with left translation, and $\pi^*_\rho(x,z) = z\rho^*(x)$. We have

$$\overset{\#}{W}_u(\rho^*(y)\phi)(x) = \bar{w}zW_u(\pi^*_\rho(y,w)\phi)(x,z) = \bar{w}zL_{(y,w)}W_u(\phi)(x,z)$$

$$=\overline{\sigma(y,y^{-1})}\sigma(y^{-1},x)L_y \overset{\#}{W}_u (\phi)(x)$$
$$=L_{\#}(y) \overset{\#}{W}_u (\phi)(x).$$

(4) If we denote the isometry isomorphism between B_u and $\hat{B}_u$ by Λ, then $\overset{\#}{W}_u = \Lambda^{-1}W_u : \mathrm{Co}_\rho^u B \to B_u^{\#}$ and the result is obtained. □

In the following theorem, we prove that the twisted coorbit space is independent of the choice of the ρ-analyzing vector under some assumptions.

Theorem 5.17. *Assume that u_1 and u_2 are ρ-analyzing vectors for $\mathcal{S}$ which satisfy Assumption 5.10, and the following properties are hold for $i,j \in \{1,2\}$*

1. *there are nonzero constants $C_{i,j}$ such that $\overset{\#}{W}_{u_i} (\lambda) \# \overset{\#}{W}_{u_j} (u_i) = C_{i,j} \overset{\#}{W}_{u_j} (\lambda)$ for all $\lambda \in \mathcal{S}^*$*
2. *the mapping $B_{u_i} \ni f \mapsto f \# \overset{\#}{W}_{u_j} (u_i) \in B$ is continuous.*

Then $Co_\rho^{u_1} B = Co_\rho^{u_2} B$ with equivalent norms.

Proof. Consider the space $\widehat{B}$ and the Mackey group $G \times \mathbb{T}$. Since u_1 and u_2 are ρ-analyzing vectors for $\mathcal{S}$ that satisfying Assumption 5.10, they are also π_ρ-analyzing vectors for $\mathcal{S}$ that satisfying Assumption 3.12 (see Theorem 5.14). Also for $i,j \in \{1,2\}$ and $\lambda \in \mathcal{S}^*$, we have

$$\begin{aligned} W_{u_i}(\lambda) * W_{u_j}(u_i)(x,t) =& \bar{t} \overset{\#}{W}_{u_i} (\lambda) \# \overset{\#}{W}_{u_j} (u_i)(x) \\ =& \bar{t} C_{i,j} \overset{\#}{W}_{u_j} (\lambda)(x) \\ =& C_{i,j} W_{u_j}(\lambda)(x,t) \end{aligned}$$

Moreover, the mapping $\widehat{B}_{u_i} \ni \widetilde{f} \mapsto \widetilde{f} * W_{uj}(u_i) \in \widehat{B}$ is continuous, indeed, $|\widetilde{f} * W_{uj}(u_i)| = |f \# \overset{\#}{W}_{u_j} (u_i)| \leq C\|f\|_B = C\|\widetilde{f}\|_{\widehat{B}}$. Therefore, by Theorem 3.15, $Co_{\pi_\rho}^{u_1} \widehat{B} = Co_{\pi_\rho}^{u_2} \widehat{B}$. Since $Co_{\pi_\rho}^{u_i} \widehat{B} = Co_\rho^{u_i} B$, the result is obtained. □

We finish this section by the following version of Duflo-Moore theorem for square integrable projective representation.

Theorem 5.18. *Let $(\rho, \mathcal{H})$ be a square-integrable projective representation of G.*

1. *There exists a positive self adjoint operator A_ρ which is defined on a dense subset D of $\mathcal{H}$, such that $u \in \mathcal{H}$ is ρ-admissible if and only if $u \in D$. Moreover, the orthogonality relation*

$$\int_G (v_1, \rho(x)u_1)\,(\rho(x)u_2, v_2)\;dx = (A_\rho u_2, A_\rho u_1)\,(v_1, v_2)$$

holds for all $u_1, u_2 \in D$ and $v_1, v_2 \in \mathcal{H}$.

2. *In addition, if G is a unimodular, then $D = \mathcal{H}$ and $A_\rho = c_\rho Id_{\mathcal{H}}$. Thus, all vectors of $\mathcal{H}$ are ρ-admissible and*

$$\int_G (v_1, \rho(x)u_1)\,(\rho(x)u_2, v_2)\;dx = c_\rho^2 (u_2, u_1)\,(v_1, v_2)$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$. The constant c_ρ is called the formal dimension of ρ.

Proof. Consider the Mackey group $G \times \mathbb{T}$ with the corresponding representation $\pi_\rho(x,t) = t\rho(x)$. The representation π_ρ is square integrable. Indeed, if W is a π_ρ invariant subspace of $\mathcal{H}$, then $\rho(x)W \subset W$ for all $x \in G$, so $W = 0$ or $W = \mathcal{H}$. Also, $\langle \pi_\rho(x,t)v, \pi_\rho(x,t)u\rangle = \langle t\rho(x)v, t\rho(x)u\rangle = \langle v, u\rangle$, thus π_ρ is an irreducible unitary representation of $G \times \mathbb{T}$. Let u be a ρ-admissible vector, then

$$\begin{aligned}\iint |\langle u, \pi_\rho(x,t)u\rangle|^2\;dt\;dx &= \iint |\langle u, t\rho(x)u\rangle|^2\;dt\;dx \\ &= \int_G |\langle u, \rho(x)u\rangle|^2\;dx < \infty.\end{aligned}$$

By Theorem 2.16, there is a positive self adjoint operator A_{π_ρ} with domain $D \subset \mathcal{H}$ such that the orthogonality relation

$$\iint (v_1, \pi_\rho(x,t)u_1)\,(\pi_\rho(x,t)u_2, v_2)\;dt\;dx = (A_{\pi_\rho}u_2, A_{\pi_\rho}u_1)\,(v_1, v_2)$$

holds for all $u_1, u_2 \in D$ and $v_1, v_2 \in \mathcal{H}$. Now,

$$\iint (v_1, \pi_\rho(x,t)u_1)\,(\pi_\rho(x,t)u_2, v_2)\; dt\; dx = \iint (v_1, t\rho(x)u_1)\,(t\rho(x)u_2, v_2)\; dt\; dx$$
$$= \int_G (v_1, \rho(x)u_1)\,(\rho(x)u_2, v_2)\; dx.$$

If we define $A_\rho := A_{\pi_\rho}$, then the orthogonality relation holds for ρ. For the second part, note that if G is unimodular, then $G \times \mathbb{T}$ is also a unimodular, therefore the formal dimension c_ρ is the same as c_{π_ρ} and the relation holds for ρ. □

5.3 Discretization on a Twisted Convolutive Coorbit

After constructing the twisted convolutive coorbit spaces, we are ready to state the theory of discretizing such spaces. With some modification, we will see that the theory of convolutive coorbits cam be transformed to projective representations.

From now on, we assume B satisfying Assumption 3.26. We define ρ-weakly differentiable vectors for a given projective representation $(\rho, \mathcal{S})$ on G in the same way that we defined the π-weakly differentiable vectors for a given representation $(\pi, \mathcal{S})$. We start by obtaining the relation between the ρ-weak differentiability and the corresponding π_ρ-weak differentiability. Let $\{E_1, E_2, ..., E_{\dim G}\}$ be a basis for $\mathfrak{g}$. Then $\{\widetilde{E}_1, \widetilde{E}_2, ..., \widetilde{E}_{\dim G+1}\}$ forms a basis of $\mathfrak{g} \times i\mathbb{R}$ where $\widetilde{E}_j = (E_j, 0)$ for $j = 1, ..., \dim G$ and $\widetilde{E}_{\dim G+1} = (0, i)$. Note that $\widetilde{E}_\alpha := \widetilde{E}_{\alpha(N)}\widetilde{E}_{\alpha(N-1)}...\widetilde{E}_{\alpha(1)} = (E_{\beta(N-s)}E_{\beta(N-s-1)}...E_{\beta(1)}, i^s)$ with $0 \le \beta(j) \le \dim G$ and $|\beta| = |\alpha| - s$.

Proposition 5.19. *(i) If a vector $u \in \mathcal{S}$ is ρ-weakly differentiable up to order $\dim G + 1$, then it is π_ρ-weakly differentiable up to order $\dim G \times \mathbb{T}$. Moreover, $\pi_\rho(\widetilde{E}_\alpha)u = i^s \rho(E_{\alpha'})u$ for some α' with $|\alpha'| = |\alpha| - s$ and $0 \le \alpha'(j) \le \dim G$.*

(ii) If a distribution $\lambda \in \mathcal{S}^*$ is ρ^*-weakly differentiable up to order $\dim G + 1$, then it is π_ρ^*-weakly differentiable up to order $\dim G \times \mathbb{T}$. Moreover, $\pi_\rho^*(\widetilde{E}_\alpha)u = i^s \rho^*(E_{\alpha'})u$ for some α' with $|\alpha'| = |\alpha| - s$ and $0 \le \alpha'(j) \le \dim G$.

Proof. We will prove the second part, and the first part is proved in the same way. Assume that $\lambda \in \mathcal{S}^*$ is a ρ^*-weakly differentiable up to order $\dim G + 1$. We can easily see that for $X \in \mathfrak{g}$, and for all $v \in \mathcal{S}$

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} \langle \pi_\rho^*(\exp(s(X,0))\lambda, v\rangle &= \frac{d}{ds}\bigg|_{s=0} \langle \lambda, \pi_\rho(\exp(s(X,0))^{-1}v\rangle \\
&= \frac{d}{ds}\bigg|_{s=0} \langle \lambda, \pi_\rho(\exp(sX), a(s))^{-1}v\rangle \\
&= \frac{d}{ds}\bigg|_{s=0} \langle \lambda, \overline{a(s)}\rho\left(\exp(sX)\right)^{-1} v\rangle \\
&= \frac{d}{ds}\bigg|_{s=0} a(s)\langle \lambda, \rho\left(\exp(sX)\right)^{-1} v\rangle \\
&= \langle \rho^*(X)\lambda, v\rangle,
\end{aligned}$$

where $a(s)$ is a curve on $\mathbb{T}$ such that $a(0) = 1$ and $a'(0) = 0$. Hence $\pi_\rho^*(X,0)\lambda = \rho^*(X)\lambda$. Similarly we have $\pi_\rho^*(0,i)\lambda = i\lambda$, and inductively we conclude that

$$\pi_\rho^*(\widetilde{E}_\alpha)\lambda = i^s \rho^*(E_{\alpha'})\lambda$$

with $|\alpha| = |\alpha'| + s$. Therefore, u is π^*-weakly differentiable up to order $\dim(G\times\mathbb{T})$. $\square$

As we mentioned before, we used the space $\widehat{B}$ to connect the coorbits with the twisted coorbits. Nevertheless, the space $\widehat{B}$ is no longer solid, and therefore we need to study the space $\widetilde{B}$ to see the relation between the frames and the atomic decompositions of the coorbits and the ones of the twisted coorbits. We summarize some of the properties of the space $\widetilde{B}$ in the following theorem.

Theorem 5.20. *Let B be a BF-space on G, and let $\widetilde{B}$ defined as before, then the following are true:*

1. *If the space B satisfies Assumption 3.26, then the space $\widetilde{B}$ satisfies the same assumption.*

2. *The spaces $\widetilde{B}_u$ and $\widehat{B}_u$ are equal, and hence the spaces $\widetilde{B}_u$ and $B_u^{\#}$ are isometrically isomorphic via $\Lambda : B_u^{\#} \to \widetilde{B}_u$, which is given by $\Lambda(f)(x,t) = \bar{t}f(x)$. In particular, $\widetilde{f} \in \widetilde{B}_u$ if and only if $\widetilde{f}(x,t) = \bar{t}f(x)$ for some $f \in B_u^{\#}$.*

3. *Assume that B^d and $\widetilde{B}^d$ are the corresponding sequence spaces of B and $\widetilde{B}$ respectively. Assume that $\{x_i\}_{i\in I}$ is a U-well spread set in G, choose V and $\{t_j\}_{j=1}^N$ as in Example 3.21. Then, for $N \in \mathbb{N}$, the set $\{(x_i,t_j)\}_{i\in I, j=1,2,\ldots,N}$ is a $U\times V$-well spread set for $G\times\mathbb{T}$, and $\{\lambda_{i,j}\} \in \widetilde{B}^d$ if and only if $\{\sum_{j=1}^N |\lambda_{i,j}|\} \in B^d$, in this case $\|\{\lambda_{i,j}\}\|_{\widetilde{B}^d} = \|\{\sum_{j=1}^N |\lambda_{i,j}|\}\|_{B^d}$.*

4. *Let $u \in \mathcal{S}$ be a ρ-analyzing vector for $\mathcal{S}$. If B and u satisfy Assumption 3.12, then u is a π_ρ-analyzing vector and the space $\widetilde{B}$ and u satisfy Assumption 5.10.*

It is worth now to remind the reader that the assumption $(R'1/2)$ implies that the mapping

$$f \mapsto \int_G f(y) \overset{\#}{W}_u (u)^\vee(y)\overline{\sigma(y,y^{-1})}\, dy$$

is continuous on B. As a result, the mapping

$$f \mapsto \int_G f(y) L_{\#}(x) \overset{\#}{W}_u (u)^\vee(y)\sigma(x,x^{-1}y)\overline{\sigma(y,y^{-1}x)}\, dy$$

is continuous for every $x \in G$, it follows that for every $x \in G$ this map defines a functional on B and we will keep denoting it by $L_{\#}(x) \overset{\#}{W}_u (u)^\vee$ when there is no confusion. Note that

$$\langle L_{\#}(x) \overset{\#}{W}_u (u)^\vee, f\rangle = \int_G f(y) L_{\#}(x) \overset{\#}{W}_u (u)^\vee(y)\sigma(x,x^{-1}y)\overline{\sigma(y,y^{-1}x)}\, dy.$$

In particular, for any $f \in B_u^{\#}$, we have $\langle L_{\#}(x) \overset{\#}{W}_u (u)^\vee, f\rangle = f(x)$. Before we construct a Banach frame for the twisted coorbit space $\mathrm{Co}_\rho^u B$, we will construct a Banach frame for $B_u^{\#}$.

Theorem 5.21. *Let $(\rho,\mathcal{S})$ be a projective representation of G, and let B be a BF-space on G satisfies Assumption 3.26. Assume that $u \in \mathcal{S}$ is ρ-weakly and ρ^*-weakly* *differentiable up to order of* $\dim G + 1$*, and satisfies Assumption 5.10 as well. Assume that* $\overset{\#}{W}_u(u) \in B$*, and assume that the mappings*

$$f \mapsto f * |\overset{\#}{W}_{\rho(E_\alpha)u}(u)| \quad \text{ and } \quad f \mapsto f * |\overset{\#}{W}_u(\rho^*(E_\alpha)u)|$$

are continuous on B *for all* α *with* $|\alpha| \le \dim G + 1$*. Then*

1. *We can choose* ϵ *small enough such that for any* U_ϵ*- well spread set* $\{x_i\}$ *and any* U_ϵ*-BUPU* $\{\psi_i\}$ *with* $\mathrm{supp}\psi_i \subset x_i U_\epsilon$ *the following three operators*

$$S_1 f := \sum_i f(x_i)(\psi_i \# \overset{\#}{W}_u(u))$$

$$S_2 f := \sum_i \lambda_i(f) L_\#(x_i) \overset{\#}{W}_u(u) , \qquad \left(\lambda_i(f) = \int f(x)\psi_i(x)\, dx\right)$$

$$S_3 f := \sum_i c_i f(x_i) L_\#(x_i) \overset{\#}{W}_u(u) , \qquad \left(c_i = \int \psi_i(x)\, dx\right)$$

   are all invertible on $B_u^\#$*. The convergence of the sums above is pointwise and, if the continuous compactly supported functions are dense in* B*, then the convergence is also in norm.*

2. *(Frame) The family* $\{L_\#(x_i) \overset{\#}{W}_u(u)^\vee\}$ *is a Banach frame for the space* $B_u^\#$ *with respect to the sequence space* B^d*, with reconstruction operator* $R = S_1^{-1}A$*, where* $A : B^d \to B_u^\#$ *given by*

$$A(\{\eta_i\}) = \sum_i \eta_i \psi_i \# \overset{\#}{W}_u(u).$$

3. *(Atomic decomposition) The families* $\{\lambda_i \circ S_2^{-1}, L_\#(x_i) \overset{\#}{W}_u(u)\}$ *and* $\{c_i L_\# \overset{\#}{W}_u(u)^\vee \circ S_3^{-1}, L_\#(x_i) \overset{\#}{W}_u(u)\}$ *are atomic decompositions of* $B_u^\#$ *with respect to the sequence space* B^d*, with reconstruction representations given by* $f = \sum_i \lambda_i(S_2^{-1}(f)) L_\#(x_i) \overset{\#}{W}_u(u)$ *and* $f = \sum_i c_i\, S_3^{-1} f(x_i) L_\#(x_i) \overset{\#}{W}_u(u)$ *respectively.*

Proof. (1) Consider the group $G \times \mathbb{T}$ and the space $\widetilde{B}$. By Theorem 5.20 the space $\widetilde{B}$ satisfies Assumption 3.26 and the vector u satisfies Assumption 3.12. By Theorem 5.19 the vector u is π_ρ and π_ρ^*-weakly differentiable up to order $\dim G \times \mathbb{T}$, also $W_u(u)$ is an element in $\widetilde{B}$ because $W_u(u)(x,t) = \bar{t} \overset{\#}{W}_u (u)(x)$ with $\overset{\#}{W}_u (u) \in B$. The continuity of $f \mapsto f * | \overset{\#}{W}_{\rho(E_\alpha)u} (u)|$ and $f \mapsto f * | \overset{\#}{W}_u (\rho^*(E_\alpha)u)|$ on B implies the continuity of $\widetilde{f} \mapsto \widetilde{f} * |W_{\pi_\rho(\widetilde{E}_\beta)u}(u)|$ and $\widetilde{f} \mapsto \widetilde{f} * |W_u(\pi_\rho^*(\widetilde{E}_\beta)u)|$ on $\widetilde{B}$, for $|\beta| \le \dim G \times \mathbb{T}$. Indeed for $|\beta| \le \dim G \times \mathbb{T} = \dim G + 1$ with $0 \le \beta(j) \le \dim G \times \mathbb{T}$ we have

$$
\begin{aligned}
|W_{\pi_\rho(\widetilde{E}_\beta)u}(u)(x,t)| &= \left| \langle u, \pi_\rho(x,t)(\pi_\rho(\widetilde{E}_\beta)u) \rangle \right| \\
&= |\langle u, \rho(x)(\rho(E_\alpha)u) \rangle| \\
&= | \overset{\#}{W}_{\rho(E_\alpha)u} (u)(x)|
\end{aligned}
$$

for some α with $|\alpha| \le |\beta|$ and $0 \le \alpha(j) \le \dim G$. Now, for $\widetilde{f} \in \widetilde{B}$ we have

$$
\begin{aligned}
\left| \widetilde{f} * |W_{\pi_\rho(\widetilde{E}_\beta)u}(u)| \right| &= \left| \iint \widetilde{f}(y,w) |W_{\pi_\rho(\widetilde{E}_\beta)u}(u) \left((y,w)^{-1}(x,t) \right) | \, dw \, dy \right| \\
&= \left| \iint \widetilde{f}(y,w) |W_{\pi_\rho(\widetilde{E}_\beta)u}(u) \left(y^{-1}x, \bar{w} t \sigma(y, y^{-1}x) \right) | \, dw \, dy \right| \\
&= \left| \iint \widetilde{f}(y,w) | \overset{\#}{W}_{\rho(E_\alpha)u} (u)(y^{-1}x)| \, dw \, dy \right| \\
&= \left| \int_G \left(\int_{\mathbb{T}} \widetilde{f}(y,w) \, dw \right) | \overset{\#}{W}_{\rho(E_\alpha)u} (u)(y^{-1}x)| \, dy \right| \\
&\le C \left\| \int_{\mathbb{T}} |\widetilde{f}(\cdot, w)| \, dw \right\|_B \\
&= C \|\widetilde{f}\|_{\widetilde{B}}.
\end{aligned}
$$

The same calculations show the other continuity condition on $\widetilde{B}$. It follows that the vector u and the space $\widetilde{B}$ satisfy all the assumptions of Theorem 3.28. For a fixed N, let $\{t_j\}_{j=1}^N$ and V be defined as in Example 3.21, then $\{(x_i, t_j)\}_{i,j}$ is a $U_\epsilon \times V$-well spread set(see [10]). If we define $\widetilde{\psi}_{i,j}(x,t) := \psi_i(x) 1_{t_j V}(t)$, then $\widetilde{\psi}_{i,j}$ is a $U_\epsilon \times V$-BUPU. There exists an ϵ such that for large enough N, the operators T_1, T_2, and T_3 (which defined in Theorem 3.28) are invertible. Assume $\widetilde{f}(x,t) = \bar{t} f(x)$ with $f \in B_u^\#$, then simple calculations show that

$$\sum_{i,j} \widetilde{f}(x_i, t_j)\tilde{\psi}_{i,j} * W_u(u)(x,t) = \bar{t}\ \frac{\sin(\pi/N)}{\pi/N} \sum_i f(x_i)\psi_i \# \overset{\#}{W}(x).$$

If we denote the isomorphism between $B_u^{\#}$ and $\widetilde{B}_u$ by Λ, then the operator

$$S_1 := \frac{\pi/N}{\sin(\pi/N)} \Lambda^{-1} T_1 \Lambda$$

is invertible on $B_u^{\#}$ and

$$S_1 f = \sum_i f(x_i)\psi_i \# \overset{\#}{W}(x).$$

If $C_c(G)$ is dense in B, the sum converges in norm as proved in [38, 5]. Indeed, if we define $\Theta f = (\sum_i f(x_i)\psi_i) \# \overset{\#}{W}_u (u)$, then Θ is bounded operator. Moreover, $S_1 f = \Theta f$, for $f \in \mathcal{C}_c(G)$, because $\sum_i f(x_i)\psi_i$ is finite. Now, assume that $f \in B_u$. For $\epsilon > 0$, choose $g \in \mathcal{C}_c(G)$ such that $\|f - g\|_B$. Then

$$\begin{aligned} \|S_1 f - \Theta f\|_B \leq & \|S_1 f - S_1 g\|_B + \|S_1 g - \Theta f\|_B \\ = & \|S_1 f - S_1 g\|_B + \|\Theta g - \Theta f\|_B \leq C\|f - g\|_B. \end{aligned}$$

The last term can be made as small as we please, so the convergence is in norm. For the operator S_2, the operator

$$T_2 \widetilde{f} = \sum_{i,j} \tilde{\lambda}_{i,j}(\widetilde{f}) L_{(x_i,t_j)} W_u(u)$$

is invertible on $\widetilde{B}$, where $\tilde{\lambda}_{i,j}(\widetilde{f}) = \iint \widetilde{f}(x,t)\tilde{\psi}_{i,j}(x,t)\ dt\, dx$. Note that

$$\begin{aligned} \tilde{\lambda}_{i,j}(\widetilde{f}) = & \iint \widetilde{f}(x,t)\tilde{\psi}_{i,j}(x,t)\ dt\, dx \\ = & \iint \bar{t}\ \ f(x)\psi_i(x)\, 1_{t_j V}(t)\ dt\, dx \\ = & \lambda_i(f) \int_{\mathbb{T}} \bar{t}\ \ 1_{t_j V}(t)\ dt \\ = & \frac{-\sin(\pi/N)}{\pi/N}\ \ \bar{t}_j\ \ \ \lambda_i(f). \end{aligned}$$

If we define

$$S_2 := \frac{-\pi/N}{\sin(\pi/N)}\Lambda^{-1}T_2\Lambda,$$

then S_2 is invertible on $B_u^{\#}$, with

$$S_2 f = \sum_i \lambda_i(f) L_{\#}(x_i) \overset{\#}{W}_u (u) \qquad \Big(\lambda_i(f) := \int_G f(x)\psi_i(x)\,dx\Big).$$

The convergence statement follows as above. The invertibility of S_3 is done by the same calculations above.

(2) To show that the family $\{L_{\#}(x_i) \overset{\#}{W}_u (u)^{\vee}\}$ forms a Banach frame for $B_u^{\#}$ we apply again Theorem 3.28 on the space $\widetilde{B}$. It follows that the family $\{L_{(x_i,t_j)}W_u(u)^{\vee}\}$ is a frame for $\widetilde{B}_u$ with reconstruction operator $\widetilde{R} = T_1^{-1}\widetilde{A}$, where

$$\widetilde{A}(\{\widetilde{\eta_i}\}) = \sum_{i,j} \widetilde{\eta}_{i,j}\, \widetilde{\psi}_{i,j} * W_u(u)$$

for any $\{\widetilde{\eta}_{i,j}\} \in \widetilde{B}$. For any $\{\eta_i\} \in B^d$ define $\widetilde{\eta}_{i,j} = \bar{t}_j\eta_i$. By Theorem 5.20, we have $\{\widetilde{\eta}_{i,j}\} \in \widetilde{B}$ and $\|\{\widetilde{\eta}_{i,j}\}\|_{\widetilde{B}^d} = N\|\{\eta_i\}\|_{B^d}$. Also

$$\begin{aligned}
\widetilde{A}(\{\widetilde{\eta_i}\})(x,t) =& \sum_{i,j} \widetilde{\eta}_{i,j}\, \widetilde{\psi}_{i,j} * W_u(u)(x,t) \\
=& \sum_{i,j} \bar{t}_j\eta_i \iint 1_{t_jV}(w)\psi_i(y)W_u(u)(y^{-1}x, \bar{w}t\sigma(y,y^{-1}x))\ dw\,dy \\
=& \sum_{i,j} \bar{t}_j\eta_i \int_{\mathbb{T}} w1_{t_jV}(w)\ dw \int_G \psi_i(y) \overset{\#}{W}_u (u)(y^{-1}x)\overline{\sigma(y,y^{-1}x)}\ dy \\
=& \bar{t}\sum_{i,j} \bar{t}_j\eta_i\, t_j \frac{\sin(\pi/N)}{\pi}\psi_i\# \overset{\#}{W}_u (u)(x) \\
=& \bar{t}\ \frac{\sin(\pi/N)}{\pi/N} \sum_i \eta_i\, \psi_i\# \overset{\#}{W}_u (u)(x).
\end{aligned}$$

If we define $A(\{\eta_i\}) = \sum_i \eta_i\, \psi_i\# \overset{\#}{W}_u (u)$, then the correspondence between $B_u^{\#}$ and $\widetilde{B}_u$ in Theorem 5.20 implies that $A : B^d \to B_u^{\#}$ is a well defined and bounded operator. Also, for any $f \in B_u^{\#}$, we have $\widetilde{f} \in \widetilde{B}_u$ with $\|f\|_B = \|\widetilde{f}\|_{\widetilde{B}}$. On the other hand

$\widetilde{f}(x_i, t_j) = \bar{t}_j f(x_i)$, and hence $\{f(x_i)\} \in B^d$ with $\|\{\widetilde{f}(x_i, t_j)\}\|_{\widetilde{B}^d} = N\|\{f(x_i)\}\|_{B^d}$. It follows that $\|f\|_B$ and $\|\{f(x_i)\}\|_{B^d}$. Finally, if we define $R = S_1^{-1}A$, then $R(\{f(x_i)\}) = f$ for all $f \in B_u^{\#}$, which completes our proof.

(3) We only have to prove the reconstruction formula

$$f = \sum_i \lambda_i(S_2^{-1}(f))L_{\#}(x_i) \overset{\#}{W}_u (u).$$

By applying the operator S_2 for $S_2^{-1}f$, the reconstruction formula is obtained. Similarly, we can show that $\{c_i L_{\#} \overset{\#}{W}_u (u)^{\vee} \circ S_3^{-1}, L_{\#}(x_i) \overset{\#}{W}_u (u)\}$ is also an atomic decomposition for $\mathbb{B}_u^n$. $\square$

The following is the main theorem of this section. It provides frames and atomic decompositions for coorbits.

Theorem 5.22. *Let $(\rho, \mathcal{S})$ be a projective representation of G, and let B be a BF-space on G satisfying Assumption 3.26. Assume that $u \in \mathcal{S}$ is a ρ-analyzing vector satisfying Assumption 5.10, which is both ρ-weakly and ρ^*-weakly differentiable. Furthermore, assume that $\overset{\#}{W}_u (u) \in B$ and the mappings*

$$f \mapsto f * |\overset{\#}{W}_{\rho(E_\alpha)u} (u)| \quad \textit{and} \quad f \mapsto f * |\overset{\#}{W}_u (\rho^*(E_\alpha)u)|$$

are continuous on B for all α with $|\alpha| \leq \dim G + 1$.

Then, we can choose ϵ small enough such that for any U_ϵ-well spread set $\{x_i\}$ the family $\{\rho^*(x_i)u\}$ is a Banach frame for $\mathrm{Co}_\rho^u B$ with respect to the sequence space B^d, and the families $\{\lambda_i \circ S_2^{-1} \circ \overset{\#}{W}_u, \rho^*(x_i)u\}$ and $\{c_i S_3^{-1} \circ \overset{\#}{W}_u, \rho^*(x_i)u\}$ are atomic decompositions for $\mathrm{Co}_\rho^u B$ with respect to the sequence space B^d. In particular, $\phi \in \mathrm{Co}_\rho^u B$ can be reconstructed by

$$\phi = (\overset{\#}{W}_u)^{-1} S_1^{-1}\Big(\sum_i \overset{\#}{W}_u (\phi)(x_i)\psi_i \# \overset{\#}{W}_u (u)\Big)$$

$$\phi = \sum_i \lambda_i\big(S_2^{-1} \overset{\#}{W}_u (\phi)\big)\rho^*(x_i)u$$

$$\phi = \sum_i c_i \; S_3^{-1} \overset{\#}{W}_u (\phi)\, \rho^*(x_i)u$$

with convergence in $\mathcal{S}^*$. The convergence is in $\mathrm{Co}_\rho^u B$ if $C_c(G)$ is dense in B.

Proof. Our assumptions are the same as the assumptions of Theorem 5.21, which guarantee the invertibility of the operators S_1, S_2, and S_3 for a small enough ϵ. Also the family $\{L_\#(x_i) \overset{\#}{W}_u (u)^\vee\}$ forms a frame for $B_u^\#$ with respect to B^d. For any $\phi \in \mathrm{Co}_\rho^u B$, we have $\overset{\#}{W}_u (\phi) \in B_u^\#$. Hence, $\{\overset{\#}{W}_u (\phi)(x_i)\} \in B^d$ such that $\| \overset{\#}{W}_u (\phi)\|_B = \|\phi\|_{\mathrm{Co}_\rho^u B}$ and $\| \overset{\#}{W}_u (\phi)(x_i)\|_{B^d} = \|\langle\phi, \rho(x_i)u\rangle\|_{B^d}$ are equivalent. Next, we show the reconstruction formula. For $\phi \in \mathrm{Co}_\rho^u B$, the reconstruction formula

$$\overset{\#}{W}_u (\phi) = S_1^{-1}\Big(\sum_i \overset{\#}{W}_u (\phi)(x_i)\psi_i \# \overset{\#}{W}_u (u)\Big)$$

holds. Therefore,

$$\phi = (\overset{\#}{W}_u)^{-1} S_1^{-1}\Big(\sum_i \overset{\#}{W}_u (\phi)(x_i)\psi_i \# \overset{\#}{W}_u (u)\Big)$$

is true for all $\phi \in \mathrm{Co}_\rho^u B$.

Now, let us show that the family $\{\lambda_i \circ S_2^{-1} \circ \overset{\#}{W}_u , \rho^*(x_i)u\}$ forms an atomic decomposition for $\mathrm{Co}_\rho^u B$ with respect to B^d. By Theorem 5.21, the family $\{\lambda_i \circ S_2^{-1}, L_\#(x_i) \overset{\#}{W}_u (u)\}$ forms an atomic decomposition for $B_u^\#$. That is, $\|\{\lambda_i \circ S_2^{-1}(\phi)\}\|_{B^d}$ and $\| \overset{\#}{W}_u (\phi)\|_B = \|\phi\|_{\mathrm{Co}_\rho^u B}$ are equivalent, and for any $\phi \in \mathrm{Co}_\rho^u B$, the function $\overset{\#}{W}_u (\phi)$ can be reconstructed by

$$\overset{\#}{W}_u (\phi) = \sum_i \lambda_i\big(S_2^{-1} \overset{\#}{W}_u (\phi)\big) L_\#(x_i) \overset{\#}{W}_u (u).$$

It follows that

$$\phi = \sum_i \lambda_i\big(S_2^{-1} \overset{\#}{W}_u (\phi)\big)(\overset{\#}{W}_u)^{-1} \left(L_\#(x_i) \overset{\#}{W}_u (u)\right).$$

By Theorem 5.16, $\overset{\#}{W}_u$ intertwines ρ^* and the twisted left translation, that is $L_{\#}(x_i) \overset{\#}{W}_u (u) = \overset{\#}{W}_u (\rho^*(x_i)u)$. Therefore, $\phi = \sum_i \lambda_i \big(S_2^{-1} \overset{\#}{W}_u (\phi)\big)\rho^*(x_i)u$. A similar argument shows that the family $\{c_i S_3^{-1} \circ \overset{\#}{W}_u, \rho^*(x_i)u\}$ forms an atomic decomposition for $\mathrm{Co}_\rho^u B$ with respect to the sequence space B^d. $\square$

Chapter 6
Bergman Spaces on the Unit Ball

As we motivated in the discretization of Bergman spaces on the unit disc, Bergman space are discretized through the simply connected subgroup S with the corresponding family of representations $(\pi_s, \mathcal{H}_s)$ for $s > 1$. Recently, in [6], Christensen, Gröchenig, and Ólafsson, obtained more general results not only for Bergman spaces on the unit disc, but also for Bergman spaces on the unit ball in $\mathbb{C}^n$. The authors describe Bergman spaces on the unit ball as coorbits of L^p spaces. Moreover they constructed Banach frames and atomic decompositions for Bergman spaces on the unit ball through a finite covering group of the group $SU(n,1)$ with restricting $s > n$ to be rational. Discretization of Bergman spaces through the group $SU(n,1)$ is valid for integer values of the parameter $s > n$. Nevertheless, the definition of the representation is no longer gives a single valued function for non-integer values of s. We dedicate this chapter to generate a Banach frame and an atomic decomposition of Bergman spaces on the unit ball via the group $SU(n,1)$. In this chapter we collected most of our facts from [6] and we use the same technique that used to prove the results for the regular coorbit with some modifications that needed to the projective representation. For more references we encourage the reader to see [4, 22, 29, 32, 45, 50].

6.1 Bergman Spaces on the Unit Ball

In this section we collect facts about Bergman spaces on the unit ball. Let $\mathbb{C}^n$ be equipped with the usual inner product $(z, w) = z_1\overline{w_1} + z_2\overline{w_2} + ... + z_n\overline{w_n}$ and define the unit ball by

$$\mathbb{B}^n := \left\{ z \in \mathbb{C}^n : |z|^2 := |z_1|^2 + |z_2|^2 + ... + |z_n|^2 < 1 \right\}.$$

Let $\mathrm{d}v$ be a normalized volume measure on the unit ball under identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. For $\alpha > -1$, define the measure

$$\mathrm{d}v_\alpha(z) := C_\alpha(1-|z|^2)^\alpha \mathrm{d}v(z)$$

where C_α is a constant such that $\mathrm{d}v_\alpha$ is a probability measure. Notice that the measure $\mathrm{d}v_\alpha$ is finite measure on $\mathbb{B}^n$ if and only if $\alpha > -1$.

We define the α-weighted L^p space on the unit ball as

$$L^p_\alpha(\mathbb{B}^n) = \{f : \mathbb{B}^n \to \mathbb{C} : \int_{\mathbb{B}^n} |f(z)|^p \mathrm{d}v_\alpha(z) < \infty\}$$

with norm

$$\|f\|_{L^p_\alpha} = \left(\int_{\mathbb{B}^n} |f(z)|\, \mathrm{d}v_\alpha(z)\right)^{1/p},$$

where $1 \le p < \infty$. For $\alpha > -1$, we define the weighted Bergman spaces on the unit ball to be

$$A^p_\alpha(\mathbb{B}^n) := L^p_\alpha(\mathbb{B}^n) \cap \mathcal{O}(\mathbb{B}^n)$$

with norm inherited from $L^p_\alpha(\mathbb{B}^n)$, where $\mathcal{O}(\mathbb{B}^n)$ is the space of holomorphic functions on the unit ball. We have the condition $\alpha > -1$ to construct a non-trivial Bergman spaces, in fact, if $\alpha \le -1$, then the only holomorphic function in $L^p_\alpha(\mathbb{B}^n)$ is the zero function.

As we have seen in the special case on the unit disc, Bergman spaces are closed subspaces of $L^p_\alpha(\mathbb{B}^n)$, *i.e.*, Bergman spaces are Banach spaces. In the case $p = 2$, the space $A^2_\alpha(\mathbb{B}^n)$ is a Hilbert space with the inner product

$$(f, g)_\alpha = \int_{\mathbb{B}^n} f(z)\overline{g(z)}\, \mathrm{d}v_\alpha(z).$$

The orthogonal projection of $L^2_\alpha(\mathbb{B}^n)$ on the space $A^2_\alpha(\mathbb{B}^n)$ is given by

$$P_\alpha f(z) = \int_{\mathbb{B}^n} f(w) K_\alpha(z, w)\, \mathrm{d}v_\alpha(w),$$

where

$$K_\alpha(z,w) = \frac{1}{(1-(z,w))^{n+1+\alpha}}$$

is the reproducing kernel for $A^2_\alpha(\mathbb{B}^n)$.

The group $SU(n,1)$ is defined to be the group of all $(n+1)\times(n+1)$-matrices x of determinant 1 and $x^* J_{(n,1)} x = J_{(n,1)}$, where

$$J_{(n,1)} = \begin{pmatrix} -I_n & 0 \\ 0 & 1 \end{pmatrix}.$$

We always write x in the block form $x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$, where A is an $n \times n$ matrix, and b, c are vectors in $\mathbb{C}^n$, and $d \in \mathbb{C}$. Simple calculations show that

$$x^{-1} = \begin{pmatrix} A^* & -\bar{c} \\ -\bar{b}^t & \bar{d} \end{pmatrix}.$$

The fact that $xx^{-1} = I$ implies

$$|d|^2 - |b|^2 = 1 \tag{6.1}$$

Form now on, we write $G = SU(n,1)$. This group acts transitively on $\mathbb{B}^n$ by

$$x \cdot z = (Az+b)((c,\bar{z})+d)^{-1}.$$

If we define the subgroup K of G as

$$K = \left\{ \begin{pmatrix} k & 0 \\ 0 & \overline{\det(k)} \end{pmatrix} \mid k \in U(n) \right\},$$

then the stabilizer of the origin $o \in \mathbb{C}^n$ is K and $\mathbb{B}^n \simeq G/K$. It follows that there is a one to one correspondence between the K-right invariant functions on G and the functions on $\mathbb{B}^n$ via

$$\widetilde{f}(x) = f(x \cdot 0).$$

This correspondence defines an G-invariant measure on $\mathbb{B}^n$ which is given by $(1-|z|^2)^{-n-1}dz$. The compactness of K ensures that we can normalize this measure so that, for any $\widetilde{f}$ K-right invariant function on G, we have

$$\int_G \widetilde{f}(x)\,dx = \int_{\mathbb{B}^n} f(z)\,(1-|z|^2)^{-n-1}dz. \tag{6.2}$$

Let $v_\alpha(x) = (1-|x\cdot o|^2)^\alpha$, then we define the v_α-weighted L^p spaces on G to be

$$L^p_\alpha(G) = \left\{ F : G \to \mathbb{C} : \|F\|_{L^p_\alpha(G)} := \left(c_\alpha \int_G |F(x)|^p\,(1-|x\cdot o|^2)^\alpha\,dx \right)^{1/p} < \infty \right\}$$

If we denote by $L^p_\alpha(G)^K$ the space of K-right invariant functions in the space $L^p_\alpha(G)$, then it is easy to see that $L^p_\alpha(\mathbb{B}^n)$ and $L^p_{\alpha+n+1}(G)^K$ are isometric. That is,

$$\|f\|_{L^p_\alpha(\mathbb{B}^n)} = \|\widetilde{f}\|_{L^p_{\alpha+n+1}(G)}. \tag{6.3}$$

For $s > n$, the action of G on $\mathbb{B}^n$ defines an irreducible unitary projective representation of G on the space $\mathcal{H}_s = A^2_{s-n-1}$ by

$$\rho_s(x)f(z) = (-(z,b)+\bar{d})^{-s} f(x^{-1}\cdot z), \tag{6.4}$$

where $x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$, which also defines a representation for the universal covering group of G. We denote the twisted wavelet transform on $\mathcal{H}_s$ by

$$\overset{\#}{W}{}^s_u\,(\lambda)(x) = (\lambda, \rho_s(x)u)_{\mathcal{H}_s}.$$

Let $\mathcal{P}_k$ be the space of all homogeneous polynomials of degree k on $\mathbb{C}^n$. In the following theorem we summarize some properties of the space of smooth vectors for ρ_s and its conjugate dual space, which will be the candidate Fréchet space $\mathcal{S}$ for constructing the coorbits of $L^p_{\alpha+n+1-sp/2}(G)$.

Theorem 6.1. *Let $s > n$ and let $(\rho_s, \mathcal{H}_s)$ be the projective representation of G which is defined in 6.4. The following are true:*

1. *Every polynomial is a smooth vector for ρ_s.*

2. *Every smooth vector for ρ_s is bounded.*

3. *Assume $v \in \mathcal{H}_s$, then $v \in \mathcal{H}_s^\infty$ if and only if $v = \sum_k v_k$, $v_k \in \mathcal{P}_k$, and for all $N \in \mathbb{N}$ there exists a constant $C > 0$ such that $\|v_k\|_{\mathcal{H}_s} \leq C(1+k)^{-N}$.*

4. *A vector $\phi \in \mathcal{H}_s^{-\infty}$ if and only if $\phi = \sum_k \phi_k$, $\phi_k \in \mathcal{P}_k$, and there exist $N \in \mathbb{N}$ and $C > 0$ such that $\|\phi_k\|_{\mathcal{H}_s} \leq C(1+k)^N$. Moreover, the dual pairing is given by*

$$\langle \phi, v \rangle_s = \sum_k (\phi_k, v_k)_{\mathcal{H}_s}.$$

Proof. The proof is done by noting that ρ_s is a unitary representation of the universal covering group of G, so the smooth vectors are the same for both, where the smooth vectors for ρ_s, as a representation, satisfy all the above properties as proved in [4] and [6]. □

6.2 Bergman Spaces as Twisted Convolutive Coorbits

As before, we assume $G = SU(n,1)$ and $(\mathcal{H}_s^\infty, \rho_s)$ is the subrepresentation of $(\mathcal{H}_s, \rho_s)$ on the group G for $s > n$. In this section we show that Bergman spaces are twisted convolutive coorbits of weighted L^p spaces, which allows us to discretize Bergman spaces using the full group $SU(n,1)$. For this goal we need the following results which already proved for the linear representation in [6]. The same proof will work (with minor differences) for the projective representation case. For completeness we will provide a full proof for each of these results.

Lemma 6.2. *Assume u and v are smooth vectors for ρ_s. There is a constant C depending on u and v such that*

$$|\overset{\#}{W}{}_u^s(v)(x)| \leq C(1-|x\cdot o|^2)^{s/2}\left(1-\log(1-|x\cdot o|^2)\right).$$

Proof. Let $x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$. Since smooth vectors for ρ_s are bounded (see Theorem 6.1), we can define $C = \sup_{z,w\in\mathbb{B}^n} |u(z)||v(w)|$. Now, note that

$$\begin{aligned} |\rho_s(x)u(z)| =& |\bar{d} - (z,b)|^{-s}|u(x^{-1}\cdot z)| \\ =& |d|^{-s}|1-(z,bd^{-1})|^{-s}|u(x^{-1}\cdot z)| \\ =& |d|^{-s}|1-(z,x\cdot 0)|^{-s}|u(x^{-1}\cdot z)|. \end{aligned}$$

Also, by (6.1) we have $|d|^{-2} = 1 - |bd^{-1}|^2 = 1 - |x\cdot o|^2$. In other words, $|d|^{-s} = (1-|x\cdot o|^2)^{s/2}$. It follows that

$$\begin{aligned} |\overset{\#}{W_u^s}(v)(x)| =& \left|\int_{\mathbb{B}^n} v(z)\overline{\rho_s(x)u(z)}(1-|z|^2)^{s-n-1}\,dz\right| \\ \leq& |d|^{-s}\int_{\mathbb{B}^n} |v(z)||u(x^{-1}\cdot z)||1-(z,x\cdot 0)|^{-s}(1-|z|^2)^{s-n-1}\,dz \\ \leq& C(1-|x\cdot o|^2)^{s/2}\int_{\mathbb{B}^n} |1-(z,x\cdot 0)|^{-s}(1-|z|^2)^{s-n-1}\,dz. \end{aligned}$$

The last integral is comparable to $1-\log(1-|x\cdot o|^2)$ (see [50, Theorem 1.12]), which proves our assertion. □

Proposition 6.3. *Let $\alpha > -1$, $1 \leq p < \infty$, and $s > n$ be chosen. Assume that u and v are smooth vectors for ρ_s. Then $\overset{\#}{W_u^s}(v) \in L_t^p(G)$ for $t + ps/2 > n$.*

Proof. Assume that $u, v \in \mathcal{H}_s^\infty$. Since for any $\epsilon > 0$ the limit

$$\lim_{t\to 1}(1-t)^\epsilon(1-\log(1-t)) = 0$$

where $t \in (0,1)$, one can find $C_1 > 0$ such that

$$1-\log(1-|x\cdot o|^2) \leq C_1(1-|x\cdot o|^2)^{-\epsilon/2}$$

for all $x \in G$. Now, for $t + ps/2 > n$ choose ϵ small enough so that $t + p(s-\epsilon)/2 > n$. By Lemma 6.2, there is a constant C_2 such that

$$|\overset{\#}{W_u^s}(v)(x)| \leq C_2(1-|x\cdot o|^2)^{s/2}\left(1-\log(1-|x\cdot o|^2)\right) \leq C(1-|x\cdot o|^2)^{(s-\epsilon)/2}.$$

Since the function $x \mapsto (1-|x\cdot o|^2)$ is K-right invariant on G, we can write the following integral as an integral on $\mathbb{B}^n$ as in (6.2)

$$\begin{aligned}\int_G |\overset{\#}{W_u^s}(v)(x)|^p(1-|x\cdot o|)^t\,dx \leq& C\int_G (1-|x\cdot o|^2)^{(s-\epsilon)p/2+t}\,dx\\ =& C\int_{\mathbb{B}^n}(1-|z|^2)^{(s-\epsilon)p/2+t-n-1}\,dz.\end{aligned}$$

For $t+sp/2 > t+(s-\epsilon)p/2 > n$, we have $(s-\epsilon)p/2+t-n-1 > -1$. Therefore, the last integral is finite, and hence, $\overset{\#}{W_u^s}(v) \in L^p_t(G)$ for $t+sp/2 > n$. □

Now we show that the twisted coorbits of the spaces $L^p_{\alpha+n+1-sp/2}(G)$ generated by any nonzero smooth vector $u \in \mathcal{H}_s^\infty$ are well defined nonzero spaces under the assumptions in the following theorem.

Theorem 6.4. *Let $1 \leq p < \infty$, and $s > n$. Assume that $-1 < \alpha < p(s-n)-1$. For a nonzero smooth vector $u \in \mathcal{H}_s^\infty$, the coorbit space $\mathrm{Co}^u_{\rho_s}L^p_{\alpha+n+1-sp/2}(G)$ is a nonzero well defined Banach space.*

Proof. Let us show that the nonzero smooth vector $v \in \mathcal{H}_s^\infty$ satisfies Assumption 5.10. First, u is ρ-cyclic because $\mathcal{H}_s^\infty$ is irreducible projective representation. Next, since ρ_s is square integrable and G is unimodular, every smooth vector for ρ_s is ρ-admissible, *i.e.*, u is in the domain of the operator A_ρ, which is given in Theorem 5.18. It follows that

$$\begin{aligned}\overset{\#}{W_u^s}(v)\#\overset{\#}{W_u^s}(u)(x) =& \int_G (v,\rho(y)u)_{\mathcal{H}_s}(u,\rho(y^{-1}x)u)_{\mathcal{H}_s}\overline{\sigma(y,y^{-1}x)}\,dy\\ =& \int_G (v,\rho(y)u)_{\mathcal{H}_s}(u,\rho(y^{-1})\rho(x)u)_{\mathcal{H}_s}\overline{\sigma(y^{-1},x)}\,\overline{\sigma(y,y^{-1}x)}\,dy\\ =& \int_G (v,\rho(y)u)_{\mathcal{H}_s}(u,\rho(y)^{-1}\rho(x)u)_{\mathcal{H}_s}\sigma(y,y^{-1})\,\overline{\sigma(y,y^{-1})}\,dy\\ =& \int_G (v,\rho(y)u)_{\mathcal{H}_s}(\rho(y)u,\rho(x)u)_{\mathcal{H}_s}\,dy\\ =& c_\rho^2(v,\rho(x)u)_{\mathcal{H}_s}(u,u)_{\mathcal{H}_s}\end{aligned}$$

$$=C \overset{\#}{W}{}_u^s (v)(x)$$

for all $v \in \mathcal{H}_s^\infty$. Let us show that this formula extends for all $\phi \in \mathcal{H}_s^{-\infty}$. Assume that $v \in \mathcal{H}_s^\infty$. By the orthogonality relation in Theorem 5.18, we have

$$\int_G (v, \rho(x)u)_{\mathcal{H}_s} (\rho(x)u, u)_{\mathcal{H}_s} \, dx = c_\rho^2 \|u\|_{\mathcal{H}_s} (v, u)_{\mathcal{H}_s}.$$

If we define $\eta := c_\rho^2 \|u\|_{\mathcal{H}} v$, then $\eta \in \mathcal{H}_s^\infty$, and hence $\langle \phi, \eta \rangle_s$ is well defined for all $\phi \in \mathcal{H}_s^{-\infty}$. By Theorem 6.1, we have

$$\begin{aligned}
\langle \phi, \eta \rangle_s &= \sum_k (\phi_k, \eta)_{\mathcal{H}_s} \\
&= \sum_k c_\rho^2 \|u\|_{\mathcal{H}} (\phi_k, u)_{\mathcal{H}_s} \\
&= \sum_k \int_G (\phi_k, \rho(x)u)_{\mathcal{H}_s} (\rho(x)u, u)_{\mathcal{H}_s} \, dx \\
&= \int_G (\phi, \rho(x)u)_{\mathcal{H}_s} (\rho(x)u, u)_{\mathcal{H}_s} \, dx.
\end{aligned}$$

The interchanging of the integral and the sum is valid by Tonelli's theorem, because

$$\sum_k \int_G (\phi_k, \rho(x)u)_{\mathcal{H}_s} (\rho(x)u, u)_{\mathcal{H}_s} \, dx = \langle \phi, \eta \rangle_s$$

exists. Therefore, the mapping

$$\phi \mapsto \int_G (\phi, \rho(x)u)_{\mathcal{H}_s} (\rho(x)u, u)_{\mathcal{H}_s} \, dx$$

is weakly continuous on $\mathcal{H}_s^{-\infty}$. Hence, the reproducing formula extends for all $\phi \in \mathcal{H}_s^{-\infty}$. This shows that u is a ρ-analyzing vector.

Now, we show that the mapping $(f, v) \mapsto \int_G f(x) \overset{\#}{W}{}_u^s (x^{-1}) \overline{\sigma(x, x^{-1})} \, dx$ is continuous on $L^P_{\alpha+n+1-sp/2}(G)$. By Remark 5.11, it is enough to show that

$$\overset{\#}{W}{}_u^s (v) \in (L^p_{\alpha+n+1-sp/2}(G))^* = L^q_{sq/2-(\alpha+n+1)q/p}(G),$$

where $1/p+1/q=1$. This is done by Proposition 6.3, because $(sq/2-(\alpha+n+1)q/p)+sp/2>n$, whenever $-1<\alpha<p(s-n)-1$. Therefore, the space $Co^u_{\rho_s}L^p_{\alpha+n+1-sp/2}(G))$ is well defined. Finally, note that $\overset{\#}{W}{}^s_u(u)\in L^p_{\alpha+n+1-sp/2}(G)$ again by Proposition 6.3, hence it is nonzero Banach space. □

Our goal now is to describe Bergman spaces as twisted coorbits generated by any nonzero smooth vector $u\in\mathcal{H}^\infty_s$. First we describe Bergman spaces as twisted coorbits by the special ρ-analyzing vector $u=1_{\mathbb{B}^n}$, then we show that this coorbit is independent of the choice of u.

Theorem 6.5. *Let $\alpha>-1$, $1\le p<\infty$, and $u=1_{\mathbb{B}^n}$. The Bergman space $A^p_\alpha(\mathbb{B}^n)$ is the twisted coorbit space of $L^p_{\alpha+n+1-sp/2}(G)$ that corresponds to the projective representation $(\mathcal{H}^\infty_s,\rho_s)$. i.e., $A^p_\alpha(\mathbb{B}^n)=Co^u_{\rho_s}L^p_{\alpha+n+1-sp/2}(G)$ for $\alpha<p(s-n)-1$.*

Proof. As in [6, Theorem 3.6], the space $A^p_\alpha(\mathbb{B}^n)\subset\mathcal{H}^{-\infty}_s$ for all $p\ge1$, which is still valid in the case of smooth vectors for ρ. The reason behind that is that the smooth vectors for the projective representation ρ_s are the same smooth vectors of the representation of the universal covering of G. So we have only to show that for a holomorphic function f, the function $f\in L^p_\alpha(\mathbb{B}^n)$ if and only if $\overset{\#}{W}{}^s_u(f)\in L^p_{\alpha+n+1-sp/2}(G)$. To this end, assume $x=\begin{pmatrix}A & b\\ c^t & d\end{pmatrix}$. Then we have

$$\begin{aligned}|\overset{\#}{W}{}^s_u(f)(x)| &=\left|\sum_k \overset{\#}{W}{}^s_u(f_k)(x)\right|\\ &=\left|\sum_k(\bar{d})^{-s}f(bd^{-1})\right|\\ &=|d|^{-s}\left|\sum_k f_k(x\cdot o)\right|\\ &=|d|^{-s}|f(x\cdot o)|.\end{aligned}$$

As we have seen before, $|d|^{-s} = (1 - |x \cdot o|^2)^{s/2}$. It follows that

$$|f(x \cdot o)| = (1 - |x \cdot o|^2)^{-s/2} |\overset{\#}{W}{}^s_u(f)(x)|$$

by the isometry in (6.3). We conclude that $f \in \text{Ł}^p_\alpha(\mathbb{B}^n)$ if and only if $\overset{\#}{W}{}^s_u(f) \in L^p_{\alpha+n+1-sp/2}(G)$. $\square$

To prove our main result in this section, which says that Bergman spaces are twisted coorbits for weighted L^p spaces generated by any smooth vector, we need the following theorem. It will be used in the subsequent section to generate a Banach frame and atomic decomposition for Bergman spaces.

Theorem 6.6. *Let $1 \leq p < \infty$, $-1 < \alpha < p(s-n) - 1$, and let v and u be smooth vectors. The convolution operator $f \mapsto f * |\overset{\#}{W}{}^s_u(v)|$ is continuous on $L^p_{\alpha+n+1-sp/2}(G)$. In particular, $f \mapsto f \# \overset{\#}{W}{}^s_u(v)$ is continuous on $L^p_{\alpha+n+1-sp/2}(G)$.*

Proof. Let $F \in L^p_{\alpha+n+1-sp/2}(G)$ and define

$$\widetilde{f}(x) := \int_K F(xk)\mathrm{d}k.$$

Then $\widetilde{f}$ is K-right invariant function on G. Therefore, there is a corresponding $f \in L^p_{\alpha-sp/2}(\mathbb{B}^n)$. Now, for ϵ small enough such that $-(s-\epsilon)p/2 < \alpha - sp/2 + 1 < p\left((s-\epsilon)/2 - n\right) - 1$ whenever $-sp/2 < \alpha - sp/2 + 1 < p\left(s/2 - n\right) - 1$, we have

$$\begin{aligned}
|F| * |\overset{\#}{W}{}^s_u(v)|(x) =& \int_G |F(y)|\, |\overset{\#}{W}{}^s_u(v)(y^{-1}x)|\, dy \\
\leq& C \int_G |F(y)|(1 - |y^{-1}x \cdot o|^2)^{s/2} |1 - \log(1 - |y^{-1}x \cdot o|^2)|\, dy \\
\leq& C \int_G |F(y)|(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2}\, dy \\
=& C \int_{G/K} |\widetilde{f}(y)|(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2}\, dy.
\end{aligned}$$

If we assume that $x = \begin{pmatrix} A_x & b_x \\ c_x^t & d_x \end{pmatrix}$, $y = \begin{pmatrix} A_y & b_y \\ c_y^t & d_y \end{pmatrix}$, $w = x \cdot o = b_x d_x^{-1}$, and $z = y \cdot o = b_y d_y^{-1}$, then

$$d_{y^{-1}x} = \bar{d}_y d_x (1 - (w, z))$$

and

$$|d_x|^{-(s-\epsilon)} = (1 - |x \cdot o|^2)^{(s-\epsilon)/2}.$$

Therefore,

$$\begin{aligned}(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2} =& |d_{y^{-1}x}|^{-(s-\epsilon)} \\ =& (1 - |x \cdot o|^2)^{(s-\epsilon)/2} (1 - |y \cdot o|^2)^{(s-\epsilon)/2}.\end{aligned}$$

Thus,

$$\begin{aligned}|F| * |\overset{\#}{W_u^s}(v)|(x) =& C \int_{G/K} |\widetilde{f}(y)| \frac{(1 - |x \cdot o|^2)^{(s-\epsilon)/2} (1 - |y \cdot o|^2)^{(s-\epsilon)/2}}{|1 - (w, z)|^{(s-\epsilon)}} \, dy \\ =& C(1 - |w|^2)^{(s-\epsilon)/2} \int_{\mathbb{B}^n} |f(z)| \frac{(1 - |z|^2)^{(s-\epsilon)/2 - n - 1}}{|1 - (x \cdot o, y \cdot o)|^{(s-\epsilon)}} \, dz.\end{aligned}$$

According to [50, Theorem 2.10], the operator S which is given by

$$Sf(z) = (1 - |w|^2)^{(s-\epsilon)/2} \int_{\mathbb{B}^n} |f(z)| \frac{(1 - |z|^2)^{(s-\epsilon)/2 - n - 1}}{|1 - (w, z)|^{(s-\epsilon)}} \, dz$$

is continuous on $L^p_{\alpha - sp/2}(\mathbb{B}^n)$ whenever $-(s-\epsilon)p/2 < \alpha - sp/2 + 1 < p\left((s-\epsilon)/2 - n\right) - 1$, which is equivalent to $-1 < \alpha < p(s-n) - 1$. Since

$$\|f\|_{L^p_{\alpha - sp/2}(\mathbb{B}^n)} = \|\widetilde{f}\|_{L^p_{\alpha + n + 1 - sp/2}(G/K)} = \|F\|_{L^p_{\alpha + n + 1 - sp/2}(G)},$$

the operator $F \mapsto F * |\overset{\#}{W_u^s}(v)|$ is continuous on $L^p_{\alpha + n + 1 - sp/2}(G)$. The second part is clear from the relation $|F \# \overset{\#}{W_u^s}(v)(x)| \leq |F| * |\overset{\#}{W_u^s}(v)(x)|$. $\square$

We conclude our section with the following main result.

Theorem 6.7. *Let $1 \leq p < \infty$ and $-1 < \alpha < p(s-n) - 1$, and let $v \in \mathcal{H}_s^\infty$ be a nonzero smooth vector. The Bergman space $A^p_\alpha(\mathbb{B}^n)$ is the twisted coorbit space of $L^p_{\alpha+n+1-sp/2}(G)$ via the projective representation $(\mathcal{H}_s^\infty, \rho_s)$. That is, $A^p_\alpha(\mathbb{B}^n) = Co^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ for $\alpha < p(s-n) - 1$.*

Proof. Assume $u = 1_{\mathbb{B}^n}$. By Theorem 6.6, we have $A^p_\alpha(\mathbb{B}^n) = Co^u_\rho L^p_{\alpha+n+1-sp/2}(G)$. We will show that the twisted coorbit $Co^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ does not depend on the analyzing vector v, by applying Theorem 5.17. First, according to Theorem 6.6, the operators $f \mapsto f \# \overset{\#}{W^s_u}(v)$ and $f \mapsto f \# \overset{\#}{W^s_v}(u)$ are continuous on $L^p_{\alpha+n+1-sp/2}(G)$. Next, we show that $\overset{\#}{W^s_u}(\phi) \# \overset{\#}{W^s_v}(u) = C \overset{\#}{W^s_v}(\phi)$ for all $\phi \in \mathcal{H}_s^{-\infty}$. For $f \in \mathcal{H}_s^\infty$, we can use the orthogonality relation in Theorem 5.18 to get $\overset{\#}{W^s_u}(f) \# \overset{\#}{W^s_v}(u) = C \overset{\#}{W^s_v}$ (f). To extend this relation to the dual of the smooth vectors, it is enough to show that

$$\phi \mapsto \int_G \langle \phi, \rho(x)u \rangle \ \langle \rho(x)v, u \rangle \ dx$$

is weakly continuous. Same argument, as in the proof of Theorem 6.4, can be made to show our claim. Therefore, the twisted coorbit spaces $Co^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ are all equal to the space $A^p_\alpha(\mathbb{B}^n)$. □

6.3 Discretization on Bergman Spaces

In this section we generate a wavelet frame and an atomic decomposition of Bergman spaces depending on the coorbit theory, where this discretization would work for all projective representations with $s > n$, including the non-integrable cases. Also, we have more freedom in choosing the wavelet u. That is we show that any nonzero smooth vector is a good candidate to generate a Banach frame and an atomic decomposition for Bergman spaces.

Theorem 6.8. *Assume that $1 \leq p < \infty$, $s > n$, and $-1 < \alpha < p(s-n) - 1$. For a nonzero smooth vector u for ρ_s, we can choose ϵ small enough such that for every U_ϵ-well spread set $\{x_i\}_{i \in I}$ in G the following hold.*

1. *(Twisted wavelet frame) The family $\{\rho_s(x_i)u : i \in I\}$ is a Banach frame for $A^p_\alpha(\mathbb{B}^n)$ with respect to the sequence space $\ell^p_{\alpha+n+1-ps/2}(I)$. That is, there exist constants $A, B > 0$ such that for all $f \in A^p_\alpha(\mathbb{B}^n)$ we have*

$$A\|f\|_{A^p_\alpha(\mathbb{B}^n)} \leq \|\{\langle f, \rho_s(x_i)u\rangle\}\|_{\ell^p_{\alpha+n+1-sp/2}(I)} \leq B\|f\|_{A^p_\alpha(\mathbb{B}^n)},$$

and f can be reconstructed by

$$f = (\overset{\#}{W}{}^s_u)^{-1} S_1^{-1}\Big(\sum_i \overset{\#}{W}{}^s_u\,(f)(x_i)\psi_i \# \overset{\#}{W}{}^s_u\,(u)\Big)$$

where $\{\psi_i\}$ is any U_ϵ-BUPU with $\mathrm{supp}\psi_i \subset x_iU_\epsilon$.

2. *(Atomic decomposition) There exists a family of functionals $\{\gamma_i\}_{i\in I}$ on $A^p_\alpha(\mathbb{B}^n)$ such that the family $\{\gamma_i, \rho_s(x_i)u\}$ forms an atomic decomposition for $A^p_\alpha(\mathbb{B}^n)$ with respect to the sequence space $\ell^p_{\alpha+n+1-ps/2}(I)$, so that any $f \in A^p_\alpha(\mathbb{B}^n)$ can be reconstructed by*

$$f = \sum_i \gamma_i(f)\,\rho_s(x_i)u.$$

Proof. We show that the assumptions of Theorem 5.22 are satisfied. Under the conditions on p and s, the twisted coorbit of $L^p_{\alpha+n+1-ps/2}(G)$ is well defined and u satisfies Assumption 5.10 as we have seen in Theorem 6.4, and it is equal to $A^p_\alpha(\mathbb{B}^n)$. Since u is smooth vector for $\mathcal{H}_s$, and $\mathcal{H}_s^\infty$ is continuously embedded in its dual $\mathcal{H}_s^{-\infty}$, the vector u is ρ - and ρ^*-weakly differentiable. According to Theorem 6.6, the mappings

$$f \mapsto f * |\overset{\#}{W}_{\rho(E_\alpha)u}\,(u)| \quad \text{ and } \quad f \mapsto f * |\overset{\#}{W}_u\,(\rho^*(E_\alpha)u)|$$

are continuous on $L^p_{\alpha+n+1-ps/2}(G)$. Therefore, we can choose ϵ small enough such that the family $\{\rho_s(x_i)u\}$ forms a frame and an atomic decomposition for $A^p_\alpha(\mathbb{B}^n)$ with reconstruction operators that are given in Theorem 5.22. $\square$

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Vita

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