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Monotone clutters

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Abstract


A clutter is k-monotone, completely monotone or threshold if the corresponding Boolean function is k-monotone, completely monotone or threshold, respectively. A characterization of k-monotone clutters in terms of excluded minors is presented here. This result is used to derive a characterization of 2-monotone matroids and of 3-monotone matroids (which turn out to be all the threshold matroids).

1. Introduction

A clutter $H$ is an ordered pair $(V(H), E(H))$ where $V(H)$ is a finite set and $E(H)$ is a set of subsets of $V(H)$ such that $A_1 \not\subseteq A_2$ for distinct $A_1, A_2 \in E(H)$. We define the blocker of a clutter $H$ to be a clutter $b(H)$ with $V(b(H)) = V(H)$ and $E(b(H))$ the set of all minimal subsets of $V(H)$ which meets every member of $E(H)$. It is well known [2] that $b(b(H)) = H$ for any clutter $H$. Let $X \subseteq V(H)$ where $H$ is a clutter; we define

$$H \setminus X = (V(H) - X, \{A \in E(H) : A \cap X = \emptyset\}) \quad \text{and} \quad H / X = b(b(H) \setminus X).$$

Clearly, both $H \setminus X$ and $H / X$ are clutters and by [7], we have $H \setminus X \setminus Y = H \setminus Y \setminus X$, $H \setminus X / Y = H / Y \setminus X$ and $H / X / Y = H / Y / X$ for any disjoint subsets $X, Y$ of $V(H)$. If $H, J$ are clutters with $J = H \setminus X / Y$ for a pair of disjoint subsets $X, Y$ of $V(H)$, then $J$ is called a minor of $H$.

There is a natural correspondence between clutters and monotone Boolean functions. (A monotone Boolean function on $n$ variables is a mapping $f$ from $\{0, 1\}^n$ to $\{0, 1\}$ such that for all $x, y \in \{0, 1\}^n$, $x \leq y$ implies $f(x) \leq f(y)$.) If $H$ is a clutter, we define $f_H$ with variables $\{x_i : i \in V(H)\}$ such that $f_H(x) = 1$ if and only if $\{i : x_i = 1\} \supseteq A$ for some $A \in E(H)$. Obviously, $f_H$ is a monotone Boolean function. Conversely, it is not difficult to see that, for any monotone Boolean function $f$, there exists a unique clutter $H$ with $f_H = f$. Moreover, one can easily check that if $H$ is a clutter and $W \subseteq V(H)$, then
(i) $f_{H|W} = (f_H)^d$, (ii) $f_{H-W}$ can be obtained from $f_H$ by setting $x_i = 0$ for $i \in W$, and (iii) $f_{H-W}$ can be obtained from $f_H$ by setting $x_i = 1$ for $i \in W$. From this point of view, clutters and monotone Boolean functions are the same mathematical objects. In this paper, we are going to study, in terms of clutters, a special class of monotone Boolean functions, namely $k$-monotone Boolean functions.

$k$-monotone Boolean functions were introduced in early 1960s because of the study of threshold Boolean functions. Classical results about $k$-monotone Boolean functions can be found in [5]. In the rest of this paper, we will not mention Boolean functions any more. However, people who are familiar with Boolean functions will find out very easily that a clutter $H$ is $k$-monotone, completely monotone or threshold if and only if the corresponding Boolean function $f_H$ is $k$-monotone, completely monotone or threshold, respectively.

In Section 2, a characterization of $k$-monotone clutters is given, and some applications are discussed. In Section 3, the class of 2-monotone matroids are characterized. Finally, in Section 4, the class of 3-monotone matroids (which turn out to be all the threshold matroids) are characterized.

2. $k$-Monotone clutters

If $H$ is a clutter and $A \in E(H)$, we will write $A \in H$ for brevity. Let $H, J$ be two clutters. We define $J < H$ if, for every $X \in H$, there exists $Y \in J$ with $Y \subseteq X$. Then the following lemma is clear.

**Lemma 2.1.** Let $H, J$ be two clutters. Then $J \leq H$ if and only if $b(H) \leq b(J)$.

A clutter $H$ is $k$-monotone, where $k \geq 1$ is an integer if, for all disjoint subsets $X, Y$ of $V(H)$ with $|X| + |Y| \leq k$, either $H \setminus X/Y \leq H/X \setminus Y$ or $H/X \setminus Y \leq H \setminus X/Y$ holds. Since $H/x \leq H \setminus x$ for all clutters $H$ and $x \in V(H)$, it follows that all the clutters are 1-monotone. The goal of this section is to characterize $k$-monotone clutters. We first establish the following lemmas.

**Lemma 2.2.** A clutter is $k$-monotone if and only if its blocker is $k$-monotone.

**Proof.** This is clear by Lemma 2.1.

**Lemma 2.3.** If a clutter $H$ is $k$-monotone and $J$ is a minor of $H$, then $J$ is $k$-monotone.

**Proof.** Suppose $J = H \setminus Z/Z'$, where $Z, Z'$ are disjoint subsets of $V(H)$. By Lemma 2.2, we may assume that $Z' = \emptyset$. Let $X, Y$ be disjoint subsets of $V(H) - Z$ with $|X| + |Y| \leq k$. Then at least one of $H \setminus X/Y \leq H/X \setminus Y$ and $H/X \setminus Y \leq H \setminus X/Y$ holds since $H$ is $k$-monotone. By symmetry, we may assume $H \setminus X/Y \leq H/X \setminus Y$. Now, for any $A \in J \setminus X/Y$, since $A \in H \setminus X/Y$ and $A \cap Z = \emptyset$, it follows that there exists $A' \in H \setminus X/Y$ with $A' \subseteq A$. Thus, $A' \in J \setminus X/Y$ which implies that $J \setminus X/Y \leq J \setminus X/Y$. 


Lemma 2.4. Let $H$ be a clutter and let $X, Y$ be disjoint subsets of $V(H)$. Then $H\setminus X/Y \not\subseteq H/X \setminus Y$ if and only if there exist $A \in H$ and $B \in b(H)$ such that $(A \cup B) \cap Y = \emptyset$, $A \cap B \subseteq X$.

Proof. 

\begin{align*}
H\setminus X/Y & \not\subseteq H/X \setminus Y \\
\iff & \exists A \in H \setminus X/Y \text{ such that } A \not\supseteq A' \text{ for all } A' \in H \setminus X/Y \\
\iff & \exists A \in H, A \cap Y = \emptyset \text{ such that } A - X \not\supseteq A' - Y \text{ for all } A' \in H, \text{ with } A' \cap X = \emptyset \\
\iff & \exists A \in H, A \cap Y = \emptyset \text{ such that } A' \cap (V(H) - A - Y) \not= \emptyset, \text{ for all } A' \in H, \text{ with } A' \cap X = \emptyset \\
\iff & \exists A \in H, A \cap Y = \emptyset \text{ such that } X \cup (V(H) - A - Y) \supseteq B \text{ for some } B \in b(H) \\
\iff & \exists A \in H, B \in b(H) \text{ such that } (A \cup B) \cap Y = \emptyset \text{ and } A \cap B \subseteq X. \quad \Box
\end{align*}

This lemma suggests the following definition. Let $H$ be a clutter and let $A_1, A_2 \in H$, $B_1, B_2 \in b(H)$. The quadruple $Q = (A_1, A_2, B_1, B_2)$ is bad if $(A_1 \cup B_1) \cap A_2 \cap B_2 = (A_2 \cup B_2) \cap A_1 \cap B_1 = \emptyset$. The order of $Q$ is $\min \{|A_1 \cap B_2|, |A_1 \cap B_1| + |A_2 \cap B_1|\}$.

Lemma 2.5. If $H$ is not $k$-monotone, then $H$ has a bad quadruple of order at most $k$.

Proof. If $H$ is not $k$-monotone, then there exist disjoint subsets $X, Y$ of $V(H)$ with $|X| + |Y| \leq k$ such that neither $H\setminus X/Y \subseteq H/X \setminus Y$ nor $H/X \setminus Y \subseteq H\setminus X/Y$ holds. By Lemma 2.4, there exist $A_1, A_2 \in H$ and $B_1, B_2 \in b(H)$ with $(A_1 \cup B_1) \cap Y = \emptyset$, $(A_2 \cup B_2) \cap X = \emptyset$, $A_1 \cap B_1 \subseteq X$ and $A_2 \cap B_2 \subseteq Y$. Thus, $Q = (A_1, A_2, B_1, B_2)$ is a bad quadruple of order at most $|X| + |Y|$, which is at most $k$. \quad \Box

A refinement of this structure is the following. A clutter $H$ is called partitionable if there exists a proper partition $(V_1, V_2, V_3, V_4)$ of $V(H)$ such that $V_1 \cup V_2, V_3 \cup V_4 \in H$ and $V_1 \cup V_3, V_2 \cup V_4 \in b(H)$. Note that $A \cap B \not= \emptyset$ for any $A \in H, B \in b(H)$; thus, none of $V_1, V_2, V_3, V_4$ is empty. The order $k$ of a partitionable clutter $H$ is defined to be

\[ k = \min \{ |V_1| + |V_4|, |V_2| + |V_3| \} : (V_1, V_2, V_3, V_4) \text{ is a proper partition of } V(H) \}

Lemma 2.6. If $H$ has a bad quadruple of order at most $k$, then $H$ has a minor $J$ such that $J$ is partitionable of order at most $k$.

Proof. Let $J$ be a minor of $H$ such that $J$ is minor-minimal with the property that $J$ has a bad quadruple $Q = (C_1, C_2, D_1, D_2)$ of order at most $k$. Then we claim that $J$ is partitionable with proper partition $(C_1 \cap D_1, C_1 \cap D_2, C_2 \cap D_1, C_2 \cap D_2)$ and hence of
order at most \( k \). Because, otherwise, \( J \backslash (V(H) - C_1 - C_2)/(C_1 \cup C_2 - D_1 - D_2) \) also has a bad quadruple of order at most \( k \) and this is contrary to the minimality of \( J \). \( \square \)

**Theorem 2.7.** The following are equivalent for any clutter \( H \):

(i) \( H \) is not \( k \)-monotone;

(ii) \( H \) has a bad quadruple of order at most \( k \);

(iii) \( H \) has a partitionable clutter of order at most \( k \) as a minor.

**Proof.** Because of the previous lemmas, it is enough to show that if \( H \) is a partitionable clutter of order at most \( k \), then \( H \) is not \( k \)-monotone. This is clear because it is easy to check that if \((V_1, V_2, V_3, V_4)\) is a proper partition of \( V(H) \) with, say, \(|V_1| + |V_4| \leq k\), then by Lemma 2.4 both \( H \backslash V_1 \leq H \backslash V_4 \) and \( H \backslash V_1 \leq H \backslash V_4 \) hold. \( \square \)

This is the best theorem we have been able to find to characterize \( k \)-monotone clutters by excluded minors. It seems quite difficult to exhibit all the partitionable clutters even for order \( k = 2 \) because there are too many of them. Nevertheless, as observed in \([6]\), one can easily recognize 2-monotone clutters in polynomial (in \(|V(H)| + |E(H)|\)) time.

Now we shall use Theorem 2.7 to study another class of clutters. We call a clutter \( H \) completely monotone if \( H \) is \( k \)-monotone for all \( k \geq 1 \). Completely monotone clutters can be characterized in the following ways.

**Theorem 2.8.** The following are equivalent for any clutter \( H \):

1. \( H \) is completely monotone;
2. \( b(H) \) is completely monotone;
3. \( H \) has no partitionable clutter as a minor;
4. \( H \) has no bad quadruples;
5. \( H \) is \( k \)-monotone for \( k = \max \{|A| : A \in H\} \);
6. either \( J \leq b(J) \) or \( b(J) \leq J \) holds for all minors \( J \) of \( H \).

Before proving this theorem, let us establish the following trivial lemma.

**Lemma 2.9.** Let \( H \) be a clutter. Then \( b(H) \leq H \) if and only if \( A \cap A' \neq \emptyset \) for all \( A, A' \in H \).

**Proof of Theorem 2.8.** The equivalences of (1)–(4) are clear by Lemma 2.2 and Theorem 2.7. The implication (1) to (5) is trivial and the implication (6) to (3) is clear by Lemma 2.9. Thus, we only need to show the implication (5) to (6).

Suppose that there exists a minor \( J = H \backslash X/Y \) of \( H \) such that neither \( J \leq b(J) \) nor \( b(J) \leq J \) holds, where \( X, Y \) are disjoint subsets of \( V(H) \). By Lemma 2.9, there exist \( C_1, C_2 \in J \) and \( D_1, D_2 \in b(J) \), such that \( C_1 \cap C_2 = \emptyset \), \( D_1 \cap D_2 = \emptyset \). Let \( A_1, A_2 \in H \) and \( B_1, B_2 \in b(H) \) such that \( A_i \leq C_i \cup Y \), \( B_i \leq D_i \cup X \) where \( i = 1, 2 \). Then it is easy to see
that \((A_1, A_2, B_1, B_2)\) is a bad quadruple of order \(\min\{|A_1 \cap B_1| + |A_2 \cap B_2|, |A_1 \cap B_2| + |A_2 \cap B_1|\}\), which is at most \(k\). Thus, by Theorem 2.7, \(H\) is not \(k\)-monotone, contrary to (5). \(\square\)

**Remark 2.10.** (a) The equivalence of (1) and (4), the implication of (1) to (6) and Lemma 2.9 can also be found in [5].

(b) Clutters with property (4) are called 2-asummable (see [5]). This property (together with [6, Theorem 6]) provides a polynomial (in \(|V(H)| + |E(H)|\)) time algorithm to test if a clutter is completely monotone.

(c) From this theorem, it follows that a graph \(G\) (a clutter with \(|A| = 2\) for all \(A \in G\)) is completely monotone if and only if \(G\) has no induced subgraphs \(2K_2, P_4 \) and \(C_4\). This was already known and, in fact, it was proved [1] that both these properties are equivalent to being a threshold graph.

### 3. 2-Monotone matroids

As we mentioned earlier, it is not easy to characterize \(k\)-monotone clutters by exhibiting all the partitionable clutters even for order \(k=2\). However, we will show in this section that this can be done for a special class of clutters, namely the circuit clutters of matroids.

A **matroid** is a pair \(M=(E, \mathcal{F})\) where \(E\) is a set and \(\mathcal{F}\) is a collection of subsets of \(E\) with the following properties: (i) \(\emptyset \in \mathcal{F}\), (ii) \(X \subseteq Y \in \mathcal{F}\) implies that \(X \in \mathcal{F}\) and (iii) for any subset \(X\) of \(E\), all maximal subsets of \(X\) belonging to \(\mathcal{F}\) have the same cardinality. Members of \(\mathcal{F}\) are called *independent sets* of \(M\) and the other subsets of \(E\) are called *dependent sets*. We call the minimal dependent sets *circuits* of \(M\) and the maximal independent sets *bases* of \(M\). The collections of circuits and bases of \(M\) are denoted by \(\mathcal{C}(M)\) and \(\mathcal{B}(M)\), respectively. We say \(M\) is \(k\)-monotone if the clutter \(C(M)=(E, \mathcal{C}(M))\) is \(k\)-monotone. In this section, we are going to characterize the class of 2-monotone matroids.

We begin with introducing some terminology in matroid theory which will be used in this paper. For any undefined terminology, the reader is referred to [8]. Let \(M=(E, \mathcal{F})\) be a matroid. The **dual matroid** of \(M\) is denoted by \(M^*\). For \(Z \subseteq E\), we define \(sp(Z)=\{z \in E: z \in Z \text{ or } \exists C \in \mathcal{C}(M) \text{ with } z \in C \subseteq Z \cup \{z\}\}\). Let \(x \in E\). The **deletion** of \(x\) from \(M\), denoted by \(M \setminus x\), is the matroid on \(E \setminus \{x\}\) such that a subset of \(E \setminus \{x\}\) is independent in \(M \setminus x\) if and only if it is independent in \(M\). The **contraction** of \(x\) from \(M\), denoted by \(M/x\), is the matroid \((M^* \setminus x)^*\). We say that \(N\) is a **minor** of \(M\) if \(N\) can be obtained from \(M\) by a series of deletions and contractions.

**Lemma 3.1.** Let \(M\) be a matroid and \(x \in E(M)\). Then

(i) \(\mathcal{C}(M \setminus x) = \mathcal{C}(M) \setminus x\); (ii) \(\mathcal{C}(M/x) = \mathcal{C}(M)/x\) provided \(x\) is not a loop.
Proof. (i) \( C \in \mathcal{C}(M \setminus x) \) if and only if \( C \in \mathcal{C}(M) \) and \( x \notin C \), that is, if and only if \( C \in \mathcal{C}(M) \setminus \{x\} \).

(ii) \( C \in \mathcal{C}(M/x) \) if and only if either \( C \in \mathcal{C}(M) \) and \( x \notin \text{sp}(C) \) or \( C \cup \{x\} \in \mathcal{C}(M) \), that is, if and only if \( C \in \mathcal{C}(M)/x \). \( \square \)

Lemma 3.2. Let \( M \) be a matroid. Then (i) if \( N \) is a minor of \( M \), then \( \mathcal{C}(N) \) is a minor of \( \mathcal{C}(M) \), (ii) if \( H \) is a minor of \( \mathcal{C}(M) \) with \( \emptyset \notin H \), then \( H = \mathcal{C}(N) \) for a minor \( N \) of \( M \).

Proof. (i) Obviously, we may assume that \( N = M \setminus x \) or \( N = M/x \) for some \( x \in E(M) \) and, moreover, if \( N = M/x \) we may assume \( x \) is not a loop, since if \( x \) is a loop, \( M/x = M \setminus x \). Therefore, by Lemma 3.1, \( \mathcal{C}(N) = \mathcal{C}(M) \setminus \{x\} \), or \( \mathcal{C}(M)/x \), respectively.

(ii) Similarly, we may assume that \( H = \mathcal{C}(M) \setminus x \) or \( H = \mathcal{C}(M)/x \) for some \( x \in E(M) \). Then if \( H = \mathcal{C}(M)/x \), \( x \) is not a loop since \( \emptyset \notin H \). Therefore, by Lemma 3.1, \( H = \mathcal{C}(M \setminus x) \) or \( \mathcal{C}(M/x) \), respectively. \( \square \)

Now let us define matroids \( M_n \) on \( \{1, 2, \ldots, 2n\} \), where \( n \geq 2 \), with independent sets all sets of cardinality at most \( n \) except \( \{1, 2, \ldots, n\} \) and \( \{n+1, n+2, \ldots, 2n\} \). Then we have the following lemma.

Lemma 3.3. If \( M \) has no \( M_n \) minor for all \( n \geq 2 \), then for any two circuits \( C_1, C_2 \) of \( M \), either \( C_1 \subseteq \text{sp}(C_2) \) or \( C_2 \subseteq \text{sp}(C_1) \).

Proof. Suppose that there exists a counterexample, that is, a matroid \( M \) with no \( M_n \) minor for all \( n \geq 2 \) such that there are two circuits \( C_1, C_2 \) of \( M \) with \( C_1 \notin \text{sp}(C_2) \) and \( C_2 \notin \text{sp}(C_1) \). We choose \( M \) to be a minor-minimal counterexample. Then \( C_1 \cap C_2 = \emptyset \) and \( C_1 \cup C_2 = E(M) \) because, otherwise, \( M \setminus (E(M) - C_1 - C_2)/(C_1 \cup C_2) \) is a smaller counterexample. Let \( x \in C_1 - \text{sp}(C_2) \) and \( y \in C_2 - \text{sp}(C_1) \). Then \( X = \{x\} \cup (C_2 - \{y\}) \) and \( Y = \{y\} \cup (C_1 - \{x\}) \) are independent. Moreover, \( X, Y \) are bases of \( M \) because, otherwise, there exists (say) \( x' \in C_2 - \{y\} \) with \( \{x'\} \cup Y \) independent, when \( M/x' \) would be a smaller counterexample. Now we claim that \( C_1, C_2 \) are hyperplanes of \( M \) because, otherwise, there exists (say) \( x' \in C_2 - \{y\} \) such that \( \{x'\} \cup (C_1 - \{x\}) \) contains a unique circuit \( C \) of \( M \). But if we take \( y' \in C \cap C_1 \), and we take circuit \( C' \subseteq (C_1 \cup C) - \{x'\} \) of \( M \) (it is clear that \( x, x' \in C' \)), then \( M \setminus y'/x' \) would be a smaller counterexample since it contains the circuits \( C' - \{x'\} \) and \( C_2 - \{x'\} \). Finally, we claim that for any \( Z \subseteq E(M) \) with \( |Z| \leq |C_1| \) and \( Z \neq C_1, C_2, Z \) is independent. Because, otherwise, we may choose a circuit \( Z \neq C_1, C_2 \) with \( |Z| \leq |C_1| \) and with \( Z \cap C_2 \) minimal. Since \( C_1, C_2 \) are hyperplanes, it follows that \( |Z \cap C_1|, |Z \cap C_2| \geq 2 \), and hence \( M' = M/(Z \setminus C_1) \) is a smaller counterexample because of \( C_1 - Z, C_2 \cap Z \in \mathcal{C}(M') \).

Therefore, the matroid \( M \) is nothing but \( M_n \) for \( n = |C_1| \), a contradiction. \( \square \)

Theorem 3.4. The following are equivalent for any matroid \( M \):

1. \( M \) is 2-monotone;
2. \( M \) has no \( M_n \) minor for any \( n \geq 2 \);
3. for any two circuits \( C_1, C_2 \) of \( M \), either \( C_1 \subseteq \text{sp}(C_1) \) or \( C_2 \subseteq \text{sp}(C_1) \).
Proof. (1) ⇒ (2): Suppose that \( M \) has an \( M_n \) minor for some \( n \geq 2 \). Then \( C(M) \) has a \( C(M_n) \) minor by Lemma 3.2, contrary to Lemma 2.3, since \( M_n \) is not 2-monotone.

(2) ⇒ (3): This is the assertion of Lemma 3.3.

(3) ⇒ (1): If \( M \) is not 2-monotone, then by Theorem 2.7, \( C(M) \) has a bad quadruple \((C_1, C_2, B_1, B_2)\) of order 2. Let \( \{x\} = C_1 \cap B_1 \) and \( \{y\} = C_2 \cap B_2 \). Then we claim that \( x \notin sp(C_2) \) and hence \( C_1 \notin sp(C_2) \). For if \( x \in sp(C_2) = sp(C_2 - \{y\}) \), then there exist a circuit \( C \) of \( M \) with \( C \subseteq (C_2 - \{y\}) \cup \{x\} \subseteq E(M) - B_2 \), contrary to the independence of \( E(M) - B_2 \). Similarly, \( C_2 \notin sp(C_1) \), contrary to (3). \( \square \)

Corollary 3.5. A matroid \( M \) is 2-monotone if and only if its dual \( M^* \) is 2-monotone.

Proof. It is clear by Theorem 3.4 since \( M^*_n = M_n \). \( \square \)

Corollary 3.6. Let \( B(M) = (E(M), \mathcal{B}(M)) \). Then \( B(M) \) is 2-monotone if and only if \( C(M) \) is 2-monotone.

Proof. \( B(M) \) is 2-monotone if and only if \( b(B(M)) = C(M^*) \) is 2-monotone, that is, if and only if \( C(M) \) is 2-monotone. \( \square \)

We close this section by pointing out that there is no polynomial-time oracle algorithm to test if a matroid is 2-monotone. First we have to explain what an oracle algorithm is. It is clear that it is impossible to store an arbitrary matroid on \( n \) elements in \( O(n^c) \) space, where \( c \) is a constant. Thus, for any matroid \( M \), we assume that \( M \) is represented by \( E \), on which \( M \) is defined, and an oracle, with which we can tell, for any \( X \subseteq E \), if \( X \) is independent in \( M \) in unit time. But the oracle is a 'black box', we cannot use of its internal properties in designing our algorithm. In other words, our algorithm can only use the oracle as a subroutine to get the information of a matroid. This kind of algorithm is called an oracle algorithm.

Theorem 3.7. There is no polynomial-time oracle algorithm to test if a matroid is 2-monotone.

Proof. Let \( E = \{1, 2, \ldots, 2n\} \) and let \( X \subseteq W \) with \( |X| = n \). We define \( M_X \) to be the matroid on \( E \) with independent sets all sets of cardinality at most \( n \) except \( X \) and \( E - X \). Then \( M_X \) is not 2-monotone. Suppose that there is an oracle algorithm which test if a matroid is 2-monotone. Plug in the matroid \( M_X \), then we claim that the algorithm must ask for the independence of \( X \) or \( E - X \). For, otherwise, plug in matroid \( M \), the uniform matroid on \( E \) of rank \( n \). It is clear that the only difference between \( M \) and \( M_X \) is that \( X, E - X \) are independent in \( M \) but dependent in \( M_X \). Since the algorithm does not ask for the independence of \( X \) and \( E - X \), so the algorithm does the same with \( M \) as with \( M_X \) and hence reaches the same conclusion, contrary to the fact that \( M \) is 2-monotone. Thus, we deduce that the algorithm must ask for the
independence of $X$ or $E - X$ for all $X \subseteq E$ with $|X| = n$. Therefore, the running time of this algorithm is at least $\left(\frac{2^n}{2}\right)/2$, which is not polynomial.

4. Threshold clutters and matroids

A clutter $H$ is threshold if there exist a function $w: V(H) \to \mathbb{N} = \{1, 2, \ldots\}$ and $t \in \mathbb{N}$ such that $X \subseteq V(H)$ contains some $A \in H$ if and only if $w(X) = \sum_{x \in X} w(x) \geq t$. The pair $(w, t)$ is called a representation of $H$. The following lemmas are well known (see [5]), but we present them here for completeness.

**Lemma 4.1.** If a nonempty clutter $H$ (i.e. $E(H) \neq \emptyset$) is threshold with representation $(w, t)$, then $b(H)$ is threshold with representation $(w, w(V(H)) - t + 1)$.

**Proof.** Let $X$ be a subset of $V(H)$. Then $X \supseteq B$ for some $B \in b(H) \iff V(H) - X \not\supseteq A$ for all $A \in H \iff w(V(H) - X) \leq t - 1 \iff w(X) \geq w(V(H)) - t + 1$.

**Lemma 4.2.** Let $H$ be threshold with representation $(w, t)$ and let $x \in V(H)$. Suppose that $w'$ is the restriction of $w$ to $V(H) - \{x\}$, then (i) $H \setminus x$ is threshold with representation $(w', t)$, and (ii) $H/x$ is threshold with representation $(w', t - w(x))$ provided $\{x\} \notin H$.

**Proof.** (i) Let $X$ be a subset of $V(H) - \{x\}$, then $X \supseteq A$ for some $A \in H \setminus x$ if and only if $X \supseteq A$ for some $A \in H$ if and only if $w(X) \geq t$ if and only if $w'(X) \geq t$.

(ii) The assertion is clear if $E(H) = \emptyset$, so we may assume that $E(H)$ is not empty. Therefore, $b(H)$ is threshold with representation $(w, w(V(H)) - t + 1)$ and hence $b(H) \setminus x$ is threshold with representation $(w', w(V(H)) - t + 1)$. But $E(b(H) \setminus x)$ is not empty since $\{x\} \notin H$. Thus, $H/x = b(b(H) \setminus x)$ is threshold with representation $(w', w(V(b(H) \setminus x)) - w(V(H)) + t - 1 + 1)$ which is $(w', t - w(x))$.

**Remark 4.3.** From Theorem 2.8(3), and Lemmas 4.1 and 4.2, it follows that all the threshold clutters are completely monotone. But the converse is not true. A counterexample with $|V(H)| = 9$ can be found in [3].

Let us now turn to matroids. A matroid $M$ is threshold if $C(M)$ is threshold. It was shown in [9] that, for any 2-monotone clutter $H, H = C(M)$ for some threshold matroid $M$ if and only if $H$ has a unique ceiling (see [9] for details about ceiling). With this result, Giles and Kannan [4] proved that a matroid is threshold if and only if it is 3-monotone. In the rest of this paper we will characterize threshold matroids in terms of forbidden minors. This characterization implies the result in [4], but the proof is independent of the results in [4] and [9].

We first define two matroids $N_1$ and $N_2$ on $\{1, 2, 3, 4, 5, 6\}$ such that (i) $X$ is independent in $N_1$ if and only if $|X| \leq 3$ and $\{1, 2\}$ is not a subset of $X$, (ii) $X$ is
independent in $N_2$ if and only if $|X| \leq 3$ and none of $\{1, 2\}$, $\{1, 3, 5\}$, $\{2, 3, 5\}$ is a subset of $X$. These two matroids are not 3-monotone (therefore not threshold) because both of $C(N_1)$ and $C(N_2)$ are partitionable of order 3 with proper partition $\{1\}, \{2\}, \{3, 4\}$ and $\{5, 6\}$.

**Lemma 4.4.** Let $M = (E, \mathcal{F})$ be a 2-monotone matroid with no loop or coloop. If the rank of $M$ is $r$ and $M$ has no minors $N_1$ and $N_2$, then there exist a partition $(X, Y)$ of $E$ with $|Y| \geq 2$ such that either

$$\mathcal{F} = \mathcal{F}_1 = \{Z \subseteq E: |Z| \leq r \text{ and } |Z \cap X| \leq r-1\}$$

or

$$\mathcal{F} = \mathcal{F}_2 = \{Z \subseteq E: |Z| \leq r \text{ and } |Z \cap X| \leq |Y| + 1\} \text{ with } |Y| \leq r.$$ 

**Proof.** If $M$ is a uniform matroid, take $X = \emptyset$, $Y = E$. Then $|Y| \geq 2$ (since $M$ has no loop or coloop), $\mathcal{F} = \mathcal{F}_1$, and hence we are done. Suppose that $M$ is not uniform, then there exist a circuit $C$ with $|C| \leq r$. We choose such a circuit $C$ with $|C|$ as big as possible. Let $X = sp(C)$ and let $Y = E - X$. Clearly, $|Y| \geq 2$ since $M$ has no coloop. Moreover, we have the following results.

**Claim 1.** For every $y \in Y$, if $y \in C' \in \mathcal{C}(M)$, then $|C'| = r + 1$.

Since $M$ is 2-monotone, it follows from Lemma 3.4 that $C \subseteq sp(C')$ and hence $|C'| = r(C') + 1 > r(C) + 1 = |C|$. By the maximality of $|C|$, we have $|C'| = r + 1$.

A consequence of this claim is that, for any subset $Z$ of $E$, $Z$ is independent if and only if $|Z| \leq r$ and $Z \cap X$ is independent.

**Claim 2.** $M\setminus Y$ is a uniform matroid.

Suppose that $M\setminus Y$ is not uniform, then we want to show that $M$ has a minor $N_2$ and hence this is a contradiction. Let $r'$ be the rank of $M\setminus Y$. We first prove that there are two circuits $C', C''$ of $M\setminus Y$ such that $|C''| = r' + 1 > |C'|$ and $|C' - C''| = 1$. Since $M\setminus Y$ is not uniform, there exists a circuit $C'$ of $M\setminus Y$ with $|C'| \leq r'$. We choose such a circuit with $C' - C$ minimal. Let $x \in C - C$. Then $C' - \{x\}$ is independent and therefore there exist a subset $Z$ of $C - C'$ such that $Z \cup (C' - \{x\})$ is a base of $M\setminus Y$. Let $y \in C - Z - C'$ and let $C'' \subseteq \{y\} \cup Z \cup (C' - \{x\})$ be the unique circuit of $M\setminus Y$. Then from the minimality of $C'$, we conclude that $|C''| = r' + 1$ and hence $C'' - \{y\} \cup Z \cup (C' - \{x\})$ is the circuit we are looking for. Now let $C' - C'' = \{1\}; \{2\} \subseteq C' \cap C''$ (we may assume this since $M$ has no loop); $\{3, 5\} \subseteq C'' - C'$; $\{4, 6\} \subseteq Y$; $Z_1 = C'' - \{2, 3, 5\}$; $Z_2 \subseteq Y - \{4, 6\}$ with $|Z_2| = r - r' - 1$ (we can do this since $M$ has no coloop and hence $|Y| \geq r - r' + 1$), and $Z_3 = E(M) - \{1, 2, 3, 4, 5, 6\} - Z_1 - Z_2$. Then it is not difficult to check that $M\setminus Z_3/Z_1/Z_2 = N_2$. 
Proof of Lemma 4.4 (Conclusions). From Claims 1 and 2, it follows that $Z \subseteq E$ is independent if and only if $|Z| \leq r$ and $|Z \cap X| \leq r'$. Now we claim that either $r - r' \leq 1$ or $r - r' \geq |Y| - 1$. For, otherwise, $2 \leq r - r' \leq |Y| - 2$. Let $\{1, 2\} \subseteq C$ (we may assume this since $M$ has no loop); $\{3, 4, 5, 6\} \subseteq Y$ (we may assume this since $2 \leq |Y| - 2$); $Z_1 = C - \{1, 2\}; Z_2 \subseteq Y - \{3, 4, 5, 6\}$ with $|Z_2| = r - r' - 2$ (we may assume this since $r - r' \leq |Y| - 2$) and $Z_3 = E(M) - Z_1 - Z_2 - \{1, 2, 3, 4, 5, 6\}$. Then it is not difficult to check that $M \setminus Z_3 / Z_1 / Z_2 = N_1$.

Now if $r - r' \leq 1$, then $r - r' = 1$ by the definition of $r'$. It is clear that in this case $\mathcal{F} = \mathcal{F}_1$. If $r - r' \geq |Y| - 1$, then $r - r' = |Y| - 1$ since $r - r' \leq |Y|$ and $M$ has no coloop.

Again it is clear that in this case $\mathcal{F} = \mathcal{F}_2$. □

Theorem 4.5. The following are equivalent for any matroid $M$:

(i) $M$ is threshold;

(ii) $M$ is 3-monotone;

(iii) $M$ has no minors $N_1, N_2$ and $M_n$ for all $n \geq 2$.

(iv) $M$ can be obtained from $N = (E, \mathcal{F})$, where $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F}_2$ as defined in Lemma 4.4, by adding loops and coloops.

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): This is clear since all the matroids $N_1, N_2$ and $M_n (n \geq 2)$ are not 3-monotone.

(iii) $\Rightarrow$ (iv): This is clear by Lemma 4.4 and Theorem 3.4.

(iv) $\Rightarrow$ (i): Obviously, we may assume that $M$ has no loop or coloop. Let $r, r'$ be the rank of $M$ and $M \setminus Y$, respectively. We define $w(x) = 2r(r - r')$ for $x \in X$; $w(y) = 2r - 1$ for $y \in Y$ and $t = (2r - 1)(2r')$. Then we want to show that $M$ is threshold with representation $(w, t)$. Clearly, $r' \leq r - 1$ and hence $2r(r - r') > 2r - 1$. Thus, for any base $B$ of $M$, $w(B) \leq 2r(r - r')r'(2r - 1)$.

To finish the whole proof, we only need to show that, for any circuit $C$ of $M$, $w(C) > t$. This is clear if $C \subseteq Y$ because $|C| = r + 1$ and hence $w(C) = 2r(r - r')(r + 1) > t$. Therefore, we may assume that $C \not\subseteq X$ and hence $|C| = r + 1$. If $\mathcal{F} = \mathcal{F}_1$, then $r' = r - 1$ and so $w(C) \geq (2r - 1)(r + 1) > 2r^2 - 1 = t$. If $\mathcal{F} = \mathcal{F}_2$, then $r' = r - |Y| + 1$ and so $w(C) \geq 2r(r - r')(r - |Y| + 1) + (2r - 1)|Y| > t$. □

Since the blocker of $C(M)$ is $B(M^*)$, then by Lemma 4.5 we have the following corollary.

Corollary 4.6. Let $M = (E, \mathcal{F})$ be a matroid with no loop or coloop. Then $B(M)$ is threshold if and only if there exist a partition $(X, Y)$ of $E$ such that either

$$\mathcal{F} = \{Z \subseteq E: |Z| \leq r \text{ and } Y \not\subseteq Z\}$$

or

$$\mathcal{F} = \{Z \subseteq E: |Z| \leq r \text{ and } |Z \cap Y| \leq 1\} \text{ with } |Y| \leq |E| - r,$$

where $r$ is the rank of $M$. 

Finally, we would like to point out that there is no polynomial-time oracle algorithm to test if a matroid is threshold since the proof of Theorem 3.7 is also a proof of this assertion.

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References