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Damage Evolution in Pressurized Domain: A Gradient Based Variational Approach

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DAMAGE EVOLUTION IN PRESSURIZED DOMAIN: A GRADIENT BASED VARIATIONAL APPROACH

A Thesis

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NOTATION

The following notation is used throughout this thesis.

- All scalar variables are shown by italic letters.
- Subscripts t refers to specific variable at time t.
- In n dimension, $n > 1$, the boldfaces are used to indicate vectors and second order tensors. As an example $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are used to show displacement vector, strain tensor and stress tensor respectively. Their components are shown by italic letters without boldfaces like u_i, ε_{ij} and σ_{ij} .
- Fourth order tensors are shown by double struck letters like $\mathbb{E}$ and its components are shown as $\mathbb{E}_{ijkl}$. Fourth order tensors act as linear maps on vectors and second order tensors. Multiplication of tensors are denoted without dots as an example ij-component of $\mathbb{E}\boldsymbol{\varepsilon}$ is shown by $\mathbb{E}_{ijkl}\varepsilon_{kl}$. The summation convention is used implicitly on repeated indices. The inner product between two vectors or two tensors of the same order is indicated by a dot. Therefore, $\boldsymbol{a}.\boldsymbol{b}$ stands for $a_i b_i$ and $\boldsymbol{\sigma}.\boldsymbol{\varepsilon}$ stands for $\sigma_{ij}\varepsilon_{ij}$.
- The reference configuration of a material point is shown by x and its Cartesian coordinates in $\mathbb{R}^n$ are indicated by $(x_1, \dots, x_n)$. The orthonormal basis of $\mathbb{R}^n$ is $(e_1, \dots, e_n)$ and $\boldsymbol{M}_s^n$ shows the space of $n \times n$ symmetric tensors as linear maps on $\mathbb{R}^n$. Moreover, $\mathbb{I}$ represents identity tensor in $\mathbb{R}^n$
- The symbol $\otimes$ denotes the tensor product and $\otimes_s$ denotes its symmetrized form, i.e. $2e_1 \otimes_s e_2 = e_1 \otimes e_2 + e_2 \otimes e_1$.

- In one – dimension, $n = 1$, all the scalar quantities or fields are indicated by italic letters, like u, ε, σ or $E(\alpha)$ for the displacement, strain, stress or damaged Young modulus. The prime stands for either the derivative with respect to the coordinate x or the derivative with respect to the damage parameter, e.g. $u' = \partial u / \partial x$, $E'(\alpha) = dE(\alpha)/d\alpha$.
- Time derivative is shown by dote, e.g. $\dot{\alpha} = \partial \alpha / \partial t$.
- The qualifier increasing stands for strictly increasing , e.g. increasing β with respect to time means $\frac{\partial \beta}{\partial t} > 0$.
- The qualifier decreasing stands for strictly decreasing , e.g. decreasing β with respect to time means $\frac{\partial \beta}{\partial t} < 0$.
- The qualifier positive stands for strictly positive , e.g. β is positive means $\beta > 0$.
- The qualifier negative stands for strictly negative , e.g. β is negative means $\beta < 0$.
- Orders of magnitude is used in its classical sense: $o(h^n)$ denotes functions of h such that each term order is greater that n, therefore, $lim_{h \to 0}\, o(h)/h = 0$.

ABSTRACT

Construction of appropriate models through mathematical analysis for materials in order to find their main properties and ingredients and enhance the numerical simulations to predict their behavior under specific conditions is in interest even in mathematics departments rather than material science and engineering branches. Among these models, gradient damage models have reached to the specific stage because of their ability to bring the effects of micro cracks propagation into conventional continuum mechanics formulation and approximate brittle fracture as one of the most phenomena in the area of material behavior simulation.

This thesis includes the application and extension of a previously proposed gradient damage model through the mathematical analysis on a specific 2D problem i.e. axisymmetric domain with internal pressure, which is in interest for designing reservoirs and investigating crack propagation around oil wells. To accomplish this task, this thesis is organized as following. In the first chapter, general framework and fundamentals of damage models is discussed in details including standard models and incorporation of gradient term into standard models through variational approach. Main properties of gradient damage models are derived and all details of derivations including proofs of some propositions are added to show the flow of the presentation. In the second chapter, presented formulation is applied on a desired problem in details to show the application of the model in 2D. Discussion on main results and some recommendations are given in the last chapter.

1. GRADIENT DAMAGE MODELS PROPERTIES AND ITS CONSTRUCTION

1.1 Introduction

In this chapter, damage propagation in elastic region is investigated through the construction of brittle damage laws for general materials regardless of hardening or softening behavior in inelastic region. First, general class of local damage models based on the concept of yield criterion is considered. It is shown that Drucker – Ilyushin postulate can justify these models, but the convexity properties of the strain work as a state function depend on the hardening or softening properties of the material. Pham and Marigo (K. Pham, Amor, Marigo, & Maurini, 2011; K. Pham & Marigo, 2011) have proved that the evolution problem can be formulated through the variational approach reinforced by the concepts of stability and conservation. These types of models have ill-posed mathematical problems due to lack of damage localization limiting terms specially for the case of softening materials. Therefore, enhancement of damage models by introducing gradient terms accompanied by a length scale parameters has become an interesting topic to overcome aforementioned problems. Researchers used the principle of irreversibility, stability and energy balance to formulate the damage evolution problem enhanced by gradient term, which is discussed in details in this chapter (K. Pham et al., 2011; Kim Pham, Marigo, & Maurini, 2011; K. Pham & Marigo, 2011). The main ingredients of this chapter is borrowed from the work by Marigo and Maurini (Marigo & Maurini). Main nomenclature of this chapter is given in Table 1.

Table 1 - Main nomenclature of chapter one

State variables and state functions	
$\boldsymbol{u}$	Displacement vector with components u_i
$\boldsymbol{\varepsilon}$	Second order strain tensor with components ε_{ij}
$\boldsymbol{\sigma}$	Second order stress tensor with components σ_{ij}
$\mathbb{E}$	Fourth order stiffness tensor with components $\mathbb{E}_{ijkl}$
u'	Derivative of u w.r.t x in one dimension i.e. $\partial u/\partial x$
$E'(\alpha)$	Derivative of E w.r.t α in one dimension i.e. $dE/d\alpha$
$\psi(\boldsymbol{\varepsilon},\alpha)$	Elastic strain energy of local models
$\mathbb{S}\,(\alpha)$	Fourth order compliance tensor (Inverse of stiffness tensor)
$\phi(\boldsymbol{\varepsilon},\alpha)$	Damage yield function
$\mathcal{R}(\alpha)$	Elastic domain in strain space
$\mathcal{R}^*(\alpha)$	Elastic domain in stress space
$\boldsymbol{M}_s^n$	Space of symmetric $n \times n$ tensors
Y	Thermodynamics force – elastic energy release
$\mathrm{W}_0(\boldsymbol{\varepsilon},\alpha)$	Strain work – state function
$\omega(\alpha)$	Energy dissipation during damage process
D	Dissipated power
$\kappa(\alpha)$	Damage threshold
α_m	Ultimate damage state (max)
$\boldsymbol{f}$	Body force
$\boldsymbol{F}$	Applied external force on boundary (Neumann BC’s)

Table 1 continued

$\boldsymbol{U}$	Applied external displacement on boundary (Dirichlet BC's)
Ω	Domain occupied by damaging material
$\partial_F \Omega$	Part of the boundary with Neumann BC's
$\partial_D \Omega$	Part of the boundary with Dirichlet BC's
$\mathcal{C}$	Space of kinematically admissible displacement fields
H^1	Sobolev space
$\mathcal{D}$	Set of admissible damage fields
$\mathcal{E}\ (\boldsymbol{u}, \alpha)$	Total energy of the system
W^e	External work done by external applied fields
$\mathcal{E}'(\boldsymbol{u}, \alpha)(\boldsymbol{v}, \beta)$	Directional derivative of $\mathcal{E}(\boldsymbol{u}, \alpha)$ in the direction of $(\boldsymbol{v}, \beta)$
$u_{i,j}$	$\partial u_i / \partial x_j$
$\dot{\boldsymbol{U}}$	Rate of applied displacement on boundary
$\dot{W}$	Rate of applied work done by external forces on boundary

1.2 Local damage models: properties and formulation

Any damage model consists of three main ingredients as follows:

1- Definition of damage variable

2- Constitutive stress – strain relation as a function of damage variable

3- Specific damage evolution law

In this section, the focus would be on the general framework for constructing any brittle damage model and fundamental concepts is the main interest. Therefore, to avoid complexity following assumptions are made in the formulation:

1- For the sake of simplicity and in order to give a general framework, the damage level of a material point is described by a scalar α. This confirms that the material damage is isotropic and is not a function of specific direction. α grows from 0 to α_m where 0 corresponds to the undamaged state and α_m shows the fully damaged state and α_m is not restricted to a specific value to keep the generality of the formulation i.e. $0 < \alpha_m \leq +\infty$

2- At a given damage level α, material behavior is elastic. Its elasticity (stiffness) depends on the damage variable through the elastic strain energy $\psi(\boldsymbol{\varepsilon}, \alpha)$. It is assumed that ψ is a quadratic function of $\boldsymbol{\varepsilon}$ at a given α, i.e.

$$\psi(\boldsymbol{\varepsilon}, \alpha) = \frac{1}{2}\, \mathbb{E}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \tag{1}$$

Where $\mathbb{E}(\alpha)$ denotes the fourth order stiffness tensor. Eq. (1) implies that the material behavior is linearly elastic at given α and the stress – strain relation reads as:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) = \mathbb{E}(\alpha)\boldsymbol{\varepsilon} \tag{2}$$

Damage growth leads to decrease of the stiffness when α increases. Therefore, stiffness function $\alpha \longmapsto \mathbb{E}(\alpha)$ satisfies the following properties:

$$\mathbb{E}(0) > 0 \quad \mathbb{E}'(\alpha) < 0 \quad \mathbb{E}(\alpha_m) = 0 \tag{3}$$

Inequalities in Eq. (3) show the positivity of a fourth order tensor. Fourth order tensor $\mathbb{A}$ is said positive if the following relation holds:

$$\mathbb{A}\,\boldsymbol{\beta}.\boldsymbol{\beta} > 0 \qquad \forall\, \boldsymbol{\beta} \in \boldsymbol{M}_s^n \;\; \boldsymbol{\beta} \neq 0$$

Previous relation can be written in the following form:

$$\mathbb{A}_{ijkl}\, \beta_{kl}\beta_{ij} > 0$$

As an example, Laplace operator in the following form satisfies the previous condition although it does have negative components:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ 0 & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} > 0$$

Based on Eq. (3), the material point loses its rigidity and its stiffness becomes zero when it is fully damaged. As long as $\alpha < \alpha_m$, the stiffness tensor $\mathbb{E}(\alpha)$ is positive and invertible. Compliance tensor $\mathbb{S}\,(\alpha)$ is defined as the invers of the stiffness tensor. It reads:

$$\mathbb{S}\,(\alpha) = \mathbb{E}^{-1}(\alpha) \tag{4}$$

Using Eq. (4) into Eq. (2) gives the strain in terms of stress as:

$$\boldsymbol{\varepsilon} = \mathbb{E}^{-1}(\alpha)\,\boldsymbol{\sigma} = \mathbb{S}\,(\alpha)\boldsymbol{\sigma} \tag{5}$$

Damage is known as an irreversible process and its growth can be obtained by yield criterion. Damage yield function $\phi(\boldsymbol{\varepsilon}, \alpha)$ is expressed in terms of strain to take into account for softening behaviors. In the case of simple uniaxial test which is depicted in Fig. 1.1and Fig 1.2, expressing yield function in terms of stress leads to two different strains (one in the elastic region and the other in the softening region), but expressing it in terms of strain leads to the unique value for stress which is shown in Fig. 1.1 and Fig 1.2. For this reason, strain based formulation is always preferred specially in mathematical analysis of plasticity and damage models.

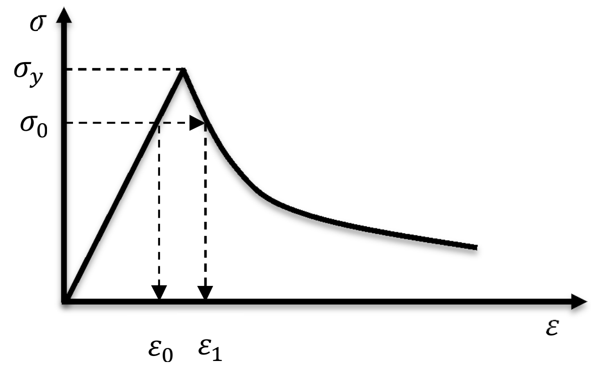

Figure 1 - Two strains for a given stress

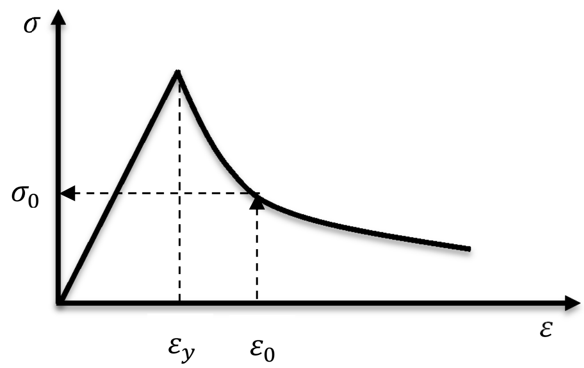

Figure 2 - One stress value for a given strain

Damage evolution i.e. the evolution of α is obtained by the Kuhn – Tucker condition as:

$$\dot{\alpha} \geq 0 \quad \phi(\boldsymbol{\varepsilon}, \alpha) \leq 0 \quad \dot{\alpha}\, \phi(\boldsymbol{\varepsilon}, \alpha) = 0 \tag{6}$$

In Eq. (6), the first condition shows the irreversibility and the second one is the damage yield criterion while the third is consistency condition which shows that the damage can grow only when the strain state is on the yield surface. Damage variable α plays the role of hardening parameter in addition to unique internal variable. It worth to mention that the yield criterion function (ϕ) would be less than zero ($\phi < 0$) for all values of damage variable in unstrained (unstressed) state ($\boldsymbol{\varepsilon} = \mathbf{0}\ \ or\ \ \boldsymbol{\sigma} = \mathbf{0}$)**.** The function ϕ is assumed to be sufficiently smooth so that $\phi(\boldsymbol{\varepsilon}, \alpha) \leq 0$ corresponds for every $\alpha \in [0, \alpha_m)$, to a closed connected set in $\boldsymbol{M}_s^n$ which contains the unstrained state i.e. $\boldsymbol{\varepsilon} = \boldsymbol{0}$. This set, denoted by $\mathcal{R}(\alpha)$, shows the elastic domain in the strain space when the material point is in the damaged state α.

$$\mathcal{R}(\alpha) = \{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \quad : \ \phi(\boldsymbol{\varepsilon}, \alpha) \leq 0\} \tag{7}$$

Associated to $\mathcal{R}(\alpha)$, the elastic domain in stress space can be defined as following using Eq. (5):

$$\mathcal{R}^*(\alpha) = \{\boldsymbol{\sigma} \in \boldsymbol{M}_s^n \quad : \ \phi(\mathbb{S}\,(\alpha)\boldsymbol{\sigma}, \alpha) \leq 0\} \tag{8}$$

1.3 Standard Models

1.3.1 Drucker – Ilyushin postulate

Standard laws consist in defining the scalar damage yield function ϕ from the thermodynamic force (Y), which is defined as follows:

$$Y := -\frac{\partial \psi}{\partial \alpha} \tag{9}$$

This force corresponds to elastic energy release rate in the present context as a scalar. It has been proved (Marigo & Maurini) that standard model properties can be derived from Drucker – Ilyushin postulate which is stated as follows:

Let α_0 be the initial damage state and let $t \longmapsto \boldsymbol{\varepsilon}(t)$ be a cycle in strain space, i.e. a path in $\boldsymbol{M}_s^n$ parameterized by $t \in [0,1]$ such that $\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}(1)$. During this cycle imposed to the material point the damage evolves, its evolution $t \longmapsto \alpha(t)$ being governed by Kuhn – Tucker condition (Eq. (6)). The Strain work W in this cycle is given by:

$$\mathrm{W} := \int_0^1 \boldsymbol{\sigma}(t).\dot{\boldsymbol{\varepsilon}}(t)dt = \int_0^1 \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(t), \alpha(t)).\dot{\boldsymbol{\varepsilon}}(t)dt \tag{10}$$

Drucker – Ilyushin postulate consists in requiring that $\mathsf{W} \geq 0$ whatever the initial state α_0 and whatever the strain cycle are considered.

1.3.2 Damage standard Law

The strain work W is non-negative for every initial damage state and every strain cycle only if the damage criterion corresponds to a critical elastic energy release rate criterion. Specifically, there necessarily exists $\kappa(\alpha) > 0$ such that $\mathcal{R}(\alpha)$ (Eq. (7)) can read as:

$$\mathcal{R}(\alpha) = \left\{ \boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \quad : -\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) \leq \kappa(\alpha) \right\} \tag{11}$$

Comparing two elastic domains in Eq. (7) and Eq. (11) leads to the following definition for yield function:

$$\phi(\boldsymbol{\varepsilon}, \alpha) = -\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) - \kappa(\alpha) \tag{12}$$

Eq. (12) is valid even if ψ is not a quadratic in $\boldsymbol{\varepsilon}$ and material obeys nonlinear elasticity rules. Therefore, the strain work can be considered as a state function, i.e. the work done in order that the state of material point evolves from its unstrained and undamaged state $(\mathbf{0}, 0)$ to the state $(\boldsymbol{\varepsilon}, \alpha)$ is independent of the strain path. Specifically, one gets:

$$\mathsf{W} = \mathsf{W}_0(\boldsymbol{\varepsilon}, \alpha) := \psi(\boldsymbol{\varepsilon}, \alpha) + \omega(\alpha) \tag{13}$$

Where $\alpha \longmapsto \omega(\alpha)$ is the anti-derivative of $\alpha \longmapsto \kappa(\alpha)$ vanishing at $\alpha = 0$ and following relation holds:

$$\omega(\alpha) = \int_0^\alpha \kappa(\beta)\, d\beta \qquad \rightarrow \quad \omega'(\alpha) = \kappa(\alpha) \tag{14}$$

From Eq. (13), following results can be deduced:

$$\frac{\partial \mathsf{W}_0}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) \tag{15}$$

$$\frac{\partial \mathrm{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) = \frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) + \omega'(\alpha)$$

Comparing Eq. (2) and Eq. (15), stress – strain relation can be obtained from strain work as follows since the dissipation is not a function of strain state:

$$\boldsymbol{\sigma} = \frac{\partial \mathrm{W}_0}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) \tag{16}$$

In Eq. (13), ω corresponds to the energy dissipation during damage process when the damage grows from 0 to α. Based on Eq. (14), $\omega' = \kappa > 0$, the dissipated energy is an increasing function of α and hence Clausius – Duhem inequality is automatically satisfied. If the free energy is given by the elastic energy ψ (Eq. (2)), then time derivative of free energy reads:

$$\dot{\psi} = \frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha)\dot{\alpha} + \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) \,.\, \dot{\boldsymbol{\varepsilon}} \tag{17}$$

Therefore, the dissipated power D reads as:

$$D := \boldsymbol{\sigma} \,.\, \dot{\boldsymbol{\varepsilon}} - \dot{\psi} = \boldsymbol{\sigma} \,.\, \dot{\boldsymbol{\varepsilon}} - \left(\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha)\dot{\alpha} + \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha) \,.\, \dot{\boldsymbol{\varepsilon}} \right) \tag{18}$$

Using Eq. (2) into Eq. (18) leads to:

$$D = -\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha)\dot{\alpha} \tag{19}$$

Substituting Eq. (12) into Eq. (19) results in:

$$D = (\phi(\boldsymbol{\varepsilon}, \alpha) + \kappa(\alpha))\,\dot{\alpha} \tag{20}$$

Furthermore, substituting Eq. (14) into Eq. (20) leads to:

$$D = (\phi(\boldsymbol{\varepsilon}, \alpha) + \omega'(\alpha))\, \dot{\alpha} \tag{21}$$

Using consistency condition (Eq. (6)) in Eq. (21) results in:

$$D = \omega'(\alpha)\, \dot{\alpha} \tag{22}$$

Comparing Eq. (19) and Eq. (22) indicates that the elastic energy release rate is equal to dissipated power as:

$$-\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha)\dot{\alpha} = \omega'(\alpha)\, \dot{\alpha}$$

Therefore, the damage evolution law of standard models can be summarized as follows which is valid until the damage variable reaches to the final value ($\alpha < \alpha_m$):

$$\dot{\alpha} \geq 0, \quad \frac{\partial \mathrm{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) \geq 0, \quad \dot{\alpha}\, \frac{\partial \mathrm{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) = 0 \tag{23}$$

Another interesting result can be obtained by changing the variable in the previous formulation. Let us consider a case in which a new damage variable is defined as a dissipation function in previous formulation. More precisely, consider a model with damage variable (d) defined as:

$$d := \omega(\alpha) \tag{24}$$

Eq. (24) turns out that the derivative of new damage variable (d) with respect to previous damage variable (α) is equal to the bound of elastic domain (damage threshold, $\kappa(\alpha)$) using Eq. (14) :

$$d' = \omega'(\alpha) = \kappa(\alpha) \tag{25}$$

Then, strain energy would be a new function of variables $\boldsymbol{\varepsilon}$ and d as $\tilde{\psi}\ (\varepsilon, d)$ and strain work can be expressed as:

$$\widetilde{W}_0(\boldsymbol{\varepsilon}, d) := \tilde{\psi}(\boldsymbol{\varepsilon}, d) + d \tag{26}$$

Taking derivative of new strain energy function with respect to previous damage variable along with using Eq. (25) and substituting the result into definition of elastic domain (Eq. (11)) results in:

$$\frac{\partial \tilde{\psi}}{\partial \alpha}(\boldsymbol{\varepsilon}, d) = \left(\frac{\partial \tilde{\psi}}{\partial d}(\boldsymbol{\varepsilon}, d)\right) d' = \left(\frac{\partial \tilde{\psi}}{\partial d}(\boldsymbol{\varepsilon}, d)\right) \kappa(\alpha) \tag{27}$$

Therefore, new damage criterion can be obtained as:

$$\mathcal{R}(\alpha) = \left\{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \quad : -\frac{\partial \tilde{\psi}}{\partial d}(\boldsymbol{\varepsilon}, d) \leq 1\right\} \tag{28}$$

Since there is no restriction on the definition of dissipation function, previous result shows that *the definition of dissipation as a function of damage variable is the most important task in formulation of the damage problem.* The damage variable definition can change the bound of elastic domain without causing any specific change in the formulation. Elastic domain (Eq. (11)) in standard models can be expressed in strain space in terms of strain work function by using Eq. (14) into Eq. (15) and substituting the result into Eq. (11):

$$\mathcal{R}(\alpha) = \left\{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \quad : \frac{\partial W_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) \geq 0\right\} \tag{29}$$

Legendre transform of $W_0(\varepsilon, \alpha)$ with respect to $\boldsymbol{\varepsilon}$ leads to the definition of strain work in stress space:

$$W_0^*(\boldsymbol{\sigma}, \alpha) = \sup_{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n} \{\boldsymbol{\sigma}.\boldsymbol{\varepsilon} - W_0(\boldsymbol{\varepsilon}, \alpha)\} \qquad (30)$$

Therefore, the elastic domain in stress space can be expressed as:

$$\mathcal{R}^*(\alpha) = \left\{\boldsymbol{\sigma} \in \boldsymbol{M}_s^n \quad : \frac{\partial W_0^*}{\partial \alpha}(\boldsymbol{\sigma}, \alpha) \leq 0\right\} \qquad (31)$$

The important property of elastic domains defined by Eq. (29) and Eq. (31) is their size rather than their shape. This size is controlled by damage parameter (α) and affect qualitative properties of the damage evolution problem.

1.4 The variational properties of standard models

1.4.1 The evolution problem

In this section, fundamental variational properties of standard models are investigated. A n – dimensional body made of one brittle damage material is considered. It is assumed that the damage behavior of material can be described by a single (scalar) damage variable like the previous section. More precisely, the damage behavior is assumed to be isotropic even if the material is anisotropic in its nature. Natural reference configuration of the body is an open set Ω of $\mathbb{R}^n$. It worth to mention that if the body is made of heterogeneous material, the strain work W_0, the damage variable α, the ultimate damage state α_m and all other material quantities depend on the material point x, i.e. should be considered as a function of position. The body undergoes the time dependent loading which is parameterized by the time parameter t. Initial state is shown by $t = 0$ and time parameter is positive $t \geq 0$. Quasi-static problem is considered which means that the effects of rate dependency, inertia, acceleration and dynamics are neglected. Therefore, damage evolution problem consists of finding three fields in the body including: 1- stress field $\boldsymbol{\sigma}_t$, 2- displacement field $\boldsymbol{u}_t$ and 3- damage field α_t at each time $t \geq 0$. These fields have to satisfy four set of equation simultaneously over the domain and its boundary like other solid mechanics problems including 1- equilibrium equation, 2- boundary conditions, 3- constitutive equation (Hook's law in the case of linear elasticity) and 4- damage evolution law. These equations can be summarized as:

$$
\begin{array}{rll}
\text{Equilibrium:} & div\, \boldsymbol{\sigma}_t + \boldsymbol{f}_t = 0 & in\ \Omega \\
\text{Neumann boundary conditions:} & \boldsymbol{\sigma}_t . n = \boldsymbol{F}_t & on\ \partial_{\mathrm{F}}\Omega \\
\text{Dirichlet boundary conditions} & \boldsymbol{u}_t = \boldsymbol{U}_t & on\ \partial_{\mathrm{D}}\Omega \\
\text{Constitutive relations} & \boldsymbol{\sigma}_t = \mathbb{E}\,(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t) & in\ \Omega \\
\text{Compatibility conditions} & 2\boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \nabla \boldsymbol{u}_t + \nabla \boldsymbol{u}_t^T & in\ \Omega \\
\text{Kuhn – Tucker conditions} & \begin{cases} \dot{\alpha}_t \geq 0 \\ -\dfrac{1}{2}\mathbb{E}'\,(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) \leq \omega'(\alpha_t) \\ \dot{\alpha}_t \left(\dfrac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\right) = 0 \end{cases} & in\ \Omega
\end{array}
\tag{32}
$$

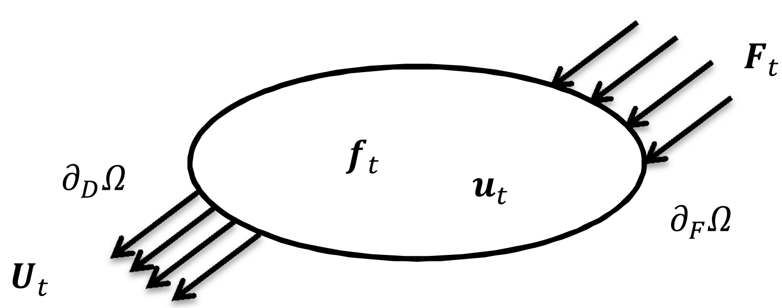

Figure 3 - Domain configuration, applied loads and boundary conditions

As it is shown in Fig. 3, the boundary of the domain in the previous formulation is divided into two parts, Dirichlet boundary ($\partial_D\Omega$) in which external displacements ($\boldsymbol{U}_t$) are prescribed on this part and Neumann boundary ($\partial_F\Omega$) in which external forces ($\boldsymbol{F}_t$) are prescribed on this part. $\boldsymbol{U}_t$ is the imposed displacement on the boundary part $\partial_D\Omega$ and $\boldsymbol{F}_t$ is the external forces on the complementary boundary part $\partial\Omega\backslash\partial_D\Omega = \partial_F\Omega$ and none of these parts are function of time. $\boldsymbol{f}_t$ is the body forces over the whole domain Ω. It worth to mention that the problem is formulated for small deformation therefore, the equilibrium equation is written in the reference configuration (effects of large deformation and equilibrium in deformed configuration is neglected) and the relation between strains and displacements are linearized. In this regard, strain field can be considered as symmetric part of the gradient of displacement field as mentioned.

1.4.2 Admissible fields and total energy

Space of kinematically admissible displacement fields at time t is defined as:

$$\mathcal{C}_t(\boldsymbol{U}_t) = \{\boldsymbol{v} \in H^1(\Omega) \quad : \boldsymbol{v} = \boldsymbol{U}_t \quad on \quad \partial_D\Omega\} \tag{33}$$

The associated linear space reads:

$$\mathcal{C}_0(\boldsymbol{U}_t) = \{\boldsymbol{v} \in H^1(\Omega) \quad : \boldsymbol{v} = \boldsymbol{0} \quad on \quad \partial_D\Omega\} \tag{34}$$

It is assumed that rigid body motion does not exist on the boundary. The set of admissible damage fields is the convex subset of Sobolev space:

$$\mathcal{D}_0 = \{\alpha \in H^1(\Omega) \quad : \alpha(x) \in [0, \alpha_m] \qquad in \ \Omega\} \tag{35}$$

Irreversibility condition implies that $\dot{\alpha} \geq 0$ which means that damage variable can only increase. Therefore, with $\alpha \in \mathcal{D}_0$ there is an associate set $\mathcal{D}\,(\alpha)$, which denotes all available damage values starting from current damage step (α) as follows:

$$\mathcal{D}\ (\alpha) = \{\beta\ \in H^1(\Omega)\quad : \alpha \le \beta \le \alpha_m \qquad in\ \Omega\} \tag{36}$$

Total energy $\mathcal{E}_t\ (\boldsymbol{v},\beta)$ of damaging body at time t associated with the pair $(\boldsymbol{v},\beta)\ \in\ \mathcal{C}_t \times \mathcal{D}_0$ in admissible fields read:

$$\mathcal{E}_t\ (\boldsymbol{v},\beta) = \int_\Omega \mathrm{W}_0(\boldsymbol{\varepsilon}(\boldsymbol{v}),\beta)dx - W_t^e(\boldsymbol{v}) \tag{37}$$

In Eq. (37), $\varepsilon(\boldsymbol{v})$ is the symmetric part of the gradient of displacement field and $W_t^e(\boldsymbol{v})$ is the work done by the external forces at time t.

$$W_t^e(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}_t.\boldsymbol{v}\ d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\boldsymbol{v}\ d\Gamma \tag{38}$$

Substituting Eq. (1), Eq. (13) and Eq. (38) into Eq. (37) leads to:

$$\mathcal{E}_t\ (\boldsymbol{v},\beta) = \int_\Omega (\frac{1}{2}\ \mathbb{E}(\beta)\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{v}) + \omega(\beta))d\Omega - \int_\Omega \boldsymbol{f}_t.\boldsymbol{v}\ d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t.\boldsymbol{v}\ d\Gamma \tag{39}$$

It is assumed that at time 0 (beginning of the loading) the body is undamaged, is free of all type of external forces and body forces. Therefore, the body is in its natural reference configuration at time 0. These assumptions can be written in the following form:

$$\boldsymbol{U}_0 = \boldsymbol{0},\ \ \boldsymbol{F}_0 = \boldsymbol{0},\ \ \boldsymbol{f}_0 = \boldsymbol{0},\ \ \boldsymbol{u}_0 = \boldsymbol{0}\ and\ \boldsymbol{\sigma}_0 = \boldsymbol{0}$$

Previous assumptions and definitions enable us to formulate the evolution problem of standard models, which is outlined in the following section.

1.4.3 Damage evolution problem (Strong Formulation)

The evolution problem for time $t > 0$ is to find a pair $(\boldsymbol{u}_t,\alpha_t)\ \in\ \mathcal{C}_t \times \mathcal{D}_0$ satisfying the following three items:

$$\begin{aligned} &\text{Irreversibility (ir):} && \dot{\alpha}_t \geq 0 \\ &\text{Stability (st):} && \mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v} - \boldsymbol{u}_t, \beta - \alpha_t) \geq 0 \quad \forall\, (\boldsymbol{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}(\alpha_t) \\ &\text{Energy balance (eb):} && \mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{0}, \dot{\alpha}_t) = 0 \end{aligned} \tag{40}$$

Following observations are deduced by looking at the three items in Eq. (40). It can be seen that since both $\boldsymbol{v}$ and $\boldsymbol{u}_t$ are in admissible set $\mathcal{C}_t$ (Eq. (33)), therefor $\boldsymbol{v} - \boldsymbol{u}_t$ would be in associated linear space $\mathcal{C}_0$ (Eq. (34)) which simply means this perturbation does not impose any external displacement on boundary. This condition is a necessity of a variational formulation of the problem. Also, β is in admissible set $\mathcal{D}(\alpha_t)$ which is defined in Eq. (36). Therefore, the increment $\beta - \alpha_t$ is positive ($\beta - \alpha_t \geq 0$) which implies that the damage level increases and it is an irreversible process. It is obvious that irreversibility and energy balance depends on rate of fields but the stability condition involves the state of the body at a given time t. In the problem stated above (Eq. (40)), $\mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta)$ denotes the directional derivative (Gateaux derivative) of $\mathcal{E}_t(\boldsymbol{u}_t, \alpha_t)$ in the direction of $(\boldsymbol{v}, \beta)$ which is defined as follows:

$$\mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) = \lim_{\gamma \to 0} \frac{d}{d\gamma} \mathcal{E}_t(\boldsymbol{u}_t + \gamma \boldsymbol{v}, \alpha_t + \gamma \beta) \tag{41}$$

For standard models with total energy in the form of Eq. (39) and aforementioned assumptions, directional derivative (Eq. (41)) reads:

$$\begin{aligned} &\mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) \\ &\quad = \int_\Omega \left(\boldsymbol{\sigma}_t \,.\, \boldsymbol{\varepsilon}(\boldsymbol{v}) + \left(\frac{1}{2}\, \mathbb{E}'(\alpha_t) \boldsymbol{\varepsilon}(\boldsymbol{u}_t).\, \boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\right) \beta\right) d\Omega \\ &\quad - W_t^e(\boldsymbol{v}) \end{aligned} \tag{42}$$

Proof

Eq. (39) indicates the total energy of the system at current state $(\boldsymbol{u}, \alpha)$:

$$\mathcal{E}_t\,(\boldsymbol{u}, \alpha) = \int_{\Omega} (\frac{1}{2}\,\mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega(\alpha))d\Omega - \int_{\Omega} \boldsymbol{f}_t.\boldsymbol{u}\,d\Omega - \int_{\partial_{\mathrm{F}}\Omega} \boldsymbol{F}_t.\boldsymbol{u}\,d\Gamma$$

Therefore, total energy in the direction of $(\boldsymbol{v}, \beta)$ with perturbation γ reads:

$$\mathcal{E}_t(\boldsymbol{u} + \gamma\boldsymbol{v}, \alpha + \gamma\beta)$$

$$= \int_{\Omega} (\frac{1}{2}\,\mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + \omega(\alpha + \gamma\beta))\,d\Omega$$

$$- \int_{\Omega} \boldsymbol{f}_t\,.(\boldsymbol{u} + \gamma\boldsymbol{v})d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t\,.(\boldsymbol{u} + \gamma\boldsymbol{v})d\Gamma$$

Small strain is assumed, therefore, strain field is a symmetric part of displacement field as:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Since strain is linear (it contains first order derivatives of displacement), its perturbation reads:

$$\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) = \frac{1}{2}\big((u + \gamma v)_{i,j} + (u + \gamma v)_{i,j}\big) = \frac{1}{2}\big(u_{i,j} + \gamma v_{i,j} + u_{j,i} + \gamma v_{j,i}\big)$$
$$= \boldsymbol{\varepsilon}(\boldsymbol{u}) + \gamma\boldsymbol{\varepsilon}(\boldsymbol{v})$$

If the prime sign stands for derivative with respect to γ, derivative of strain field reads:

$$\big(\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v})\big)' = \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) + \gamma\boldsymbol{\varepsilon}(\boldsymbol{v})\big)' = \boldsymbol{\varepsilon}(\boldsymbol{v})$$

Accordingly, derivative of each term in total energy ($\mathcal{E}_t(\boldsymbol{u} + \gamma\boldsymbol{v}, \alpha + \gamma\beta)$) with respect to

γ can be calculated as follows:

$$\big(\mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v})\big)' = \big(\mathbb{E}(\alpha + \gamma\beta)\big)'\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + \mathbb{E}(\alpha + \gamma\beta)\big(\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v})\big)'.\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + \mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\big(\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v})\big)' = \beta\,\mathbb{E}'(\alpha + \gamma\beta)\,\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + \mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + \mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{v}) = \beta\,\mathbb{E}'(\alpha + \gamma\beta)\,\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v}) + 2\,\mathbb{E}(\alpha + \gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u} + \gamma\boldsymbol{v})$$

$$(\omega(\alpha + \gamma\beta))' = \beta\,\omega'(\alpha + \gamma\beta)$$

$$\left(\int_{\Omega} \boldsymbol{f}_t\,.(\boldsymbol{u} + \gamma\boldsymbol{v})d\Omega\right)' = \int_{\Omega} \boldsymbol{f}_t\,.\boldsymbol{v}d\Omega$$

$$\left(\int_{\partial_F\Omega} \boldsymbol{F}_t\,.(\boldsymbol{u} + \gamma\boldsymbol{v})d\Gamma\right)' = \int_{\partial_F\Omega} \boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma$$

Finally, by putting previous results all together and take the limit, the directional derivative can be obtained:

$$\mathcal{E}_t'\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta) = \int_{\Omega}\left(\frac{1}{2}\beta\,\mathbb{E}'(\alpha)\,\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\,\omega'(\alpha)\right)d\Omega - \int_{\Omega}\boldsymbol{f}_t\,.\boldsymbol{v}d\Omega - \int_{\partial_F\Omega}\boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma$$

Considering the stress at current step as $\boldsymbol{\sigma} = \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u})$ and the definition of work done by external forces (Eq. (38)), previous equation can be written as:

$$\mathcal{E}_t'\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta) = \int_{\Omega}\left(\boldsymbol{\sigma}\,.\boldsymbol{\varepsilon}(\boldsymbol{v}) + \left(\frac{1}{2}\,\mathbb{E}'(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega'(\alpha)\right)\beta\right)d\Omega \;-\; W_t^e(\boldsymbol{v})$$

Which is the same as Eq. (42) and the proof is complete.

Based on Eq. (42), two different cases can be considered. The first one is the case that damage does not occur. Therefore, $\beta = 0$, $\boldsymbol{v} = \boldsymbol{u}_t$ and Eq. (41) and Eq. (42) implies that:

$$\mathcal{E}_t'\,(\boldsymbol{u}_t,\alpha_t)(\boldsymbol{u}_t,0) = \int_{\Omega}\boldsymbol{\sigma}_t\,.\boldsymbol{\varepsilon}(\boldsymbol{u}_t)d\Omega \;-\; W_t^e(\boldsymbol{u}_t) \geq 0 \qquad (43)$$

Inequality (43) is equivalent to the equilibrium equation in variational form i.e. equality holds. To show this case, integrating by parts of the first term in inequality (43) and using the definition given in Eq. (38) leads to:

$$\int_{\Omega}\boldsymbol{\sigma}_t\,.\boldsymbol{\varepsilon}(\boldsymbol{u}_t)d\Omega \;-\; W_t^e(\boldsymbol{u}_t) = \int_{\partial_F\Omega}\boldsymbol{F}_t\,.\boldsymbol{u}_t d\Gamma - \int_{\Omega} div\,\boldsymbol{\sigma}_t\,.\boldsymbol{u}_t d\Omega - \int_{\Omega}\boldsymbol{f}_t.\boldsymbol{u}_t\,d\Omega - \int_{\partial_F\Omega}\boldsymbol{F}_t.\boldsymbol{u}_t\,d\Gamma$$

Using equilibrium equation given in Eq. (32) in previous relation leads to:

$$\mathcal{E}_t'\,(\boldsymbol{u}_t,\alpha_t)(\boldsymbol{u}_t,0) = 0$$

The other case is when damage occurs $\beta \neq 0,\ \boldsymbol{v} \neq 0$, one can obtain the following result by substituting Eq. (43) into Eq. (42):

$$\mathcal{E}_t'\,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) = \int_{\Omega} \left(\frac{1}{2}\,\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\right) \beta\, d\Omega \quad \geq 0 \qquad (44)$$

Eq. (44) shows the damage criterion for local models.

1.5 The revised formulation of the evolution problem

For path independent systems (conservative systems), stable states go through the local minimum energy. This concept can be extended for systems in which energy is dissipated through the process like a damage model. Following definition is introducing a new concept (directional stability) based on local minima of energy function.

1.5.1 Directional stability

At a given time t a state $(\boldsymbol{u}, \alpha)$ of the body is called stable if the state is in admissible set and if in any accessible direction, there exist a neighborhood where every accessible direction has an energy which is no less than the energy of the state $(\boldsymbol{u}, \alpha)$.

$$\text{Global stability (ST)} \quad \begin{cases} \forall\,(\boldsymbol{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}\,(\alpha_t) \quad \exists \bar{h} > 0 \quad \forall h \in [0, \bar{h}] \\ \mathcal{E}_t(\boldsymbol{u} + h(\boldsymbol{v} - \boldsymbol{u}), \alpha + h(\beta - \alpha)) \;\geq\; \mathcal{E}_t(\boldsymbol{u}, \alpha) \end{cases} \qquad (45)$$

Expansion of $\mathcal{E}_t(\boldsymbol{u}, \alpha)$ to first order results in:

$$\mathcal{E}_t(\boldsymbol{u} + h(\boldsymbol{v} - \boldsymbol{u}), \alpha + h(\beta - \alpha)) \;\approx\; \mathcal{E}_t(\boldsymbol{u}, \alpha) + h\,\mathcal{E}_t'\,(\boldsymbol{u}, \alpha)(\boldsymbol{v} - \boldsymbol{u}, \beta - \alpha) \qquad (46)$$

Substituting Eq. (46) into Eq. (45), dividing by $h > 0$ and passing the limit when h approaches to zero lead to the local stability condition in Eq. (40). Therefore, local stability (Eq. (40)) can be obtained from Global stability (Eq. (45)). It worth to mention that these two stability conditions are not the same specially in the case of stress –

softening behavior. Global stability enables us to develop a new criterion for selection of the evolution direction in the cases where uniqueness of the solution fails.

1.5.2. The energy balance

Local energy balance (eb) (third item in Eq. (40)) states that during damage evolution elastic energy release rate is equal to dissipated power. This can be shown by the following analysis. Local energy balance reads:

$$\mathcal{E}_t' \,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{0}, \dot{\alpha}_t) = 0$$

Moreover, directional derivative (Eq. (42)) is as follows:

$$\mathcal{E}_t' \,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta)$$

$$= \int_{\Omega} \left(\boldsymbol{\sigma}_t \,.\, \boldsymbol{\varepsilon}(\boldsymbol{v}) + \left(\frac{1}{2}\,\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\right)\beta\right) d\Omega \;-\; W_t^e(\boldsymbol{v})$$

Considering a special state in which displacement field does not change, i.e. $\boldsymbol{v} = \boldsymbol{0}$ and looking at a damage level equal to increment of damage in admissible set, i.e. $\beta = \dot{\alpha}_t$ and substituting into directional derivative leads to:

$$\mathcal{E}_t' \,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{0}, \dot{\alpha}_t) = \int_{\Omega} \left(\frac{1}{2}\,\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\right) \dot{\alpha}_t \, d\Omega = 0$$

Based on Eq. (1), one can obtain:

$$\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}_t, \alpha_t) = \frac{1}{2}\,\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$$

Substituting back this result into local energy balance results in:

$$-\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}_t, \alpha_t)\dot{\alpha}_t = \omega'(\alpha_t)\dot{\alpha}_t$$

This is the same as Eq. (19) and Eq. (22). Local energy balance can be extended to Global energy balance as stated in the following section.

1.5.3 Global energy balance

During smooth damage evolution *i.e.* $t \longmapsto \alpha_t$ is absolutely continuous, the evolution of the total energy satisfies the following global energy balance:

$$\text{(EB)} \qquad \begin{aligned} \mathcal{E}_t(\boldsymbol{u}_t, \alpha_t) = {} & \mathcal{E}_t(\boldsymbol{u}_0, \alpha_0) \\ & + \int_0^t \left(\int_\Omega \boldsymbol{\sigma}_{t'} \,.\, \boldsymbol{\varepsilon}\left(\dot{\boldsymbol{U}}_{t'}\right) d\Omega - W^e_{t'}\left(\dot{\boldsymbol{U}}_{t'}\right) \right. \\ & \left. - \dot{W}^e_{t'}(\boldsymbol{u}_{t'}) \right) dt' \end{aligned} \qquad (47)$$

Where $\dot{\boldsymbol{U}}_{t'}$ and $\dot{W}^e_{t'}$ denote the rate of applied displacement field over the boundary and the rate of work done by the external forces respectively. Therefore, $\dot{W}^e_t(\boldsymbol{u}_t)$ reads:

$$\dot{W}^e_t(\boldsymbol{u}_t) = \int_\Omega \dot{\boldsymbol{f}}_t . \boldsymbol{u}_t \, d\Omega + \int_{\partial_F \Omega} \dot{\boldsymbol{F}}_t . \boldsymbol{u}_t \, d\Gamma$$

Proof

Total energy (Eq. (39)) reads:

$$\mathcal{E}_t\,(\boldsymbol{u}_t, \alpha_t) = \int_\Omega \left(\frac{1}{2}\, \mathbb{E}(\alpha_t) \boldsymbol{\varepsilon}(\boldsymbol{u}_t) . \boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega(\alpha_t)\right) d\Omega - \int_\Omega \boldsymbol{f}_t . \boldsymbol{u}_t \, d\Omega - \int_{\partial_F \Omega} \boldsymbol{F}_t . \boldsymbol{u}_t \, d\Gamma$$

Small strain is assumed, therefore, strain field is a symmetric part of displacement field as:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Therefore, time derivative of strain field reads:

$$\frac{d}{dt}(\boldsymbol{\varepsilon}(\boldsymbol{u})) = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) = \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)$$

$\mathbb{E}(\alpha_t)$ and $\omega(\alpha_t)$ are time independent functions. Therefore, time derivative of first term considering that $\boldsymbol{\sigma}_t = \mathbb{E}(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$ reads:

$$\frac{d}{dt}\left(\frac{1}{2}\, \mathbb{E}(\alpha_t) \boldsymbol{\varepsilon}(\boldsymbol{u}_t) . \boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega(\alpha_t)\right) = \boldsymbol{\sigma}_t . \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)$$

Using previous result in the derivative of total energy with respect to time results in:

$$\frac{d}{dt}\mathcal{E}_t\,(\boldsymbol{u}_t, \alpha_t) = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)\,d\Omega - \int_\Omega \dot{\boldsymbol{f}}_t.\boldsymbol{u}_t\,d\Omega - \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{u}}_t\,d\Omega$$

$$- \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t.\boldsymbol{u}_t\,d\Gamma - \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{u}}_t\,d\Gamma \qquad \text{(I)}$$

Integration by part for the first term in the above relation reads:

$$\int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)\,d\Omega = -\int_\Omega div\,\boldsymbol{\sigma}_t.\dot{\boldsymbol{u}}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{u}}_t\,d\Gamma + \int_{\partial_D\Omega} (\boldsymbol{\sigma}_t.n).\dot{\boldsymbol{U}}_t\,d\Gamma \qquad \text{(II)}$$

Applying integration by parts on the last term in the above equation reads:

$$\int_{\partial_D\Omega} (\boldsymbol{\sigma}_t.n).\dot{\boldsymbol{U}}_t\,d\Gamma = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega + \int_\Omega div\,\boldsymbol{\sigma}_t.\dot{\boldsymbol{U}}_t\,d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma \qquad \text{(III)}$$

Equilibrium equation reads:

$$div\,\boldsymbol{\sigma}_t + \boldsymbol{f}_t = 0 \qquad in\ \Omega$$

$$\boldsymbol{\sigma}_t.n = \boldsymbol{F}_t \qquad on\ \partial_F\Omega$$

Substituting equilibrium equation into Eq. (II) and Eq. (III) respectively results in:

$$\int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)\,d\Omega = \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{u}}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{u}}_t\,d\Gamma + \int_{\partial_D\Omega} (\boldsymbol{\sigma}_t.n).\dot{\boldsymbol{U}}_t\,d\Gamma \qquad \text{(IV)}$$

$$\int_{\partial_D\Omega} (\boldsymbol{\sigma}_t.n).\dot{\boldsymbol{U}}_t\,d\Gamma = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega - \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{U}}_t\,d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma \qquad \text{(V)}$$

Substituting Eq. (V) into Eq. (IV) leads to:

$$\int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)\,d\Omega = \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{u}}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{u}}_t\,d\Gamma + \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega -$$

$$\int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{U}}_t\,d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma \qquad \text{(VI)}$$

Finally, substituting Eq. (VI) into Eq. (I) gives the result as:

$$\frac{d}{dt}\mathcal{E}_t\,(\boldsymbol{u}_t,\alpha_t) = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega - \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{U}}_t\,d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma - \int_\Omega \dot{\boldsymbol{f}}_t.\boldsymbol{u}_t\,d\Omega - \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t.\boldsymbol{u}_t\,d\Gamma \qquad \text{(VII)}$$

The work done by external forces (Eq. (38)) reads:

$$W_t^e(\boldsymbol{u}_t) = \int_\Omega \boldsymbol{f}_t.\boldsymbol{u}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\boldsymbol{u}_t\,d\Gamma$$

This implies that:

$$W_t^e(\dot{\boldsymbol{U}}_t) = \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{U}}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma$$

And rate of prescribed loading as mentioned before reads:

$$\dot{W}_t^e(\boldsymbol{u}_t) = \int_\Omega \dot{\boldsymbol{f}}_t.\boldsymbol{u}_t\,d\Omega + \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t.\boldsymbol{u}_t\,d\Gamma$$

Substituting these terms into (VII) concludes that:

$$\frac{d}{dt}\mathcal{E}_t\,(\boldsymbol{u}_t,\alpha_t) = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega - W_t^e(\dot{\boldsymbol{U}}_t) - \dot{W}_t^e(\boldsymbol{u}_t)$$

The proof is complete.

Global energy balance (EB) has a specific advantage comparing to local energy balance (eb). Both are equivalent when the damage evolution is smooth (continuous) in time, but since global energy balance concerns about only the loading through the terms $t \longmapsto \boldsymbol{U}_t$ and $t \longmapsto W_t^e$ and not the response it can be used to formulate the problem when the damage evolution is not continuous.

1.5.4 The revised formulation

The new evolution problem can be obtained by changing the local stability item (st) with its more restrictive version, global stability (ST) and local energy balance (eb) with its new extended version, global energy balance (EB) in Eq. (40) as follows:

$$\begin{cases} (IR) : t \longmapsto \alpha_t \; must\ be\ non\ decreasing \\ (ST) : \; (\boldsymbol{u}_t, \alpha_t) \; must\ be\ stable\ in\ the\ sense\ of\ Eq. (43) \\ (EB) : The\ energy\ balance\ must\ be\ satisfied\ in\ the\ sense\ of\ Eq. (45) \end{cases} \tag{46}$$

Although the proposed formulation is more restrictive than the standard models since it admits only stable states but it does have the benefit of handling the problems when the damage evolution is not continuous.

1.6 Introduction of damage gradient term to the model

In order to penalize the localization of the damage, damage gradient term is introduced to the strain work function. Therefore, the gradient damage vector ($\nabla\alpha$) is considered as another internal state variable in addition to the damage variable (α) at the given material point. The strain work changes to the following state function:

$$\mathsf{W} : \; \boldsymbol{M}_s^n \times [0, \alpha_m] \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \longmapsto \mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha).$$

Taylor expansion of a multivariable function to approximate its value in the neighborhood of a desired point is valid only if the distance between points is small. Since the change in stiffness of the material is not small when the damage evolves from the beginning ($\alpha = 0$) to the final value ($\alpha = \alpha_m$), strain work function can not be expanded on damage variable. In other words, the range of change in strain work due to change in damage variable is much larger than the change due to strain and damage gradient. Therefore, the point $(0, \alpha, 0)$ is considered as "*thermo-dynamical equilibrium state*" and the strain work function ($\mathsf{W}(\varepsilon, \alpha, \nabla\alpha)$) is expanded up to the second order derivatives on both $\boldsymbol{\varepsilon}$ and $\nabla\alpha$ using the following truncated Taylor expansion of two variable function near the equilibrium state $(0, \alpha, 0)$:

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b))$$

Therefore, strain work function reads:

$$\mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \approx \mathrm{W}(\mathbf{0}, \alpha, 0) + \frac{\partial \mathrm{W}}{\partial \boldsymbol{\varepsilon}}(\mathbf{0}, \alpha, 0).\boldsymbol{\varepsilon} + \frac{\partial \mathrm{W}}{\partial \nabla\alpha}(\mathbf{0}, \alpha, 0).\nabla\alpha + \frac{1}{2!}(\frac{\partial^2 \mathrm{W}}{\partial \boldsymbol{\varepsilon}^2}(\mathbf{0}, \alpha, 0)\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \frac{\partial^2 \mathrm{W}}{\partial \nabla\alpha^2}(\mathbf{0}, \alpha, 0)\nabla\alpha\,.\nabla\alpha + 2\frac{\partial^2 \mathrm{W}}{\partial \boldsymbol{\varepsilon} \partial \nabla\alpha}(\mathbf{0}, \alpha, 0).(\boldsymbol{\varepsilon} \otimes \nabla\alpha)) \tag{47}$$

In Eq. (47), $\otimes$ stands for tensor product while inner product of same order tensors is denoted by dot. By looking at the definitions of elastic potential (Eq. (1)), definition of stress (Eq. (2)) and definition of strain work function (Eq. (13)), following results can be deduced:

$$\mathrm{W}(\mathbf{0}, \alpha, 0) = \omega(\alpha)$$

$$\frac{\partial \mathrm{W}}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \longrightarrow \frac{\partial \mathrm{W}}{\partial \boldsymbol{\varepsilon}}(\mathbf{0}, \alpha, 0) = \boldsymbol{\sigma}_0\,(\alpha)$$

$$\frac{\partial^2 \mathrm{W}}{\partial \boldsymbol{\varepsilon}^2} = \mathbb{E}(\alpha) \longrightarrow \frac{\partial^2 \mathrm{W}}{\partial \boldsymbol{\varepsilon}^2}(\mathbf{0}, \alpha, 0) = \mathbb{E}(\alpha)$$

Other tensors can be defined as follows:

$$\frac{\partial \mathrm{W}}{\partial \nabla\alpha}(\mathbf{0}, \alpha, 0) = \boldsymbol{\tau}(\alpha) \in \mathbb{R}^n$$

$$\frac{\partial^2 \mathrm{W}}{\partial \nabla\alpha^2}(\mathbf{0}, \alpha, 0) = \boldsymbol{\Gamma}\,(\alpha) \in \boldsymbol{M}_s^n$$

$$\frac{\partial^2 \mathrm{W}}{\partial \boldsymbol{\varepsilon} \partial \nabla \alpha} = \boldsymbol{\Lambda}(\alpha) \in \boldsymbol{M}_s^n \otimes \mathbb{R}^n$$

Using previous results and definitions into Eq. (47) leads to:

$$\mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \approx \omega(\alpha) + \boldsymbol{\sigma}_0(\alpha).\boldsymbol{\varepsilon} + \boldsymbol{\tau}(\alpha).\nabla\alpha + \frac{1}{2}\mathbb{E}(\alpha)\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\Gamma}(\alpha)\nabla\alpha.\nabla\alpha + \boldsymbol{\Lambda}(\alpha).(\boldsymbol{\varepsilon} \otimes \nabla\alpha) \tag{48}$$

$\boldsymbol{\sigma}_0(\alpha)$ denotes the stress in the absence of strain (damage dependent prestress) which was assumed equal to zero in the local model. Therefore, it is omitted in the following formulation as well:

$$\boldsymbol{\sigma}_0(\alpha) = \mathbf{0} \tag{49}$$

Furthermore, objectivity principle (frame invariance) states that:" The constitutive laws governing the internal conditions of a physical system and the interactions between its parts should not depend on whatever external frame of reference is used to describe them."(Noll, 2006). This principle means that if a scalar variable u, vector $\boldsymbol{p}$ and second order tensor $\boldsymbol{m}$ are shown by u^*, $\boldsymbol{p}^*$ and $\boldsymbol{m}^*$ in a moving frame respectively, following relations must be hold for any orthogonal transformation Q:

$$\boldsymbol{m}^* = Q\,\boldsymbol{m}\,Q^T, \quad \boldsymbol{p}^* = Q\,\boldsymbol{p}, \quad u^* = u \tag{50}$$

In Eq. (48), W and α are scalars, $\nabla\alpha$ is a vector and $\boldsymbol{\varepsilon}$ is a second order tensor. Therefore, applying objectivity principle leads to:

$$\mathrm{W}^*(\boldsymbol{\varepsilon}^*, \alpha^*, \nabla\alpha^*) = \mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \tag{51}$$

Substituting aforementioned transformations (Eq. (50)) results in the following relation, which must be hold for arbitrary orthogonal transformation:

$$\mathsf{W}(Q\boldsymbol{\varepsilon}Q^T, \alpha, Q\nabla\alpha) = \mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \quad \forall\, Q \in \mathbb{Q}^n, \quad \forall\, (\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \in \boldsymbol{M}_s^n \times [0, \alpha_m] \times \mathbb{R}^n \tag{52}$$

Substituting Eq. (49) and Eq. (50) in Eq. (48) leads to:

$$\omega(\alpha) + \boldsymbol{\tau}(\alpha). Q\nabla\alpha + \frac{1}{2}\mathbb{E}(\alpha)Q\boldsymbol{\varepsilon}Q^T. Q\boldsymbol{\varepsilon}Q^T + \frac{1}{2}\,\boldsymbol{\Gamma}\,(\alpha)Q\nabla\alpha\,.\,Q\nabla\alpha + \boldsymbol{\Lambda}\,(\alpha).(Q\boldsymbol{\varepsilon}Q^T \otimes Q\nabla\alpha) = \omega(\alpha) + \boldsymbol{\tau}(\alpha).\nabla\alpha + \frac{1}{2}\mathbb{E}(\alpha)\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \frac{1}{2}\,\boldsymbol{\Gamma}\,(\alpha)\nabla\alpha\,.\nabla\alpha + \boldsymbol{\Lambda}\,(\alpha).(\boldsymbol{\varepsilon} \otimes \nabla\alpha) \tag{53}$$

where $\boldsymbol{\Gamma}\,(\alpha)$ is a second order tensor. Eq. (53) is written in index notation as follows:

$$\tau_i Q_{ij}\nabla\alpha_j + \frac{1}{2}\,\mathbb{E}_{ijkl}\,Q_{pm}Q_{im}\varepsilon_{pj}Q_{qn}Q_{kn}\varepsilon_{ql} + \frac{1}{2}\Gamma_{ki}\,Q_{im}\nabla\alpha_m\,Q_{kj}\nabla\alpha_j + \Lambda_{ijk}\,Q_{pm}Q_{im}\varepsilon_{pj}\,Q_{kq}\nabla\alpha_q = \tau_i\nabla\alpha_i + \frac{1}{2}\,\mathbb{E}_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}\Gamma_{ki}\,\nabla\alpha_i\,\nabla\alpha_k + \Lambda_{ijk}\,\varepsilon_{ij}\,\nabla\alpha_k \tag{54}$$

Also, note that $Q_{pm}Q_{im} = \delta_{pi}$. Therefore, for second term in Eq. (54), one can write:

$$\frac{1}{2}\,\mathbb{E}_{ijkl}\,Q_{pm}Q_{im}\varepsilon_{pj}Q_{qn}Q_{kn}\varepsilon_{ql} = \frac{1}{2}\,\mathbb{E}_{ijkl}\,\delta_{pi}\varepsilon_{pj}\delta_{qk}\varepsilon_{ql} = \frac{1}{2}\,\mathbb{E}_{pjql}\,\varepsilon_{pj}\varepsilon_{ql} = \frac{1}{2}\,\mathbb{E}_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl}$$

Moreover, consider that:

$$\nabla\alpha_i = \delta_{mi}\nabla\alpha_m, \qquad \nabla\alpha_k = \delta_{jk}\nabla\alpha_j$$

Substituting previous relations in the third term and equating those terms results in:

$$\frac{1}{2}\Gamma_{ki}\,Q_{im}\nabla\alpha_m\,Q_{kj}\nabla\alpha_j = \frac{1}{2}\Gamma_{ki}\,\delta_{mi}\nabla\alpha_m\,\delta_{jk}\nabla\alpha_j$$

Therefore,

$$(\Gamma_{ki}\, Q_{im}\, Q_{kj} - \Gamma_{ki}\, \delta_{mi}\delta_{jk})\nabla\alpha_m\nabla\alpha_j = 0$$

Since damage gradient vector is arbitrary, previous relation implies that:

$$\Gamma_{ki}\, Q_{im}\, Q_{kj} - \Gamma_{ki}\, \delta_{mi}\delta_{jk} = 0$$

The only way to satisfy the previous relation is considering the second order tensor $\mathbf{\Gamma}\,(\alpha)$ as multiplication of a scalar with an identity tensor ($\mathbf{\Gamma}(\alpha) = \ell(\alpha)\mathbf{I} \;\; \longrightarrow \Gamma_{ki} = \ell(\alpha)\delta_{ki}$). Substituting this relation into previous relation leads to:

$$\ell(\alpha)\delta_{ki}\, Q_{im}\, Q_{kj} - \ell(\alpha)\delta_{ki}\, \delta_{mi}\delta_{jk} = 0 \quad \longrightarrow \;\; Q_{km}\, Q_{kj} - \delta_{ki}\, \delta_{mi}\delta_{jk} = 0$$

$$\longrightarrow \;\; \delta_{mj} - \delta_{mj} = 0\; ok$$

Therefore, second and third terms in both sides of Eq. (54) are equal considering the fact that $\mathbf{\Gamma}(\alpha) = \ell(\alpha)\mathbf{I}$. Also note that:

$$\Lambda_{ijk}\, Q_{pm}Q_{im}\varepsilon_{pj} = \; \Lambda_{ijk}\, \delta_{ip}\varepsilon_{pj} = \Lambda_{ijk}\, \varepsilon_{ij}$$

Therefore, Eq. (54) can be summarized as:

$$\tau_i(Q_{ij} - \delta_{ij})\nabla\alpha_j + \Lambda_{ijk}\varepsilon_{ij}(\, Q_{kq} - \delta_{kq})\nabla\alpha_q = 0 \tag{55}$$

In general, $\varepsilon_{ij} \neq 0, \nabla\alpha_j \neq 0, Q_{ij} \neq \delta_{ij}$. Therefore, it concludes that:

$$\tau_i = \; \Lambda_{ijk} = 0 \;\; \longrightarrow \;\; \boldsymbol{\tau}(\alpha) = \mathbf{0}, \qquad \mathbf{\Lambda}\,(\alpha) = \mathbf{0}$$

In previous proof, it is shown that second order tensor $\mathbf{\Gamma}\,(\alpha)$ is proportional to identity tensor and acts as a scalar. Therefore, from now on second order tensor $\mathbf{\Gamma}\,(\alpha)$ is considered as $\mathbf{\Gamma}(\alpha) = \ell(\alpha)\mathbf{I}$. Finally, using all obtained results in Eq. (48) the strain work function for isotropic gradient damage models can be written as follows:

$$\mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = \omega(\alpha) + \frac{1}{2}\mathbb{E}(\alpha)\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \frac{1}{2}\,\ell(\alpha)\nabla\alpha\,.\,\nabla\alpha \tag{56}$$

It can be seen that the only difference between new definition of strain work function (Eq. (56)) and W_0 (Eq. (13)) is the involvement of scalar function through a quadratic form of a damage gradient term (∇α). Considering this fact, total energy of a gradient damage model for isotropic damage can be written in the following form and it shows that the effect of non local damage can be introduced to the formulation by adding a scalar function of damage gradient:

$$\mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = \mathrm{W}_0(\boldsymbol{\varepsilon}, \alpha) + \frac{1}{2}\,\ell(\alpha)\nabla\alpha\,.\,\nabla\alpha \tag{57}$$

It worth to mention that the dissipation function ($\omega(\alpha)$) indicates the density of dissipated energy during a homogeneous damage process.

1.7 The variational formulation of the damage evolution problem

As it is mentioned in standard local models, three principles of irreversibility, stability and energy balance can be used in the variational approach to the damage problem. Therefore, associated total energy $\mathcal{E}_t\,(\boldsymbol{v}, \beta)$ of the body at time t with an admissible pair $(\boldsymbol{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}_0$ can be defined by subtracting the work done by body forces and external forces from the integration of strain work of the system over the domain as following:

$$\mathcal{E}_t\,(\boldsymbol{v}, \beta) = \int_\Omega \mathrm{W}(\boldsymbol{\varepsilon}(\boldsymbol{v}), \beta, \nabla\beta)dx - W_t^e(\boldsymbol{v}) \tag{58}$$

Global stability (Eq. (43)) must hold:

$$\text{Global stability (ST)} \quad \begin{cases} \forall\,(\boldsymbol{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}\,(\alpha_t) \quad \exists\bar{h} > 0 \quad \forall h \in [0, \bar{h}] \\ \mathcal{E}_t(\boldsymbol{u} + h(\boldsymbol{v} - \boldsymbol{u}), \alpha + h(\beta - \alpha)) \geq \mathcal{E}_t(\boldsymbol{u}, \alpha) \end{cases} \tag{59}$$

Global energy balance (Eq. (44)) must hold as well:

$$\text{(EB)} \quad \begin{aligned} &\mathcal{E}_t(\boldsymbol{u}_t, \alpha_t) = \\ &\mathcal{E}_t(\boldsymbol{u}_0, \alpha_0) + \int_0^t \left(\int_\Omega \boldsymbol{\sigma}_{t'}\,.\,\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{U}}_{t'}\right)d\Omega - W_{t'}^e\left(\dot{\boldsymbol{U}}_{t'}\right) - \dot{W}_{t'}^e(\boldsymbol{u}_{t'})\right)dt' \end{aligned} \tag{60}$$

Therefore, the evolution problem reads as:

$$\begin{cases} (IR) : t \ \longmapsto \alpha_t \ must\ be\ non\ decreasing \\ (ST) : \ (\boldsymbol{u}_t, \alpha_t)\ must\ be\ stable\ in\ the\ sense\ of\ Eq. (59) \\ (EB) : The\ energy\ balance\ must\ be\ satisfied\ in\ the\ sense\ of\ Eq. (60) \end{cases} \tag{61}$$

Local problem (Eq. (46)) and variational problem (Eq. (61)) do not admit the same solutions because of appearance of damage gradient term in strain work function although they are formally the same. In the following section properties of gradient damage model through variational approach is investigated.

1.7.1 First order optimality condition

Dividing global stability (Eq. (59)) by $h > 0$ and evaluating the result when h tends to zero ($h \longrightarrow 0$) leads to the same definition of first order optimality condition at time t as it was derived for local models (Eq. (43)):

$$\mathcal{E}_t' \ (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v} - \boldsymbol{u}_t, \beta - \alpha_t) \geq 0 \qquad \forall \ (\boldsymbol{v}, \beta) \in \mathcal{C}_t \times \ \mathcal{D}\ (\alpha_t) \tag{62}$$

Where $\mathcal{E}_t' \ (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta)$ denotes the directional derivative (Gateaux derivative) of $\mathcal{E}_t$ at $(\boldsymbol{u}_t, \alpha_t)$ in the $(\boldsymbol{v}, \beta)$ direction which is defined as follows:

$$\mathcal{E}_t' \ (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) = \lim_{\gamma \longrightarrow 0} \frac{d}{d\gamma} \ \mathcal{E}_t(\boldsymbol{u}_t + \gamma \boldsymbol{v}, \alpha_t + \gamma \beta) \tag{63}$$

For gradient damage models with total energy in the form of Eq. (58) and aforementioned assumptions, directional derivative (Eq. (63)) reads:

$$\begin{aligned} &\mathcal{E}_t' \ (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) \\ &\qquad = \int_\Omega \ ((\boldsymbol{\sigma}_t \,.\, \boldsymbol{\varepsilon}(\boldsymbol{v}) + \ \frac{1}{2} \ \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\, \boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t))\beta \\ &\qquad + \ell \ \nabla\alpha_t \,.\, \nabla\beta) d\Omega \ \ - \ \ W_t^e(\boldsymbol{v}) \end{aligned} \tag{64}$$

Proof

Eq. (56) indicates the total energy of the system at current state $(\boldsymbol{u}, \alpha)$:

$$\mathcal{E}_t\,(\boldsymbol{u},\alpha) = \int_\Omega (\mathrm{W}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\alpha,\nabla\alpha))d\Omega - W_t^e(\boldsymbol{u})$$

Therefore, total energy in the direction of $(\boldsymbol{v},\beta)$ with perturbation γ reads:

$$\mathcal{E}_t(\boldsymbol{u}+\gamma\boldsymbol{v},\alpha+\gamma\beta) = \int_\Omega (\mathrm{W}(\boldsymbol{\varepsilon}(\boldsymbol{u}+\gamma\boldsymbol{v}),(\alpha+\gamma\beta),\nabla(\alpha+\gamma\beta)))d\Omega - W_t^e(\boldsymbol{u}+\gamma\boldsymbol{v}) =$$

$$\int(\tfrac{1}{2}\,\mathbb{E}(\alpha+\gamma\beta)\boldsymbol{\varepsilon}(\boldsymbol{u}+\gamma\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u}+\gamma\boldsymbol{v}) + \omega(\alpha+\gamma\beta) + \tfrac{1}{2}\,\ell\nabla(\alpha+\gamma\beta)\,.\nabla(\alpha+\gamma\beta))\,d\Omega\, -$$

$$\int_\Omega \boldsymbol{f}_t\,.(\boldsymbol{u}+\gamma\boldsymbol{v})d\Omega - \int_{\partial_{\mathrm{F}}\Omega}\boldsymbol{F}_t\,.(\boldsymbol{u}+\gamma\boldsymbol{v})d\Gamma$$

Comparing previous equation with the total energy of the local models (Eq. (42)) shows that the only additional term is $\frac{1}{2}\,\ell\nabla(\alpha+\gamma\beta)\,.\nabla(\alpha+\gamma\beta)$. Therefore, directional derivative of this term is calculated and added to the previous terms of local models as follows (prime denotes the derivative with respect to γ):

$$\left(\frac{1}{2}\,\ell\nabla(\alpha+\gamma\beta)\,.\nabla(\alpha+\gamma\beta)\right)'$$

$$= \frac{1}{2}\,\ell\,\left[\left(\nabla(\alpha+\gamma\beta)\right)'\,.\nabla(\alpha+\gamma\beta) + \nabla(\alpha+\gamma\beta).\left(\nabla(\alpha+\gamma\beta)\right)'\right]$$

$$= \frac{1}{2}\,\ell\,\left[\nabla\beta.\nabla(\alpha+\gamma\beta) + \nabla(\alpha+\gamma\beta).\nabla\beta\right]$$

Taking the limit of the previous term, when γ tends to zero results in:

$$\lim_{\gamma\to 0}\frac{1}{2}\,\ell\,\left[\nabla\beta.\nabla(\alpha+\gamma\beta) + \nabla(\alpha+\gamma\beta).\nabla\beta\right] = \ell\,\nabla\alpha.\nabla\beta$$

Finally, by putting results of other terms in local model and new gradient term all together, the directional derivative can be obtained:

$$\mathcal{E}_t'\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta) = \int\left(\frac{1}{2}\,\mathbb{E}'(\alpha)\,\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega'(\alpha))\beta + \ell\,\nabla\alpha.\nabla\beta\right)d\Omega - \int_\Omega \boldsymbol{f}_t\,.\boldsymbol{v}d\Omega - \int_{\partial_{\mathrm{F}}\Omega}\boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma$$

Considering the stress at current step as $\boldsymbol{\sigma} = \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u})$ and the definition of work done by external forces (Eq. (38)), previous equation can be written as:

$$\mathcal{E}'_t\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta)$$

$$= \int_\Omega \left(\boldsymbol{\sigma}\,.\,\boldsymbol{\varepsilon}(\boldsymbol{v}) + \frac{1}{2}\,\mathbb{E}'(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u}).\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega'(\alpha)\right)\beta + \ell\;\nabla\alpha.\nabla\beta\;)d\Omega$$

$$-\;W_t^e(\boldsymbol{v})$$

Which is the same as Eq. (64) and the proof is complete.

Testing variational inequality (Eq. (62)) for the case that there is no damage at the specific level (i.e. $-\alpha_t = 0 \rightarrow \beta = \alpha_t$) with a displacement field $\overline{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{u}_t \in \mathcal{C}_0$ leads to the variational formulation of equilibrium equation at any damage level. In this case Eq. (62) reads:

$$\mathcal{E}'_t\,(\boldsymbol{u}_t,\alpha_t)(\overline{\boldsymbol{v}},0) \geq 0 \qquad \forall\;(\boldsymbol{v},\beta) \in \mathcal{C}_t \times\; \mathcal{D}\;(\alpha_t) \tag{65}$$

Using Eq. (64) and substituting $\beta = 0$ leads to:

$$\mathcal{E}'_t\,(\boldsymbol{u}_t,\alpha_t)(\overline{\boldsymbol{v}},0) = \int_\Omega \boldsymbol{\sigma}_t\,.\,\boldsymbol{\varepsilon}(\overline{\boldsymbol{v}})d\Omega \;-\; W_t^e(\overline{\boldsymbol{v}}) = 0 \qquad \forall\;\overline{\boldsymbol{v}} \in \mathcal{C}_0 \tag{66}$$

Substituting work done by external forces and the definition of stress ($\boldsymbol{\sigma}_t = \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$) in Eq. (66) leads to:

$$\mathcal{E}'_t\,(\boldsymbol{u}_t,\alpha_t)(\overline{\boldsymbol{v}},0) = \int_\Omega \mathbb{E}(\alpha)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\overline{\boldsymbol{v}})\mathrm{d}\Omega - \int_\Omega f_t\,.\,\overline{\boldsymbol{v}}d\Omega - \int_{\partial_F\Omega} F_t\,.\,\overline{\boldsymbol{v}}d\Gamma \tag{67}$$

Integration by parts on the first term in Eq. (67) leads to:

$$\mathcal{E}'_t\,(\boldsymbol{u}_t,\alpha_t)(\overline{\boldsymbol{v}},0) = -\int_\Omega (div\,\boldsymbol{\sigma}_t + f_t).\overline{\boldsymbol{v}}d\Omega \;+\; \int_{\partial_F\Omega}(\boldsymbol{\sigma}_t.n - F_t)\,.\,\overline{\boldsymbol{v}}d\Gamma \;\geq 0 \tag{68}$$

Since $\overline{\boldsymbol{v}}$ is an arbitrary displacement field, each integrand in Eq. (63) needs to be equal to zero in order to satisfy inequality. (68). This results in equilibrium equation with damaged elasticity tensor as:

$$div\, \boldsymbol{\sigma}_t + f_t = 0 \qquad on\ \Omega$$
$$\boldsymbol{\sigma}_t . n = F_t \qquad on\ \partial_F\Omega \tag{69}$$

Non local damage problem is the result of testing Eq. (62) for arbitrary damage value (β) in the convex cone $\mathcal{D}$ with $\boldsymbol{v} = \boldsymbol{u}_t$. This means that displacement field does not change but damage evolves which can be obtained by:

$$\mathcal{E}_t' (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{0}, \beta) \geq 0 \qquad \forall\ (\boldsymbol{0}, \beta) \in \mathcal{C}_0 \times \mathcal{D}\ (\alpha_t) \tag{70}$$

Using Eq. (64) into Eq. (70) leads to:

$$\mathcal{E}_t' (\boldsymbol{u}_t, \alpha_t)(\boldsymbol{0}, \beta) = \int_\Omega ((\frac{1}{2}\ \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t))\beta + \ell\, \nabla\alpha_t . \nabla\beta)\, d\Omega \geq 0 \tag{71}$$

Applying first Green formula on the last term of Eq. (71) leads to:

$$\int_\Omega \ell\, \nabla\alpha_t . \nabla\beta d\Omega = \ell(\int_{\partial_F\Omega} \beta \frac{\partial\alpha_t}{\partial n} d\Gamma - \int_\Omega \beta\nabla^2\alpha_t d\Omega) \tag{72}$$

∇^2 stands for Laplacian of a scalar i.e. $\nabla^2(b) = \nabla.\nabla(b)$. Substituting Eq. (72) into Eq. (71) results in:

$$\int_{\partial_F\Omega} \ell\beta \frac{\partial \alpha_t}{\partial n} d\Gamma + \int_{\Omega} \frac{1}{2}\, \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\beta d\Omega + \int_{\Omega} \omega'(\alpha_t)\beta d\Omega$$
$$- \int_{\Omega} \ell\beta\nabla^2\alpha_t d\Omega \geq 0 \tag{73}$$

Which must be hold for all $\beta \in \mathcal{D}$. Inequality (Eq. (73)) can be written in the following form:

$$\int_{\partial_F\Omega} \ell\beta \frac{\partial \alpha_t}{\partial n} d\Gamma + \int_{\Omega} (\frac{1}{2}\, \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ell\nabla^2\alpha_t)\beta d\Omega \geq 0 \tag{74}$$

In Eq. (74), n denotes the unit outer normal to Ω. Therefore, from the Eq. (74) one gets:

$$\ell \frac{\partial \alpha_t}{\partial n} \geq 0 \qquad on\ \partial_F\Omega$$
$$(\frac{1}{2}\, \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ell\nabla^2\alpha_t) \geq 0 \qquad in\ \Omega \tag{75}$$

Eq. (75) denotes the constitutive equations for gradient damage models which contains second spatial derivatives of the damage field over the domain comparing to the local damage formulation. Global energy balance (EB) which is given in Eq. (60) needs to be satisfied in the proposed variational formulation of the problem as it has been proved in the associated local model. Therefore, differentiating (EB) with respect to time t results in:

$$\frac{d}{dt}\mathcal{E}_t(\boldsymbol{u}_t, \alpha_t) = \frac{d}{dt}\, \mathcal{E}_t(\boldsymbol{u}_0, \alpha_0)$$
$$+ \frac{d}{dt}(\int_0^t (\int_{\Omega} \boldsymbol{\sigma}_{t'} \,.\, \boldsymbol{\varepsilon}\,(\dot{\boldsymbol{U}}_{t'}) d\Omega - W^e_{t'}(\dot{\boldsymbol{U}}_{t'}) - \dot{W}^e_{t'}(\boldsymbol{u}_{t'}))dt'\ ) \tag{76}$$

Eq. (76) leads to:

$$\frac{d}{dt}\mathcal{E}_t(\boldsymbol{u}_t,\alpha_t) - \int_\Omega \boldsymbol{\sigma}_t \,.\, \boldsymbol{\varepsilon}\left(\dot{\boldsymbol{U}}_t\right) d\Omega + W_t^e\left(\dot{\boldsymbol{U}}_t\right) + \dot{W}_t^e(\boldsymbol{u}_t) = 0 \tag{77}$$

Since small strain as symmetric part of the gradient of displacement field is assumed, following relation holds as it is shown before:

$$\frac{d}{dt}\boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t)$$

Differentiating $\mathcal{E}_t(\boldsymbol{u}_t,\alpha_t)$ with respect to time for gradient damage model using Eq. (56) and Eq. (58) leads to:

$$\frac{d}{dt}\mathcal{E}_t(\boldsymbol{u}_t,\alpha_t) = \frac{d}{dt}\left(\int_\Omega \left(\frac{1}{2}\mathbb{E}(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega(\alpha_t) + \frac{1}{2}\,\ell\nabla\alpha_t\,.\,\nabla\alpha_t\right) d\Omega - W_t^e(\boldsymbol{u}_t)\right) =$$

$$\int_\Omega \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \mathbb{E}(\alpha_t)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t)\dot{\alpha}_t + \ell\nabla\alpha_t\,.\,\nabla\dot{\alpha}_t\right) d\Omega -$$

$$W_t^e(\dot{\boldsymbol{u}}_t) - \int_\Omega \dot{\boldsymbol{f}}_t\,.\,\boldsymbol{u}_t d\Omega - \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t\,.\,\boldsymbol{u}_t d\Gamma = \int_\Omega \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \boldsymbol{\sigma}_t.\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t) +\right.$$

$$\left.\omega'(\alpha_t)\dot{\alpha}_t + \ell\nabla\alpha_t\,.\,\nabla\dot{\alpha}_t\right) d\Omega - W_t^e(\dot{\boldsymbol{u}}_t) - \int_\Omega \dot{\boldsymbol{f}}_t\,.\,\boldsymbol{u}_t d\Omega - \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t\,.\,\boldsymbol{u}_t d\Gamma$$

Substituting previous result into Eq. (77) and using the fact that $\dot{W}_t^e$ is the rate of prescribed loading (see Eq. (47)), leads to:

$$\int_\Omega \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \boldsymbol{\sigma}_t.\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_t) + \omega'(\alpha_t)\dot{\alpha}_t + \ell\nabla\alpha_t\,.\,\nabla\dot{\alpha}_t\right) d\Omega - W_t^e(\dot{\boldsymbol{u}}_t) -$$

$$\int_\Omega \dot{\boldsymbol{f}}_t\,.\,\boldsymbol{u}_t d\Omega - \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t\,.\,\boldsymbol{u}_t d\Gamma - \int_\Omega \boldsymbol{\sigma}_t\,.\,\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{U}}_t\right) d\Omega + W_t^e\left(\dot{\boldsymbol{U}}_t\right) + \dot{W}_t^e(\boldsymbol{u}_t) =$$

$$\int_\Omega \boldsymbol{\sigma}_t.\,\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{u}}_t - \dot{\boldsymbol{U}}_t\right) - W_t^e\left(\dot{\boldsymbol{u}}_t - \dot{\boldsymbol{U}}_t\right) + \int_\Omega \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \omega'(\alpha_t)\dot{\alpha}_t +\right.$$

$$\left.\ell\nabla\alpha_t\,.\,\nabla\dot{\alpha}_t\right) d\Omega = 0$$

In previous expansion, the first two terms cancel out because of the equilibrium in variational form (Eq. (66)). Therefore, Eq. (77) leads to the following result:

$$\int_{\Omega} \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \omega'(\alpha_t)\dot{\alpha}_t + \ell\nabla\alpha_t . \nabla\dot{\alpha}_t\right) = 0 \tag{78}$$

Applying integration by parts on the last term of Eq. (78) leads to:

$$\int_{\Omega} \ell \nabla\alpha_t . \nabla\dot{\alpha}_t d\Omega = \ell(\int_{\partial_F\Omega} \dot{\alpha}_t \frac{\partial\alpha_t}{\partial n} d\Gamma - \int_{\Omega} \dot{\alpha}_t \nabla^2 \alpha_t d\Omega) \tag{79}$$

Substituting Eq. (79) into Eq. (78) leads to:

$$\int_{\Omega} \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ell\nabla^2\alpha_t\right)\dot{\alpha}_t d\Omega + \int_{\partial_F\Omega} \ell\dot{\alpha}_t \frac{\partial\alpha_t}{\partial n} d\Gamma = 0 \tag{80}$$

Considering the irreversibility condition ($\dot{\alpha}_t \geq 0$) and the damage criterion for gradient model (Eq. (75)), following consistency conditions can be obtained from Eq. (80):

$$\begin{gathered} \ell\dot{\alpha}_t \frac{\partial\alpha_t}{\partial n} = 0 \qquad on\ \partial_F\Omega \\ \left(\frac{1}{2}\mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ell\nabla^2\alpha_t\right)\dot{\alpha}_t = 0 \qquad in\ \Omega \end{gathered} \tag{81}$$

Finally, derived constitutive equation, boundary conditions and damage criterion in the form of Kuhn-Tucker condition for gradient damage models can be summarized as follows by virtue of the variational formulation (Eq. (61)):

$$
\begin{aligned}
&\text{Equilibrium:} && div\, \boldsymbol{\sigma}_t + \boldsymbol{f}_t = 0 && in\ \Omega \\
&\text{Neumann boundary conditions:} && \boldsymbol{\sigma}_t . n = \boldsymbol{F}_t\ ,\ \ \ell \dot{\alpha}_t \frac{\partial \alpha_t}{\partial n} = 0\ ,\ \ell \frac{\partial \alpha_t}{\partial n} \geq 0 && on\ \partial_{\mathrm{F}}\Omega \\
&\text{Dirichlet boundary conditions:} && \boldsymbol{u}_t = \boldsymbol{U}_t && on\ \partial_{\mathrm{D}}\Omega \\
&\text{Constitutive relations:} && \boldsymbol{\sigma}_t = \mathbb{E}\,(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t) && in\ \Omega \\
&\text{Compatibility conditions:} && 2\boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \nabla \boldsymbol{u}_t + \nabla \boldsymbol{u}_t^T && in\ \Omega \\
&\text{Kuhn – Tucker conditions:} && \begin{cases} \dot{\alpha}_t \geq 0 \\ (\frac{1}{2}\ \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ell \nabla^2 \alpha_t) \geq 0 \\ \dot{\alpha}_t \left(\frac{1}{2} \mathbb{E}'(\alpha_t)\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - \ \ell \nabla^2 \alpha_t \right) = 0 \end{cases}
\end{aligned}
\tag{82}
$$

Eq. (82) indicates the complete set of equations, constitutive relation, boundary conditions and appropriate compatibility conditions for general class of gradient damage models through variational approach. It can be seen that all physical restrictions including thermodynamics principles such as irreversibility can be satisfied through this mathematical approach. Also, comparing the set equations (Eq. (82)) to standard local models (Eq. (32)) shows appropriate boundary conditions and the way that involvement of gradient term affects the formulation of the problem.

2. GRADIENT DAMAGE MODEL IN INFINITE DOMAIN WITH CIRCULAR HOLE

2.1 Introduction

In this chapter, specific problem is considered to show the application of the formulation which is presented in full details in previous chapter. The aim of this chapter is to develop a gradient damage model for an axisymmetric domain with a circular hole pressurized by internal pressure. This work is inspired by application of gradient damage models on 1D traction test (Benallal & Marigo, 2007; K. Pham & Marigo, 2011) and thermal shock problem (Bourdin, Marigo, Maurini, & Sicsic, 2014; Sicsic, Marigo, & Maurini, 2013). Therefore, this formulation can be considered as an application of gradient damage model in 2D problems. Therefore, we start to solve the problem for a ring shown in Fig. 2.1 and we intent to find the solution for an infinite domain, which can be obtained when the external radius tends to infinity and external pressure vanishes simultaneously. For the sake of generality, a ring under internal and external pressure is considered and wherever algebraic calculation becomes tedious two special cases are considered including 1- ring without external pressure and 2- infinite domain with circular hole (case in which outer radius tends to infinity and external pressure tends to zero simultaneously). Two different prescribed boundary conditions on internal and external circumference of the ring are considered including: 1- prescribed displacement 2- prescribed internal stress in terms of applied pressure on the boundary to complete the analysis and show practical applicability of the solution. Main nomenclature of this chapter is given in Table 2.

Table 2 - Main nomenclature of chapter two

State variables and state functions	
$\boldsymbol{u}$	Displacement field with components u_r , u_θ and u_z
$\boldsymbol{\varepsilon}$	Second order strain tensor with components ε_{ij}
$\boldsymbol{\sigma}$	Second order stress tensor with components σ_{ij}
$f(t)$	Function to represent applied internal pressure
$g(t)$	Function to represent applied external pressure
U_i	Internal applied displacement
U_{i-cr}	Critical internal applied displacement
U_e	External applied displacement
P_i	Internal applied pressure
P_{i-cr}	Critical internal applied pressure
P_e	External applied pressure
λ, μ	Lame constants
E	Modulus of elasticity
G	Shear modulus
ν	Poisson ratio
Y	Energy release rate - thermodynamic conjugate force due to damage
$\boldsymbol{q}$	Damage flux vector
σ_1 , σ_2 , σ_3	Principle stresses
Ω	Domain occupied by damaging material
$\partial_F \Omega$	Part of the boundary with Neumann BC’s

Table 2 continued

$\partial_D \Omega$	Part of the boundary with Dirichlet BC's
$\mathcal{C}$	Space of kinematically admissible displacement fields
H^1	Sobolev space
$\mathcal{D}$	Set of admissible damage fields
$\mathcal{E}\ (u, \alpha)$	Total energy of the system
W^e	External work done by external applied fields
$\mathcal{E}'(\boldsymbol{u}, \alpha)(\boldsymbol{v}, \beta)$	Directional derivative of $\mathcal{E}(\boldsymbol{u}, \alpha)$ in the direction of $(\boldsymbol{v}, \beta)$
$u_{i,j}$	$\partial u_i / \partial x_j$
$\dot{\boldsymbol{U}}$	Rate of applied displacement on boundary
$\dot{W}$	Rate of applied work done by external forces on boundary
$\omega(\alpha)$	Dissipation function
$S(\alpha)$	Function to describe reduction in stiffness due to damage

2.2 Problem setting

2.2.1 The body and its loading

We shall use general constraints for the problem as much as algebraic analysis allows us to do. The natural reference configuration of a ring under internal and external pressure is depicted in Fig. 4.

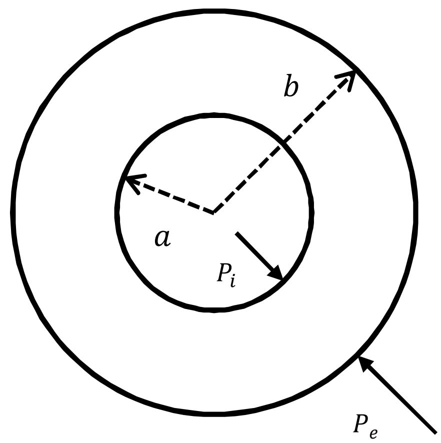

Figure 4 - Domain configuration and applied loading

Mathematical representation of the domain in cylindrical coordinate 2D system reads:

$$\Omega = \{(r, \theta) \mid a \le r \le b\} \tag{1}$$

It is assumed that internal radius of the body is much greater than the internal length (ℓ) of the material and body forces are neglected. Inner and outer circumference of the cylinder is subjected to given internal and external pressure in two different forms: 1- radial stresses and 2- radial displacements so other components are zero over the boundary. Therefore, mechanical boundary conditions read:

1- Prescribed displacement (Dirichlet boundary condition)

$$u_r(r = a) = U_i$$
$$u_r\,(r = b) = -U_e \tag{2-A}$$

$$u_\theta = 0 \qquad (r = a)$$

$$u_\theta = 0 \qquad (r = b)$$

2- Prescribed pressure (Neumann boundary condition)

$$\begin{aligned} \sigma_r(r = a) &= P_i \\ \sigma_r\,(r = b) &= -P_e \\ \sigma_\theta = \sigma_{r\theta} = 0 &\qquad (r = a) \\ \sigma_\theta = \sigma_{r\theta} = 0 &\qquad (r = b) \end{aligned} \tag{2-B}$$

It is obvious that the solution for the problem of infinite domain with circular hole can be obtained when the outer radius of the cylinder tend to infinity ($b \to \infty$) and external pressure approaches to zero ($U_e \to 0 \;\; or \; P_e \to 0$) simultaneously. Boundary condition is not imposed on the damage field at inner and outer boundary of the cylinder ($r = a$ and $r = b$) to enable it to evolve freely. The cylinder is at its reference configuration and neither internal pressure nor external pressure is applied at time 0 ($t = 0$). Therefore, displacement, strain and stress fields at time 0 read:

$$\boldsymbol{u}_t(r,\theta) = \boldsymbol{0}, \qquad \boldsymbol{\varepsilon}_t(r,\theta) = \boldsymbol{0}, \qquad \boldsymbol{\sigma}_t(r,\theta) = \boldsymbol{0} \qquad \forall\,(r,\theta) \in \Omega \quad \forall\, t \le 0 \tag{3}$$

From time 0, internal and external pressure are applied on the internal and external circumferences of the cylinder which are not functions of damage field. Since it is desirable to analyze the effects of internal and external pressure on damage field, it is assumed that both internal and external applied boundary conditions are monotonically increasing functions of time in the following form:

$$U_i = f(t)\,U_{it} \qquad U_{it} = const. \quad f(t) > 0, \qquad \dot{f} > 0$$
$$U_e = g(t)\,U_{et} \qquad U_{et} = const. \quad g(t) > 0, \qquad \dot{g} > 0 \tag{4-A}$$

$$P_i = f(t)\,P_{it} \qquad P_{it} = const. \quad f(t) > 0, \qquad \dot{f} > 0$$
$$P_e = g(t)\,P_{et} \qquad P_{et} = const. \quad g(t) > 0, \qquad \dot{g} > 0 \tag{4-B}$$

Throughout the formulation, superscript dot stands for derivative with respect to time. Eqs. (1), (2) and (4) imply that the problem definition is independent of θ, so axisymmetric properties hold. Moreover, it enables us to investigate the effects of time on evolution of damage in cylindrical domain caused by internal and external pressure. Therefore, strain field is just based on regular elasticity equations and is symmetrized part of the gradient of displacement field. Elastic solution of the problem for undamaged material is considered first and the solution is entered to the formulation of the damage problem in the subsequent sections.

2.2.2 Elastic solution of the problem

In this section, elasticity equations are solved for the cylinder and energy terms are calculated which are capable to be used in the simplest form into the damage problem formulation. Plane stress condition is assumed along with a homogeneous cylindrical domain made of isotropic damaging material. By symmetry of the problem, it is obvious that ($u_\theta = 0 \; at \; r = a \; and \; r = b$) and radial displacement is a function of radius only ($u_r = f(r)$). Therefore, displacement field and strain field reads:

$$\boldsymbol{u}_t(r, \theta, z) = \boldsymbol{u}_t\,(r) \tag{5}$$

$$\nabla \boldsymbol{u}_t = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \begin{bmatrix} \frac{\partial u_r}{\partial r} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Stress strain relation (Hook's law) can be written as follows:

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\delta_{ij} + 2\mu\, \varepsilon_{ij} \tag{8}$$

Where λ and μ are lame constants and are related to the modulus of elasticity and shear modulus as follows:

$$\mu = G = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{E\nu}{(1-\nu^2)} \tag{9}$$

Based on the components of strain field, stress field components can be calculated as follows:

$$\begin{gathered} \varepsilon_{kk} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \\ \sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{u_r}{r} \\ \sigma_{\theta\theta} = \lambda\frac{\partial u_r}{\partial r} + (\lambda + 2\mu)\frac{u_r}{r} \\ \sigma_{zz} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) \\ \sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0 \end{gathered} \tag{10}$$

Therefore, stress tensor is obtained in the following form:

$$\boldsymbol{\sigma} = \begin{bmatrix} (\lambda + 2\mu)\,\frac{\partial u_r}{\partial r} + \lambda\,\frac{u_r}{r} & 0 & 0 \\ 0 & \lambda\,\frac{\partial u_r}{\partial r} + (\lambda + 2\mu)\frac{u_r}{r} & 0 \\ 0 & 0 & \lambda\,(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}) \end{bmatrix} \tag{11}$$

Equilibrium equation for aforementioned problem (Cylinder under internal and external applied boundary conditions) reads:

$$div\,\boldsymbol{\sigma} = 0 \qquad in\,\Omega \quad \text{with appropriate boundary conditions (Eq. (2))} \tag{12}$$

Divergence operator in cylindrical coordinate system reads:

$$div\,\boldsymbol{\sigma} = \begin{bmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\ \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{\theta r}}{r} \\ \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} \end{bmatrix} \tag{13}$$

Since off diagonal components of stress tensor is equal to zero, equilibrium equation reduces to the following form for cylindrical problem:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{14}$$

Substituting stress components in terms of displacement from Eq. (10) into Eq. (14) leads to:

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} = 0 \tag{15}$$

Since radial displacement is a function of radius only, Eq. (15) is written as follows:

$$\frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0 \tag{16}$$

The general solution of Eq. (16) follows the following form:

$$u_r = C_1 r + \frac{C_2}{r} \tag{17}$$

Therefore, stress components read:

$$\sigma_{rr} = (2\lambda + 2\mu)C_1 - \frac{2\mu C_2}{r^2}$$
$$\sigma_{\theta\theta} = (2\lambda + 2\mu)C_1 + \frac{2\mu C_2}{r^2} \tag{18}$$
$$\sigma_{zz} = 2\lambda C_1$$

Applying boundary conditions (Eq. (2)) on Eq. (18) leads to find the constants for different cases as follows:

1- Prescribed displacement (Dirichlet boundary condition)

Substituting Eq. (2-A) into Eq. (17) leads to:

$$C_1 = -\frac{aU_i + bU_e}{(b^2 - a^2)}$$
$$C_2 = \frac{ab(bU_i + aU_e)}{(b^2 - a^2)} \tag{19}$$

In the case of prescribed displacement, coefficients in Eq. (19) reduce to the following form for the special cases.

Case 1: ring without external displacement ($U_e \rightarrow 0$)

$$C_1 = -\frac{aU_i}{(b^2 - a^2)}$$
$$C_2 = \frac{ab^2 U_i}{(b^2 - a^2)} \tag{20}$$

Case 2: infinite domain with circular hole ($U_e \rightarrow 0$ and $b \longrightarrow \infty$)

$$C_1 = 0$$
$$C_2 = aU_i \tag{21}$$

2- Prescribed pressure (Neumann boundary condition)

Substituting Eq. (2-B) into Eq. (17) leads to:

$$\sigma_{rr}(r = a) = (2\lambda + 2\mu)C_1 - \frac{2\mu C_2}{a^2} = P_i$$
$$\sigma_{rr}(r = b) = (2\lambda + 2\mu)C_1 - \frac{2\mu C_2}{b^2} = -P_e \tag{22}$$

Solving equations (Eq. (22)) simultaneously leads to find the constants C_1 and C_2 as follows:

$$C_1 = \frac{a^2 P_i + \ b^2 P_e}{2(\lambda + \mu)(a^2 - b^2)}$$
$$C_2 = \frac{a^2 b^2 \ (P_i + \ P_e)}{2\mu(a^2 - b^2)} \tag{23}$$

In the case of prescribed pressure, coefficients in Eq. (23) reduce to the following forms for two special cases.

Case 1: ring without external pressure ($P_e \ \rightarrow 0$)

$$C_1 = \frac{a^2 P_i}{2(\lambda + \mu)(a^2 - b^2)}$$
$$C_2 = \frac{a^2 b^2 \ P_i}{2\mu(a^2 - b^2)} \tag{24}$$

Case 2: infinite domain with circular hole ($P_e \longrightarrow \ 0$ and $b \ \longrightarrow \ \infty$)

$$C_1 = 0$$
$$C_2 = \frac{-a^2 \ P_i}{2\mu} \tag{25}$$

Obtained coefficients i.e. C_1 and C_2 for special cases are summarized in tables 3 and 4.

Table 3 - Displacement field coefficients with Dirichlet BCs

Description	C_1	C_2
Ring	$-\frac{aU_i + bU_e}{(b^2 - a^2)}$	$\frac{ab(bU_i + aU_e)}{(b^2 - a^2)}$
Ring without external pressure $U_e \rightarrow 0$	$-\frac{aU_i}{(b^2 - a^2)}$	$\frac{ab^2U_i}{(b^2 - a^2)}$
Infinite domain $U_e \rightarrow 0 \; and \; b \rightarrow \infty$	0	aU_i

Table 4 - Displacement field coefficients with Neumann BCs

Description	C_1	C_2
Ring	$\frac{a^2P_i + b^2P_e}{2(\lambda + \mu)(a^2 - b^2)}$	$\frac{a^2b^2 (P_i + P_e)}{2\mu(a^2 - b^2)}$
Ring without external pressure $P_e \rightarrow 0$	$\frac{a^2P_i}{2(\lambda + \mu)(a^2 - b^2)}$	$\frac{a^2b^2 P_i}{2\mu(a^2 - b^2)}$
Infinite domain $P_e \rightarrow 0 \; and \; b \rightarrow \infty$	0	$\frac{-a^2 P_i}{2\mu}$

Using Eq. (17) into Eq. (7) and Eq. (11) result in strain, strain rate and stress fields as a function of calculated coefficients:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \begin{bmatrix} C_1 - \frac{C_2}{r^2} & 0 & 0 \\ 0 & C_1 + \frac{C_2}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$$\dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}_t) = \begin{bmatrix} \dot{C}_1 - \frac{\dot{C}_2}{r^2} & 0 & 0 \\ 0 & \dot{C}_1 + \frac{\dot{C}_2}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} (2\lambda + 2\mu)\, C_1 - \frac{2\mu\, C_2}{r^2} & 0 & 0 \\ 0 & (2\lambda + 2\mu)\, C_1 + \frac{2\mu\, C_2}{r^2} & 0 \\ 0 & 0 & 2\lambda\, C_1 \end{bmatrix} \tag{28}$$

Therefore, stress components for two different types of boundary conditions can be obtained as follows:

1- Stress components in the body under prescribed displacement:

$$\sigma_{rr} = -\frac{(2\lambda + 2\mu)(aU_i + bU_e)}{(b^2 - a^2)} - \frac{2\mu ab(bU_i + aU_e)}{r^2(b^2 - a^2)}$$

$$\sigma_{\theta\theta} = -\frac{(2\lambda + 2\mu)(aU_i + bU_e)}{(b^2 - a^2)} + \frac{2\mu ab(bU_i + aU_e)}{r^2(b^2 - a^2)} \tag{29}$$

$$\sigma_{zz} = -\frac{2\lambda(aU_i + bU_e)}{(b^2 - a^2)}$$

Stress components in Eq. (29) reduce to the following form for the special cases i.e. ring without external displacement or infinite domain with circular hole.

Case 1: ring without external displacement ($U_e \rightarrow 0$)

$$\sigma_{rr} = -\frac{(2\lambda + 2\mu)(aU_i)}{(b^2 - a^2)} - \frac{2\mu ab(bU_i)}{r^2(b^2 - a^2)}$$

$$\sigma_{\theta\theta} = -\frac{(2\lambda + 2\mu)(aU_i)}{(b^2 - a^2)} + \frac{2\mu ab(bU_i)}{r^2(b^2 - a^2)} \tag{30}$$

$$\sigma_{zz} = -\frac{2\lambda(aU_i)}{(b^2 - a^2)}$$

Since $\lambda > 0$ and $\mu > 0$, following relation between stress components hold:

$$\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr} \tag{31}$$

Case 2: infinite domain with circular hole ($U_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$\begin{aligned} \sigma_{rr} &= -\frac{2\mu a U_i}{r^2} \\ \sigma_{\theta\theta} &= +\frac{2\mu a U_i}{r^2} \\ \sigma_{zz} &= 0 \end{aligned} \tag{32}$$

Since $\lambda > 0$ and $\mu > 0$, following relation between stress components hold:

$$\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr} \tag{33}$$

2- Stress components in the body under prescribed pressure:

$$\begin{aligned} \sigma_{rr} &= \frac{a^2 P_i + b^2 P_e}{(a^2 - b^2)} - \frac{a^2 b^2 (P_i + P_e)}{r^2 (a^2 - b^2)} \\ \sigma_{\theta\theta} &= \frac{a^2 P_i + b^2 P_e}{(a^2 - b^2)} + \frac{a^2 b^2 (P_i + P_e)}{r^2 (a^2 - b^2)} \\ \sigma_{zz} &= \frac{\lambda (a^2 P_i + b^2 P_e)}{(\lambda + \mu)(a^2 - b^2)} \end{aligned} \tag{34}$$

Stress components in Eq. (34) reduce to the following form for two special cases.

Case 1: ring without external pressure ($P_e \rightarrow 0$)

$$\begin{aligned} \sigma_{rr} &= \frac{a^2 P_i (b^2 - r^2)}{r^2 (b^2 - a^2)} \\ \sigma_{\theta\theta} &= -\frac{a^2 P_i (b^2 + r^2)}{r^2 (b^2 - a^2)} \\ \sigma_{zz} &= -\frac{\lambda a^2 P_i}{(\lambda + \mu)(b^2 - a^2)} \end{aligned} \tag{35}$$

Since $\lambda > 0$ and $\mu > 0$, following relation between stress components hold:

$$\sigma_{rr} > \sigma_{zz} > \sigma_{\theta\theta} \tag{36}$$

Case 2: infinite domain with circular hole ($P_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$\sigma_{rr} = \frac{a^2\, P_i}{r^2}$$

$$\sigma_{\theta\theta} = -\frac{a^2\, P_i}{r^2} \tag{37}$$

$$\sigma_{zz} = 0$$

Since $a^2 > 0$ and $P_i > 0$, following relation between stress components hold:

$$\sigma_{rr} > \sigma_{zz} > \sigma_{\theta\theta} \tag{38}$$

2.2.3 Elastic energy density

Elastic energy density can be calculated using Eq. (26) and Eq. (28):

$$U = \frac{1}{2}\,\boldsymbol{\sigma} \,.\, \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\,\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) = (2\lambda + 2\mu)C_1^2 + \frac{2\mu C_2^2}{r^4} \tag{39}$$

Introducing two new constants as follows into Eq. (39) leads to:

$$A = (2\lambda + 2\mu)C_1^2, \quad B = 2\mu C_2^2$$

$$U = A + \frac{B}{r^4} \tag{40}$$

Coefficients A and B for two different types of boundary conditions can be obtained as follows:

1- The body under prescribed displacement using Eq. (19):

$$A = \frac{(2\lambda + 2\mu)(aU_i + bU_e)^2}{(b^2 - a^2)^2}$$

$$B = \frac{2\mu a^2 b^2 (bU_i + aU_e)^2}{(b^2 - a^2)^2} \tag{41}$$

These coefficients for the special cases can be obtained as following:

Case 1: ring without external displacement ($U_e \rightarrow 0$)

$$A = \frac{(2\lambda + 2\mu)(aU_i)^2}{(b^2 - a^2)^2}$$

$$B = \frac{2\mu a^2 b^2 (bU_i)^2}{(b^2 - a^2)^2} \tag{42}$$

Case 2: infinite domain with circular hole ($U_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$A = 0$$

$$B = 2\mu a^2 U_i^2 \tag{43}$$

2- The body under prescribed pressure, coefficients A and B using Eq. (21) read:

$$A = \frac{(a^2 P_i + b^2 P_e)^2}{2(\lambda + \mu)(a^2 - b^2)^2}$$

$$B = \frac{a^4 b^4 \ (P_i + P_e)^2}{2\mu(a^2 - b^2)^2} \tag{44}$$

These coefficients for two special cases can be obtained as following:

Case 1: ring without external pressure ($P_e \rightarrow 0$)

$$A = \frac{a^4 P_i^2}{2(\lambda + \mu)(a^2 - b^2)^2}$$

$$B = \frac{a^4 b^4 P_i^2}{2\mu(a^2 - b^2)^2} \tag{45}$$

Case 2: infinite domain with circular hole ($P_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$A = 0$$

$$B = \frac{a^4 P_i^2}{2\mu} \tag{46}$$

Obtained stress components i.e. σ_{rr} and $\sigma_{\theta\theta}$ as well as elastic energy density i.e. U for special cases are summarized in tables 5 and 6.

Table 5 – Stresses and elastic energy density for ring without external loading

Items	Dirichlet BCs	Neumann BCs
σ_{rr}	$-\frac{(2\lambda+2\mu)(aU_i)}{(b^2-a^2)}-\frac{2\mu ab(bU_i)}{r^2(b^2-a^2)}$	$\frac{a^2P_i(b^2-r^2)}{r^2(b^2-a^2)}$
$\sigma_{\theta\theta}$	$-\frac{(2\lambda+2\mu)(aU_i)}{(b^2-a^2)}+\frac{2\mu ab(bU_i)}{r^2(b^2-a^2)}$	$-\frac{a^2P_i(b^2+r^2)}{r^2(b^2-a^2)}$
σ_{zz}	$-\frac{2\lambda(aU_i)}{(b^2-a^2)}$	$-\frac{\lambda a^2P_i}{(\lambda+\mu)(b^2-a^2)}$
U	$\frac{(2\lambda+2\mu)(aU_i)^2}{(b^2-a^2)^2}+\frac{2\mu a^2b^2(bU_i)^2}{r^4(b^2-a^2)^2}$	$\frac{a^4P_i^2}{2(\lambda+\mu)(a^2-b^2)^2}+\frac{a^4b^4P_i^2}{2\mu r^4(a^2-b^2)^2}$

Table 6 – Stresses and elastic energy density for infinite domain with circular hole

Items	Dirichlet BCs	Neumann BCs
σ_{rr}	$-\frac{2\mu aU_i}{r^2}$	$\frac{a^2\,P_i}{r^2}$
$\sigma_{\theta\theta}$	$\frac{2\mu aU_i}{r^2}$	$-\frac{a^2\,P_i}{r^2}$
σ_{zz}	0	0
U	$\frac{2\mu a^2U_i^2}{r^4}$	$\frac{a^4P_i^2}{2\mu r^4}$

2.2.4 Total elastic energy

Total elastic energy (E_T) of the cylinder can be obtained by integration of the elastic energy density (Eq. (39)) over the domain as follows:

$$E_T = \int_0^{2\pi}\int_{r=a}^{b} U\,r\,dr\,d\theta = \int_0^{2\pi}\int_{r=a}^{b}\left(A+\frac{B}{r^4}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_{r=a}^{b}\left(Ar+\frac{B}{r^3}\right)\,dr\,d\theta$$
$$= \pi\left(Ar^2-\frac{B}{r^2}\right)_a^b = \pi\left(A(b^2-a^2)-\frac{B(b^2-a^2)}{b^2a^2}\right)$$

$$E_T = \pi \left(A(b^2 - a^2) - \frac{B(b^2 - a^2)}{b^2 a^2} \right) \quad (47)$$

2.3 Damage problem setting

2.3.1 General setting

We recall here the main steps of the construction of gradient damage model by varational approach which was investigated in details in first chapter. As it is mentioned in the last part, plane stress condition and a homogenous cylinder made of a damaging isotropic material is considered with the following specification:

1- Isotropic damage is considered which is defined by a scalar variable α. Damage variable can increase from 0 (corresponding to undamaged state. i.e. beginning of the loading) to final value 1 (corresponding to fully damaged state).

2- Material points are characterized by a triplet $(\boldsymbol{\varepsilon}(\boldsymbol{u}), \alpha, \nabla\alpha)$ where, $\boldsymbol{\varepsilon}$ denotes elastic strain field as symmetrized part of the displacement field gradient, α shows the damage variable and $\nabla\alpha$ denotes the damage gradient vector.

3- The bulk energy density of the material (strain work function) describes the state of material at each point $\mathrm{W}: (\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \longmapsto \mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha)$. Since strain work function depends on the gradient damage vector, non local behavior of material is considered. As it is mentioned in previous chapter, strain work function consist of three terms including the stored elastic strain energy $\psi(\boldsymbol{\varepsilon}, \alpha)$, energy dissipation during damage $\omega(\alpha)$ and the incorporation of non local damage $\frac{1}{2} w\, \ell^2\, \nabla\alpha . \nabla\alpha$. Therefore,

$$\mathrm{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = \psi(\boldsymbol{\varepsilon}, \alpha) + \omega(\alpha) + \frac{1}{2} w\, \ell^2\, \nabla\alpha . \nabla\alpha \quad (48)$$

Local behavior of the material is characterized by strain work function in the absence of gradient term ($\mathrm{W}_0(\boldsymbol{\varepsilon},\alpha) = \mathrm{W}(\boldsymbol{\varepsilon},\alpha,0)$ which is defined as follows:

$$\mathrm{W}_0(\boldsymbol{\varepsilon},\alpha) = \psi(\boldsymbol{\varepsilon},\alpha) + \omega(\alpha) \tag{49}$$

Strongly brittle materials obey the state function $\mathrm{W}_0(\boldsymbol{\varepsilon},\alpha)$ in which they do not show damage gradient effects (non local) behavior i.e. $\frac{1}{2} w\, \ell^2\, \nabla\alpha . \nabla\alpha = 0$. Each term in Eq. (48) maintains the following properties:

1- Elastic energy reads:

$$\psi(\boldsymbol{\varepsilon},\alpha) = \frac{1}{2} S(\alpha)\, \mathbb{E}\, \boldsymbol{\varepsilon} . \boldsymbol{\varepsilon} \tag{50}$$

In Eq. (50), $\mathbb{E}$ is the elasticity tensor of undamaged material and is not a function of damage variable. Hence, $S(\alpha)\, \mathbb{E}$ shows the stiffness of material in the damage state α. It should be noted that material compliance tensor will be denoted by $\mathbb{S}$ and is defined as invers of elasticity tensor $\mathbb{S} = \mathbb{E}^{-1}$.

2- Dissipation due to local damage reads:

$$\omega(\alpha) \quad \textit{to be chosen} \tag{51}$$

For the sake of generality, dissipation function due to local damage is introduced in a general form in Eq. (51). As a general requirement in Eq. (51), energy dissipated is an increasing function which is zero at the beginning of the loading (undamaged state).

1- Stress tensor ($\boldsymbol{\sigma}$), thermodynamic conjugate force due to damage (energy release rate density) (Y), and damage flux vector ($\boldsymbol{q}$) associated with the state variables $(\boldsymbol{\varepsilon},\alpha,\nabla\alpha)$ defined by strain work function $\mathrm{W}(\boldsymbol{\varepsilon},\alpha,\nabla\alpha)$ in Eq. (48) read:

$$\boldsymbol{\sigma} = \frac{\partial \mathrm{W}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\alpha,\nabla\alpha), \quad Y = -\frac{\partial \mathrm{W}}{\partial \alpha}(\boldsymbol{\varepsilon},\alpha,\nabla\alpha), \qquad \boldsymbol{q} = \frac{\partial \mathrm{W}}{\partial(\nabla\alpha)}(\boldsymbol{\varepsilon},\alpha,\nabla\alpha) \tag{52}$$

Substituting Eq. (50) into Eq. (48) leads to:

$$\mathsf{W}(\boldsymbol{\varepsilon},\alpha,\nabla\alpha) = \frac{1}{2}S(\alpha)\,\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega(\alpha) + \frac{1}{2}w\,\ell^2\,\nabla\alpha.\nabla\alpha \tag{53}$$

Therefore, dual quantities defined by Eq. (52) read:

$$\boldsymbol{\sigma} = S(\alpha)\mathbb{E}\,\boldsymbol{\varepsilon}, \quad Y = -\frac{1}{2}S'(\alpha)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} - \omega'(\alpha), \qquad \boldsymbol{q} = w\,\ell^2\,\nabla\alpha \tag{54}$$

4- Elastic domain is independent of gradient effects and it depends only on local behavior of material. This local behavior characterized by $\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)$ function requires that elastic domain as an increasing function of α in strain space $\mathcal{R}(\alpha)$ corresponding to elastic domain as decreasing function of α in stress space $\mathcal{R}^*(\alpha)$. These domains are defined as:

$$\mathcal{R}(\alpha) = \left\{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \ : \frac{\partial \mathsf{W}_0}{\partial\alpha}(\boldsymbol{\varepsilon},\alpha) \geq 0\right\}, \qquad \mathcal{R}^*(\alpha) = \left\{\boldsymbol{\sigma} \in \boldsymbol{M}_s^n \ : \frac{\partial \mathsf{W}_0^*}{\partial\alpha}(\boldsymbol{\sigma},\alpha) \leq 0\right\} \tag{55}$$

Where $\mathsf{W}_0^*(\boldsymbol{\sigma},\alpha)$ is the Legendre transformation of $\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)$ with following definition which was discussed in the previous chapter:

$$\mathsf{W}_0^*(\boldsymbol{\sigma},\alpha) = sup_{\boldsymbol{\varepsilon}\in\boldsymbol{M}_s^n}\{\boldsymbol{\sigma}.\boldsymbol{\varepsilon} - \mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)\} \tag{56}$$

Using Eq. (50) and Eq. (51) into Eq. (49) leads to:

$$\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha) = \frac{1}{2}S(\alpha)\,\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega(\alpha) \tag{57}$$

Therefore, appropriate expression for $\mathsf{W}_0^*(\boldsymbol{\sigma},\alpha)$ can be derived as follows:

$$\mathsf{W}_0^*(\boldsymbol{\sigma},\alpha) = \boldsymbol{\sigma}.\boldsymbol{\varepsilon}_0 + \frac{1}{2S(\alpha)}\,\mathbb{S}\,\boldsymbol{\sigma}.\boldsymbol{\sigma} - \omega(\alpha) \tag{58}$$

Substituting Eq. (57) and Eq. (58) into definitions of elastic domain (Eq. (55)) results in:

$$\begin{aligned} \mathcal{R}(\alpha) &= \left\{\boldsymbol{\varepsilon} \in \boldsymbol{M}_s^n \quad : \ \mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} \geq \frac{2\omega'(\alpha)}{S'(\alpha)}\right\} \\ \mathcal{R}^*(\alpha) &= \left\{\boldsymbol{\sigma} \in \boldsymbol{M}_s^n \quad : \ \mathbb{S}\,\boldsymbol{\sigma}.\boldsymbol{\sigma} \geq \frac{-2\omega'(\alpha)S^2(\alpha)}{S'(\alpha)}\right\} \end{aligned} \tag{59}$$

As it can be seen in the definition of the elastic domain in stress or strain space (Eq. (59)), specific domain needs to be defined to show the region in which material behaves elastically without damage growth. This can be done by finding a critical stress that causes inelastic deformation and assuming that damage will start once the level of stress reaches to this value at a specific point. Critical stress that represents the yield stress can be obtained in two different ways, which is given in next section. First stage of damage process is considered so the damage variable (α) changes from zero to 1 which corresponds to loss of rigidity of the material (zero stiffness). Consequently, set of admissible damage field ($\mathcal{D}$) and the set of kinematically admissible displacement field ($\mathcal{C}$) are defined as:

$$\mathcal{D} := \{\beta \in H^1(\Omega) : 0 \leq \beta < 1 \;\; in \;\; \Omega\}$$
$$\mathcal{C} := \{\boldsymbol{v} = (v_r \, , v_\theta) \in H^1(\Omega)^2 \; : \; v_r = v_\theta = 0 \;\; on \;\; r = a \; and \; b\} \tag{60}$$

$H^1(\Omega)$ is the Sobolev space of functions with the specification that these functions and their distributional gradients are both square integrable over Ω. The spaces $\mathcal{D}$ and $\mathcal{C}$ are time independent. Considering the external work done by prescribed internal pressure in the form of either prescribed displacement or prescribed stress over the boundary, associated total energy of the body at time t with every pair $(\boldsymbol{v}, \beta) \in \mathcal{C} \times \mathcal{D}$ reads:

$$\mathcal{E}_t\,(\boldsymbol{v}, \beta) \coloneqq \int_\Omega \mathsf{W}(\boldsymbol{\varepsilon}(\boldsymbol{v}), \beta, \nabla\beta)\, d\Omega - W_t^e(\boldsymbol{v}) \;=\; \int_\Omega \Big(\frac{1}{2}\, S(\beta)\mathbb{E}\, \boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{v}) + \omega(\beta) + \frac{w\ell^2}{2} \nabla\beta.\nabla\beta\Big)\, d\Omega - W_t^e(\boldsymbol{v}) \tag{61}$$

As it is mentioned in previous chapter, external work reads:

$$W_t^e(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}_t.\boldsymbol{v}\, d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\boldsymbol{v}\, d\Gamma \tag{62}$$

$\boldsymbol{\varepsilon}(\boldsymbol{v})$ denotes the symmetrized gradient of $\boldsymbol{v}$. Strain field depends on time implicitly since the applied load is a function of time based on previous definitions. Derivative of total energy with respect to time can be obtained as:

$$\dot{\mathcal{E}}_t\,(\boldsymbol{v},\beta) := \int_\Omega \Big(S(\beta)\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}).\dot{\boldsymbol{\varepsilon}}(\boldsymbol{v}) + \frac{1}{2}\,S'(\beta)\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{v})\dot{\beta} + \omega'(\beta)\dot{\beta} + w\ell^2\nabla\beta.\nabla\dot{\beta}\Big)\,d\Omega - \frac{d}{dt}W_t^e(\boldsymbol{v}) \tag{63}$$

First directional derivative of $\mathcal{E}_t\,(\boldsymbol{u},\alpha)$ in the direction $(\boldsymbol{v},\beta)$ as discussed in previous chapter reads:

$$\mathcal{E}_t'\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta) = \int_\Omega (S'(\alpha)\,\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + S(\alpha)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega'(\alpha))\beta + w\ell^2\,\nabla\alpha.\nabla\beta)d\Omega - \int_\Omega \boldsymbol{f}_t\,.\boldsymbol{v}d\Omega - \int_{\partial_F\Omega}\boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma \tag{64}$$

Substituting constitutive relations (Eq. (54)) into Eq. (64) results in:

$$\mathcal{E}_t'\,(\boldsymbol{u},\alpha)(\boldsymbol{v},\beta) = \int_\Omega (\boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\boldsymbol{v}) - Y\beta + \boldsymbol{q}.\nabla\beta)d\Omega - \int_\Omega \boldsymbol{f}_t\,.\boldsymbol{v}d\Omega - \int_{\partial_F\Omega}\boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma \tag{65}$$

2.3.2 Critical pressure

Two different yield criterions are used to define critical pressure, in both forms of applied displacement or stress as following:

Tresca yield criterion

If the Tresca yield criterion is used, then it reads:

$$\frac{1}{2} max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = \frac{1}{2}\ \sigma_y \tag{66}$$

It is obvious that $\sigma_{rr} - \sigma_{\theta\theta}$ is the maximum stress based on either Eq. (30) or Eq. (34). Aforementioned two problem settings i.e. prescribed displacement or prescribed pressure on body can be recognized as following:

1- For the general case of prescribed displacement using Eq. (30) this criterion reads:

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{4\mu ab(bU_i + aU_e)}{r^2(b^2 - a^2)} \tag{67}$$

And its maximum achieved at $r = a$:

$$\sigma_y = \frac{4\mu b(bU_i + aU_e)}{a(b^2 - a^2)} \tag{68}$$

Therefore, for the special cases this criterion reads:

Case 1: ring without external displacement ($U_e \rightarrow 0$)

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{4\mu ab^2 U_i}{r^2(b^2 - a^2)} \qquad for\ all\ r, \theta \tag{69}$$

In this case, $max|\sigma_{rr} - \sigma_{\theta\theta}|$ achieved at $r = a$:

$$\sigma_y = \frac{4\mu b^2 U_i}{a(b^2 - a^2)} \rightarrow U_{i-cr} = \frac{\sigma_y a(b^2 - a^2)}{4\mu b^2} \tag{70}$$

Case 2: infinite domain with circular hole ($U_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{4\mu a U_i}{r^2} \qquad for\ all\ r, \theta \tag{71}$$

In this case, $max|\sigma_{rr} - \sigma_{\theta\theta}|$ achieved at $r = a$:

$$\sigma_y \ = \frac{4\mu U_i}{a} \ \rightarrow \ U_{i-cr} \ \ = \frac{\sigma_y a}{4\mu} \tag{72}$$

2- For the general case of prescribed pressure, this criterion reads:

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{2a^2 b^2 \ (P_i \ + \ P_e)}{r^2(b^2 - a^2)} \tag{73}$$

And its maximum achieved at $r = a$:

$$\sigma_y = \frac{2b^2 \ (P_i \ + \ P_e)}{(b^2 - a^2)} \tag{74}$$

Therefore, for two different cases this criterion reads:

Case 1: ring without external pressure ($P_e \ \rightarrow 0$)

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{2a^2 b^2 P_i}{r^2(b^2 - a^2)} \qquad for\ all\ r, \theta \tag{75}$$

In this case, $max|\sigma_{rr} - \sigma_{\theta\theta}|$ achieved at $r = a$:

$$\sigma_y \ = \frac{2b^2 P_i}{(b^2 - a^2)} \ \rightarrow \ P_{i-cr} \ \ = \frac{\sigma_y (b^2 - a^2)}{2b^2} \tag{76}$$

Case 2: infinite domain with circular hole ($P_e \longrightarrow \ 0$ and $b \ \longrightarrow \ \infty$)

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{2a^2 P_i}{r^2} \qquad for\ all\ r, \theta \tag{77}$$

In this case, $max|\sigma_{rr} - \sigma_{\theta\theta}|$ i.e. yielding takes place at $r = a$:

$$\sigma_y \ = 2P_i \ \rightarrow \ P_{i-cr} \ \ = \frac{\sigma_y}{2} \tag{78}$$

Von Mises yield criterion

If the Von Mises yield criterion is used, then for principle stresses in three dimensions it reads:

$$(\sigma_1 - \sigma_2)^2 + \ (\sigma_2 - \sigma_3)^2 + \ (\sigma_3 - \sigma_1)^2 = 2\ \sigma_y^2 \tag{79}$$

Or, in general state of stress this criterion reads:

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{31})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 6k^2$$
$$k = \frac{\sigma_y}{\sqrt{3}} \tag{80}$$

General state of stress given in Eq. (18) demonstrates that three stress components are principle stresses. Therefore, Eq. (79) reads the following general form:

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 = 8\mu^2 C_1^2 + \frac{24\mu^2 C_2^2}{r^4} \tag{81}$$

For different boundary conditions, this criterion can be evaluated using tables 2 and 3 as following.

1- For the general case of prescribed displacement using Eq. (30) this criterion reads:

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2$$
$$= \frac{8\mu^2(aU_i + bU_e)^2}{(b^2 - a^2)^2} + \frac{24\mu^2 a^2 b^2 (bU_i + aU_e)^2}{r^4 (b^2 - a^2)^2} \quad for\ all\ r, \theta \tag{82}$$

And its maximum achieved at $r = a$:

$$\sigma_y = \frac{2\mu}{a(b^2 - a^2)} \sqrt{a^2(aU_i + bU_e)^2 + 3b^2(bU_i + aU_e)^2} \tag{83}$$

Therefore, for the special cases this criterion reads:

Case 1: ring without external displacement ($U_e \to 0$)

$$\sigma_y = \frac{2\mu}{a(b^2 - a^2)} \sqrt{a^4 U_i^2 + 3b^4 U_i^2} \to U_{i-cr} = \frac{\sigma_y a(b^2 - a^2)}{2\mu\sqrt{a^4 + 3b^4}} \tag{84}$$

Case 2: infinite domain with circular hole ($U_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr} - \sigma_{zz})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2$$
$$= \frac{24\mu^2 a^2 U_i^2}{r^4} \quad for\ all\ r, \theta \tag{85}$$

In this case, maximum is achieved at $r = a$:

$$\sigma_y = \frac{2\sqrt{3}\mu U_i}{a} \rightarrow U_{i-cr} = \frac{\sigma_y a}{2\sqrt{3}\mu} \tag{86}$$

2- For the general case of prescribed pressure, this criterion reads:

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2$$
$$= \frac{2\mu^2(a^2P_i + b^2P_e)^2}{(\lambda+\mu)^2(b^2-a^2)^2} + \frac{6a^4b^4(P_i+P_e)^2}{r^4(b^2-a^2)^2} \qquad for\ all\ r, \theta \tag{87}$$

And its maximum achieved at $r = a$:

$$\sigma_y = \frac{1}{(\lambda+\mu)(b^2-a^2)}\sqrt{\mu^2(a^2P_i + b^2P_e)^2 + 3(\lambda+\mu)^2b^4(P_i+P_e)^2} \tag{88}$$

Therefore, for two different cases this criterion reads:

Case 1: ring without external pressure ($P_e \rightarrow 0$)

$$\sigma_y = \frac{1}{(\lambda+\mu)(b^2-a^2)}\sqrt{\mu^2a^4P_i^2 + 3(\lambda+\mu)^2b^4P_i^2}$$
$$P_{i-cr} = \frac{(\lambda+\mu)(b^2-a^2)\sigma_y}{\sqrt{\mu^2a^4 + 3(\lambda+\mu)^2b^4}} \tag{89}$$

Case 2: infinite domain with circular hole ($P_e \longrightarrow 0$ and $b \longrightarrow \infty$)

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr} - \sigma_{zz})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 = \frac{6a^4P_i^2}{r^4} \qquad for\ all\ r, \theta \tag{90}$$

Yielding takes place at $r = a$ in this case as:

$$\sigma_y = \sqrt{3}P_i \rightarrow P_{i-cr} = \frac{\sigma_y}{\sqrt{3}} \tag{91}$$

Summary of these results are given in tables 7 and 8.

Table 7 - Critical pressure based on Tresca yield criterion

Description	Dirichlet BCs	Neumann BCs
Ring without external pressure $U_e \rightarrow 0$	$U_{i-cr} = \frac{\sigma_y a(b^2 - a^2)}{4\mu b^2}$	$P_{i-cr} = \frac{\sigma_y(b^2 - a^2)}{2b^2}$
Infinite domain $U_e \rightarrow 0 \; and \; b \rightarrow \infty$	$U_{i-cr} = \frac{\sigma_y a}{4\mu}$	$P_{i-cr} = \frac{\sigma_y}{2}$

Table 8 - Critical pressure based on Von Mises yield criterion

Description	Dirichlet BCs	Neumann BCs
Ring without external pressure $P_e \rightarrow 0$	$U_{i-cr} = \frac{\sigma_y a(b^2 - a^2)}{2\mu\sqrt{a^4 + 3b^4}}$	$P_{i-cr} = \frac{(\lambda + \mu)(b^2 - a^2)\sigma_y}{\sqrt{\mu^2 a^4 + 3(\lambda + \mu)^2 b^4}}$
Infinite domain $P_e \rightarrow 0 \; and \; b \rightarrow \infty$	$U_{i-cr} = \frac{\sigma_y a}{2\sqrt{3}\mu}$	$P_{i-cr} = \frac{\sigma_y}{\sqrt{3}}$

2.4 Damage evolution problem

As discussed in previous chapter, damage evolution in body is governed using variational approach along with three fundamental principles of irreversibility, stability and energy balance in global form rather than local form. Governing damage law enjoys the following conditions as it was stated in Eq. (46) of previous chapter:

$$(IR)\ Irreversibility: t \longmapsto \alpha_t \ must\ be\ non\ decreasing\ and\ at\ each\ time\ t\ \dot{\alpha}_t \geq 0, \ \alpha_t \in \mathcal{D}$$

$$(ST)\ Stability: (\boldsymbol{u}_t, \alpha_t) \in \mathcal{C} \times \mathcal{D}\ must\ be\ stable\ for\ all\ (\boldsymbol{v}, \beta) \in \mathcal{C} \times \mathcal{D} \tag{92}$$

$$such\ that\ \beta \geq \alpha_t, there\ exists\ \bar{h} > 0\ such\ that\ for\ all\ h \in [0, \bar{h}]$$

$$\mathcal{E}_t(\boldsymbol{u}_t + h(\boldsymbol{v} - \boldsymbol{u}_t), \alpha_t + h(\beta - \alpha_t)) \geq \mathcal{E}_t(\boldsymbol{u}_t, \alpha_t)$$

(EB) *Energy balance*:

At each time $t \geq 0$ *the following energy balance must hold*

$$\dot{\mathcal{E}}_t\,(\boldsymbol{u}_t, \alpha_t) = \int_\Omega \boldsymbol{\sigma}_t.\boldsymbol{\varepsilon}(\dot{\boldsymbol{U}}_t)\,d\Omega - W_t^e(\dot{\boldsymbol{U}}_t) - \dot{W}_t^e(\boldsymbol{u}_t)$$

In Eq. (92), as it was discussed in previous chapter, work done by external forces and its time derivative read respectively:

$$W_t^e(\dot{\boldsymbol{U}}_t) = \int_\Omega \boldsymbol{f}_t.\dot{\boldsymbol{U}}_t\,d\Omega + \int_{\partial_F\Omega} \boldsymbol{F}_t.\dot{\boldsymbol{U}}_t\,d\Gamma \tag{93}$$

And

$$\dot{W}_t^e(\boldsymbol{u}_t) = \int_\Omega \dot{\boldsymbol{f}}_t.\boldsymbol{u}_t\,d\Omega + \int_{\partial_F\Omega} \dot{\boldsymbol{F}}_t.\boldsymbol{u}_t\,d\Gamma \tag{94}$$

Substituting Eq. (93), Eq. (94) into Eq. (92) and considering Eq. (63) leads to:

$$\int_\Omega \left(\frac{1}{2}S'(\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t)\dot{\alpha}_t + \omega'(\alpha_t)\dot{\alpha}_t + w\ell^2\nabla\alpha_t.\nabla\dot{\alpha}_t\right)\,d\Omega = 0 \tag{95}$$

As it was derived in the previous chapter (Eq. (80) of chapter one), global energy balance is satisfied only if the following relation holds:

$$\int_\Omega \left(\frac{1}{2}S'(\alpha_t)\,\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - w\ell^2\nabla^2\alpha_t\right)\dot{\alpha}_t d\Omega + \int_{\partial_F\Omega} w\ell^2\dot{\alpha}_t\frac{\partial\alpha_t}{\partial n}d\Gamma = 0 \tag{96}$$

With the same arguments that have been made in previous chapter (Eq. (76)), Eq. (96) leads to the following criterion:

$$\dot{\alpha}_t\frac{\partial\alpha_t}{\partial n} = 0 \qquad on\ \partial_F\Omega \tag{97}$$

$$\left(\frac{1}{2}S'(\alpha_t)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - w\ell^2\nabla^2\alpha_t\right)\dot{\alpha}_t = 0 \quad in\ \Omega$$

Directional stability inequality (Eq. (92)) must hold for sufficiently small h for given admissible direction $(\boldsymbol{v}, \beta)$. Therefore, expanding total energy $\mathcal{E}_t(\boldsymbol{u}_t, \alpha_t)$ of the perturbed state with respect to h up to the second order as it was done in the previous chapter leads to:

$$0 \leq \mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v} - \boldsymbol{u}_t, \beta - \alpha_t) + \frac{h}{2}\mathcal{E}_t''(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v} - \boldsymbol{u}_t, \beta - \alpha_t) + o(h) \tag{98}$$

Limit of the expression in Eq. (98) when h tends to zero and considering that $\mathcal{C}$ is a linear space leads to the following definition:

$$\mathcal{E}_t'(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v} - \boldsymbol{u}_t, \beta - \alpha_t) \geq 0 \tag{99}$$

Using directional derivative that is derived in Eq. (65) into inequality (99) with the same arguments that have been made in previous chapter leads to the following two criterions including equilibrium and damage criterion which must be satisfied simultaneously:

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\boldsymbol{v})d\Omega = 0 \quad \forall\boldsymbol{v} \in \mathcal{C} \tag{100}$$

$$\int_\Omega (-Y_t(\beta - \alpha_t) + \boldsymbol{q}_t.\nabla(\beta - \alpha_t))d\Omega \geq 0 \quad \forall\beta \in \mathcal{D} : \ \beta \geq \alpha_t \tag{101}$$

Eq. (100) denotes the equilibrium of the body and Eq. (101) denotes the damage criterion. These two first order stability conditions are satisfied if the following local conditions hold. The first one (Eq. (100)) leads to equilibrium equation as follows, which is the same as equilibrium equation of the infinite domain with circular whole:

$$\begin{aligned} div\,\boldsymbol{\sigma} &= 0 \qquad in\ \Omega \\ \sigma_{rr} &= P_i \qquad on\ r = a \end{aligned} \tag{102}$$

$$\sigma_{rr} = -P_e \quad on\, r = b$$

The second one (Eq. (101)) can be simplified using Eq. (54):

$$\int_{\Omega} \left(\frac{1}{2} S'(\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega'(\alpha_t))\beta + w\,\ell^2\,\nabla\alpha_t.\nabla\beta\right) d\Omega \geq 0 \quad \forall\beta \in \mathcal{D} :\ \beta \geq \alpha_t \qquad (103)$$

The third term in inequality (103) reads:

$$\int_{\Omega} w\,\ell^2\,\nabla\alpha_t\,.\nabla\beta d\Omega = w\,\ell^2 (\int_{\partial_F\Omega} \beta \frac{\partial\alpha_t}{\partial n} d\Gamma - \int_{\Omega} \beta\nabla^2\alpha_t d\Omega) \qquad (104)$$

Substituting Eq. (104) into (103) leads to the following criterion:

$$\frac{1}{2} S'(\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega'(\alpha_t) - w\,\ell^2\,\nabla^2\alpha_t \leq 0 \;\; in\,\Omega, \quad \frac{\partial\alpha_t}{\partial n} \geq 0 \;\; on\;\; \partial_F\Omega \qquad (105)$$

It worth to mention that gradient vector and Laplacian term in Eqs. (96) - (105) are in the appropriate form of cylindrical coordinates. Damage is independent of z component of coordinate and because of the symmetry of the problem it is not a function of θ as well. Therefore,

$$\alpha_t(r, \theta, z) = \alpha_t(r) \qquad (106)$$

Consequently, damage gradient vector and Laplacian of damage as a function of one variable reads:

$$\nabla\alpha_t = \frac{d\alpha_t}{dr}\,\vec{e}_r$$
$$\nabla^2\alpha_t = \frac{1}{r}\frac{d}{dr}(r\frac{d\alpha_t}{dr}) \qquad (107)$$

Finally, all criterions and equilibrium equations can be summarized as following:

1. $The\ Kuhn - Tucker\ conditions\ in\ the\ bulk$

$$In\ \Omega: \begin{cases} \dot{\alpha}_t \geq 0 \\ \frac{1}{2} S'(\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega'(\alpha_t) - w\,\ell^2\,\nabla^2\alpha_t \leq 0 \\ \left(\frac{1}{2} S'(\alpha_t)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - w\ell^2\nabla^2\alpha_t\right)\dot{\alpha}_t = 0 \end{cases} \tag{108}$$

2. $The\ Kuhn - Tucker\ conditions\ on\ the\ boundary$

$$On\ \partial_F\Omega: \quad \dot{\alpha}_t \geq 0, \qquad \frac{\partial \alpha_t}{\partial n} \geq 0, \qquad \dot{\alpha}_t \frac{\partial \alpha_t}{\partial n} = 0 \tag{109}$$

3. $The\ equlibrium\ equation\ and\ the\ static\ boundary\ conditions$

$$div\,\boldsymbol{\sigma} = 0 \qquad in\ \Omega$$

$$\sigma_{rr} = P_i \qquad on\ r = a \tag{110}$$

$$\sigma_{rr} = -P_e \qquad on\ r = b$$

4. $The\ stress - strain\ relation$

$$\boldsymbol{\sigma}_t = S'(\alpha_t)\,\mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}_t) \qquad in\ \Omega \tag{111}$$

2.5 Fundamental Branch

Homogeneous response of the infinite domain at every time t can be obtained by examining the directional derivative $\mathcal{E}'_t\,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta)$ (Eq. (65)) for two different cases which will be described. Therefore, based on Eq. (65) which is repeated here, one can obtain both equilibrium and a specific criterion:

$$\mathcal{E}'_t\,(\boldsymbol{u}_t, \alpha_t)(\boldsymbol{v}, \beta) = \int_\Omega (\boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\boldsymbol{v}) - Y\beta + \boldsymbol{q}.\nabla\beta)d\Omega - \int_\Omega \boldsymbol{f}_t\,.\boldsymbol{v}d\Omega - \int_{\partial_F\Omega} \boldsymbol{F}_t\,.\boldsymbol{v}d\Gamma \geq 0 \tag{112}$$

Inequality (112) leads to the variational form of the equilibrium if the damage does not evolve i.e. $\beta = 0$ as follows:

$$\int_{\Omega} \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\boldsymbol{v})d\Omega - \int_{\Omega} \boldsymbol{f}_t \,.\boldsymbol{v}d\Omega - \int_{\partial_{\mathrm{F}}\Omega} \boldsymbol{F}_t \,.\boldsymbol{v}d\Gamma = \int_{\Omega} div\boldsymbol{\sigma}.\boldsymbol{v}d\Omega + \int_{\partial_{\mathrm{F}}\Omega}(\boldsymbol{\sigma}.\boldsymbol{n})\,.\boldsymbol{v}d\Gamma - \int_{\Omega} \boldsymbol{f}_t \,.\boldsymbol{v}d\Omega - \int_{\partial_{\mathrm{F}}\Omega} \boldsymbol{F}_t \,.\boldsymbol{v}d\Gamma \tag{113}$$

Which leads to:

$$\begin{aligned} div\boldsymbol{\sigma} &= \boldsymbol{f}_t \qquad in\ \Omega \\ \boldsymbol{\sigma}.n &= \boldsymbol{F}_t \qquad on\ \partial_F\Omega \end{aligned} \tag{114}$$

And, following criterion can be obtained if the damage evolves but the displacement field does not change i.e. $\boldsymbol{v} = 0$:

$$\int_{\Omega} \left(\frac{1}{2}S'(\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + \omega'(\alpha_t)\right)\beta d\Omega \geq 0 \qquad for\ \alpha_t = 0\ and\ \beta > 0 \tag{115}$$

To demonstrate the crack initiation, specific function for stiffness reduction is considered i.e. $S(\alpha) = (1-\alpha)^2$. With this definition of stiffness reduction function, energy balance equation (Eq. (97)) in cylindrical coordinate changes to the following form using Eq. (107):

$$\left(-(1-\alpha_t)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + \omega'(\alpha_t) - w\ell^2\left(\frac{d^2\alpha_t}{dr^2} + \frac{1}{r}\frac{d\alpha_t}{dr}\right)\right)\dot{\alpha}_t = 0 \tag{116}$$

In order to show the effect of the dissipation function on analysis i.e. function $\omega(\alpha)$ two different types of functions are considered in separate sections. Obviously, the type of boundary conditions affects inequality (115). Therefore, two specific cases are considered to analyze for infinite domain with circular hole in each section in order to complete the analysis of the problem.

Linear form of dissipation function $\omega(\alpha) = w\alpha$

1- Prescribed displacement on boundary

Substituting from Eq. (40) and Eq. (43) into Eq. (115) leads to:

$$\int_{\Omega} (w - \frac{4\mu a^2\, U_i^2}{r^4})\beta d\Omega \geq 0 \qquad \beta > 0 \tag{117}$$

For all r inequality (117) is possible only if $U_i \leq \frac{a}{2}\sqrt{\frac{w}{\mu}}$. Inequality (117) holds for specific radius and prescribed displacement, and elastic solution is stable if $w \geq \frac{4\mu a^2\, U_i^2}{r^4}$. Otherwise, stability is not satisfied. Therefore, at $r = a$ constant w can be defined using the previously obtained critical internal displacement (Eq. (66)) as follows:

$$w = \frac{\sigma_y^2}{4\mu} \tag{118}$$

Interestingly, constant w is a function of material properties and not the geometry. Loading and the geometry of the problem do not depend on θ, so the damage evolution will depend only on r and t. In this case, three new dimensionless variables are introduced including one spatial variable ($\bar{r}$), one variable regarding geometry and internal length of material (λ) and a loading parameter (q) considering the loading process and a stationary evolution $(\boldsymbol{u}_t\,, \alpha_t)$ such that:

$$\alpha_t(r) = \bar{\alpha}_\tau(\bar{r}),\;\; q = \frac{4\mu U_i}{a\sigma_y}, \bar{r} = \left(\frac{r}{a}\right), \lambda = \frac{a}{\ell} \tag{119}$$

Displacement field does not change with this change of variable since it satisfies the equilibrium equation in the transformed form along with appropriate boundary conditions as follows:

$$\frac{d^2u_{\bar{r}}}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{du_{\bar{r}}}{d\bar{r}} - \frac{u_{\bar{r}}}{\bar{r}^2} = 0 \qquad u_{\bar{r}}(\bar{r} = 1) = U_i$$
$$u_{\bar{r}} = \bar{C}_1\bar{r} + \frac{\bar{C}_2}{\bar{r}} \qquad \bar{C}_1 = 0\,, \bar{C}_2 = U_i \tag{120}$$

Strain field does change with this change of variable and reads:

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau) = \begin{bmatrix} -\frac{U_i}{\bar{r}^2} & 0 & 0 \\ 0 & \frac{U_i}{\bar{r}^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{a^2U_i}{r^2} & 0 & 0 \\ 0 & \frac{a^2U_i}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = a\boldsymbol{\varepsilon}(\boldsymbol{u}_t) \tag{121}$$

Stress field changes since damage occurs during the loading:

$$\bar{\boldsymbol{\sigma}}_\tau = S(\bar{\alpha}_\tau)\,\mathbb{E}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau) \tag{122}$$

Differentiation with respect to the new spatial variable reads:

$$\frac{d\alpha_t}{dr} = \frac{d\bar{\alpha}_\tau}{dr} = \frac{d\bar{\alpha}_\tau}{d\bar{r}}\frac{d\bar{r}}{dr} = \frac{1}{a}\frac{d\bar{\alpha}_\tau}{d\bar{r}} \tag{123}$$

$$\frac{d^2\alpha_t}{dr^2} = \frac{d^2\bar{\alpha}_\tau}{dr^2} = \frac{d}{dr}\left(\frac{d\bar{\alpha}_\tau}{dr}\right) = \frac{d}{dr}\left(\frac{1}{a}\frac{d\bar{\alpha}_\tau}{d\bar{r}}\right) = \frac{1}{a^2}\frac{d^2\bar{\alpha}_\tau}{d\bar{r}^2} \tag{124}$$

Therefore, new damage parameter ($\bar{\alpha}_\tau$) needs to be determined in the interval $[1, \delta_\tau)$. Using energy balance equation (Eq. (116)), damage parameter ($\bar{\alpha}_\tau$) must satisfy the following differential equation:

$$\frac{1}{\lambda^2}\left(\frac{d^2\bar{\alpha}_\tau}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d\bar{\alpha}_\tau}{d\bar{r}}\right) + \frac{q^2}{\bar{r}^4}(1 - \bar{\alpha}_\tau) = 1 \tag{125}$$

The first derivative of $\bar{\alpha}_\tau$ vanishes at $\bar{r} = 1$ due to Kuhn-Tucker condition (Eq. (109)). Continuity of $\bar{\alpha}_\tau$ and its first derivative at $\bar{r} = \delta_\tau$ leads to equality of both quantities to zero. Therefore, appropriate boundary conditions for Eq. (125) read:

$$\frac{d\bar{\alpha}_\tau}{d\bar{r}}(\bar{r} = 1) = 0\,, \quad \frac{d\bar{\alpha}_\tau}{d\bar{r}}(\bar{r} = \delta_\tau) = 0, \;\; \bar{\alpha}_\tau(\bar{r} = \delta_\tau) = 0 \tag{126}$$

2- Prescribed pressure on boundary

Substituting from Eq. (40) and Eq. (46) into Eq. (107) leads to:

$$\int_{\Omega} (w - \frac{a^4 P_i^2}{\mu r^4})\beta d\Omega \geq 0 \quad \beta > 0 \tag{127}$$

For all r inequality (127) is possible only if $P_i \leq \sqrt{\mu w}$. Inequality (127) holds for specific radius and internal pressure and elastic solution is stable if $w \geq \frac{a^4 P_i^2}{\mu r^4}$. Otherwise, stability is not satisfied. Therefore, at $r = a$ constant w can be defined using the previously obtained critical internal pressure (Eq. (72)) as follows:

$$w = \frac{\sigma_y^2}{4\mu} \tag{128}$$

Interestingly, constant w is a function of material properties and not the geometry and also it is the same as the solution in the case of prescribed internal displacement. Loading and the geometry of the problem does not depend on θ, so the damage evolution will depend only on r and t. In this case three new dimensionless variables are introduced including one spatial variable ($\bar{r}$), one variable regarding geometry and internal length of material (λ) and a loading parameter (q) as follows:

$$q = \frac{2P_i}{\sigma_y} \ , \quad \bar{r} = \left(\frac{r}{a}\right), \quad \lambda = \frac{a}{\ell} \tag{129}$$

Change in the displacement field scales up with the coefficient $\frac{1}{a}$.

$$\frac{d^2 u_{\bar{r}}}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{du_{\bar{r}}}{d\bar{r}} - \frac{u_{\bar{r}}}{\bar{r}^2} = 0 \qquad \sigma_{\bar{r}\bar{r}}(\bar{r} = 1) = P_i \quad \sigma_{\bar{r}\bar{r}}(\bar{r} \to \infty) = 0$$

$$u_{\bar{r}} = \bar{C}_1 \bar{r} + \frac{\bar{C}_2}{\bar{r}} \qquad \bar{C}_1 = 0 \, , \bar{C}_2 = -\frac{P_i}{2\mu} \tag{130}$$

Therefore, the relation between displacement field in new configuration and the original one reads:

$$\bar{\boldsymbol{u}}_t = \boldsymbol{u}_t(\bar{r}) = \frac{1}{a}\boldsymbol{u}_t\ (r) \tag{131}$$

Following the steps in elastic solution of the problem shows that the strain field does not change with this change of variable. Strain field reads:

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_t) = \begin{bmatrix} \frac{P_i}{2\mu\bar{r}^2} & 0 & 0 \\ 0 & \frac{-P_i}{2\mu\bar{r}^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{a^2 P_i}{2\mu r^2} & 0 & 0 \\ 0 & \frac{-a^2 P_i}{2\mu r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{\varepsilon}(\boldsymbol{u}_t) \tag{132}$$

Stress field changes since damage occurs during the loading:

$$\bar{\boldsymbol{\sigma}}_\tau = S(\bar{\alpha}_\tau)\ \mathbb{E}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_\tau) \tag{133}$$

Following the differentiation of new damage parameter in the same way as it was done in the displacement based problem, new damage parameter ($\bar{\alpha}_\tau$) needs to be determined in the interval $[1, \delta_\tau)$. Using energy balance equation (Eq. (116)), damage parameter ($\bar{\alpha}_\tau$) must satisfy the following differential equation:

$$\frac{1}{\lambda^2}\left(\frac{d^2\bar{\alpha}_\tau}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d\bar{\alpha}_\tau}{d\bar{r}}\right) + \frac{q^2}{\bar{r}^4}(1 - \bar{\alpha}_\tau) = 1 \tag{134}$$

With the same argument as before, appropriate boundary conditions for Eq. (134) are given as:

$$\frac{d\bar{\alpha}_\tau}{d\bar{r}}\ (\bar{r} = 1) = 0\,, \quad \frac{d\bar{\alpha}_\tau}{d\bar{r}}(\bar{r} = \delta_\tau) = 0, \ \ \bar{\alpha}_\tau(\ \bar{r} = \delta_\tau) = 0 \tag{135}$$

It can be seen that by appropriate change of variables the same equation can be obtained for both cases of applied external forces i.e. Eq. (125) and Eq. (134).

Quadratic form of dissipation function $\omega(\alpha) = w\alpha^2$

In the following part, previous problem is solved using another type of dissipation function to find the possibility of the damage localization. Dissipation function is considered as $\omega(\alpha) = w\alpha^2$. Inequality (115) changes to the following form using quadratic form of dissipation function:

$$\int_{\Omega} (-(1-\alpha_t)\mathbb{E}\,\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon} + 2w\alpha_t)\beta d\Omega \geq 0 \qquad for\ \alpha_t = 0\ and\ \beta > 0 \tag{136}$$

Substituting $\alpha_t = 0$ into inequality (136) shows that the elastic phase is not stable since there is not any specific region that holds this inequality. Therefore, damage evolves from the beginning of the loading and elastic domain cannot be specified. Also, energy balance equation (Eq. (116)) becomes:

$$\left(-(1-\alpha_t)\mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_t).\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + 2w\alpha_t - w\ell^2\left(\frac{d^2\alpha_t}{dr^2} + \frac{1}{r}\frac{d\alpha_t}{dr}\right)\right)\dot{\alpha}_t = 0 \quad in\ \Omega \tag{137}$$

The type of boundary conditions affects inequality (136). Therefore, two specific cases are considered to analyze for infinite domain with circular hole:

1- Prescribed displacement on boundary

Substituting from Eq. (40) and Eq. (43) along with $\alpha = 0$ into Eq. (136) leads to:

$$\int_{\Omega} \left(-\frac{4\mu a^2\, U_i^2}{r^4}\right)\beta d\Omega \geq 0 \quad \beta > 0 \tag{138}$$

As mentioned before, inequality (138) is not satisfied for specific radius and prescribed displacement, and elastic solution is not stable and damage starts right at the beginning of the loading. In this case, three new dimensionless variables are introduced including one spatial variable ($\bar{r}$), one variable regarding geometry and internal length of material (λ)

and a loading parameter (q) considering the loading process and a stationary evolution $(\boldsymbol{u}_t\,, \alpha_t)$ such that:

$$\alpha_t(r) = \bar{\alpha}_\tau(\bar{r}), \;\; \bar{r} = \left(\frac{r}{a}\right), q = \frac{\sqrt{2}\sqrt{\mu}U_i}{\sqrt{w}a}, \lambda = \frac{\sqrt{2}a}{\ell} \tag{139}$$

Change in displacement, strain and stress fields obey the same changes for the previous case i.e. Eqs. (120) – (122) are valid. Therefore, new damage parameter ($\bar{\alpha}_\tau$) needs to be determined in the interval $[1, \delta_\tau)$. Using energy balance equation (Eq. (137)), damage parameter ($\bar{\alpha}_\tau$) must satisfy the following differential equation:

$$\frac{1}{q^2\lambda^2}\left(\frac{d^2\bar{\alpha}_\tau}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d\bar{\alpha}_\tau}{d\bar{r}}\right) - \left(\frac{1}{\bar{r}^4} + \lambda^2\right)\bar{\alpha}_\tau + \frac{1}{\bar{r}^4} = 0 \tag{140}$$

Appropriate boundary conditions for Eq. (140) are the same as Eq. (126).

2- Prescribed pressure on boundary

Substituting from Eq. (40) and Eq. (46) into Eq. (136) leads to:

$$\int_\Omega \left(-\frac{a^4\, P_i^2}{\mu r^4}\right)\beta d\Omega \;\geq 0 \quad \beta > 0 \tag{141}$$

As mentioned before, inequality (141) is not satisfied for specific radius and prescribed displacement, and elastic solution is not stable. In this case, three new dimensionless variables are introduced including one spatial variable ($\bar{r}$), one variable regarding geometry and internal length of material (λ) and a loading parameter (q) considering the loading process as following:

$$\bar{r} = \left(\frac{r}{a}\right), \quad q = \frac{P_i}{w}, \lambda = \frac{\sqrt{2}a}{\ell} \tag{142}$$

Change of displacement, strain and stress fields obey the same changes for the previous case i.e. Eqs. (129) – (132) are valid. Therefore, new damage parameter ($\bar{\alpha}_\tau$) needs to be determined in the interval $[1, \delta_\tau)$. Using energy balance equation (Eq. (137)), damage parameter ($\bar{\alpha}_\tau$) must satisfy the following differential equation:

$$\frac{1}{\lambda^2 q^2}\left(\frac{d^2\bar{\alpha}_\tau}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{d\bar{\alpha}_\tau}{d\bar{r}}\right) - \left(\frac{1}{\bar{r}^4} + \lambda^2\right)\bar{\alpha}_\tau + \frac{1}{\bar{r}^4} = 0 \tag{144}$$

Appropriate boundary conditions for Eq. (144) are given in Eq. (135). As it was obtained in the linear form of dissipation function, it can be seen that by appropriate change of variables the same equation can be obtained for both cases of applied external forces i.e. Eq. (140) and Eq. (143).

3. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

In the first chapter of this thesis, properties of local and standard damage models as a classical approach to the problem is considered. Consequently, it has been shown that standard model can be described through variational approach. Then, an appropriate way to add gradient term to state function consistent with thermodynamics and mechanics principles is investigated. Finally, construction of general gradient damage model through variational approach considering three main specifications of the problem i.e. irreversibility, stability and energy balance is presented in full details. Main properties of gradient damage models through variational approach have been reviewed. This approach has been successfully applied on thermal shock problem (Sicsic et al., 2013) which was the motivation of the second chapter. Since damage propagation and fracture mechanism is in interest in various area of engineering a general case of this problem is considered in the second chapter. Nucleation and growth of cracks in pressurized domain has applications in engineering from designing reservoirs to oil wells. Therefore, in second chapter general gradient damage model formulation is used to investigate damage evolution in pressurized cavity. Different cases of geometry and applied external pressure as boundary conditions are considered to increase the capability of the analysis. Elastic damage evolution equations are obtained with different types of dissipation function. An appropriate way to solve these equations is proposed and damage profile in a ring around the boundary is obtained. It is shown that gradient damage model through variational approach can be applied on a 2D problem and is capable enough to find the damage profile.

This work can be extended with the following recommendations.

Stability analysis of the elastic solution needs to be done and possible way of bifurcation should be characterized through the appropriate use of material properties and geometry of the problem. Softening or hardening behavior of the material after the elastic phase needs to be taken into account. This task enhances the formulation to be capable enough for using various types of material. Obtained general formulations in this thesis and aforementioned tasks needs to be implemented on computer as an energy minimizing problem using the Finite Element Method.

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VITA

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