Group theoretical approach to pairing and non-linear phenomena in atomic nuclei

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GROUP THEORETICAL APPROACH TO PAIRING AND NON-LINEAR PHENOMENA IN ATOMIC NUCLEI

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Physics and Astronomy

by

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Where can we find Him? Not on earth, for He is not here. And not in heaven, for we are not there. But in our hearts we can find Him. He ascended to heaven openly so that He could come back to us inwardly, and never leave us again.

–St. Augustine
To my family,
for their tremendous love and support
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Abstract

The symplectic $\mathfrak{sp}(4)$ algebra provides a natural framework for studying proton-neutron ($pn$) and like-nucleon pairing correlations as well as higher-$J$ $pn$ interactions in nuclei when protons and neutrons occupy the same shell. While these correlations manifest themselves most clearly in the binding energies of $0^+$ ground states, they also have a large effect on the spectra of excited isobaric analog $0^+$ states. With a view towards nuclear structure applications, a fermion realization of $\mathfrak{sp}(4)$ is explored and its $q$-deformed extension, $\mathfrak{sp}_q(4)$, is constructed for single and multiple shells. The $\mathfrak{su}_q(2)$ substructures that enter are associated with isospin symmetry and with identical-particle and $pn$ pairing.

We suggest a non-deformed as well as a $q$-deformed algebraic descriptions of pairing for even-$A$ nuclei of the mass $32 \leq A \leq 164$ region. A Hamiltonian with a symplectic dynamical symmetry is constructed and its eigenvalues are fit to the relevant Coulomb corrected experimental $0^+$ state energies in both the “classical” and “deformed” cases. While the non-deformed microscopic theory yields results that are comparable to other models for light nuclei, the present approach succeeds in providing a reasonable estimate for interaction strength parameters as well as a detailed investigation of isovector pairing, symmetry energy and symmetry breaking effects. It also reproduces the relevant ground and excited $0^+$ state energies and predicts some that are not yet measured. The model successfully interprets fine features driven by pairing correlations and higher-$J$ nuclear interactions. In a classification scheme that is inherent to the $\mathfrak{sp}(4)$ algebraic approach, a finite energy difference technique is used to investigate two-particle separation energies, irregularities found around the $N = Z$ region, and like-particle and $pn$ isovector pairing gaps. The analysis identifies a prominent staggering behavior between groups of even-even and odd-odd nuclides that is due to discontinuities in the pairing and symmetry terms. While the “classical” limit of the theory provides good overall results, the analysis also shows that $q$-deformation can be used to gain a better understanding of higher-order effects in the interaction within each individual nucleus.
Chapter 1

Introduction

In the universe we live, the little we do know is the foundation for understanding nature at a deeper level. We are able to ‘see’ effects, more of them the sharper our ‘eyes’ become. And while we know that there is something that causes the sun to move, hydrogen to burn, nuclei to fuse or neutrinos to oscillate between flavors, do we really comprehend this something or do we need to admit like St. Augustine (4th century AD) on the concept of time: “What then is time? If no one asks of me, I know; if I wish to explain to him who asks, I know not.” As an avalanche, the deeper our knowledge goes the more we need to understand the four fundamental forces of nature and search deeper for the elementary (structureless) constituents of matter. While this avalanche may drag us onto a seemingly infinite path like Zeno’s arrow in the ancient paradox (450 BC) that always travels just the first half of the distance to the end point, never reaching its aim, we must continue for such is the nature of scientific discovery.

Presently, our knowledge goes beyond matter discretized in atoms and each atom consisting of a nucleus and electrons. It extends to the very nature of nuclear structure that results from a complicated strong interaction between nucleons, protons and neutrons. And it does not stop there but seeks inside a nucleon to redefine it as a complex composition of quarks. The structure and interaction of nucleons can be then described by the quantum chromodynamics (QCD) of quarks and gluons but, alas, even this theory confronts a great many difficulties in providing reasonable answers for low-energy nuclear structure physics. However, one can still move forward because for low-energy phenomena of the type we consider here, one can ignore the quark substructure of nucleons and consider the protons and neutrons as basic constituents of nuclei.

The long-lasting interest in nuclear structure physics is fueled by the fact that the nuclear problem we are left with, the many-nucleon non-relativistic Schrödinger equation, cannot be treated exactly even if only two-body interactions are assumed between the particles. Two major themes help. The analysis of empirical evidence gives rise to simplified or idealized models of physical systems and the recognition of symmetries, exact and approximate, often yield tractable model spaces and exact solutions. A group theoretical approach makes use of exact as well as almost exact symmetries in nuclear dynamics and thus leads to a powerful and elegant description of particular sets of phenomena in a mathematically robust and
A corner stone of many nuclear structure studies up to the present time is a marvelous model that produces a shell structure and, above all, major shell closures at the empirical magic proton (neutron) numbers ($Z(N) = 2, 8, 20, 28, 50, 82, \ldots$). This is the successful harmonic oscillator shell-model of independently moving nucleons, each with total angular momentum $j$, that fill up single-particle $j$-levels [1]. In the framework of the shell-model, the residual interactions in nuclei are dominated by short-range pairing correlations and a long-range quadrupole-quadrupole interaction. The first accounts for the pairing gap in nuclear energy spectra triggered by the formation of fermion pairs, while the latter is responsible for the strong enhanced electric quadrupole transitions in collective rotational bands. Both limits, pairing and quadrupole, can be clearly described by symplectic symmetry structures. In the quadrupole limit, the non-compact symplectic $Sp(2n, \mathbb{R})$ group [2, 3, 4, 5] governs a shape-determined collective dynamics. On the other hand, in the pairing limit, the nuclear energy spectra are generated by the conventional $U(2(2j + 1)) \supset Sp(2j + 1)$ seniority scheme [6, 7], or alternatively by the symplectic $Sp(2j + 1)$ group together with its dual, the symplectic $Sp(4)$ group ($\sim SO(5)$) [8, 9, 10, 11, 12, 13, 14, 15]. The latter is an extension to two types of nucleons of Kerman’s quasi-spin $SU(2)$ group [16] to incorporate proton-neutron pairing correlations.

A recent renaissance of studies on pairing is related to the search for a reliable microscopic theory for a description of medium and heavy nuclei around the $N = Z$ line, where protons and neutrons occupy the same major shells and their mutual interactions are expected to influence significantly the structure and decay of these nuclei. Such a microscopic framework is also essential for astrophysical applications, for example the description of the $rp$-process in nucleosynthesis, which runs close to the proton-rich side of the valley of stability through reaction sequences of proton captures and competing $\beta$ decays [17, 18]. The revival of interest in pairing correlations is also prompted by the initiation of radioactive beam experiments [19], which advance towards exploration of ‘exotic’ nuclei, such as neutron-deficient or $N \approx Z$ nuclei far off the valley of stability.

The objective of the present dissertation research is to obtain a microscopic description and investigate properties of pairing-governed $0^+$ states in the energy spectra of even-$A$ nuclei with mass numbers $32 \leq A \leq 164$ with protons and neutrons filling the same major shell. This is achieved by employing the group theoretical $Sp(4)$ approach together with its $q$-deformed generalization, the construction of which is realized in Chapter 2. Since the dawn of the $q$-deformed (quantum) algebra concept [20, 21, 22], the recognition of two major features makes the approach very attractive for physical applications. The first is that in the $q \to 1$ limit of the deformation parameter the $q$-algebra reverts back to the “classical” Lie algebra. Second, the $q$-deformation introduces non-linear terms into the theory while preserving the underlying symmetry. The specific $q$-deformation of $sp(4)$ we employ is one needed for the present work and yet it retains the properties of the standard construction of the $q$-deformed algebra [23, 24, 25].
In Chapter 3 we suggest two models, namely a “classical” one, designed according to the \( \mathfrak{sp}(4) \) algebraic approach, and a \( q \)-deformed one in the framework of the \( \mathfrak{sp}_q(4) \) algebra. Their model spaces are described and the associated model Hamiltonians are presented with emphasis on the physical essence of the distinct types of interactions they include. The “classical” model Hamiltonian is a two-body interaction, including proton-neutron and like-particle pairing plus symmetry terms, while the \( q \)-deformed Hamiltonian adds higher-order many-body terms. The shell structure and its dimension play an important role in the construction of the fermion pairs and their interaction is in accordance with the Pauli Exclusion principle.

The last Chapter 4, which employs both models, weaves a story that only the real internucleon interaction knows how to tell. In both the “classical” and \( q \)-deformed settings, we present comparisons of the theoretical results with experimental values and examine in detail their outcome. The investigation reveals the advantage of both approaches among other theoretical studies [26, 27, 28, 15, 29, 30] (for the “classical” model) and [31, 32, 33, 34] (for the \( q \)-deformed model). We also include a discussion of isospin symmetry breaking and isospin mixing — a recent focus of many novel high-precision experiments [35, 36, 37, 38] and theoretical investigations [39, 40, 41], which comprise a precise test of the standard particle model. A more detailed examination of the nuclear structure, including \( N = Z \) anomalies, pairing gaps and staggering effects, is approached through a finite energy difference method in a useful \( \text{Sp}(4) \) systematics (Section 4.4). In Section 4.6, the ongoing question on the physical meaning behind the \( q \)-parameter is attacked within the framework of the non-linear \( \mathfrak{sp}_q(4) \) algebraic approach. The concept of quantum deformation is linked to the smooth behavior of physical phenomena in atomic nuclei.

A summary of our findings and the main conclusions are discussed in the final Chapter 5. The symplectic \( \text{Sp}(4) \) scheme allows not only for an extensive systematic study of various experimental patterns of the even-\( A \) nuclei, it also offers simple \( \mathfrak{sp}(4) \) and \( \mathfrak{sp}_q(4) \) algebraic models for interpreting the results. Moreover, the present investigation serves as a test for the validity and reliability of the models with respect to the interactions they include. This work is our attempt to find a cause behind observed effects. The search is not over. In the universe we live, the little we do know serves to inspire an even deeper understanding of nature.
Chapter 2

Symplectic $\mathfrak{sp}(4)$ Algebra and $q$-Deformations of $\mathfrak{sp}(4)$

Symplectic algebras enter naturally in the description of many-particle systems when the number of particles or couplings between the particles change in a pairwise fashion from one configuration to the next. Both compact and non-compact versions of the symplectic algebra provide for a powerful and useful algebraic tool for understanding the collective dynamics of a many-body physical system. In general, the compact $\mathfrak{sp}(2n)$ symplectic algebra can be used to explore pairing correlations in systems with $n$ different types of particles. The non-compact algebra, $\mathfrak{sp}(2n, \mathbb{R})$, is suitable to describe collective vibrational excitations of a system of particles in an $n$-dimensional harmonic oscillator potential.

In particular, the growing interest in symplectic symmetries is related to physical applications in nuclear and atomic structure, like the applications of the compact version to pairing correlations in nuclei [8, 13, 15] and in superconductors [42], as well as the applications of the non-compact one to monopole excitations in the harmonic oscillator shell model [43] and to shell-model core excitations associated with the nuclear quadrupole degree of freedom [2, 3, 4, 5]. A further interest in the symplectic algebras is related to their use in mapping methods from the fermion space to the space spanned by collective bosons and ideal fermions [44] with a primary purpose to simplify the Hamiltonian of the initial problem.

The simplest non-trivial case ($n = 2$) of the compact symplectic algebra, $\mathfrak{sp}(4)$, has been related to three distinct nuclear applications, namely, (1) a charge-independent pairing model [10, 11, 12, 13, 14, 15, 45], (2) a monopole-plus-pairing model and (3) a two-dimensional vibration-rotation model (see [46] and references there). The physical interpretation of the underlying quantities and operators follows straightforwardly from the model that is adopted, which for our investigation of nuclear pairing correlations is the first of these, that is, a charge-independent pairing model. However, we try to keep the algebraic approach as general as possible so that the mathematical apparatus can also be applied to the other two models if they are of interest. Furthermore, it is rather easy to generalize this work to higher rank algebras while the algebraic techniques are illustrated by the comparatively simple $\mathfrak{sp}(4)$ example [46].
2.1 Fermion Realization of the $\mathfrak{sp}(4)$ Algebra

The fermion realization of the $\mathfrak{sp}(4)$ Lie algebra (Appendix A) [10, 11, 12, 13, 46], isomorphic to the $\mathfrak{so}(5)$ Lie algebra of the five-dimensional rotation group SO(5) ($\mathfrak{sp}(4) \sim \mathfrak{so}(5)$) (see for example [47]), is constructed in terms of the fermion creation and annihilation operators, $c_{jm\sigma}^\dagger$ and $c_{jm\sigma}$. Each operator $c_{jm\sigma}^\dagger$ ($c_{jm\sigma}$) creates (annihilates) a particle of type $\sigma$ ($\frac{1}{2}$ for proton and $-\frac{1}{2}$ for neutron), in a single-particle state of total angular momentum $j$ (half-integer) with projection $m$ along the $z$-axis ($-j \leq m \leq j$). These operators are Hermitian conjugates of one another, $(c_{jm\sigma}^\dagger)^\dagger = c_{jm\sigma}$, and satisfy the standard Fermi anticommutation relations\(^1\):

$$\{c_{jm'\sigma'}, c_{jm\sigma}^\dagger\} = \delta_{m',m} \delta_{\sigma',\sigma}, \quad \{c_{jm'\sigma'}, c_{jm\sigma}^\dagger\} = \{c_{jm'\sigma'}, c_{jm\sigma}\} = 0.$$

In a model with a degenerate single-particle $j$-level, the dimension of the fermion space for given $\sigma$ is $2\Omega_j = 2j + 1$.

The $\mathfrak{sp}(4)$ algebra is realized as a bilinear product of the second-quantized fermion operators\(^2\) coupled to total angular momentum and parity $J^\pi = 0^+$ [10, 13, 46]:

$$A_k^{(j)} = \frac{1}{\sqrt{2\Omega_j (1 + \delta_{\sigma\sigma'})}} \sum_{m=-j}^j (-1)^{-m} c_{jm\sigma}^\dagger c_{jm,-m,\sigma'}^\dagger = (A_k^{-})^\dagger, \quad k = \sigma + \sigma', \quad (2.2)$$

$$A_k^{(j)} = \frac{1}{\sqrt{2\Omega_j (1 + \delta_{\sigma\sigma'})}} \sum_{m=-j}^j (-1)^{-m} c_{jm,-m,\sigma} c_{jm,\sigma'}^\dagger, \quad k = \sigma + \sigma', \quad (2.3)$$

$$\tau_{\pm}^{(j)} = \frac{1}{\sqrt{2\Omega_j}} \sum_{m=-j}^j c_{jm,\pm1/2}^\dagger c_{jm,\pm1/2}, \quad (2.4)$$

$$N_{\pm}^{(j)} = \sum_{m=-j}^j c_{jm,\pm1/2}^\dagger c_{jm,\pm1/2}. \quad (2.5)$$

A direct computation of the commutators\(^3\) of (2.2)-(2.5), obtained by means of (2.1), yields the commutation relations of the $\mathfrak{sp}(4)$ algebra [13]. The operators, (2.2) and (2.3), which are also denoted as

$$A^{\dagger} = A^+, \quad A = A^- \quad (2.6)$$

\(^1\)The anticommutator of two operators $X$ and $Y$ is defined as $\{X, Y\} = XY + YX$.

\(^2\)In terms of four irreducible tensor operators, $(t_\sigma)_m = c_{jm\sigma}$ and $(t_\sigma^\dagger)_m = (-)^{-m} c_{jm,-m,\sigma}$, $\sigma = \pm1/2$, of degree $j$ with respect to the total angular momentum operator $J$, the ten basis operators in $\mathfrak{sp}(4)$ are obtained as all possible scalar (zero total angular momentum) tensor products of $(t_{1/2}),(t_{-1/2}),(t_{1/2}^\dagger)$ and $(t_{-1/2}^\dagger)$ and they commute with $J$ [13].

\(^3\)The commutator given by the Lie multiplication of two operators $X$ and $Y$ is defined as $[X, Y] = XY - YX$. 
are symmetric with respect to the interchange of $\sigma$ and $\sigma'$. They create (annihilate) a pair of fermions coupled to total angular momentum $J = 0$ [10] and thus constitute boson-like objects according to the spin-statistics theorem [48]. $A_1^{\pm(j)}$ create (annihilate) in the $j$-level a pair of protons $(pp)$, $A_0^{\pm(j)}$ – a pair of a proton and a neutron $(pn)$, and $A_{-1}^{\pm(j)}$ – a pair of neutrons $(nn)$. The operators $\tau_{\pm}^{(j)}$ and $N_{\pm 1}^{(j)}$ preserve the number of fermions, where $\tau_{\pm}^{(j)}$ change the isospin coordinate of a nucleon.

The two commuting operators, $N_{\pm 1}^{(j)}$ (2.5), that form a basis in the Cartan subalgebra of $\mathfrak{sp}(4)$ (Appendix A), are the operators of the total number of fermions of each kind (proton number/neutron number) in a single $j$-level. The action of these operator on the fermion creation and annihilation operators is given by

$$N_{2\sigma}^{(j)} c_{j m \sigma}^\dagger = c_{j m \sigma'}^\dagger (N_{2\sigma}^{(j)} + \delta_{\sigma,\sigma'}),$$

$$N_{2\sigma}^{(j)} c_{j m \sigma'} = c_{j m \sigma'} (N_{2\sigma}^{(j)} - \delta_{\sigma,\sigma'}), \quad \sigma, \sigma' = \pm 1/2,$$  

(2.7)

and together with the anticommutation relations (2.1) yield the equality

$$\sum_{m=-j}^{j} c_{j m \pm 1/2}^\dagger c_{j m \pm 1/2}^\dagger = 2\Omega_j - N_{\pm 1}^{(j)}. $$

(2.8)

The linear combinations of (2.5) are also in the Cartan subalgebra of $\mathfrak{sp}(4)$, two of them are of particular interest, the operator that counts the total number of particles,

$$\hat{N}^{(j)} = N_{1}^{(j)} + N_{-1}^{(j)} = \sum_{\sigma=-1/2}^{1/2} \sum_{m=-j}^{j} c_{j m \sigma}^\dagger c_{j m \sigma},$$

(2.9)

and the third projection of the isospin,

$$\tau_{0}^{(j)} = \frac{1}{2} (N_{1}^{(j)} - N_{-1}^{(j)}) = \sum_{\sigma=-1/2}^{1/2} \sum_{m=-j}^{j} \sigma c_{j m \sigma}^\dagger c_{j m \sigma}.$$ 

(2.10)

Every transformation in the vector space of the single-particle fermion operators generated by the operators (2.2)-(2.5) is an element of the $\text{Sp}(4)$ Lie group. In this context the basis operators (2.2)-(2.5) of the $\mathfrak{sp}(4)$ algebra are referred to as generators of the $\text{Sp}(4)$ group. The symplectic group, $\text{Sp}(4)$, is a ten parameter group and hence the number of the generators, (2.2)-(2.5), is ten. $\text{Sp}(n)$ is of rank two (Table A.1) and thus there exist two commuting operators, namely $N_{\pm 1}^{(j)}$ (2.5) or $\{ \hat{N}^{(j)} (2.9) \text{ and } \tau_{0}^{(j)} (2.10) \}$, and two invariant operators. The latter are operators in the enveloping $\mathfrak{sp}(4)$ algebra (polynomials of the $\text{Sp}(4)$ generators) and commute with all the generators of $\text{Sp}(4)$. The first invariant is the
second-order Casimir operator,

\[
C_2^{(j)}(\mathfrak{sp}(4)) = A^{(j)} \cdot A^{(j)} + A^{(j)} \cdot A^{(j)} + \frac{1}{\Omega_j} \left( (\tau^{(j)})^2 + \left( \hat{\Omega}^{(j)} - 2\Omega_j \right)^2 \right) \tag{2.11}
\]

\[
= \left\{ A_{-1}^{(j)}, A_{1}^{(j)} \right\} + \left\{ A_{0}^{(j)}, A_{0}^{-1} \right\} + \left\{ A_{1}^{(j)}, A_{-1}^{(j)} \right\} + \left\{ \tau_+^{(j)}, \tau_-^{(j)} \right\} + \frac{1}{\Omega_j} \left( \tau_0^{(j)} \tau_0^{(j)} + \left( \hat{\Omega}^{(j)} - 2\Omega_j \right)^2 \right), \tag{2.12}
\]

where the notations are defined as follows, \( A^{(j)} = \sum_{k=-1}^1 A_k^{(j)} A_k^{-1} \) and \( (\tau^{(j)})^2 = \Omega_j \left\{ \tau_+^{(j)}, \tau_-^{(j)} \right\} + \tau_0^{(j)} \tau_0^{(j)}. \) The other invariant operator is of fourth order in the group generators and is given in [13].

In a model with a single-\( j \) shell, the index \( (j) \) is redundant as it takes on only one value and it can be dropped from the notation introduced above. However, one should still keep in mind that the operators realized as (2.2)-(2.5) are single-level operators.

### 2.1.1 Subalgebras of \( \mathfrak{sp}(4) \)

Four different subsets of the generators (2.2)-(2.5) close on an \( \mathfrak{u}(2) \) unitary subalgebra of \( \mathfrak{sp}(4) \) (Table 2.1). The reductions yield four distinct limiting cases described by the algebra

<table>
<thead>
<tr>
<th>symmetry ( \mu )</th>
<th>( \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau ) (isospin)</td>
<td>( \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) )</td>
</tr>
<tr>
<td>( \mathfrak{u}_{C_2}(1) )</td>
<td>( \mathfrak{su}^\mu(2) )</td>
</tr>
<tr>
<td>( \tau_0 )</td>
<td>( \tau_+ ), ( \tau_0 = \frac{1}{2}(N_1 - N_{-1}) ), ( \tau_- )</td>
</tr>
<tr>
<td>( \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) )</td>
<td></td>
</tr>
<tr>
<td>( \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) )</td>
<td></td>
</tr>
</tbody>
</table>

chains

\[
\mathfrak{sp}(4) \supset \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) \supset \mathfrak{su}^\mu(2) \supset \mathfrak{u}^\mu(1), \mu = \{ \tau, 0, \pm \}. \tag{2.13}
\]

The first-order invariant of \( \mathfrak{u}^\mu(2) \), \( C_1^{\{\tau, 0, \pm\}} = \{ \hat{N}, \tau_0, N_{\pm 1} \} \), realizes the decomposition \( \mathfrak{u}^\mu(2) = \mathfrak{u}_{C_2}(1) \oplus \mathfrak{su}^\mu(2) \) as it commutes with the rest of the operators of \( \mathfrak{u}^\mu(2) \) (Table 2.2). The latter (second column in Table 2.1) close on the \( \mathfrak{su}(2) \) algebra, which is isomorphic to the \( \mathfrak{so}(3) \) algebra of three-dimensional rotations and both have an identical algebraic structure to that of \( \mathfrak{sp}(2) \). The operators of each \( \mathfrak{su}^\mu(2) \) subalgebra are associated with the polar components of a “spin” vector operator that generates rotations in an abstract space. This is in analogy with the spin operator \( \mathbf{S} \) (and the angular momentum operator \( \mathbf{J} \)), which is a...
vector in coordinate space with three components, \( S_1 = \frac{S_+ + S_-}{2} \), \( S_2 = \frac{S_+ - S_-}{2i} \), and \( S_3 = S_0 \), where \( S_{+,-} \) form a basis in the spin algebra, \( \mathfrak{su}(2) \sim \mathfrak{so}(3) \), with standard commutation relations \( [S_+, S_-] = 2S_0 \) and \( [S_0, S_\pm] = \pm S_\pm \). The latter can be compared to the \( \mathfrak{su}^\mu(2) \) commutation relations (Table 2.2) and can be reproduced after a trivial renormalization of the corresponding raising/lowering “spin” operators. The third projection of the “spin” operator (middle operator in the third column of Table 2.1) further defines the \( \mathfrak{su}^\mu(2) \supset \mathfrak{u}^{\mu}(1) \) reduction in the usual way. The \( \mathfrak{su}(2) \) algebra is of first rank and therefore there exists only one invariant, the second-order Casimir operator, \( C_2(\mathfrak{su}^\mu(2)) = C_2^\mu \) (Table 2.2).

Table 2.2: Commutation relations of the basis operators of the unitary \( \mathfrak{u}^\mu(2) \) subalgebras of \( \mathfrak{sp}(4) \), \( \mu = \tau, 0, \pm \), along with the Casimir invariants of \( \mathfrak{su}^\mu(2) \). The repeated commutation relations are in parenthesis.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>unitary ( \mathfrak{u}^\mu(2) ) subalgebra</th>
<th>( C_2(\mathfrak{su}^\mu(2)) = C_2^\mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>( [\hat{N}, \tau_0, \pm] = 0 )</td>
<td>( [\tau_+, \tau_-] = \frac{2\tau_0}{2\Omega} ), ( [\tau_0, \tau_\pm] = \pm \tau_\pm )</td>
</tr>
<tr>
<td>0</td>
<td>( [\tau_0, A_0^{(1)}] = 0 ) ( ([\tau_0, N^0] = 0) )</td>
<td>( [A_0^{(1)}, A_0] = \frac{2N^0}{2\Omega} ), ( [N^0, A_0^{(1)}] = \pm A_0^{(1)} )</td>
</tr>
<tr>
<td>+</td>
<td>( [N^-, A_{\pm 1}^+] = 0 ) ( ([N^-, N^+] = 0) )</td>
<td>( [A_{\pm 1}^+, A_{-1}] = \frac{4N_+}{2\Omega} ), ( [N^+, A_{\pm 1}^+] = \pm A_{\pm 1}^+ )</td>
</tr>
<tr>
<td>−</td>
<td>( [N^+, A_{\pm 1}^-] = 0 ) ( ([N^+, N^-] = 0) )</td>
<td>( [A_{-1}^-, A_1] = \frac{4N_-}{2\Omega} ), ( [N^-, A_{\pm 1}^-] = \pm A_{\pm 1}^- )</td>
</tr>
</tbody>
</table>

An alternative reduction chain to (2.13) is realized through the direct sum of the mutually complementary \( \mathfrak{su}^+(2) \) and \( \mathfrak{su}^-(2) \) subalgebras,

\[
\mathfrak{sp}(4) \supset \mathfrak{su}^+(2) \oplus \mathfrak{su}^-(2),
\]

since each of the generators of the \( \text{SU}^+(2) \) symmetry commutes with all of the \( \text{SU}^-(2) \) generators. This reduction is the mathematically natural one for the \( \text{Sp}(4) \) group, which will be described in more details at the end of Section 2.1.4. In the model (1) under investigation this limit describes like-particle pairing (for both protons and neutrons).

In the charge-independent pairing model, (1), the \( \mathfrak{su}^\tau(2) \) subalgebra (Table 2.1) is associated with isospin symmetry and the vector \( \tau \) with components \( \tau_1 = \frac{\tau_+ + \tau_-}{2} \), \( \tau_2 = \frac{\tau_+ - \tau_-}{2i} \) and \( \tau_3 = \tau_0 \) is the isospin operator acting in the abstract isotopic space of proton and neutron states. In the other three limits, the \( \text{SU}^{\mu}(2) \) generators \( \{A_\mu, N^\mu, A_{-\mu}\} \) correspond to components of the so-called quasi-spin operator \( (s^\mu) \) as introduced by Kerman [16]. The similarity between the isospin/quasi-spin algebras to the angular momentum algebra, which is well-studied, can furnish the solution of the eigenvalue problem for \( \tau^2 \) and \( (s^\mu)^2 \) and yield the properties of the distinct limits of specific physical interest, namely isospin symmetry.
(\mu = \tau) and pp, pn, nn pairing (\mu = 1, 0, -1, respectively). This is the real beauty of the theory.

The pair creation (2.2) and annihilation (2.3) operators are components of two conjugate vectors \( A^\pm = \{ A^\pm_k \}_{k=0, \pm 1} \) with respect to the unitary isospin SU\(^\tau\)(2) subgroup (Table 2.1):

\[
\tau_0, A^\pm_k = k A^\pm_k, \quad \tau_l, A^\pm_k = \pm \frac{1}{\sqrt{\Omega}} A^\pm_{l+k}, \quad l = \pm 1, \quad k = 0, \pm 1. \tag{2.15}
\]

The relations (2.15) mean that the operators \( A^\pm_{0, \pm 1} \) carry (isospin) \( \tau = 1 \) and specifically in our model, where \( \tau \) is interpreted as the isospin operator, \( A^\pm_{0, \pm 1} \) create (destroy) a pair of fermions coupled to a total isospin \( \tau = 1 \).

Together with the commutation relations shown in Table 2.2 and in (2.15) the set of commutation relations for the symplectic \( \text{sp}(4) \) algebra is completed with the additions

\[
[A^\pm_l, A^\pm_k] = 0, \quad l, k = 0, \pm 1, \tag{2.16}
\]

\[
[A^\dagger_l, A_k] = \frac{1}{\sqrt{\Omega}} \tau_{l+k}, \quad l + k \neq 0. \tag{2.17}
\]

### 2.1.2 Models with Symplectic \( \text{Sp}(4) \) Symmetry

As we mentioned briefly in the beginning of this chapter, different interpretations of the quantum numbers of the fermions (2.1) used to construct the generators of the \( \text{Sp}(4) \) group, correspond to different physical meanings for the operators and therefore different physical models, like (1) charge-independent pairing model [10, 11, 12, 13, 14, 15, 45], (2) monopole-plus-pairing model and (3) two-dimensional vibration-rotation model (see [46] and references there). These can be used to describe various aspects of the nuclear interaction (different Hamiltonians) [46]. The quantum number \( \sigma = \pm \frac{1}{2} \) of the single-particle fermion operators defines the algebraic properties of the operators (2.2)-(2.5) and characterizes the various models.

1. **Charge independent pairing model.** The quantum number \( \sigma = \pm \frac{1}{2} \) distinguishes between protons and neutrons, which are assumed to occupy a single degenerate orbit in the \( jj \)-coupling shell model. The ten generators of \( \text{Sp}(4) \), (2.2)-(2.5), carry angular momentum zero. The \( \text{su}(2) \) subalgebra (Table 2.1) is associated with the isospin. The generators of the \( \text{SU}^0(2) \) subgroup describe particles of two different kinds (protons and neutrons) paired to total angular momentum zero and total isospin one. The generators of the \( \text{SU}^\pm(2) \) subgroups describe particles of the same kind coupled again to \( J = 0 \) and \( \tau = 1 \). The Hamiltonian of the model, \( H \), is restrained to commute with the third isospin projection \( \tau_0 \) (a more strict requirement is that \( H \) transforms as a scalar under the isospin \( \text{SU}^\tau(2) \) subgroup). Additionally, the Hamiltonian commutes with the operator of the total number of particles, \( \hat{N} \), as \([\hat{N}, \tau_0] = 0\). According to Noether’s Theorem both operators, \( \hat{N} \) and \( \tau_0 \), yield integrals of motion, that is, conserved quantities. In this way the Hamiltonian of the system preserves the total
Figure 2.1: Models of physical interest with symplectic Sp(4) symmetry and their limiting descriptions of the nuclear many-body interaction.

number of particles and the third isospin projection. We exploit this model (1) to investigate pairing correlations in atomic nuclei.

2. **Monopole-plus-pairing model.** The quantum number $2\sigma = \pm 1$ distinguishes between two separated single-particle levels of equal degeneracy. The model is a generalization of the Lipkin model [49] and of the two-level pairing model. The ten generators of Sp(4), (2.2)-(2.5), have angular momentum zero. The SO(4) = SU$^+$(2) $\otimes$ SU$^-$ (2) subgroup corresponds to pairing of particles of only one kind (protons or neutrons). The SU$^+$(2) subgroup gives the monopole limit, which describes scattering of particles between the levels that does not change the angular momentum [49]. The model Hamiltonian is required to be a scalar under three-dimensional rotations and to conserve number of particles.

3. **Two-dimensional vibration-rotation model.** The quantum number $2\sigma = \pm 1$ is interpreted as a unit of angular momentum along (opposite) the (spatial) $z$-axis. The operators $A^\pm_k$ create (annihilate) two particles and add (remove) $2k$ units of angular momentum in a completely degenerate orbit. The number preserving operators, $\tau_{k=0,\pm}$, add $2k$ units of angular momentum, where $2\tau_0$ represents the physical angular
momentum in the two-dimensional model. The corresponding SU\(^T\)(2) group gives the rotational limit (two-dimensional version of the Elliott SU(3) model, [50]). The SU\(^0\)(2) is associated with the neutron (proton) pairing limit of the model. The Hamiltonian is required to conserve particle number and angular momentum (\(\hat{N}\) and \(2\tau_0\)) and describes rotations and vibrations in two dimensions that can serve as the grounds to extend the approach to applications to the collective behavior of real nuclear systems.

2.1.3 Irreducible Representations of the Sp(4) Group

The fermion operators (2.1) act in a finite space \(\mathcal{E}_j\) for a particular \(j\)-level. In \(\mathcal{E}_j\) the vacuum \(|0\rangle\) is defined by \(c_{jm\sigma}^\dagger|0\rangle = 0\) and the scalar product is chosen so that \(\langle 0|0\rangle = 1\).

The states that span the finite \(\mathcal{E}_j\) spaces consist of linear combinations of different numbers of fermion creation operators acting on the vacuum state [8],

\[
\Pi_{\{m=-j,-j+1,...,j\}} \Pi_{\{\sigma=\pm 1/2\}} c_{jm\sigma}^\dagger|0\rangle.
\]  

(2.18)

Each \(\mathcal{E}_j\) space is finite according to the Pauli principle, \(c_{jm\sigma}^\dagger c_{jm\sigma}|0\rangle = 0\), that allows no more than \(2\Omega_j\) identical fermions in a single \(j\)-shell and which is implicitly taken into account through the fermion anti-commutation relations (2.1). The operator of the total number of fermions is diagonal in the \(\mathcal{E}_j\) space and yield the operator \(P = (-1)^N\), which is invariant with respect to the ten parameter symplectic transformations (that is, under Sp(4)). The eigenvalues of \(P\) label the two subspaces of \(\mathcal{E}_j\), the even \(\mathcal{E}_j^+\) space and the odd \(\mathcal{E}_j^-\) space. Here we are interested in the even \(\mathcal{E}_j^+\) space, which contains states of coupled fermions and leads to a theory that can describe pairing correlations in nuclei/superconductors.

A finite vector subspace \((\mathcal{E}_j^+)^{0}\) of the even \(\mathcal{E}_j^+\) space, which consists of fully-paired states with total angular momentum and parity \(J^\pi = 0^+\), can be constructed as the ‘boson creation operators’ (2.2) act on the vacuum state,

\[
|\Omega; n_1, n_0, n_{-1}\rangle = \left(A_1^\dagger\right)^{n_1} \left(A_0^\dagger\right)^{n_0} \left(A_{-1}^\dagger\right)^{n_{-1}} |0\rangle,
\]

(2.19)

where such states are trivially labelled by the integer parameters \(n_{0,\pm 1}\) introduced in the right-hand side of (2.19) and the value of \(\Omega\) (\(= j + \frac{1}{2}\) for a single \(j\)-level) related to the fermion construction of the \(A^\dagger\) operators. The basis is obtained by orthonormalization of a linearly independent subset of the vectors (2.19). We use the convention of denoting basis vectors with \(\ldots\) if they are orthonormalized and with \(\ldots\) if they are not.

Since the \(A_{1,0,-1}^\dagger\) operators create a pair of a certain type (\(pp, pn\) and \(nn\), respectively) the corresponding integers \(n_{1,0,-1}\) in (2.19) give the number of \(pp, pn\) and \(nn\) pairs that are created on the vacuum state.

As the rank of the symplectic Sp(4) group is two, its finite-dimensional irreducible representations (irreps) (Appendix A) are specified by two quantum numbers that are eigenvalues of the two invariant operators, which commute with all the generators of the group. For sp(4) the invariants are of second-order (the Casimir operator, (2.12)) and fourth-order in the generators [13]. Thus, representations of Sp(4) are labeled by the largest eigenvalue \(w\).
of the number operator $\hat{N}$, and the reduced isospin, $t$, of the uncoupled fermions in the corresponding state [7, 8, 9, 10, 11, 13].

In each representation $(w,t)$ of the $\text{Sp}(4)$ group in the vector space spanned over (2.19), the maximum number of particles is $w = 4\Omega$ and all the states consist of no uncoupled fermions (reduced isospin zero, $t = 0$). It follows that the two invariant operators are linearly dependent for these representations (restricted to $t = 0$) and only one quantum number is needed, namely the eigenvalue, $\Omega$, of the second-order Casimir operator (2.12)

$$C^2_2(\text{sp}(4)) |\Omega; n_1, n_0, n_{-1} \rangle = (\Omega + 3) |\Omega; n_1, n_0, n_{-1} \rangle.$$  \hspace{1cm} (2.20)

Within a representation, as $\Omega$ is fixed it will be dropped from the labelling of the states, $|n_1, n_0, n_{-1} \rangle$.

Each representation labelled by $\Omega$ is finite, because of the fermion structure of the operators $A^\dagger_{\pm 1}$: $(A^\dagger_{\pm 1})^{\Omega+1} |0\rangle = 0$ or $(A^\dagger_{\pm 1})^{\Omega} (A_0^\dagger) |0\rangle = 0$. Another consequence of the fermion realization is that some of the vectors (2.19) in the finite space $E_j^+$ are linearly dependent, for example $(A_1^\dagger)^\Omega (A_{-1}^\dagger)^\Omega |0\rangle \sim (A_0^\dagger)^{2\Omega} |0\rangle$. The states (2.19) are the common eigenvectors of

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$n$ & $i = 2$ & $i = 1$ & $i = 0$ & $i = -1$ & $i = -2$ \\
\hline
0 & |0, 0, 0| & |0, 0, 0| & |0, 0, 0| & |0, 0, 0| & |0, 0, 0| \\
2 & |1, 0, 0| & |0, 1, 0| & |0, 0, 1| & |0, 0, 0| & |0, 0, 0| \\
4 & |2, 0, 0| & |1, 1, 0| & |1, 0, 1| & |0, 1, 1| & |0, 0, 2| \\
6 & |2, 0, 1| & |1, 1, 1| & |1, 1, 1| & |1, 0, 2| & |0, 0, 1| \\
8 & |2, 0, 2| & |1, 2, 1| & |1, 0, 2| & |0, 2, 1| & |0, 0, 0| \\
\hline
\end{tabular}
\caption{Basis states labeled by $|n_1, n_0, n_{-1} \rangle$ for $\Omega_{3/2} = 2$.}
\end{table}

the fermion number operators $N_{\pm 1} = (N_{\pm 1})^\dagger$

$$N_{+1} |n_1, n_0, n_{-1} \rangle = N_+ |n_1, n_0, n_{-1} \rangle, \hspace{1cm} N_+ = 2n_1 + n_0, \hspace{1cm} (2.21)$$

$$N_{-1} |n_1, n_0, n_{-1} \rangle = N_- |n_1, n_0, n_{-1} \rangle, \hspace{1cm} N_- = 2n_{-1} + n_0, \hspace{1cm} (2.22)$$

or of their linear combinations, the operators of total number of particles $N = N_{+1} + N_{-1}$ and the third projection $\tau_0 = N_{+1} - N_{-1}$, which are both simultaneously diagonalizable

$$\hat{N} |n_1, n_0, n_{-1} \rangle = n |n_1, n_0, n_{-1} \rangle, \hspace{1cm} n = 2(n_1 + n_{-1} + n_0), \hspace{1cm} (2.23)$$

$$\tau_0 |n_1, n_0, n_{-1} \rangle = i |n_1, n_0, n_{-1} \rangle, \hspace{1cm} i = n_1 - n_{-1}. \hspace{1cm} (2.24)$$
The first two operators, \( N_{\pm 1} \), form a basis in the Cartan subalgebra of \( \mathfrak{sp}(4) \), while the second two, \( \hat{N} \) and \( \tau_0 \), are regarded as the basis operators in the Cartan subalgebra of \( \mathfrak{so}(5) \) (\( \sim \mathfrak{sp}(4) \)). Their eigenvalues, \((N_+, N_-)\) or \((n, i)\), can be used to classify the basis within a representation \( \Omega \). The classification scheme of the basis states (2.19) (labeled by \([n_1, n_0, n_{-1}]\)) for \( \Omega_{3/2} = 2 \) is shown in Table 2.3, where \( n \) enumerates the rows and \( i \) the columns. The basis vectors are degenerate in the sense that more than one of the common eigenstates of \( \hat{N} \) and \( \tau_0 \) have one and the same eigenvalues \( \{n, i\} \) and thus belong to one and the same cell of Table 2.3.

### Table 2.3: Basis states labeling scheme in each limit, \( \mu = \tau, 0, \pm \)

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \Omega )-irreps of ( \text{Sp}(4) )</th>
<th>allowed quantum number values for ( \mathcal{E}_{j=0}^{\pm} ) (2.19)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>(</td>
<td>n, \tau, i\rangle )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>( \tau = \frac{n}{2}, \frac{n}{2} - 2, \ldots, 1 ) (odd) or ( 0 ) (even), ( \text{where} \ \tilde{n} = \min {n, 4\Omega - n} )</td>
<td></td>
</tr>
<tr>
<td>( \tau )</td>
<td>( i = -\tau, -\tau + 1, \ldots, \tau )</td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>(</td>
<td>i, s^0, n\rangle )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \frac{n}{2} - \Omega = -s^0, -s^0 + 1, \ldots, s^0 )</td>
<td></td>
</tr>
<tr>
<td>( \pm )</td>
<td>(</td>
<td>N_\mp, s^\pm, N_\pm\rangle )</td>
</tr>
<tr>
<td>( \pm )</td>
<td>( \frac{N_\mp - \Omega}{2} = -s^\pm, -s^\pm + 1, \ldots, s^\pm )</td>
<td></td>
</tr>
</tbody>
</table>

In general, each \( (\mu = \tau, 0, \pm) \) realization of the reduction chain of the \( \mathfrak{sp}(4) \) algebra (2.13) describes a limiting case of a restricted symmetry. It provides for a complete labeling of the basis vectors of the irreducible representations of \( \text{Sp}(4) \) by the eigenvalues of the invariant operators of the underlying subalgebras. The first-order \( C_{1}^{(\mu = \tau, 0, \pm)} \) invariant of \( \mathfrak{u}(2) \) (Table 2.1) reduces the finite action space into a direct sum of unitary irreps of \( \mathfrak{u}(2) \) labeled by the \( C_{1}^{\mu} \) eigenvalue (\( \{n, i, N_\pm\} \)-multiplets). The next two quantum numbers are provided by the \( \text{SU}(2) \) group in a standard way: the “spin”, \( s \) (isospin \( \tau \) or quasi-spin \( s^\mu \)) and its third projection (Table 2.4). They are related, respectively, to the eigenvalue of the second-order Casimir invariant of \( \mathfrak{su}(2) \), \( s(s + 1) \), which labels the irreducible unitary representations (IUR) of \( \text{SU}(2) \), and to the eigenvalue of the middle operator in the third column in Table 2.1.

A vector with fixed quantum numbers, \((N_+, N_-)\) or \((n, i)\), corresponds to a given nucleus (a cell in Table 2.3). In this way the \( \text{Sp}(4) \supset \text{U}(2) \supset \text{SU}(2) \) symmetry provides for a natural classification scheme of nuclei that belong to a single-\( j \) level, which are mapped to algebraic \( \text{U}(2) \) multiplets. This classification also extends to the corresponding ground and excited states of the nuclei, which can be distinguished as eigenstates of the Casimir operators of \( \mathfrak{su}(2) \) in the limiting cases.
2.1.4 Description of the Symmetries Embedded in $\text{Sp}(4)$

The significant reduction chains of the $\mathfrak{sp}(4)$ algebra need special attention because, from a mathematical point of view, they provide for a complete labeling of the basis states in the action space and, from a physical point of view, they help build a model Hamiltonian and offer a physical interpretation of the underlying interactions \[10, 11, 12, 13, 46\].

Table 2.5: Isospin eigenstates labeled by $|n_1, n_0, n_{-1}\rangle$ for $(\Omega_{3/2} = 2)$-irrep of $\text{Sp}(4)$.

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$i = 2$</th>
<th>$i = 1$</th>
<th>$i = 0$</th>
<th>$i = -1$</th>
<th>$i = -2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\tau = 0$</td>
<td></td>
<td>$</td>
<td>0, 0, 0\rangle$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\tau = 1$</td>
<td></td>
<td>$</td>
<td>1, 0, 0\rangle$</td>
<td>$</td>
</tr>
<tr>
<td>4</td>
<td>$\tau = 2$</td>
<td>$\sqrt{2}</td>
<td>1, 1, 0\rangle$</td>
<td>$\frac{0.20}{\sqrt{3/2}} + \frac{1.01}{\sqrt{3/2}}$</td>
<td>$\sqrt{2}</td>
</tr>
<tr>
<td>6</td>
<td>$\tau = 1$</td>
<td></td>
<td>$2</td>
<td>1, 1, 1\rangle \equiv -\frac{2}{\sqrt{3}}</td>
<td>0, 3, 0\rangle$</td>
</tr>
<tr>
<td>8</td>
<td>$\tau = 0$</td>
<td></td>
<td>$\frac{5}{3}</td>
<td>0, 4, 0\rangle \equiv -2</td>
<td>1, 2, 1\rangle \equiv</td>
</tr>
</tbody>
</table>

**Isospin $SU^\mu = \tau (2)$ Symmetry**

In this limit, the basis states are labeled by the eigenvalues of the invariant operator of each subgroup in the reduction $\text{Sp}(4) \supset U^\tau(2) \supset U_N(1) \otimes SU^\tau(2) \supset SU^\tau_2(2) \supset U^\tau_0(1)$. As a first-order invariant of $U^\tau(2)$, the total number of particles operator $\hat{N}$ decomposes the spaces spanned by the basis vectors (2.19) into a direct sum of eigensubspaces, defined by the condition that $n$ (2.23) is fixed, $n = 0, 2, 4, \ldots, 4\Omega$ (Table 2.4), thus realizing an IUR of $U^\tau(2)$ in each row of Table 2.3. The $SU^\tau(2)$ subgroup provides the other two standard quantum numbers: the eigenvalue of the isospin operator, $\tau$, as naturally arises in the eigenvalue problem for $C_2(\mathfrak{su}^\tau(2))$ (Table 2.2), $\tau^2 |n, \tau, i\rangle = \tau (\tau + 1) |n, \tau, i\rangle$, and the eigenvalue of $\tau_0$, $i$ (2.24), where in each $n$-irrep of $U(2)$ they take on the values given in (Table 2.4). As an example, the orthonormalized basis $|n, \tau, i\rangle$ given in terms of the states $|n_1, n_0, n_{-1}\rangle$ (2.19) is shown in Table 2.5 for $\Omega_{3/2} = 2$ and Table 2.6 for $\Omega_{7/2} = 4$. The state with the maximum number of particles, $|n = 4\Omega, \tau = 0, i = 0\rangle$ (Table 2.5, $n = 8$), always has isospin zero ($\tau = 0$) and all possible states expressed in the basis $|n_1, n_0, n_{-1}\rangle$ are equivalent within a normalization factor. The linear dependence of (2.19), which is a straightforward consequence of the Pauli principle, is recognized throughout the table for $n > 2\Omega$. For fixed
Table 2.6: Isospin eigenstates labeled by $|n_1, n_0, n_{-1}\rangle$ for $(\Omega_{7/2} = 4)$-irrep of Sp(4). The table is symmetric with respect to the sign of $i$ and $n - 2\Omega$.

<table>
<thead>
<tr>
<th>$n(\tau)$</th>
<th>$i = 4$</th>
<th>$i = 3$</th>
<th>$i = 2$</th>
<th>$i = 1$</th>
<th>$i = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, $\tau = 0$</td>
<td>$</td>
<td>0, 0, 0\rangle$</td>
<td>$</td>
<td>1, 0, 0\rangle$</td>
<td>$</td>
</tr>
<tr>
<td>2, $\tau = 1$</td>
<td>$</td>
<td>2, 0, 0\rangle$</td>
<td>$</td>
<td>1, 1, 0\rangle$</td>
<td>$\frac{1}{2}\sqrt{\frac{3}{2}}(0, 2, 0) + 2(1, 0, 1)$</td>
</tr>
<tr>
<td>4, $\tau = 0$</td>
<td>$</td>
<td>3, 0, 0\rangle$</td>
<td>$</td>
<td>2, 1, 0\rangle$</td>
<td>$\frac{1}{2}\sqrt{\frac{3}{2}}(0, 3, 0) + 3(1, 1, 1)$</td>
</tr>
<tr>
<td>6, $\tau = 1$</td>
<td>$</td>
<td>4, 0, 0\rangle$</td>
<td>$</td>
<td>3, 1, 0\rangle$</td>
<td>$\frac{1}{2}\sqrt{\frac{3}{2}}(0, 4, 0) + 8(1, 2, 1) + 2(2, 0, 2)$</td>
</tr>
</tbody>
</table>

$n$ and $i$ quantum numbers (a cell in Table 2.5) the general isospin eigenstates can be derived as orthonormal to the eigenstate with the lowest $\tau$ value expressed in the form [45]

$$
|n, \tau = |i\rangle + \text{mod}(\frac{n}{2}, 2), i\rangle \sim (A_0^+ A_0^- - 2A_1^+ A_-^1) \frac{n}{2} \text{mod} - (A_1^+ A_{\text{sign(i)}}) |i\rangle (A_0^\dagger)^{\text{mod}(\frac{n}{2}, 2)} |0\rangle,
$$

(2.25)

where the lowest value is $\tau = |i\rangle$ ($\tau = |i\rangle + 1$) for $n/2$ even (odd) and follows from the vector coupling rules for the isospin one $A_k^\dagger$ objects with isospin projection $k$.

The raising (lowering) generators $\tau_{\pm 1}$ acting $2(\tau + 1)$ times on the lowest $|n, \tau, -\tau\rangle$ (highest $|n, \tau, \tau\rangle$) weight state give all the basis states of the respective $\tau$-representation according to the result

$$
\tau_{\pm} |n, \tau, i\rangle = \sqrt{\frac{(\tau + i) (\tau \pm i + 1)}{2\Omega}} |n, \tau, i \pm 1\rangle.
$$

(2.26)

This reduction to $SU^+(2)$ is usually referred to as ‘physically interesting’ since it yields a labeling scheme that includes quantities of physical interest, namely isospin and total number of particles [10, 12]. The reduction provides for an elegant approach to nuclear problems where in addition to the mass number and isospin projection, the isospin is a good quantum number.
• Identical-Particle Pairing, $SU^{\mu=\pm}(2)$, and Kerman’s Quasi-Spin $SU(2)$ Group

Through the group reduction chain, $Sp(4) \supset U^{\pm}(2) \supset U_{N_{+}}(1) \otimes SU^{\pm}(2) \supset SU^{\pm}_{(s^{\pm})^{2}}(2) \supset U^{\pm}_{N_{+}}(1)$, the action space for each $j$ level is decomposed to the subspaces defined by the conditions $N_{\mp} = 2n_{\mp 1} + n_{0} = 0, 1, \ldots, 2\Omega$ and represented by the left-and-up (right-and-down) diagonals in Table 2.3. Each $SU(2)$ multiplet is specified by the eigenvalue of the quasi-spin operator, $s^{\pm}$, which together with the third $s^{\pm}$ projection value completes the labeling of the basis states (Table 2.4).

The second-order Casimir operator $C_{2}(su^{\pm}(2)) \equiv (s^{\pm})^{2}$ (Table 2.2) is in general not diagonal in the fully-paired space $\mathcal{E}_{j,0}^{+}$ of (2.19) but one can easily find a transformation from the pair basis $|n_{1}, n_{0}, n_{-1}\rangle$ to the $s^{\pm}$ eigenstates $|N_{\mp}, s^{\pm}, N_{\pm}\rangle$. In this way we are able to find an explicit construction of the latter states in terms of the fermion operators used to construct the $sp(4)$ algebra. In fact, for fixed $(N_{-}, N_{+})$ quantum numbers the matrix representation of $(s^{\pm})^{2}$ in $\mathcal{E}_{j,0}^{+}$ is a triangular matrix, which implies that its eigenvalues coincide with its diagonal elements. The latter can be used directly to determine the allowed values for the quasi-spin in the $(N_{-}, N_{+})$ cell and can be generalized for the whole $U(2)$ multiplet. The values of $\Omega - 2s^{\pm}$ are found to differ by two, and hence $s^{\pm}$ decreases by one from its maximum value.

A special case, that is of particular physical interest, is when the quasi-spin has its maximal value, $s^{\pm} = \frac{\Omega - \text{mod}(N_{+}, 2)}{2}$ (underlined value for $s^{\pm}$ in Table 2.4). The eigenstates for this case are actually the pair states (2.19)

$$
(s^{\pm})^{2}|n_{1}, n_{0}, n_{-1}\rangle = \frac{\Omega - \nu_{1}}{2}(\frac{\Omega - \nu_{1}}{2} + 1)|n_{1}, n_{0}, n_{-1}\rangle
$$

$$
= \frac{\Omega - n_{0}}{2}(\frac{\Omega - n_{0}}{2} + 1)|n_{1}, n_{0}, n_{-1}\rangle
$$

where the seniority quantum number $\nu_{1} = \Omega - 2s^{\pm}$ is the number of the ‘uncoupled’ like-particles (protons in the $\mu = +$ limit and neutrons in the $\mu = -$ limit), that is, the number of like-particles that are not paired among themselves. For the fully-paired basis (2.19), ‘uncoupled’ like-particles, say protons, must be coupled to the other particle type (neutrons) in proton-neutron pairs. This is why in this $Sp(4)$ irrep of no uncoupled nucleons the seniority value is $\nu_{1} = n_{0}$ ($n_{0}$ being the number of $pn$ pairs) and both limits, for protons and neutrons, have the same value of the respective quasi-spin for fixed $(N_{+}, N_{-})$. Furthermore, for this special case of maximum $s^{\pm}$ value $\nu_{1} = n_{0} = \text{mod}(N_{+}, 2)$, that is zero or one $pn$ pairs, and the corresponding $s^{\pm}$ eigenstates (2.27) (placed first in each cell in Table 2.3) are either $|n_{1}, 0, n_{-1}\rangle = (A_{j}^{\dagger})^{n_{1}}(A_{j}^{-1})^{n_{-1}}|0\rangle$ (pure like-particle pairing) or $|n_{1}, 1, n_{-1}\rangle = (A_{j}^{\dagger})^{n_{1}}A_{0}^{\dagger}(A_{j}^{-1})^{n_{-1}}|0\rangle$. It means that all the nucleons in a system are coupled as like-particle pairs, $pp$ or $nn$, and only one $pn$ pair may exist, the one that is made up of the odd proton and the odd neutron. This is the maximal number of like-particle pairs case. The raising
and lowering generators of SU±(2) act along the left-and-up/right-and-down diagonals:

\[
A_{\pm 1}^\dagger |n_1, n_0, n_{-1}\rangle = |n_{\pm 1} + 1, n_0, n_{-1}\rangle, \\
A_{\mp 1} |n_1, n_0, n_{-1}\rangle = n_{\pm 1} \left(1 - \frac{n_{\pm 1} + n_0 - 1}{\Omega}\right) |n_{\pm 1} - 1, n_0, n_{-1}\rangle.
\]

Starting from the respective lowest or highest weight states, they generate the states belonging to the \(s_{\text{max}}^\pm IURs\) of the SU±(2) subgroups of Sp(4).

The limit of pure like-particle pairing (only protons or only neutrons) corresponds to the Kerman’s SU(2) group, which is the first group theoretical approach to the pairing problem [16]. It finds its applications to pairing in semi-magic nuclei, where either the proton number \((Z)\) or the neutron number \((N)\) forms a closed shell. The low-lying energy spectra of the corresponding \(Z\)-isotopes \((N\)-isotones\) are reproduced well by the SU(2) group reflecting the dynamics of the neutron (proton) pairs above the closed shell. The eigenvalue problem of the quasi-spin operator is given by the same relation \((2.27)\), which appears to be a general one for generating the pairing spectrum. For example, for neutron pairs the eigenstates are \((N_+ = 0), \nu_1 = \hat{N}_- - 2n_{-1}, n_{-1}\), which include the fully-paired ground state \((\nu_1 = 0)\) [this belongs to \(\mathcal{E}_{j,0}^+\)] and excited states with \(\nu_1 = 2, 4, \ldots, \hat{N}_-\) (where \(\hat{N}_-\) is defined in Table 2.4).

**Isovector \(pn\) Pairing, SU\(^\mu=0(2)\)**

The group reduction chain \(\text{Sp}(4) \supset U^0(2) \supset U_{\tilde{N}}(1) \otimes SU^0(2) \supset SU^0_{(s^0)^2}(2) \supset U^0_{\tilde{N}}(1)\) decomposes the space into a direct sum of eigensubspaces of the operator \(\tau_0\) at each of its fixed values \((2.24)\) (columns of Table 2.3) and further to \(s^0\)-multiplets (Table 2.4). Here again as for the \(\mu = \pm\) case, the second-order Casimir operator \(C_2(s^0_u(2)) = (s^0)^2\) (Table 2.2) should be diagonalized in the fully-paired space \(\mathcal{E}_{j,0}^+\) \((2.19)\) to obtain the explicit construction of the \(s^0\) eigenstates \(|i, s^0, n\rangle\) in terms of the fermion creation operators \((2.1)\). Since, for fixed \((i, n)\) quantum numbers the matrix representation of \((s^0)^2\) in \(\mathcal{E}_{j,0}^+\) is a triangular matrix, its diagonal elements determine the allowed values for the quasi-spin. The values of \(\Omega - s^0\) \((s^0)\) are found to increase (decrease) by two from its minimum (maximum) value.

When the quasi-spin value is maximal, \(s^0 = \Omega - |i|\) (underlined value for \(s^0\) in Table 2.4), the eigenstates are the pair states \((2.19)\) and the quasi-spin eigenvalue is found related to the seniority quantum number \(\nu_0 = \Omega - s^0\), which counts the nucleons not coupled in \(pn\) pairs

\[
(s^0)^2 |n_1, n_0, n_{-1}\rangle = \frac{2\Omega - 2\nu_0}{2} \left(\frac{2\Omega - 2\nu_0}{2}\right) + 1 |n_1, n_0, n_{-1}\rangle = \frac{2\Omega - 2(n_1 + n_{-1})}{2} \left(\frac{2\Omega - 2(n_1 + n_{-1})}{2}\right) + 1 |n_1, n_0, n_{-1}\rangle.
\]

In this particular Sp(4) representation (Flower’s isospin \(t = 0\)) the seniority value can be expressed as the number of pairs of the other two kinds, \(pp\) and \(nn\), that is \(\nu_0 = n_1 + n_{-1}\).
In addition, for this special case of $s_{\text{max}}^0 = \Omega - |i|$ the seniority quantum number is $\nu_0 = n_1 + n_{-1} = |i|$. Since $i = n_1 - n_{-1}$ (2.24), this implies that $n_1 = 0$, or $n_{-1} = 0$, or both $n_1 = n_{-1} = 0$, and that the number of $pn$ pairs is maximal in this case. These states are the last ones in each of the cells in the Table 2.3.

The raising and lowering SU$^0$(2) generators act along the columns and generate the states belonging to the $s_{\text{max}}^0$ IURs of SU$^0$(2)

\[
A_0^+ |n_1, n_0, n_{-1}) = |n_1, n_0 + 1, n_{-1}), \quad A_0 |n_1, n_0, n_{-1}) = n_0 \left(1 - \frac{2(n_{-1} + n_1) + n_0 - 1}{2\Omega}\right) |n_1, n_0 - 1, n_{-1}).
\]

We conclude this part with a few comments on the reduction chains and the basis labeling scheme. As shown above in detail, the labeling of the basis states in all four limits can be specified in the standard way by the eigenvalues (summarized in Table 2.7) of the invariant operators of the subalgebras in the reduction of $\mathfrak{sp}(4)$. In the specific irrep of Sp(4) under investigation (in a fully-paired space), for each of these basis sets there exist a transformation to the pair states (2.19), as mentioned above, and hence in physical applications it is intuitive and convenient to use the physically relevant labeling scheme in terms of the numbers of pairs. Furthermore, in the different limits we noticed that the seniority quantum number $\nu$ for the case of maximum number of ‘primary’ pairs (e.g., pure like-particle pairing in the $\mu = \pm$ limit) was equal to the number of the ‘minor’ pairs ($pn$ pairs for the $\mu = \pm$ case and $pp + nn$ pairs in the $\mu = 0$ limit) and increases by two in each of the next $\mathfrak{su}(2)$ multiplets. Now, in the pair state representation, the seniority quantum number simply differs between two states of one and the same $i$ and $n$, but different coupling scheme (e.g., Table 2.3). In this regard, we derive the matrix representation of the operators that enter in our theoretical model in this fully-paired space defined by (2.19). Of course, one cannot gain much without losing something; we are forced to work in an environment of non-orthonormalized basis states (2.19) (Appendix C.3).

In general, two reductions are employed, namely $r_1$ (2.13) in the $\mu = \tau$ description with $\Lambda_T = \{(w, t), n, \tau, i\}$ isospin labeling scheme and $r_2$ (2.14) with $\Lambda_P = \{(w, t), s^+, s^-, N_+, N_-\}$
like-particle labeling scheme. In terms of the isomorphic Lie algebras, the reductions $\mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(3) \oplus \mathfrak{so}(2)$ (corresponding to $r_1$) and $\mathfrak{so}(5) \supset \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ (corresponding to $r_2$) are related through ordinary 3-dimensional angular momentum addition, although in reality the unitary transformation matrix of the $\Lambda_P$ set to the isospin $\Lambda_T$ one is very difficult to obtain [10, 11]. Again, speaking in general for all the $(w, t)$ representations of $\text{Sp}(4)$, the reduction $r_1$ (2.13), although inherent to the physically interesting isospin $\Lambda_T$ labeling scheme, introduces serious difficulties as there is a missing quantum number associated with it. At the same time the other reduction $r_2$ (2.14) is canonical since it corresponds to a mathematically natural group decomposition. It furnishes the four commuting operators, $s^+, s^-, N_{+1}, N_{-1}$ (the corresponding $\mathfrak{su}^\pm(2)$ Casimir invariants and the third quasi-spin projections), needed to completely specify the states. The isospin $(\tau)^2$ operator is not diagonal in these states and $\tau$ is no longer a good quantum number. Fortunately, the irreps of $\text{Sp}(4)$ that are actually interesting with respect to shell model calculations are mainly the ones that $\Lambda_T$ are sufficient to label completely the states [10], among which a special interest is related to $t = 0$ and $t = 1/2$ ($t$ being the Flower’s reduced isospin of the unpaired particles). We have already presented in detail the $(\Omega, t = 0)$ $\text{Sp}(4)$ representation related to the description of even-even and odd-odd nuclei, while the $t = 1/2$-irrep describes odd-mass nuclei. We saw that for the first irrep ($t = 0$) three quantum numbers were indeed sufficient for a complete labeling of the basis state and that the eigenvalues of the two second-order $\mathfrak{su}^\pm(2)$ Casimir invariants are the same (2.27), yielding $s^+ = s^-$. In this way the labeling scheme provided by the reductions $\mathfrak{sp}(4) \supset \mathfrak{su}^\pm(2)$ in the $\mu = \pm$ limits is equivalent to the one of $r_2$ (2.14), and together with the isospin labeling scheme can be transformed to the fully-paired basis (2.19) specified by the three pair numbers.

2.2 Quantum Algebras

A quantum Lie algebra, $U_q(\mathfrak{g})$, represents a $q$-analog of the corresponding Lie algebra $\mathfrak{g}$ and is described by a $q$-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra, where $U(\mathfrak{g})$ is the algebra over the field $\mathbb{C}$ of complex numbers based on all sums and products of the basis operators of $\mathfrak{g}$ (Appendix A and B). A feature of any quantum theory is that in the $q \to 1$ limit one recovers the non-deformed (“classical”) picture. This may be recognized as an old concept in physics, namely, quantum mechanics is a deformation of classical mechanics, in which $e^\hbar \to 1$, that is, the Plank constant $\hbar$ goes to zero; the $c$-deformed Einsteinian relativity reverts to the Galilean relativity in the limit $e^{1/c} \to 1$, that is, when the speed of light $c$ goes to infinity [51].

The concept of quantum (or $q$-) deformation, formulated by Drinfeld and Jimbo [20, 21, 22], arose in physics. Originally, the $q$-analog of $\text{SU}(2)$ appeared in the application of the quantum inverse scattering method to 2-dimensional models in quantum field theory and statistical mechanics [52, 53]. Thereafter, especially following the introduction of the $q$-deformed harmonic oscillator [54, 55], considerable attention has focused on studies based on the novel and promising approach of quantum deformation in various fields of physics [22, 51]. The earliest applications of $q$-deformation in nuclear physics were related to a description
of rotational bands in axially deformed nuclei [56, 57, 58]. In the realm of the pairing correlations models the quantum deformation concept was introduced first for like-particle pairing [31, 32] based on a $SU_q(2)$ approach. A lot of effort, from a purely mathematical [23, 24, 25, 59, 60] as well as from a physical point of view [34, 61, 62] has been concentrated on the $q$-analogs of symplectic algebras (or their isomorphic algebras). In [34] $so_q(5)$ was employed for the investigation of like-particle and $pn$ pairing. In recent years, in addition to purely mathematical examinations of quantum algebraic concepts (see e.g. [63]), studies of interest include applications in string/brane theory, conformal field theory, statistical mechanics, $q$-deformed quantum mechanics and metal clusters [64, 65, 66, 67, 68].

Avoiding the formal presentation of the quantum algebra formulation (that one may find in Appendix B), we introduce the properties of $q$-deformation, which is fundamental for the quantum symmetry concept.

In quantum algebras, there is a one-to-one mapping of the ordinary real $c$-numbers into $q$-numbers over the field $\mathbb{R}$ of real numbers, which sets a correspondence of every real number $x$ to a $q$-number $[x]_k$ defined as

$$[x]_k = \frac{q^{kx} - q^{-kx}}{q^k - q^{-k}}, \quad k \in \mathbb{R},$$

for which in the transition to the non-deformed Lie algebra the limit holds

$$\lim_{q \to 1} [x]_k = x, \quad \forall k \in \mathbb{R}. \quad (2.32)$$

The definition (2.31) is symmetric with respect to the exchange $q$ to $q^{-1}$. The $q$-parameter may be real and positive and hence expressed as $q = e^{\kappa}$ or complex, $q = e^{i\kappa}$, where $\kappa$ is a real parameter. In terms of $\kappa$ the $q$-number (2.31) has the form

$$[x]_k = \begin{cases} \frac{\sinh(k\kappa x)}{\sinh(k\kappa)}, & q = e^{\kappa} \\ \frac{\sin(k\kappa x)}{\sin(k\kappa)} & q = e^{i\kappa} \end{cases}, \quad \kappa \in \mathbb{R},$$

(2.33)

The definition (2.31) (and (2.33)) can be generalized to a mapping of linear operators

$$[X]_k = \frac{q^{kX} - q^{-kX}}{q^k - q^{-k}} = \begin{cases} \frac{\sinh(k\kappa X\kappa)}{\sinh(k\kappa)}, & q = e^{\kappa} \\ \frac{\sin(k\kappa X\kappa)}{\sin(k\kappa)} & q = e^{i\kappa} \end{cases}, \quad k \in \mathbb{R},$$

(2.34)

where each linear operator $X$ is related for a given real number $k$ to a $q$-deformed operator $[X]_k$, which in the $q \to 1$ limit reverts back to the non-deformed one

$$\lim_{q \to 1} [X]_k = X, \quad \forall k \in \mathbb{R}. \quad (2.35)$$

It is common, when $k = 1$, for the $q$-bracket to be simply denoted as $[x]_{k=1} \equiv [x]$ for numbers $x$, and $[X]_{k=1} \equiv [X]$ for linear operators $X$. The properties and identities for the $q$-numbers
operators) [51] follow from their definitions, (2.31) and (2.34), for all real $k \in \mathbb{R}$, and when they are listed for the case of linear operators $X$ they can be also applied to $q$-numbers $x$:

$$[0]_k = 0; \quad (2.36)$$

$$[-X]_k = -[X]_k; \quad (2.37)$$

$$[n]_k = \sum_{l=-(n-1)/2}^{(n-1)/2} q^{2lk}, \quad n \text{ is positive integer,} \quad (2.38)$$

with examples, $[1] = 1$, $[2] = q^1 + q^{-1}$, $[3] = q^2 + 1 + q^{-2}$, $[4] = q^3 + q^1 + q^{-1} + q^{-3}$, etc., yielding as well the helpful relations

$$[n + 1]_k + [n]_k + 1 = \sum_{l=0}^{n} (q^l + q^{-l}), \quad n \text{ is positive integer,} \quad (2.39)$$

$$[n + n' + 1]_k - [n - 1]_k = \sum_{l=n/2}^{(n+n')/2} (q^{2lk} + q^{-2lk}), \quad n, n' \text{ are positive integers;} \quad (2.40)$$

$$[2x]_k = \frac{[2X]_k}{[X]_k} = q^{kx} + q^{-kx} q^{-1} 2, \quad (2.41)$$

with examples, $[4]_2 = q^2 + q^{-2}$, $[5]_2 = q^{3/2} + q^{-3/2}$, etc.;

$$[x]_k - [y]_k = \left[ \frac{x - y}{2} \right]_k \left[ \frac{x + y}{2} \right]_k; \quad (2.42)$$

$$[2x]_k + [2y]_k = \left[ \frac{2x+y}{2} \right]_k \left[ \frac{2x-y}{2} \right]_k; \quad (2.43)$$

$$[2x]_k - [2y]_k = \left[ \frac{x+y}{2} \right]_k \left[ \frac{x-y}{2} \right]_k (q^k - q^{-k})^2; \quad (2.44)$$

$$[x]_k [y - z]_k + [y]_k [z - x]_k + [z]_k [x - y]_k = 0, \quad (2.45)$$

which yields the very useful identity when $z = x + y$

$$[x]_k^2 - [y]_k^2 = [x + y]_k [x - y]_k; \quad (2.46)$$
as well as the recursive formula when \( y = 2, \ z = 1 \)

\[
[x]_k = [2]_k [x - 1]_k - [x - 2]_k, \quad [0]_k = 0, \ [1]_k = 1,
\]

which expresses any \( q \)-integer as a polynomial in \([2]_k\) of degree \( n - 1 \) with integer coefficients, e.g. \([3]_k = [2]_k^2 - 1, \ [4]_k = [2]_k^3 - 2 [2]_k, \ [5]_k = [2]_k^4 - 3 [2]_k^2 + 1, \ etc.;

\[
[kX] = [k] [X]_k ;
\]

\[
[kX] [X + 1]_k - [kX] [X - 1]_k = [2kX] ;
\]

\[
[kX] [X + 1]_k + [kX] [X - 1]_k = [2k] [X]_k^2 .
\]

In addition, the \( q \)-analog of some operations and quantities can be introduced in the quantum theory by the definitions (for all real \( k \in \mathbb{R} \))

\[
q\text{-addition:} \quad [x]_k \oplus [y]_k \doteq q^{-ky} [x]_k + q^{kx} [y]_k = [x + y]_k ,
\]

\[
q\text{-exponent:} \quad \exp_q(X) \doteq \sum_{l=0}^{\infty} \frac{X^l}{l!} ;
\]

\[
q\text{-factorial:} \quad [x]_k! \doteq [x]_k [x - 1]_k [x - 2]_k ... 1 , \quad [0]_k! \doteq 1 ,
\]

\[
q\text{-binomial coefficients (} n \text{ integer):} \quad \left[ \begin{array}{c} x \\ n \end{array} \right]_k \doteq \frac{[x]_k[x-1]_k ... [x-n+1]_k}{[n]_k!} , \quad \left[ \begin{array}{c} x \\ 0 \end{array} \right]_k \doteq 1 .
\]

Note that the \( q \)-addition of linear operators according to (2.51), as well as the identities (2.42)-(2.46) applied to linear operators, require all the operators involved to commute with one another.

For a quantum algebra over the field \( \mathbb{R} \) of real numbers there is a rule of composition (that satisfies the Lie algebra axioms (Appendix A) when \( k = 0 \)) generally defined as

\[
[X,Y]_k = XY - q^k YX, \quad k \in \mathbb{R},
\]

where \([X,Y]_k\) is referred as a \( q \)-commutator of the two \( q \)-deformed operators \( X \) and \( Y \). For non-zero values of \( k \), even if (2.55) is zero the \( X \) and \( Y \) operators do not commute, since \( XY = q^k YX \); such non-commutative coordinates are a fundamental part of the quantum algebra concept. In analogy, the \( q \)-anticommutator of two \( q \)-deformed operators \( X \) and \( Y \) is defined as

\[
\{X,Y\}_k = XY + q^k YX, \quad k \in \mathbb{R}.
\]

In the \( q \)-deformed picture the terminology and the fundamental algebraic properties remain the same with respect to non-deformed Lie algebras. Regarding the Lie group associated with the Lie algebra, a notation \( G_q \) we use should be understood as the set of all transformations
generated by the \( q \)-deformed basis operators of \( \mathcal{U}_q(\mathfrak{g}) \) and in this sense the latter are called \textit{generators} of \( G_q \). However, \( G_q \) should be distinguished from a \( q \)-analog of the Lie group \( G \).  

Quantum algebras possess a unique asset that is interesting to explore not only as a pure mathematical technique but as well in real physical applications. Specifically, the continuous quantum parameter \( q \) brings into the theory an additional degree of freedom without compromising the underlying symmetries. In general, it is not an easy task to find a \( q \)-deformed analog of a Lie algebra, especially for high algebraic dimensions. One may approach this problem by introducing a specific \( q \)-deformation at the level of the fermion operators, which realize the algebra, by choosing a suitable \( q \)-deformed anticommutation relation. Not every \( q \)-deformed relation yields a \( q \)-version of the Lie algebra. And even if the \( q \)-algebra is obtained, it is very probable that its subalgebraic structure appears only in the trivial limit when \( q \) goes to one. And finally, even if one succeeds in deriving the \( q \)-analog of the Lie algebra and its subalgebras, there is still a major problem left to be solved: What is the physical meaning and significance behind the quantum deformation? As we go through the chapters an answer to this question will be given for the specific deformations found. In the mean time, we introduce the concept of the \( q \)-deformation into our model without an attempt to explain what it really means.

\section*{2.3 \( q \)-Deformations of the Fermion Realization of \( \text{sp}(4) \)}

Consider \( q \)-deformed creation and annihilation fermion operators \( \alpha_{jm\sigma}^\dagger \) and \( \alpha_{jm\sigma} \) \([69]\), which are assumed to coincide with the "classical" fermion operators in the \( q \to 1 \) limit \( \alpha_{jm\sigma}^\dagger \to c_{jm\sigma}^\dagger \). In a complete analogy with the non-deformed picture, each of the second-quantized spin \( \frac{1}{2} \) operators \( \alpha_{jm\sigma}^\dagger \) (\( \alpha_{jm\sigma} \)) create (annihilate) a particle of type \( \sigma = \pm \frac{1}{2} \) in a single-particle state of a total angular momentum \( j \) (half integer) with projection \( m = -j, -j + 1, \ldots , j \) on the \( z \) axis. The Hermitian conjugation relation is defined as \( (\alpha_{jm\sigma}^\dagger)^* = \alpha_{jm\sigma} \) and in principle it may be different than the one for non-deformed objects (recall, \( (c_{jm\sigma}^\dagger)^\dagger = c_{jm\sigma} \)). Different deformations may arise depending on the type of anticommutation relations that the \( \alpha_{jm\sigma}^\dagger \) and \( \alpha_{jm\sigma} \) operators obey. An important requirement we impose is that the physical observables should remain non-deformed. In this way, quantities such as the number of particles, the third projection of the isospin, the angular momentum and the spin continue to have their physical meaning even though a deformation is introduced in a physical many-body system. For that reason, the particle number operators \((2.5)\) count the \( q \)-deformed operators as they did for the "classical" case \((2.7)\)

\begin{equation}
[N_{2\sigma}^{(j)}, \alpha_{jm\sigma}^\dagger] = \delta_{\sigma,\sigma'} \alpha_{jm\sigma'}^\dagger, \quad [N_{2\sigma}^{(j)}, \alpha_{jm\sigma}] = -\delta_{\sigma,\sigma'} \alpha_{m\sigma'}, \quad \sigma, \sigma' = \pm \frac{1}{2}.
\end{equation}

The \( q \)-deformed counterparts of the "classical" operators \((2.2)-(2.4)\) are built in the same way but in terms of the \( q \)-deformed fermion operators coupled to angular momentum and

\footnote{A quantum group \( G_q \) is described by a \( q \)-deformation of (the matrix representation of) the Lie group \( G \) and is a dual approach to the \( q \)-deformation of \( \mathcal{U}(\mathfrak{g}) \) \([51]\).}
parity $J^z = 0^+$

\[
B^{(j)}_k = \frac{1}{\sqrt{2\Omega_j (1 + \delta_{\sigma\sigma'})}} \sum_{m=-j}^{j} (-1)^{j-m} \alpha^\dagger_{jm\sigma} \alpha_{j,-m,\sigma'} = (B^{(j)}_{-k})^*, \tag{2.58}
\]

\[
B^{(j)}_{-k} = \frac{1}{\sqrt{2\Omega_j (1 + \delta_{\sigma\sigma'})}} \sum_{m=-j}^{j} (-1)^{j-m} \alpha_{j,-m,\sigma} \alpha_{j, m, \sigma'}, \quad k = \sigma + \sigma', \tag{2.59}
\]

\[
T^{(j)}_{\pm} = \frac{1}{\sqrt{2\Omega_j}} \sum_{m=-j}^{j} \alpha^\dagger_{jm,\pm1/2} \alpha_{jm,\mp1/2}, \tag{2.60}
\]

in addition to the two Cartan operators $N_{\pm 1}$ of $\mathfrak{sp}(4)$, which remain undeformed. The operators $B^{\pm(j)}_{k=\pm1}$ create (destroy) a $q$-deformed pair of particles of different kinds ($k = 0$) and of the same kind ($k = \pm 1$).

The calculation of various relations that involve $q$-deformed objects is clearly much more complicated than in the “classical” limit. Yet, in both cases, a very useful tool for calculating and verifying different results is a set of computer codes (Appendix D) based on a very efficient algorithm for fermion and boson realizations of non-commutative algebras written by Gueorguiev [70].

### 2.3.1 $q$-Deformation of the Anticommutation Relations of the Fermion Operators

In order for the $q$-deformed operators, (2.58)-(2.60) and (2.5), to close on a $q$-deformed version of the $\mathfrak{sp}(4)$ algebra, we need to postulate the $q$-deformed anticommutation relation of the fermion operators, $\alpha^\dagger_{jm\sigma}$ and $\alpha_{jm\sigma}$. One can choose from among various possibilities (for example, see [25, 71, 72, 73]), each of them suitable for a certain mathematical application.

We start with the usual $q$-deformed anticommutation relations for fermions, which is analogous to the $q$-deformed commutation relations for $n$ creation (annihilation) boson system that realizes the standard Drinfeld-Jimbo quantum $u_q(n)$ algebra\(^5\) [20, 21]. For our construction in terms of $\alpha^\dagger_{jm\sigma}$ and $\alpha_{jm\sigma}$ it has the form

\[
\alpha_{jm\sigma} \alpha^\dagger_{jm\sigma} + q^{\pm 1} \alpha^\dagger_{jm\sigma} \alpha_{jm\sigma} = q^{\pm N^{(j)}_{m,2\sigma}}, \tag{2.61}
\]

which holds for every $\sigma = \pm \frac{1}{2}$ and $m = -j,-j+1,\ldots,j$ and where $N^{(j)}_{m,2\sigma} = c^\dagger_{jm\sigma} c_{jm\sigma}$ such that $N^{(j)}_{2\sigma} = \sum_{m=-j}^{j} N^{(j)}_{m,2\sigma}$ are the “classical” number operators (2.5). The relation (2.61) yields

\[
\alpha^\dagger_{jm\sigma} \alpha_{jm\sigma} = [N^{(j)}_{m,2\sigma}], \tag{2.62}
\]

\[
\alpha_{jm\sigma} \alpha^\dagger_{jm\sigma} = [1 - N^{(j)}_{m,2\sigma}], \tag{2.63}
\]

\(^5\)Going forward, we will use the notation $\mathfrak{g}_q$ for a $q$-analog of the $\mathfrak{g}$ Lie algebra (even if it is not a quantum algebra).
and since the single-particle state with quantum numbers \((jm)\) cannot be filled with more than one fermion of type \(\sigma\), \(N^{(j)}_{m,2\sigma} = 0,1\) and both relations turn out to be undeformed. This essentially means that such \(q\)-deformed anticommutation relation (2.61) leads back to a non-deformed algebra.

We introduce a novel set of \(q\)-deformed relations [69] with a weight coefficient \(\omega_j = 1/(2\Omega_j)\)

\[
\{\alpha_{jms}, \alpha^\dagger_{jm's}\} = q^\frac{N_{\sigma}}{2\Omega_j} \delta_{m,m'}, \quad \{\alpha_{jms}, \alpha^\dagger_{jm's'}\} = 0, \sigma \neq \sigma', \quad \{\alpha_{jms}, \alpha_{jm's'}\} = 0,
\]

(2.64)

Using both anticommutation relations for \(m = m'\) and \(\sigma = \sigma'\), it follows that \(\alpha^\dagger_{jms} \alpha_{jm's} = \frac{N^{(j)}_{\sigma}}{2\Omega_j}\) (that is, it depends on the number of \(\sigma\)-particles summed over all \(m\) and weighted by the degeneracy of the single-particle level \(j\)) and hence

\[
\sum_m \alpha^\dagger_{jms} \alpha_{jm's} = 2\Omega_j \left[ \frac{N^{(j)}_{\sigma}}{2\Omega_j} \right],
\]

(2.65)

\[
\sum_m \alpha_{jms} \alpha^\dagger_{jm's} = 2\Omega_j \left[ 1 - \frac{N^{(j)}_{\sigma}}{2\Omega_j} \right].
\]

(2.66)

The introduction of \(\omega_j\) into (2.64) is justified because in the \(q \to 1\) limit as \(\alpha^\dagger_{jms} \to c^\dagger_{jms}\) the relations (2.65) and (2.66) revert back to the “classical” formulae (2.5) and (2.8).

The set of eight operators, (2.58)-(2.60), as defined in terms of the \(q\)-deformed creation and annihilation operators, \(\alpha^\dagger_{jms}\) and \(\alpha_{jm's}\), fulfilling anticommutation relations (2.64), together with the two Cartan non-deformed \(N_{\pm 1}\) operators close on a \(q\)-deformed \(sp_q(4)\) algebra and its subalgebraic structure is obtained in complete analogy with the “classical” case (Table 2.8). The operators in the fourth column of (Table 2.8) close an \(su_q(2) \sim so_q(3)\) algebra. Both reductions of the non-deformed \(sp(4)\) algebra, (2.13) and (2.14), hold in the \(q\)-deformed case

\[
sp_q(4) \supset u_q^\mu(2) = u_{C_T}^\mu(1) \oplus su_q^\mu(2) \supset su_q^\mu(2) \supset u^\mu(1), \quad \mu = \{\tau, 0, \pm\},
\]

(2.67)

\[
sp_q(4) \supset su_q^\mp(2) \oplus su_q^\mp(2).
\]

(2.68)

For all the limiting cases, the one-dimensional unitary algebras, \(u^\mu(1)\), remain non-deformed. The underlying \(q\)-deformed symmetries are associated with the same physical description in each limit, \(\mu = \{\tau, 0, \pm\}\), namely isospin symmetry, \(pn, pp (nn)\) pairing. However, the corresponding generators of these symmetries are \(q\)-deformed and in general their action space is also deformed. All the results in the \(q\)-deformed analysis coincide with the corresponding non-deformed ones in the \(q \to 1\) limit. Here again, the label \(j\) is dropped from the notations when only a single-\(j\) level case is considered.
Table 2.8: Realizations of the $q$-deformed $u_{C_q^1}^\mu(1) \oplus su_q^\mu(2) \subset sp_q(4)$ reductions and their analogy to the “classical” limit, $u_{C_q^1}^\mu(1) \oplus su^\mu(2) \subset sp(4)$.

<table>
<thead>
<tr>
<th>symmetry ($\mu$)</th>
<th>$u_{C_q^1}^\mu(1)$</th>
<th>$su^\mu(2)$</th>
<th>$su_q^\mu(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>isospin ($\tau$)</td>
<td>$N = N_{++}N_{--}, \tau_0 = \frac{1}{2}(N_1 - N_{-1}), \tau_-$</td>
<td>$T_+, T_0 \equiv \tau_0, T_-$</td>
<td></td>
</tr>
<tr>
<td>$pn$ pairs (0)</td>
<td>$\tau_0$</td>
<td>$A_0^\dagger, N^0 = \frac{1}{2}N - \Omega, A_0$</td>
<td>$B_0^\dagger, K_0 \equiv N^0, B_0$</td>
</tr>
<tr>
<td>$pp$ pairs (+)</td>
<td>$N_{-1}$</td>
<td>$A_{+1}^\dagger, N^+ = \frac{1}{2}(N_{+1} - \Omega), A_{-1}$</td>
<td>$B_{+1}^\dagger, K^+ \equiv N^+, B_{-1}$</td>
</tr>
<tr>
<td>$nn$ pairs (−)</td>
<td>$N_{+1}$</td>
<td>$A_{-1}^\dagger, N^- = \frac{1}{2}(N_{-1} - \Omega), A_{+1}$</td>
<td>$B_{-1}^\dagger, K^- \equiv N^-, B_{+1}$</td>
</tr>
</tbody>
</table>

As the anticommutation relations (2.64) of the $q$-deformed fermion operators has a sign choice in its definition, the structure constants in the commutation relations of $sp_q(4)$ are not uniquely determined. However, there is a set of commutation relations of the $q$-deformed symplectic algebra that is symmetric with respect to the exchange $q \leftrightarrow q^{-1}$. Together with the commutation relations for $su_q^\mu(2)$ and $su_q^\mu(2)$ given in Table 2.9 this set includes the symmetric $su_q^\mu(2)$ $q$-commutators, for which $\tilde{\rho} = \rho$ (Table 2.9), where the $\rho$ coefficients obtained in the non-symmetric and symmetric commutation relations are

$$\tilde{\rho} = \begin{cases} 
\rho_+ \equiv q^{\frac{1}{2}+\frac{2}{\hbar}} \frac{2^{2\Omega}}{4} q^{-1}, & \text{non-symmetric} \\
\rho \equiv \frac{1}{2}(\rho_+ + \rho_-) = \frac{[2\Omega + 1]}{4} q^{-1} \frac{2^{2\Omega}}{4}, & \text{symmetric}.
\end{cases}$$

Table 2.9: Commutation relations of the basis operators of the unitary $u_q^\mu(2)$ subalgebras of $sp_q(4)$, $\mu = \tau, 0, \pm$, along with the Casimir invariants of $su^\mu(2)$. The repeated commutation relations are in parenthesis. The $\rho$ coefficients are defined in text (2.69).

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>unitary $u_q^\mu(2)$ subalgebra</th>
<th>$su^\mu(2)$</th>
<th>$C_2^\mu(sp_q^\mu(2)) = C_{2,q}^\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$\hat{N}, T_0, \pm$</td>
<td>$[T_+, T_-] = [\frac{2N_0}{2\eta^2}], T_0, T_\pm = \pm T_\pm$</td>
<td>$T^2 = \Omega([T_+, T_-] + [\frac{1}{\eta^2}] [T_0]_{\frac{1}{\eta^2}})$</td>
</tr>
<tr>
<td>0</td>
<td>$T_0, B_0^{(i)}$</td>
<td>$B_0^{\dagger}, B_0 = \pm B_0^{\pm}$</td>
<td>$(s_q^0)^2 = \Omega([B_0^{\dagger}, B_0] + [\frac{1}{\eta^2}] [N_0]_{\frac{1}{\eta^2}})$</td>
</tr>
<tr>
<td>+</td>
<td>$N^-, B_{\pm 1}^{\dagger}$</td>
<td>$B_{+1}^{\dagger}, B_{-1} = \tilde{\rho} \frac{2N_{+1}}{2\eta^2}, B_{\pm 1}^{\dagger} = \pm B_{\pm 1}^{\pm}$</td>
<td>$(s_q^2)^2 = \Omega_{\frac{1}{2}}([B_{+1}^{\dagger}, B_{-1}] + \tilde{\rho} [\frac{2}{\eta^2}] [N_{+1}]_{\frac{1}{\eta^2}})$</td>
</tr>
<tr>
<td>−</td>
<td>$N^+, B_{\mp 1}^{\dagger}$</td>
<td>$B_{-1}^{\dagger}, B_{+1} = \tilde{\rho} \frac{2N_{-1}}{2\eta^2}, B_{\mp 1}^{\dagger} = \pm B_{\mp 1}^{\pm}$</td>
<td>$(s_q^2)^2 = \Omega_{\frac{1}{2}}([B_{-1}^{\dagger}, B_{+1}] + \tilde{\rho} [\frac{2}{\eta^2}] [N_{-1}]_{\frac{1}{\eta^2}})$</td>
</tr>
</tbody>
</table>
The rest of the 45 symmetric commutation relations for \(\mathfrak{sp}_q(4)\) are

\[
[T_l, B^\pm_k] = \pm \frac{1}{\sqrt{\Omega}} B^\pm_{l+k} \frac{\Psi_{\pm 1} (N_{l+k})}{2 [2]}, \quad l, k = \pm 1,
\]

\[
[T_l, B^\pm_0] = \frac{1}{\sqrt{\Omega}} B^\pm_l \frac{\Psi_{\pm 1} (N_{l+1})}{2}, \quad l = \pm 1,
\]

\[
[T_0, B^\pm_k] = \pm k B^\pm_k, \quad (2.70)
\]

\[
[B^\pm_l, B^\pm_k] = 0,
\]

\[
[B^\dagger_l, B_k] = \frac{1}{\sqrt{\Omega}} T_{l+k} \frac{\Psi_{\pm 1} (N_l)}{2 [2]}, \quad l + k \neq 0,
\]

where we defined \([2\chi]_{\frac{1}{2n}}\) in (2.41) and we define

\[
\Psi_{\pm 1} (N_p) = 2 \sqrt{\rho_+ \rho_-} \left[ 2 N_p \pm 1 - \Omega \right]^{\frac{1}{2n}}. \quad (2.71)
\]

The \(q\)-functions \(\Psi_{\pm 1} (N_p)\) can be written in terms of the \(q\)-parameter in the following way:

\[
\Psi_{+1} (N_p) = q^{N_p} + q^{-\Omega} q^{N_p+1} + q^{-\Omega} (N_p+1), \quad (2.72)
\]

\[
\Psi_{-1} (N_p) = q^{N_p-1} + q^{-\Omega} q^{N_p+1} + q^{-\Omega} (N_p-1). \quad (2.73)
\]

The first three commutation relations in (2.70) of the \(q\)-deformed pair creation (annihilation) operators, \(B^\pm_{k=0, \pm 1}\), with \(T_{0, \pm}\) (of \(\mathfrak{su}_q^2(2)\)) show that \(B^\pm_{k=0, \pm 1}\) transform like components of a vector under transformations generated by the \(T_{0, \pm}\) operators \((T_0 \equiv \tau_0)\).

As already suggested, the set of structure constants in (2.70) is not unique. Even though we only make a use of the symmetric one (2.70), for completeness we present another set of commutation relations of \(\mathfrak{sp}_q(4)\),

\[
[B^\dagger_l, B_k]_{2(l-k)} = \frac{\varphi_{l,k}}{2\sqrt{\Omega}} T_{l+k} q^{l-k} N_{l-k}, \quad l + k \neq 0,
\]

\[
[T_l, B^\dagger_k] = \frac{\chi_{l,k}}{\sqrt{\Omega}} B_{l+k} q^{-l} N_{l-k}, \quad l \neq 0, \quad (2.74)
\]

\[
[T_l, B_k] = -\frac{\phi_{l,k}}{\sqrt{\Omega}} B_{l+k} q^{l} N_{l-k}, \quad l \neq 0,
\]

\[
[T_0, B^\dagger_k] = k F^\dagger_k, \quad [T_0, B_k] = k G_k,
\]

where (2.74) are not symmetric under the exchange \(q \leftrightarrow q^{-1}\) and for a given single-\(j\) level the constants are defined as follows

\[
\varphi_{\pm 1,0} = 2 q^2 \rho_+, \quad \varphi_{0, \pm 1} = 2 q^{-\pm 2^\pm} \rho_+, \quad \varphi_{\pm 1, \pm 1} = 0,
\]

\[
\chi_{1, -1} = \rho_-, \quad \chi_{-1, 1} = \rho_+, \quad \chi_{\pm 1, 0} = 1, \quad \chi_{\pm 1, \pm 1} = 0, \quad (2.75)
\]

\[
\phi_{1, -1} = q^{-1} \rho_-, \quad \phi_{-1, 1} = q \rho_+, \quad \phi_{\pm 1, 0} = q^{\mp 1}, \quad \phi_{\pm 1, \pm 1} = 0.
\]
In each limit, using relation (2.48) and \( \omega = \frac{1}{2\Omega} \), the \( q \)-commutators of the raising and lowering \( q \)-operators (Table 2.9) can be rewritten as

\[
[T_+, T_-] = [2\omega T_0] = [\omega] [2T_0]_\omega, \quad T_\pm \to \frac{T_\pm}{\sqrt{[\omega]}},
\]

\[
[B_0^\dagger, B_0] = [2\omega N^0] = [\omega] [2N^0]_\omega, \quad B_0^\pm \to \frac{B_0^\pm}{\sqrt{[\omega]}},
\]

\[
[B_{\pm 1}^\dagger, B_{\mp 1}] = \tilde{\rho} [4\omega N^\pm] = \tilde{\rho} [2\omega] [2N^\pm]_{2\omega}, \quad B_{\pm 1} \to \frac{B_{\pm 1}}{\sqrt{\tilde{\rho}[2\omega]}},
\]

\[
B_{\pm 1} \to \frac{B_{\pm 1}}{\sqrt{\tilde{\rho}[2\omega]}},
\] (2.76)

where to the right a renormalization of the corresponding operators is given, which transforms the \( q \)-commutator into the form, \([X_+, X_-] = [2X_0]_k\), where \( X_+,0,- \) corresponds to the basis operators of \( su_q^{(2)} \) for each of the four limits and \( k(\mu = T, 0) = \omega, \ k(\mu = \pm) = 2\omega \). In this way, the latter realization of \( su_q^{(2)} \) can be related to the standard quantum \( su_q^{(2)} \) algebra [24]. Moreover, the realization of \( sp_q(4) \) after the renormalization procedure (2.76) corresponds to the standard Drinfeld-Jimbo construction for \( so_q^{(5)} \) [24], which shows the isomorphism of the \( q \)-deformed \( sp_q(4) \) algebra and its standard \( su_q^{(2)} \) subalgebras to the \( so_q^{(5)} \) algebra and its subalgebraic structure (Appendix C).

### 2.3.2 Action Space for the \( q \)-Deformed Generators

In general, the \( q \)-deformed fermion operators (2.64) act as in the “classical” case in a finite metric space \( q\mathcal{E}_j \) for each particular \( j \)-level, with a vacuum \( |0\rangle \) defined by \( \alpha^\dagger_{jm\sigma} |0\rangle = 0 \). The scalar product in \( q\mathcal{E}_j \) is chosen in such a way that \( \alpha^\dagger_{jm\sigma} \) is a Hermitian conjugate to \( \alpha_{jm\sigma} : (\alpha^\dagger_{jm\sigma})^* = \alpha_{jm\sigma}, \) and \( \langle 0|0 \rangle = 1 \). In general the \( q \)-deformed states are different from the “classical” ones, but reduce to the “classical” ones in the limit \( q \to 1 \).

In analogy to the “classical” limit (2.19), the basis in the \( q \)-deformed case can be obtained by orthonormalization of a linearly independent subset of the \( q \)-deformed vectors

\[
|n_1, n_0, n_{-1}\rangle_q = \left( B_1^\dagger \right)^{n_1} \left( B_0^\dagger \right)^{n_0} \left( B_{-1}^\dagger \right)^{n_{-1}} |0\rangle.
\] (2.77)

The states (2.77) have total angular momentum and parity \( J^\pi = 0^+ \) and define the \( q\mathcal{E}_j^+ \) space. The latter is a subspace in the even \( q\mathcal{E}_j^+ \) space spanned by all the states with even total number of particles.

In complete analogy to the “classical” picture, the representations of the \( q \)-deformed symmetries are specified by the eigenvalues of the corresponding invariant operators, in a way that if \( [x] \) is an eigenvalue, the quantum number in the labeling scheme is \( x \) (the corresponding value in the “classical” \( q \to 1 \) limit). As far as the physical observables are concerned, the related operators remain undeformed and they provide for “classical”
eigenvalues, like total number of particles \( n \), proton/neutron number \( N_{\pm} \), \( pp/pn/nn \) pair numbers \( n_{1,0,-1} \), isospin projection \( i \) (angular momentum \( J \)). This is why the states identified in (2.77), even though \( q \)-deformed, can be labeled by the non-deformed numbers of pairs, \( n_{1,0,-1} \). Hence, a \( q \)-deformed state \( |n_1,n_0,n_{-1}\rangle_q \) (2.77) does not differ from a “classical” one \( |n_1,n_0,n_{-1}\rangle \) (2.77) in the number of particles (pairs). The difference is hidden in the way the nucleons interact among themselves. This is our first attempt to probe at a deeper level the meaning of the \( q \)-deformation introduced in \( sp(4) \).

As in the “classical” case, \( \Omega \) labels the fully-paired representation for each particular \( j \)-shell. The deformed basis states are labeled by the eigenvalues (in their “classical” limit) of the invariant operators of the subalgebras in the reduction along each of the cases considered (\( \mu = \tau, 0, \pm \)). In a way that is similar to the non-deformed analysis, another complete labeling scheme, which is more physically relevant, is provided by the numbers of pairs of the three kinds, \( n_{1,0,-1} \). In the \( q \)-deformed picture, the example of Table 2.3 can still be used so long as one remembers that the pair operators that construct the basis states are \( q \)-deformed (the index \( q \) can be dropped from the notation for the basis states whenever the \( q \)-deformed space is implied in the context). The basis states together with the second-order Casimir operators and their eigenvalues are often used in the physical applications. It is in this sense that their \( q \)-deformation may lead to some interesting new results.

### 2.3.3 Description of the \( sp_q(4) \) Subalgebraic Structure

We now list in brief form the different realizations of the \( sp_q(4) \supset su_q(2) \) reduction and compare them to their “classical” counterparts in order to emphasize the similarity and differences between them.

The operators \( T_{0,\pm 1}, T_0 \equiv \tau_0 \) (2.60) close the isospin-related \( su_q(2) \sim so_q(3) \) algebra with a second-order Casimir invariant (Table 2.9)

\[
T^2 = \frac{2\Omega}{2}(T_+T_-+T_-T_+ + [\omega T_0][T_0+1]_\omega + [\omega T_0][T_0-1]_\omega)
= \Omega\left(\{T_+,T_\}\right)\left(2\omega\right)\left[T_0\right]^2_\omega
= 2\Omega(T_-T_+ + [\omega T_0][T_0+1]_\omega)
= 2\Omega(T_+T_- + [\omega T_0][T_0-1]_\omega).
\]  

(2.78)

Here the (2.49) and (2.50) identities have been used and \( \omega = \frac{1}{2\Omega} \). The corresponding \( q \)-deformed eigenvalues and eigenstates are

\[
T^2|n,\tau,i\rangle_q = 2\Omega\left[\frac{\omega}{2\Omega}\right][\tau]_\omega [\tau+1]_\omega |n,\tau,i\rangle_q = 2\Omega\left[\frac{\tau}{2\Omega}\right][\tau+1]_\omega |n,\tau,i\rangle_q,
\]

(2.79)

where (2.48) is used and \( n, \tau \) and \( i \) take on the values specified in Table 2.4. Note that \( |n,\tau,i\rangle_q \) are deformed but each state is characterized by the “classical” values of number of particles \( n \), isospin \( \tau \), and its third projection \( i \). As we already explained, this is expected for \( n \) and \( i \). Regarding the isospin, this means that a given state in analogy to the corresponding “classical” vector can be considered as having isospin \( \tau \), which is the eigenvalue
of the $\tau$ isospin operator. The operator $T$ (with value $T = \sqrt{2\Omega \left[ \frac{\tau}{2\Omega} \right] [\tau + 1]}$) is not the physical isospin operator and since it is invariant under transformations involving a change in the isospin coordinate of a nucleon we call it the isospin-related operator.

Next we note that the operators that close $su_q^0(2)$ (Table 2.8) describe $q$-deformed paired particles of two different kinds. With the help of (2.49) and (2.50), the second-order Casimir operator, $C_2(su_q^0(2))$ (Table 2.9), can be written as

$$ (s_q^0)^2 = \frac{2\Omega}{2} (B_0^\dagger B_0 + B_0 B_0^\dagger + [\omega N^0] [N^0 + 1]_\omega + [\omega N^0] [N^0 - 1]_\omega) $$

$$ = \Omega \left( \{B_0^\dagger, B_0\} + [2\omega] [N^0]_\omega^2 \right) $$

$$ = 2\Omega (B_0^\dagger B_0 + [\omega N^0] [N^0 + 1]_\omega) $$

$$ = 2\Omega (B_0^\dagger B_0 + [\omega N^0] [N^0 - 1]_\omega), \quad (2.80) $$

with eigenvalues

$$ 2\Omega \left[ \frac{1}{2\Omega} \right] \left[ \frac{2\Omega - 2n_q}{2} \right]_\omega \left[ \frac{2\Omega - 2n_q}{2} + 1 \right]_\omega = 2\Omega \left[ \frac{1}{2\Omega} \right] [s_q^0]_\omega [s_q^0 + 1]_\omega, \quad (2.81) $$

and the $q$-deformed eigenstate labeled as in Table 2.4. In (2.81) $\nu_0$ is the "classical" seniority number and in the fully-paired space (2.19) $[(\Omega, t = 0)$-irrep of the $q$-deformed symplectic symmetry] $\nu_0 = n_1 + n_{-1}$. When $s^0$ is maximum (the number of $pn$ pairs is maximal) the eigenvalue problem holds in the $q$-deformed case as well as in the non-deformed one,

$$ (s_q^0)^2 |n_1, n_0, n_{-1}\rangle = 2\Omega \left[ \frac{1}{2\Omega} \right] \left[ \frac{2\Omega - 2(n_1 + n_{-1})}{2} \right]_\omega \left[ \frac{2\Omega - 2(n_1 + n_{-1})}{2} + 1 \right]_\omega |n_1, n_0, n_{-1}\rangle, \quad (2.82) $$

where here again, $\nu_0 = n_1 + n_{-1} = |i|$ and $n_1 = 0$, or $n_{-1} = 0$, or both $n_1 = n_{-1} = 0$. In this special case the action of the $su_q^0(2)$ basis operators along the columns of Table 2.3 is

$$ B_0^\dagger |n_1, n_0, n_{-1}\rangle = |n_1, n_0 + 1, n_{-1}\rangle, $$

$$ B_0 |n_1, n_0, n_{-1}\rangle = |n_0, \frac{1}{2\Omega} \left[ 1 - \frac{2(n_{-1} + n_1) + n_{-1} - 1}{2\Omega} \right] |n_1, n_0 - 1, n_{-1}\rangle, $$

$$ N |n_1, n_0, n_{-1}\rangle = 2 (n_{-1} + n_1 + n_0) |n_1, n_0, n_{-1}\rangle. \quad (2.83) $$

In the $\mu = \pm$ limit, it is again true that the basis operators of the two subalgebras $su_q^+(2)$ and $su_q^-(2)$ (Table 2.8) commute with each other. Their commutation relations can be expressed in non-symmetric or symmetric forms (Table 2.9). In the first case, the asymmetry arises from the factors $\rho_\pm$ (2.69) and in the second case the factor $\rho$ (2.69) is obtained symmetric and do not differ for both $\mu = \pm$ limits. In nuclear structure applications, the operator $B_\pm^{\dagger}$ ($B_\pm$) creates (destroys) a $q$-deformed pair of particles of the same kind and the $\rho_\pm$ coefficients introduce an asymmetry between proton and neutron pairs. When
a model is required to possess proton-neutron symmetry, only the symmetric form of the $q$-deformation can enter.

For both symmetric, $\tilde{\rho} = \rho$, and non-symmetric, $\tilde{\rho} = \rho_{\pm}$, realizations of $\mathfrak{su}_q^\pm(2)$, the Casimir invariants of the $\mathfrak{su}_q^\pm(2)$ algebras are

$$C_2(\mathfrak{su}_q^\pm(2)) \equiv (s_q^\pm)^2 = \frac{\Omega}{2} \left( B_{\mp 1}^\dagger B_{\mp 1} + B_{\mp 1} B_{\mp 1}^\dagger + \tilde{\rho} \left[ 2\omega N^\pm \right] \left[ N^\pm + 1 \right]_{2\omega} + \right.$$

$$\left. \tilde{\rho} \left[ 2\omega N^\pm \right] \left[ N^\pm - 1 \right]_{2\omega} \right)$$

$$= \frac{\Omega}{2} \left( \left\{ B_{\mp 1}^\dagger, B_{\mp 1} \right\} + \tilde{\rho} \left[ 4\omega \right] \left[ N^\pm \right]^2 \right)$$

$$= \Omega \left( B_{\mp 1}^\dagger B_{\mp 1} + \tilde{\rho} \left[ 2\omega N^\pm \right] \left[ N^\pm + 1 \right]_{2\omega} \right)$$

$$= \Omega \left( B_{\mp 1}^\dagger B_{\mp 1} + \tilde{\rho} \left[ 2\omega N^\pm \right] \left[ N^\pm - 1 \right]_{2\omega} \right), \quad (2.84)$$

where (2.49) and (2.50) have been again used. The eigenvalues of the $(s_q^\pm)^2$ operators are

$$\tilde{\rho} \Omega \left[ \frac{1}{\Omega} \right] \left[ \frac{\Omega - \nu_1}{2} \right]_{2\omega} \left[ \frac{\Omega - \nu_1}{2} + 1 \right]_{2\omega} = \tilde{\rho} \Omega \left[ \frac{1}{\Omega} \right] \left[ s^\pm \right]_{2\omega} \left[ s^\pm + 1 \right]_{2\omega}, \quad (2.85)$$

and the corresponding eigenstates are classified according to Table 2.4. For the eigenvalues (2.85), $\nu_1$ is the “classical” seniority quantum number and in the fully-paired space $qE_{j,0}^+$ (2.19) $\nu_1 = n_0$. When $s^\pm$ is maximum (the number of like-particles pairs is maximal) the $q$-deformed eigenvalue problem holds in a similarity with the non-deformed one

$$(s_q^\pm)^2 |n_1, n_0, n_{-1}) = \tilde{\rho} \Omega \left[ \frac{1}{\Omega} \right] \left[ \frac{\Omega - n_0}{2} \right]_{2\omega} \left[ \frac{\Omega - n_0}{2} + 1 \right]_{2\omega} |n_1, n_0, n_{-1}) \quad (2.86)$$

where $n_0 = 0$ if the total number of pairs $n/2$ is even or $n_0 = 1$ if $n/2$ is odd. In this special case, the $\mathfrak{su}_q^\pm(2)$ operators transform the states along the diagonals as

$$B_{\pm 1}^\dagger |n_1, n_0, n_{-1}) = \left| n_{\pm 1} + 1, n_0, n_{\mp 1} \right>,$$

$$B_{\mp 1} |n_1, n_0, n_{-1}) = \tilde{\rho} \left[ n_{\pm 1} \right] \Omega \left[ 1 - \frac{n_{\pm 1} + n_0 - 1}{\Omega} \right] |n_{\pm 1} - 1, n_0, n_{\mp 1} \rangle,$$

$$N_{\pm} |n_1, n_0, n_{-1}) = \left( 2n_{\pm 1} + n_0 \right) |n_1, n_0, n_{-1} \rangle. \quad (2.87)$$

The eigenvalues of the invariant operators for the different $\mu = \tau, 0, \pm$ reduction limits of $\mathfrak{sp}_q(4)$ are summarized in Table 2.10.

It is important to emphasize that the $q$-deformation (2.64) may lead to basis states whose content is very different from the “classical” case since there is no known simple function that transforms the “classical” fermion operators $c_{j\mu\sigma}^\dagger$ and $c_{j\mu\sigma}$ into the $q$-deformed ones $\tilde{c}_{j\mu\sigma}^\dagger$ and $\tilde{c}_{j\mu\sigma}$. Smooth function does not exist when the anticommutation relations (2.1) hold simultaneously with both signs for one and the same $\sigma$, as they are defined in (2.64) (Appendix C).
Table 2.10: Eigenvalues of the invariant operators in each $\mu$ reduction limit of $\mathfrak{sp}_q(4)$, for the IURs in the space spanned by the $q$-deformed eigenstates labeled as in (Table 2.4).

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$C_1(u^\mu(2))$</th>
<th>$C_2(su_q^\mu(2))$</th>
<th>$C_1(u^\mu(1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$n = 2n_1 + 2n_{-1} + 2n_0$</td>
<td>$2\Omega \left[ \frac{1}{2\Omega} \right] \frac{1}{2\Omega} \left[ \tau + 1 \right]_\Omega$</td>
<td>$i$</td>
</tr>
<tr>
<td>$0$</td>
<td>$i = n_1 - n_{-1}$</td>
<td>$2\Omega \left[ \frac{1}{2\Omega} \right] \frac{1}{2\Omega} \left[ \frac{2\Omega - 2(n_1 + n_{-1})}{2} \right]_\Omega$</td>
<td>$n$</td>
</tr>
<tr>
<td>$\pm$</td>
<td>$N_\pm = 2n_{\mp} + n_0$</td>
<td>$\bar{\rho} \Omega \left[ \frac{1}{\Omega} \right] \frac{1}{2\Omega} \left[ \Omega - n_0 \right]<em>\Omega \frac{1}{2\Omega} \left[ \Omega - n_0 + 1 \right]</em>\Omega$</td>
<td>$N_\pm$</td>
</tr>
</tbody>
</table>

### 2.3.4 $q$-Deformed Transformation of the Fermion Operators

Different $q$-deformations can be introduced in a Lie algebra, however many of them compromise properties of a quantum algebra. For the one considered in this section there exists a function that maps the $q$-deformed fermion creation and annihilation operators into their “classical” counterparts, but the deformation does not lead to standard quantum $\mathfrak{su}_q(2)$ subalgebras. Also, the $q$-deformed results only differ by a $q$-phase that is number operators dependent from the non-deformed picture. In this sense this deformation is trivial; however, it is a good example that for some deformations a transformation function between the $q$- and non-deformed fermion operators exists.

There is a general class of functions that transform the “classical” operators into deformed ones [74, 55]. We use the transformation

$$\alpha_{jms} = \theta^\frac{N_{2\sigma}^{(j)}}{N_{2\sigma}} c_{jms}, \quad \alpha_{jms}^\dagger = c_{jms}^\dagger \bar{\theta}^\frac{N_{2\sigma}^{(j)}}{N_{2\sigma}}, \quad \sigma = \pm \frac{1}{2}, \quad (2.88)$$

where $\theta$ is a complex number with a complex conjugate $\bar{\theta}$ and an amplitude $|\theta| = q$, $q$ a real number, and $N_{2\sigma}^{(j)} = \sum_m N_{m,2\sigma}^{(j)}$ (2.5) are the “classical” number operators. The transformation of (2.1) leads to anticommutation relations for the $q$-deformed fermion operators,

$$\alpha_{jms}\alpha_{jms}^\dagger + q\alpha_{jms}^\dagger\alpha_{jms} = q^{N_{2\sigma}^{(j)}}, \quad (2.89)$$

and the identities

$$\sum_m \alpha_{jms}^\dagger\alpha_{jms} = N_{2\sigma}^{(j)} q^{N_{2\sigma}^{(j)}-1}, \quad (2.90)$$

$$\sum_m \alpha_{jms}\alpha_{jms}^\dagger = \left(2\Omega_j - N_{2\sigma}^{(j)}\right) q^{N_{2\sigma}^{(j)}}.$$

Dropping the $(j)$ notation for a single $j$-level, the transformation (2.88) yields the follow-
ing relations between the deformed (2.58)-(2.60) and the “classical” operators (2.2)-(2.4):

\[
\begin{align*}
B_{\pm 1}^\dagger &= A_{\pm 1}^\dagger \tilde{\theta}^{N_{\pm 1} + \frac{1}{2}}, \\
B_{0}^\dagger &= A_{0}^\dagger \tilde{\theta}^{\frac{N}{2}}, \\
B_{\pm 1} &= \theta^{N_{\pm 1} + \frac{1}{2}} A_{\pm 1}, \\
B_{0} &= \theta^{\frac{N}{2}} A_{0}, \\
\tilde{T} &= \frac{1}{2} \left( N + 1, N + 1 \right) = \frac{1}{2} \left( N, N \right).
\end{align*}
\]

Since there is a smooth transformation that depends on the Cartan operators of $\mathfrak{sp}(4)$ only and maps the $q$-deformed operators $B_{0, \pm 1}^\dagger$ to the “classical” vectors $A_{0, \pm 1}^\dagger$, the $q$-deformed states $|B_{\pm 1}^\dagger n, 0\rangle$ are equivalent within a phase to the “classical” ones (2.19). All the relations revert back to the “classical” formulae in the limit $q \to 1$.

The important reduction of $\mathfrak{sp}_{q}(4)$ algebra to compact $\mathfrak{u}(2)$ subalgebra can be used again to obtain classification schemes for the basis states. Since the four non-deformed second-order $\mathfrak{su}_{0}^{\tau, 0, \pm}(2)$ Casimir invariants (Table 2.2) commute with $N_{\pm 1}, N_{-1}, \hat{N}, \tau_{0}$, involved in the transformation (2.91), they remain invariant with respect to the corresponding $q$-deformed $\mathfrak{su}_{0}^{\tau, 0, \pm}(2)$ operators.

The generators $T_{0}, T_{\pm}$ and $N$ satisfy the $\mathfrak{u}(2)$ commutation relations:

\[
\begin{align*}
[T_{+}, T_{-}] &= \frac{T_{0}}{2} q^{-N_{-1}}, \\
[T_{0}, T_{\pm}] &= \pm T_{\pm}, \\
[\hat{N}, T_{\pm}] &= 0, \\
[\hat{N}, T_{0}] &= 0.
\end{align*}
\]

The $q$-deformed second-order operator invariant with $T_{0}, T_{\pm}$ (and coinciding with $\tau^{2}$) is

\[
\mathbf{T}^{2} = \frac{2\Omega}{2} (T_{1}T_{-1} + T_{-1}T_{1}) q^{-N_{-1}} + T_{0} T_{0}
\]

\[
= 2\Omega T_{-1}T_{1} q^{-N_{-1}} + T_{0} (T_{0} + 1)
\]

with eigenvalues that are non-deformed ($\tau (\tau + 1)$, (Table 2.7)) and eigenvectors, $|n, \tau, i\rangle_{q}$, that are linear combinations of the $q$-deformed pair basis states, (2.77), which are phase-translated to the non-deformed ones (2.19).

The basis operators of $\mathfrak{su}_{0}^{0}(2)$ commute in the following way

\[
\begin{align*}
[B_{0}^\dagger, B_{0}]_{-2} &= \frac{N_{0}}{\Omega} q^{-N_{-2}}, \\
[N_{0}, B_{0}^\pm] &= \pm B_{0}^\pm,
\end{align*}
\]

where the second-order Casimir invariant of $\mathfrak{su}_{0}^{0}(2)$ coincides with its “classical” counterpart, $s^{0}$, after the transformation (2.91)

\[
C_{2}(\mathfrak{su}_{0}^{0}(2)) = \frac{2\Omega}{2} (q^{2} B_{0}^\dagger B_{0} + B_{0} B_{0}^\dagger) q^{-N} + (N_{0})^{2}
\]

\[
= 2\Omega B_{0} B_{0}^\dagger q^{-N} + N_{0} (N_{0} + 1).
\]
The two mutually complementary subalgebras $\mathfrak{su}^+_q(2)$ and $\mathfrak{su}^-_q(2)$ of the $\mathfrak{sp}_q(4)$ algebra given through the commutation relations of their basis operators are

$$[B^\dagger_{\mp 1}, B_{\mp 1}] - 4 = \frac{2N^\pm}{\Omega} q^{2N^\pm - 3}, \quad (2.96)$$

$$[N_{\pm 1}, B^\dagger_{\mp 1}] = 2B^\dagger_{\mp 1}, \quad [N_{\pm 1}, B_{\mp 1}] = -2B_{\mp 1}. \quad (2.97)$$

The corresponding Casimir invariant (that is equivalent to the “classical” one) is

$$C_2(\mathfrak{su}^\pm_q(2)) = \frac{\Omega}{2} (q^4 B^\dagger_{\mp 1}B_{\mp 1} + B_{\mp 1}B^\dagger_{\mp 1})q^{-2N_{\mp 1} - 1} + (N^\pm)^2$$

$$= \Omega B_{\mp 1}B^\dagger_{\mp 1}q^{-2N_{\mp 1} - 1} + N^\pm (N^\pm + 1). \quad (2.98)$$

In the three limits $\mu = 0, \pm$, the eigenvalues of the $q$-deformed second-order Casimir invariant are the “classical” ones (Table 2.7) and the eigenvectors are $q$-deformed and differ by a phase from the non-deformed pair basis (2.19).

A similar $q$-deformation is based on the transformation $\alpha_{jms} = \theta^{-\frac{N_s}{2}} c_{jms}$, which yields the same relations and identities as above, but with the exchange $q \rightarrow q^{-1}$. When $\theta$ is real and positive the deformation parameter is $\theta \equiv q$.

### 2.4 Generalization to Multi-Shell Configurations of the $\mathfrak{sp}(4)$ and $\mathfrak{sp}_q(4)$ Algebras

Up to this point we considered a single $j$-orbit that is occupied by nucleons associated with either “classical” or $q$-deformed fermion creation and annihilation operators, which enter into the construction of the fermion realization of the symplectic algebra in four dimensions. The Lie algebra $\mathfrak{sp}(4)$ can be generalized to multiple levels [16, 10, 11, 27] so the particles can occupy more than a single orbit. This is not simply an interesting exercise, a multi-shell theory is needed if one wants to build a nuclear structure model for comparatively realistic description of nuclei throughout the nuclear chart.

However, the generalization of the $q$-deformed analog, $\mathfrak{sp}_q(4)$, is not a straightforward procedure [75]. In contrast with the usual formulation of $q$-deformation for the symplectic $\mathfrak{sp}_q(4)$ algebra and its $\mathfrak{su}(2)$ subalgebras that is normally used in mathematical studies [25, 23, 24] and in nuclear physics applications [31, 34, 76], the new formulation we have discovered (2.64) depends upon the dimensionality of the underlying space $\Omega_j$. Because of this dependence, a generalization of the $q$-deformed symplectic $\mathfrak{sp}_q(4)$ algebra to a multi-orbit case is not a trivial task and introduces new elements into the theory.

The multi-shell generalization of the fermion realization of $\mathfrak{sp}(4)$ follows the single-$j$ construction of the algebra in the beginning of Chapter 2 in terms of creation and annihilation fermion operators $c^\dagger_{jms}$ and $c_{jms}$ (2.1) and enlarges the single-particle occupation space to several $j$ orbits each with degeneracy $\Omega_j$. For a given $\sigma = \pm \frac{1}{2}$ type of particle, the dimension
of the fermion space is $2\Omega = \sum_j 2\Omega_j = \sum_j (2j + 1)$, where the sum $\sum_j$ is over all orbits that are considered to be active.

In the $q$-deformed case, introduction of more $j$-levels to the occupation space requires a change in the $q$-deformed anticommutation relations of the creation and annihilation single-particle operators $\alpha^\dagger_{jm\sigma}$ and $\alpha_{jm\sigma}$

$$\{\alpha_{jm\sigma}, \alpha^\dagger_{km\sigma'}\} = \frac{q^{\pm N_{\Omega}}}{2\Omega} \delta_{j,k} \delta_{m,m'}, \quad \{\alpha_{jm\sigma}, \alpha^\dagger_{km\sigma'}\} = 0, \quad \sigma \neq \sigma', \quad (2.99)$$

which compared to the single-level definition (2.64) has the same form but now the two Cartan generators,

$$N_{2\sigma} = \sum_j N^{(j)}_{2\sigma} = \sum_j \sum_{m=-j}^j c^\dagger_{jm\sigma} c_{jm\sigma}, \quad \sigma = \pm \frac{1}{2}, \quad (2.100)$$

count the total number of particles of each type $\sigma$, where $N_{2\sigma}^{(j)}$ is defined in (2.5). In physical applications, the number generators $N_{\pm 1}$ along with the total number of particles $\hat{N}$ (generalization of (2.9)),

$$\hat{N} = N_1 + N_{-1} = \sum_j \sum_{\sigma=-1/2}^{1/2} \sum_{m=-j}^j c^\dagger_{jm\sigma} c_{jm\sigma}, \quad (2.101)$$

and the third isospin projection of the nucleons in all orbitals (generalization of (2.10)),

$$\tau_0 = \frac{1}{2}(N_1 - N_{-1}) = \sum_j \sum_{\sigma=-1/2}^{1/2} \sum_{m=-j}^j \sigma c^\dagger_{jm\sigma} c_{jm\sigma}, \quad (2.102)$$

represent physical observables, which are always non-deformed.

The other eight basis operators for the symplectic $\mathfrak{sp}(4)$ algebra (compare to (2.2)-(2.4)) are

$$A^\dagger_k = \frac{1}{\sqrt{2\Omega(1 + \delta_{\sigma\sigma'})}} \sum_j \sum_{m=-j}^j (-1)^{j-m} c^\dagger_{jm\sigma} c^\dagger_{j,-m,\sigma'} = (A_{-k})^\dagger, \quad (2.103)$$

$$A_{-k} = \frac{1}{\sqrt{2\Omega(1 + \delta_{\sigma\sigma'})}} \sum_j \sum_{m=-j}^j (-1)^{j-m} c_{j,-m,\sigma} c_{jm\sigma'}, \quad k = \sigma + \sigma', \quad (2.104)$$

$$\tau_{\pm} = \frac{1}{\sqrt{2\Omega}} \sum_j \sum_{m=-j}^j c^\dagger_{jm,\pm 1/2} c_{jm,\mp 1/2}, \quad (2.105)$$
and for the q-deformed $\mathfrak{sp}_q(4)$ algebra (compare to (2.58)-(2.60)) they are

\[
B_k^\dagger = \frac{1}{\sqrt{2\Omega(1 + \delta_{\sigma\sigma'})}} \sum_j \sum_{m=-j}^j (-1)^{j-m} \alpha^\dagger_{jm\sigma} \alpha^\dagger_{j,-m,\sigma'} = (B_{-k})^*, \quad (2.106)
\]

\[
B_{-k} = \frac{1}{\sqrt{2\Omega(1 + \delta_{\sigma\sigma'})}} \sum_j \sum_{m=-j}^j (-1)^{j-m} \alpha_{j,-m,\sigma} \alpha_{jm\sigma'}, \quad k = \sigma + \sigma', \quad (2.107)
\]

\[
T_\pm = \frac{1}{\sqrt{2\Omega}} \sum_j \sum_{m=-j}^j \alpha^\dagger_{jm,\pm 1/2} \alpha_{jm,\mp 1/2}. \quad (2.108)
\]

In addition to the Cartan operators $N_{\pm 1}$ (or their linear combinations $\hat{N}$ and $T_0 \equiv \tau_0$), the operators (2.103)-(2.105) close on the $\mathfrak{sp}(4)$ algebra and their q-deformed counterparts (2.106)-(2.108) close on the q-deformed $\mathfrak{sp}_q(4)$ algebra, each of the algebras defined by the commutation relations between their basis operators given in Chapter 2 under the substitution $\Omega_j \to \Omega$ and the single-level operators with their multi-level counterparts.

In the deformed and non-deformed cases, the multi-level operators (2.103)-(2.108) are related to the corresponding single-level operators $X^{(j)}$ as $X = \sum_j \sqrt{R_j}/\sqrt{\Omega_j} X^{(j)}$, where $X = \{B^\dagger, B, T\}$ or $X \equiv 1 \{A^\dagger, A, \tau\}$. In the non-deformed limit, the ten operators $A_{0,\pm 1}^{(j)}$, $B_{0,\pm 1}^{(j)}$, $\tau_{0,\pm}^{(j)}$ and $N^{(j)}$ close on the $\mathfrak{sp}^{(j)}(4)$ algebra for each $j$-level and the direct sum holds, $\mathfrak{sp}(4) = \bigoplus_j \mathfrak{sp}^{(j)}(4)$.

A different situation occurs in the deformed case. Recall that the two sets of q-deformed fermions that construct the single-level basis operators of $\mathfrak{sp}_q(4)$ and the multi-level basis operators of $\mathfrak{sp}_q(4)$, respectively, have different q-deformed anticommutation properties, (2.64) and (2.99). Indeed, the single-level generators do not close within the $\mathfrak{sp}_q^{(j)}(4)$ algebra (e.g. $[T^+_j, T^-_j] \neq [2 T^{(j)}_j / 2\Omega_j]$) but rather within the generalized $\mathfrak{sp}_q^{(j)}(4)$ algebra, since

\[
[T^+_j, T^-_j] = \left[2 \frac{T_0}{2\Omega}, \begin{bmatrix} B^\dagger_0^{(j)}, B_0^{(j)} \end{bmatrix} = \left[ \hat{N} - \frac{2\Omega}{2\Omega}, \begin{bmatrix} B^\dagger_{1}, B_{1}^{(j)+1} \end{bmatrix} = \hat{\rho} \left[ N_{\pm 1} - \frac{\Omega}{\Omega} \right], \quad (2.109)
\right.
\]

where $\hat{\rho} = \rho$ (symmetric) or $\hat{\rho} = \rho_\pm$ (non-symmetric) is defined in (2.69). The rest of the commutation relations remain within the single-\( j \) q-deformed algebra, for example $[T^0_0^{(j)}, T^{(j)}_\pm] = \pm T^{(j)}_\pm$. However, several of these relations, like $[T^0_0^{(j)}, B^\dagger_0^{(j)}] = \frac{1}{2\sqrt{\Omega_j}} B^\dagger_0^{(j)}[2N_{\pm l}, \Omega_j] m_1$, $l = \pm 1$, include a multiplicative q-factor with a dependence on the averaged multi-level number, $N_{\pm 1}/(2\Omega)$. This behavior of $\mathfrak{sp}_q^{(j)}(4)$ can be traced back to the generalized q-deformation (2.99), where the anticommutation relations of two fermions on a \( j \)-level depend on the total number of particles of one kind averaged over the multi-shell space. Another interesting consequence of (2.99) is the q-deformed single-\( j \) quantity

\[
\sum_m \alpha^\dagger_{jm\sigma} \alpha_{jm\sigma} = 2\Omega_j \left[ \frac{N_{\pm \sigma}}{2\Omega} \right], \quad \sigma = \pm \frac{1}{2}. \quad (2.110)
\]
In the non-deformed limit, the left-hand side of (2.110) represents the single-level number operator \( N^{(j)} \), while in the deformed extension the zeroth approximation of (2.110) gives an even distribution of the particles over the entire multi-level space weighed by the single-\( j \) dimension. In this way, the \( q \)-deformation for the generalized \( \mathfrak{sp}_q(4) \) algebra introduces probability features at the constituent levels of the theory that relate to the single-\( j \) description.

The action space is spanned by completely paired states with a total angular momentum and parity \( J^\pi = 0^+ \), which are given by (2.19) in the "classical" case and (2.77) in the deformed case. However, the pair-creation vectors, \( A_1^\dagger \) and \( B_1^\dagger \), are now multi-level operators and \( n^{(j)}_{1,0,-1} \) are the total number of pairs of each kind (\( pp \), \( pn \) and \( nn \), respectively) over all the levels, where

\[
n_k = \sum_j n^{(j)}_k, \quad k = 0, \pm 1,
\]

with \( n^{(j)}_{1,0,-1} \) being the corresponding pairs in each \( j \)-orbit. The generalized basis states for a multiple-orbit space of dimension \( 2\Omega \) is a linear combination of the product of the single-level basis states which depend on what pairs occupy which levels [77, 78]

\[
|n_1, n_0, n_{-1}\rangle = \sum_{j_1} ... \sum_{j_{n/2}} \left( \prod_{k=1}^{n/2} \sqrt{\frac{\Omega_{jk}}{\Omega}} \right) A_1^{(j_1)} ... A_1^{(j_{n_1})} \times A_0^{(j_{n_1+1})} ... A_0^{(j_{n_1+n_0})} \times A_{-1}^{(j_{n_1+n_0+1})} ... A_{-1}^{(j_{n_1+n_0+n_{-1}})} |0\rangle.
\]

In the multi-\( j \) description, the example of Table 2.3 for the pair states can still be used, as well Table 2.5 and Table 2.6. The last table may be also applied to two \( j \)-levels, namely \( j_1 = 1/2 \) and \( j_2 = 5/2 \) with total \( \Omega_{1/2,5/2} = 4 \). The generalized states are eigenvectors of the total like-particle number operators \( N_{\pm 1} \) (2.100) with eigenvalues \( N_{\pm} \), where \( N_{\pm} = 2n_{\pm 1} + n_0 \). Both \( N \) and \( \tau_0 \equiv T_0 \) are diagonal in the basis with eigenvalues \( n = 2(n_1 + n_{-1} + n_0) \) and \( i = n_1 - n_{-1} \), respectively. While the single-\( j \) fermion number operators \( N_{\pm 1}^{(j)} \) project onto the single-level basis, the \( q \)-deformed analog (2.110) is diagonal in the generalized basis with eigenvalue \( 2\Omega_j \left[ \frac{2n_{2\sigma+n_0}}{2\Omega} \right], \sigma = \pm \frac{1}{2} \).

The generalized model has the same symmetry properties as the single-level realization of the theory (beginning of Chapter 2). All formulae (like action of a group generator on the basis states, Casimir invariants, eigenvalues, normalization coefficients of the basis vectors), that are constructed in terms of commutation relations of the single-level generators, coincide at the algebraic level and have the same form under the substitution \( \Omega_j \rightarrow \Omega \) (one should keep in mind that in this case the operators and their eigenvalues refer to the multi-level description). The important reduction limits of the \( \mathfrak{sp}_q(4) \) algebra to \( \mathfrak{u}_q(2) \) can be obtained in a straightforward manner for the generalized theory. Finally, the \( q \)-deformed generalized symplectic algebra and the results of the analysis in the multi-\( j \) case revert back to the "classical" limit when \( q \rightarrow 1 \).
Although the fermion generalization allows many \( j \)-orbitals to be considered, the dimension of the space should not be allowed to grow too large because the effect of the deformation diminishes as the size of the model space grows. For example, in the case of very large \( \Omega \) the anticommutation relation of the fermions (2.64) reduces to the simpler form

\[
\{ \alpha_{j,m,\sigma}, \alpha^\dagger_{j',m',\sigma} \}_{q=1} \delta_{j,j'} \delta_{m,m'} \text{ and all } q\text{-brackets } [X] \text{ or } [X]_{q=1} \text{ go to } X \text{ when } X \text{ is not a function of } \Omega.
\]

In this limit the pair-operators obey boson commutations and a boson approximation is achieved. As a direct consequence of the dependence of the deformation on the space dimension is that in this large \( \Omega \) limit the direct product of the single-\( j \) quantum symplectic algebras holds, \( sp_q(4) = \bigoplus_j sp_q^{(j)}(4) \), as for the non-deformed case for all \( \Omega \).

2.5 Matrix Representation

Along with the generalization of \( sp_{(q)}(4) \), a very necessary step is to derive the matrix representation of the generators (and their products) of the symplectic “classical” and \( q \)-deformed symmetry in the fully-paired space [75]. The results are what is needed for nuclear structure applications and an investigation into the physical significance of \( q \)-deformation.

In order to derive the matrix representation of the \( sp_{(q)}(4) \) operators for the single-\( j \) and multi-\( j \) cases, it is useful to obtain in analytical form the commutators of the operators with (2.15) \((A^\dagger_k)^n_k \), \(((B^\dagger_k)^n_k)\), \( k = 0, \pm 1 \), which enter into the construction of the non-deformed basis states (2.19) ((2.77) for the \( q \)-deformed analog). In this fully-paired space \( E_{+0}^+ \) \((qE_{+0}^+) \) we have already shown that the matrices representing the operators of like-particle numbers, (2.21) and (2.22), total particle number, (2.23), and third isospin projection, (2.24), are diagonal and they do not mix states of different pair numbers.

2.5.1 Isospin and Isospin-Related Operators

In the “classical” theory, the commutators of the raising (lowering) \( \tau_{\pm} \) operator with the pair creation operators, \((A^\dagger_{\pm})^{n_{\mp1}}\) and \((A^\dagger_0)^{n_0}\), which construct the fully-paired basis states (2.19), are

\[
\begin{align*}
\left[ \tau_{\pm}, (A^\dagger_{\mp})^{n_{\mp1}} \right] &= A^\dagger_0 (A^\dagger_{\mp})^{n_{\mp1}-1} \frac{n_{\mp1}}{\sqrt{\Omega}}, \\
\left[ \tau_{\pm}, (A^\dagger_0)^{n_0} \right] &= A^\dagger_{\pm} (A^\dagger_0)^{n_0-1} \frac{n_0}{\sqrt{\Omega}}.
\end{align*}
\]

(2.113)

The \( q \)-deformed analog of (2.114) for the raising (lowering) \( T_{\pm} \) operator and the pair creation operators, \((B^\dagger_{\pm})^{n_{\mp1}}\) and \((B^\dagger_0)^{n_0}\) (constructing the (2.77) basis), is

\[
\begin{align*}
\left[ T_{\pm}, (B^\dagger_{\mp})^{n_{\mp1}} \right] &= B^\dagger_0 (B^\dagger_{\mp})^{n_{\mp1}-1} \frac{\sqrt{\rho_+ \rho_-}}{\sqrt{\Omega} [2]} [n_{\mp1}] \frac{1}{\sqrt{N}} \left[ 2N_{\mp1+n_{\mp1}+1/2-\Omega} \right] \frac{1}{\sqrt{\Omega}}, \\
\left[ T_{\pm}, (B^\dagger_0)^{n_0} \right] &= B^\dagger_{\pm} (B^\dagger_0)^{n_0-1} \frac{1}{2\sqrt{\Omega}} \sum_{p=0}^{n_0-1} \frac{[2]^p}{2^p} [2N_{\mp1+n_0-1-p}] \frac{1}{\sqrt{N}},
\end{align*}
\]

(2.114)
where \( \rho_+ \) and \( \rho_- \) are defined in (2.69) and the factor \( \sqrt{\rho_+ \rho_-} \) appears for both the symmetric (\( \tilde{\rho} = \rho \)) and the non-symmetric (\( \tilde{\rho} = \rho_\pm \)) cases. This is true for all the formulae where this factor enters, which is indeed symmetric under the \( q \leftrightarrow q^{-1} \) exchange,

\[
\sqrt{\rho_+ \rho_-} = \frac{[2\Omega-1/2]_{\frac{1}{2\pi}}}{2}.
\]  

(2.115)

The expressions (2.113) and (2.114) coincide in the \( q \to 1 \) limit. The matrix representation of the raising and lowering operators \( \tau_\pm (T_\pm) \) in the pair basis states \( \mathcal{E}_0^+ (q, \mathcal{E}_0^+) \) can be obtained from (2.113) and (2.114) through the commutators

\[
{[\tau_\pm, (A^\dagger_{-1})^{n_1}(A^\dagger_0)^{n_0}(A^\dagger_1)^{n_1}]} |0\rangle ,
\]

(2.116)

\[
{T_\pm, (B^\dagger_{-1})^{n_1}(B^\dagger_0)^{n_0}(B^\dagger_1)^{n_1}]} |0\rangle ,
\]

(2.117)

since \( \tau_\pm |0\rangle = 0 (T_\pm |0\rangle = 0) \) and with the help of the commutator identities for any four \( V, X, Y, Z \) operators

\[
[X, Y, Z] = [X, Y]Z + Y[X, Z], \quad [X, VY, Z] = [X, V]YZ + V[X, Y]Z + VY[X, Z].
\]  

(2.118)

Through the use of (2.113) and (2.114), the general formula for the action of the \( k \)-th order product of \( T_\pm (\tau_\pm) \) on the lowest (highest) weight basis state can be determined,

\[
T^k_\pm (B^\dagger_\pm)^{n_\pm_1} |0\rangle = \sum_{i=0}^{[k/2]} \left( \frac{\sqrt{\rho_+ \rho_-}}{\sqrt{\Omega}[2]} \left[ \frac{2^{n_{\pm 1} - i - \Omega}}{2\pi} \right] \right)^{k-i} \frac{n_{\pm 1}!}{n_{\pm 1} - k + i} \theta_q(k, i) \times \langle B^\dagger_\pm \rangle^i (\langle B^\dagger_0 \rangle^k - 2i (\langle B^\dagger_\pm \rangle)^{n_{\pm 1} - k + i} |0\rangle
\]

(2.119)

\[
\downarrow q \to 1
\]

\[
\tau^k_\pm (A^\dagger_\pm)^{n_\pm_1} |0\rangle = \sum_{i=0}^{[k/2]} \left( \frac{1}{\sqrt{\Omega}} \right)^{k-i} \frac{n_{\pm 1}!}{(n_{\pm 1} - k + i)!} \theta_q=1(k, i) \times \langle A^\dagger_\pm \rangle^i (\langle A^\dagger_0 \rangle^k - 2i (\langle A^\dagger_\pm \rangle)^{n_{\pm 1} - k + i} |0\rangle
\]

(2.120)

where the positive integer \( k \leq n_{\pm 1} \), the \( q \)-factorial \([x]!\) is defined in (2.53) and the functions in the sum can be expressed recursively as

\[
\theta_q(k, 0) = 1, \forall k,
\]

(2.121)

\[
\theta_q(k, i) = \left\{ \begin{array}{ll}
\theta_q(k-1, i) \left[ \frac{2^{n_{\pm 1} - i - \frac{1}{2} - \Omega}}{2\sqrt{\Omega}} \right] + \frac{\theta_q(k-1, i-1)}{\sqrt{\Omega}} \sum_{p=0}^{k-1} \frac{2^{p} [2k-2i-p]_{\frac{1}{2\pi}}}{2p}, i \leq \left[ \frac{k}{2} \right] \\
0, i > \left[ \frac{k}{2} \right]
\end{array} \right.
\]

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In the “classical” limit, the $\theta_{q=1}(k, i)$ functions are given as follows

$$
\theta_{q=1}(k, 0) = 1, \forall k, \quad \theta_{q=1}(k, i) = \begin{cases} 
\theta_{q=1}(k - 1, i) + \theta_{q=1}(k - 1, i - 1) \frac{(k - 2i + 1)}{\sqrt{\Omega}}, & i \leq \left\lfloor \frac{k}{2} \right\rfloor \\
0, & i > \left\lfloor \frac{k}{2} \right\rfloor
\end{cases}
$$

(2.122)

This implies that $\theta_{q}(k, \frac{k}{2}) = \frac{\theta_{q(k-1,k)-1}}{\sqrt{\Omega}}$ when $k$ is even, which also holds for the “classical” $q = 1$ case where in addition one may simplify $\theta_{q=1}(k, 1) = k(k - 1)/(2\sqrt{\Omega})$. Starting from the lowest (highest) weight basis state the action of the $\tau_\pm (T_\pm)$ operator, (2.120) and (2.119), gives all the number preserving vectors with a definite maximum value of the $\tau$ quantum number (which is $\frac{n}{2}$ in a given $n$-multiplet, Table 2.4). The rest of the vectors with lower $\tau$ values and the same $(n, i)$ quantum numbers can be found as independent and orthogonal vectors to those constructed in (2.120) and (2.119). In nuclear systems, the $\tau_\pm$ generators represent the raising and lowering isospin operators and as such they generate $\beta^T$-decay transitions in an isobaric sequence. It follows that formula (2.120) derived above is used extensively in the calculation of the strength of these transitions. In the $q$-deformed case, the matrix elements of the isospin-like $T_\pm$ transition operators are generally $q$-deformed and may involve new and interesting physics. Also, in both “classical” and deformed theory, the construction of the isospin states, (2.120) and (2.119), allows one to compute overlaps with the pair states, (2.19) and (2.77), and with the eigenvectors of a model Hamiltonian.

### 2.5.2 Pair Operators

Another set of commutators, very useful for the derivation of the matrix elements of the pair operators and their products, are

$$
\begin{align*}
[A_0, (A_{\pm 1}^\dagger)^{n_{\pm 1}}] &= -\frac{n_{\pm 1}}{\sqrt{\Omega}}(A_{\pm 1}^\dagger)^{n_{\pm 1} - 1}\tau_\pm, \\
[A_{\pm 1}, (A_0^\dagger)^{n_0}] &= -\frac{n_0}{\sqrt{\Omega}}(A_0^\dagger)^{n_0 - 1}\tau_\pm - \frac{n_0(n_0 - 1)}{2\Omega} (A_0^\dagger)^{n_0 - 2} A_{\pm 1}^\dagger, \\
[A_0, (A_0^\dagger)^{n_0}] &= \frac{n_0}{2\Omega} \left( 2\Omega - 1 - \hat{N} + n_0 \right)(A_0^\dagger)^{n_0 - 1}, \\
[A_{\pm 1}, (A_{\pm 1}^\dagger)^{n_{\pm 1}}] &= \frac{n_{\pm 1}}{\Omega} (\Omega - 1 - N_{\pm 1} + n_{\pm 1}) (A_{\pm 1}^\dagger)^{n_{\pm 1} - 1},
\end{align*}
$$

(2.123)

and in the $q$-deformed case

$$
\begin{align*}
[B_0, (B_{\pm 1}^\dagger)^{n_{\pm 1}}] &= - (B_{\pm 1}^\dagger)^{n_{\pm 1} - 1} T_\pm \frac{\sqrt{\rho - \rho^-}}{[2\sqrt{\Omega}]} [n_{\pm 1}] \frac{1}{2^n} \left[ 2N_{\pm 1} + n_{\pm 1} - 1/2 - \Omega \right] \frac{1}{2^n}, \\
[B_{\mp 1}, (B_0^\dagger)^{n_0}] &= - (B_0^\dagger)^{n_0 - 1} T_\mp \frac{\sqrt{\rho - \rho^-}}{[2\sqrt{\Omega}]} \sum_{p=0}^{n_0 - 1} \frac{[2]^p}{2^p} \left[ 2N_{\pm 1} + p - 1/2 - \Omega \right] \frac{1}{2^n} \\
&\quad - (B_0^\dagger)^{n_0 - 2} B_{\mp 1}^\dagger \frac{\sqrt{\rho + \rho^-}}{2[2\Omega]} \sum_{p=0}^{n_0 - 1} S_q(N_{\pm 1}, k),
\end{align*}
$$
\[
\begin{align*}
[B_0, (B_0^\dagger)^{n_0}] &= \left[\frac{1}{2\Omega}\right] [n_0] \sum_{i=0}^{\frac{k-1}{2}} \frac{[2i]}{2^k} [2N_{\pm1}+k-\Omega-1/2] \frac{1}{\Omega} [2N_{\pm1}+k-i] q^{-1} 4k, \\
[B_{\mp1}, (B_{\mp1}^\dagger)^{n_{\mp1}}] &= \left[\frac{1}{2\Omega}\right] [n_{\pm1}] \sum_{i=0}^{\frac{k-1}{2}} \frac{[2i]}{2^k} [2N_{\pm1}+k-\Omega-1/2] \frac{1}{\Omega} [2N_{\pm1}+k-i] q^{-1} 4k, \\
\end{align*}
\]

(2.124)

where we define the \( q \)-operator

\[
S_q(N_{\pm1}, k) = [2N_{\pm1}+k-\Omega-1/2] \sum_{i=0}^{\frac{k-1}{2}} \frac{[2i]}{2^k} [2N_{\pm1}+k-i] q^{-1} 4k,
\]

(2.125)

with \( S_q(k) = [2k-\Omega-1/2] \sum_{i=0}^{\frac{k-1}{2}} \frac{[2i]}{2^k} [2k-i] q^{-1} 4k, \)

(2.126)

both of which go to an integer number in the “classical” limit when \( q \to 1 \). In analogy to the \( \tau_\pm \) (\( T_\pm \)) operators, the matrix representation of \( A_{k=0,\pm1} \) (or \( B_{k=0,\pm1} \)) in the fully-paired basis states (2.19) (or (2.77)) is calculated through commutators of the kind, \([A_{k=0,\pm1}, (A_{\mp1}^\dagger)^{n_{\mp1}} (A_0^\dagger)^{n_0} (A_1^\dagger)^{n_1}] \) (analogously for the \( q \)-deformed case), since \( A_{k=0,\pm1} |0\rangle = 0 \) (or \( B_{k=0,\pm1} |0\rangle = 0 \)). The latter commutators can be straightforward obtained via (2.123) and (2.113) ((2.124) and (2.114)) using the commutator identities (2.118). The matrix representation of the pair-creation operators is trivial by construction,

\[
A_1^\dagger |n_1, n_0, n_{-1}\rangle = A_1^\dagger (A_{-1}^\dagger)^{n_{-1}} (A_0^\dagger)^{n_0} (A_1^\dagger)^{n_1} |0\rangle
\]

\[
= |n_1 + \delta_{k,1}, n_0 + \delta_{k,0}, n_{-1} + \delta_{k,-1}\rangle, \quad k = 0, \pm 1.
\]

(2.127)

### 2.5.3 Second-Order Operators with Zero Isospin Projection

Consider the operators with zero isospin projection that enter in the \( \mathfrak{su}(2) \) Casimir operators, \( \mathfrak{N}_m = A_0^\dagger A_0 \) and \( \mathfrak{N}_{pp+nn} = A_{+1}^\dagger A_{-1} + A_{-1}^\dagger A_{+1} \). Using the commutators derived above we can find their matrix representation in the fully paired space \( E^+_0 \) (2.19)

\[
\begin{align*}
\mathfrak{N}_m |n_1, n_0, n_{-1}\rangle &= \frac{n_0 (2\Omega + 1 - n + n_{-1})}{2\Omega} |n_1, n_0, n_{-1}\rangle \\
&\quad - \frac{n_1 n_{-1}}{\Omega} |n_1 - 1, n_0 + 2, n_{-1} - 1\rangle, \tag{2.128}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{N}_{pp+nn} |n_1, n_0, n_{-1}\rangle &= \left( \frac{n_{+1} (\Omega + 1 - N_+ + n_{+1})}{\Omega} + \frac{n_{-1} (\Omega + 1 - N_- + n_{-1})}{\Omega} \right) |n_1, n_0, n_{-1}\rangle \\
&\quad - \frac{n_0 (n_0 - 1)}{\Omega} |n_1 + 1, n_0 - 2, n_{-1} + 1\rangle. \tag{2.129}
\end{align*}
\]

We are also able to find an analytical form of the \( q \)-dependent matrix representation of the deformed operators \( \mathfrak{N}_m^q = B_0^\dagger B_0 \) and \( \mathfrak{N}_{pp+nn}^q = B_{+1}^\dagger B_{-1} + B_{-1}^\dagger B_{+1} \) in the \( q \)-deformed fully paired space \( qE^+_0 \) (2.19) for an asymmetric realization (the factors \( \rho_\pm \), but not in the
coefficient \( \sqrt{\rho_+ \rho_-} \), are to be substituted with \( \rho \) for the symmetric case)

\[
\Psi_p^0 |n_1, n_0, n_{-1}\rangle = \left[ \frac{1}{2\Omega} \right] [n_0 + 1 n_0 + 2, n_{-1} - 1) \right), \quad \Psi_{pp+nn}^0 |n_1, n_0, n_{-1}\rangle = \rho_+ \left[ \frac{1}{\Omega} \right] [n_1 + 1 n_1 - N_+ + n_{+1} + 1 |n_1, n_0, n_{-1}\rangle + \rho_- \left[ \frac{1}{\Omega} \right] [n_{-1} - 1 n_{-1} - N_- + n_{-1} - 1 |n_1, n_0, n_{-1}\rangle - \frac{1}{\Omega} \sqrt{\rho_+ \rho_-} \sum_{k=1}^{n_{0-1}} S_q(k) |n_1 + 1, n_0 - 2, n_{-1} + 1\rangle, \quad (2.131)
\]

where we define the \( q \)-deformed quantity

\[
\tilde{n}_\pm \triangleq \frac{1}{2\Omega} \left( [2n_\pm - 1] + 2[n_\pm - 2\Omega] + [2\Omega] + 1 \right) \quad (2.132)
\]

\[
= \frac{1}{\Omega} \sqrt{\rho_+ \rho_-} \left( \frac{1}{2} \right) \left[ 2n_\pm - 1 - 1 \right] \times [2\Omega] + \frac{1}{2} \left( 2\Omega \right) + 1 \quad (2.133)
\]

\[
\tilde{n}_\pm = \frac{1}{\Omega} \left( \frac{1}{2} \right) \left[ 2n_\pm - 1 - 1 \right] \times [2\Omega] + \frac{1}{2} \left( 2\Omega \right) + 1 \quad (2.134)
\]

which in the “classical” limit reverts to the number of like-particle pairs. Note that the matrices for the zero isospin projection operators introduced above have non-zero elements only along two diagonals, one of which is the main one. The \( \mathfrak{su}^{\mu=0,\pm}(2) \) Casimir invariants have similar matrix representation in the fully-paired \( E_0^+ \) space. As we already mentioned in the beginning of Chapter 2, this makes easy to determine \( C_2(\mathfrak{su}^{\mu=0,\pm}(2)) \) eigenvalues since this type of matrices have eigenvalues equal to their diagonal elements.

We denote the matrix elements by \( M^\mu_{n_1', n_0', n_{-1}'} \) with \( n_1', n_0', n_{-1}' \) indicating the number of pairs of each kind added (+)/removed (−) and \( \mu \) is related to the operators considered (e.g., \( \mu = pn \) for (2.128) and (2.130) and \( \mu = ppnn \) for (2.129) and (2.131)). As obtained, the elements \( M^\mu_{n_1', n_0', n_{-1}'} \) are functions of the pair numbers

\[
M^p_{0,0,0} = \frac{n_0 (2\Omega + 1 - n + n_0)}{2\Omega}, \quad (2.135)
\]

\[
M^p_{1,0,1} = \frac{-n_1 n_{-1}}{\Omega}, \quad (2.136)
\]

\[
M^{ppnn}_{0,0,0} = \frac{n_1 (\Omega + 1 - N_+ + n_{+1})}{\Omega} + \frac{n_{-1} (\Omega + 1 - N_- + n_{-1})}{\Omega}, \quad (2.137)
\]

\[
M^{ppnn}_{1,0,1} = \frac{-n_0(n_0 - 1)}{\Omega}, \quad (2.138)
\]
and in the $q$-deformed picture they are deformed

\[
M_{0,0,0}^{p_n} = \left[ \frac{1}{2\Omega} \right] [n_0]^{1/\Omega} [2\Omega + 1 - n + n_0]^{1/\Omega}, \tag{2.139}
\]

\[
M_{-1,+2,-1}^{p_n} = -\frac{1}{\Omega} \tilde{n}_1 \tilde{n}_{-1}, \tag{2.140}
\]

\[
M_{0,0,0}^{ppn} = \left[ \frac{1}{2\Omega} \right] [n_{+1}]^{1/\Omega} [\Omega + 1 - N_+ + n_+ + 1]^{1/\Omega}
+ \left[ \frac{1}{2\Omega} \right] [n_{-1}]^{1/\Omega} [\Omega + 1 - N_- + n_- - 1]^{1/\Omega}, \tag{2.141}
\]

\[
M_{1,-2,1}^{ppn} = -\frac{1}{\Omega} \sqrt{\rho_+ \rho_-} \sum_{k=1}^{n_0-1} S_q(k), \tag{2.142}
\]

Next, consider the action of the anticommutator $\{T_+, T_-\} = T_+ T_- + T_- T_+$ (in the $q \to 1$ limit) on the basis states, which has the following analytical form

\[
\{T_+, T_-\} |n_1, n_0, n_{-1}\rangle = M_{-1,+2,-1}^T |n_1 - 1, n_0 + 2, n_{-1} - 1\rangle + M_{0,0,0}^T |n_1, n_0, n_{-1}\rangle 
+ M_{1,-2,1}^T |n_1 + 1, n_0 - 2, n_{-1} + 1\rangle, \tag{2.143}
\]

where the pair number functions $M_{n_1, n_0, n_{-1}}^T$ given in terms of $n_1, n_0, n_{-1}$ in the “classical” case are

\[
M_{-1,+2,-1}^T = \frac{2n_1 n_{-1}}{\Omega};
\]

\[
M_{0,0,0}^T = \frac{2n_0 + n_1 + n_{-1} + 2n_0(n_1 + n_{-1})}{\Omega},
\]

\[
M_{1,-2,1}^T = \frac{2n_0(n_0 - 1)}{\Omega}, \tag{2.144}
\]

and in the $q$-deformed case

\[
M_{-1,+2,-1}^T = \frac{1}{4[2]^2\Omega} \{ \Psi(n_0, n_1 - 1) \Psi(n_0 + 1, n_{-1} - 1) 
+ \Psi(n_0, n_{-1} - 1) \Psi(n_0 + 1, n_1 - 1) \},
\]

\[
M_{0,0,0}^T = \frac{1}{4[2]^2\Omega} \{ \Phi(n_0 - 1)(\Psi(n_0 - 1, n_1) + \Psi(n_0 - 1, n_{-1}) 
+ \Phi(n_0)(\Psi(n_0, n_{-1} - 1) + \Psi(n_0, n_1 - 1)) \},
\]

\[
M_{1,-2,1}^T = \frac{\Phi(n_0 - 1)\Phi(n_0 - 2)}{2\Omega}, \tag{2.145}
\]

where we define

\[
\Phi(n_0) = \sum_{k=0}^{n_0} \frac{[2]^k}{2^k} [2_{n_0-k}]^{1/\Omega} q^{-1} 2(n_0 + 1),
\]
\[
\Psi(n_0, n_{\pm 1}) = [n_0 + 2n_{\pm 1} + 1] \frac{1}{2\pi} - [n_0 - 1] \frac{1}{2\pi} + [n_0 + 2n_{\pm 1} + 2 - 2\Omega] \frac{1}{2\pi} - [n_0 - 2\Omega] \frac{1}{2\pi}
\]
\[
= 2\sqrt{\rho_+\rho_-} [n_{\pm 1} + 1] \frac{1}{2\pi} \left[ 2n_0 + n_{\pm 1} + 2/\Omega \right] \frac{q-1}{4(n_{\pm 1} + 1)}. 
\]  
(2.146)

The second expression of \(\Psi(n_0, n_{\pm 1})\) allows the non-diagonal term that scatters two identical particle pairs of opposite kinds into two non-identical particle pairs (2.144) to be rewritten as

\[
M^T_{-1, +2, -1} = \frac{\rho_+\rho_-}{[2\Omega]^2} [n_1] \frac{1}{2\pi} [n_{-1}] \frac{1}{2\pi} \times 
\]
\[
\times \left\{ \left[ 2n_0 + n_1 - \Omega - \frac{1}{2} \right] \frac{1}{2\pi} \left[ 2n_0 + n_1 + \Omega + \frac{1}{2} \right] \frac{1}{2\pi} + \left[ 2n_0 + n_1 - \Omega + \frac{1}{2} \right] \frac{1}{2\pi} \left[ 2n_0 + n_1 - \Omega - \frac{1}{2} \right] \frac{1}{2\pi} \right\}. 
\]  
(2.147)

In this way, this matrix element is proportional to \(M^m_{-1, +2, -1}\) (2.140)

\[
M^m_{-1, +2, -1} = -\frac{1}{\Omega} n_1 n_{-1} = -\frac{\rho_+\rho_-}{[2\Omega]^2} [n_1] \frac{1}{2\pi} [n_{-1}] \frac{1}{2\pi} \left[ 2n_1 - \Omega - \frac{1}{2} \right] \frac{1}{2\pi} \left[ 2n_1 - \Omega + \frac{1}{2} \right] \frac{1}{2\pi}, 
\]  
(2.148)

with a factor of \([2x]\)-numbers and in the \(q \to 1\) limit \(M^T_{-1, +2, -1}\) is twice the negative value of \(M^m_{-1, +2, -1}\) for the pair operators.

### 2.5.4 Diagonal \(q\)-Deformed Second-Order Operator for \(\mathfrak{sp}_q(4)\)

The analytical relations in the previous section allow us to find a \(q\)-deformed second-order operator, \(O_2(\mathfrak{sp}_q(4))\), that is diagonal in the \(q\)-deformed basis and that in the limit when \(q\) goes to one reverts to the second-order Casimir invariant of the \(\mathfrak{sp}(4)\) algebra (2.12),

\[
O_2(\mathfrak{sp}_q(4)) = \frac{\gamma_1}{2} \{ (B^\dagger_{-1}, B_{+1}) + (B^\dagger_{+1}, B_{-1}) \} + \gamma_0 \frac{C_2(\mathfrak{su}_q^0(2))}{\Omega} + \frac{C_2(\mathfrak{su}_q^0(2))}{\Omega}, 
\]  
(2.149)

where the \(\gamma\)-coefficients are \(q\)-functions of the pair numbers, \(\gamma_1 = \frac{M^T_{-1, +2, -1}}{2M^m_{-1, +2, -1}} \to 2\) and \(\gamma_0 = \left[ \frac{2n_0 + n_1 - \Omega - \frac{1}{2}}{\Omega} \right] + \left[ \frac{2n_0 + n_1 + \Omega + \frac{1}{2}}{\Omega} \right] + \left[ \frac{2n_0 + n_1 - \Omega + \frac{1}{2}}{\Omega} \right] + \left[ \frac{2n_0 + n_1 + \Omega - \frac{1}{2}}{\Omega} \right] \to 1\). The Casimir invariants in (2.149), \(C_2(\mathfrak{su}_q^0(2))\) and \(C_2(\mathfrak{su}_q^0(2))\), are given in Table 2.2.

The second-order operator can be written in terms of the Casimir operators of all four limits, \(+, -, 0, \tau\), as

\[
O_2(\mathfrak{sp}_q(4)) = \sum_{k=+,-,0,\tau} \gamma_k \frac{C_2(\mathfrak{su}_q^k(2))}{\Omega} - \frac{\gamma_1}{2} \left[ \frac{2}{\Omega} \right] \{ \rho_+ \left[ \frac{N_1 - \Omega}{2} \right] \frac{1}{\pi} + \rho_- \left[ \frac{N_1 - \Omega}{2} \right] \frac{1}{\pi} \}, 
\]  
(2.150)

where \(\gamma_k \equiv \gamma_1\), \(\gamma_\tau \equiv 1\) and \(C_2(\mathfrak{su}_q^k(2))\) are given in Table 2.2 (here again, the factors \(\rho_\pm\) are to be substituted with \(\rho\) for the symmetric case). Its eigenvalue in the basis set (2.77) (see
Table 2.10 and (2.139)-(2.142), (2.145)) is

\[
\langle O_2(\text{sp}_q(4)) \rangle = \gamma_1 (\rho_+ + \rho_-) \left[ \frac{1}{\Omega} \right] \left[ \frac{\Omega - n_0}{2} \right] \left[ \frac{\Omega - n_0}{2} + 1 \right]^{-\frac{1}{n}}
\]

\[- \gamma_1 \left[ \frac{2}{\Omega} \right] \left\{ \rho_+ \left[ \frac{2n_+ + n_0 - \Omega}{2} \right]^2 + \rho_- \left[ \frac{2n_- + n_0 - \Omega}{2} \right]^2 \right\}
\]

\[+ 2\gamma_0 \left[ \frac{1}{2\Omega} \right] \left[ \frac{2\Omega - 2(n_1 + n_{-1})}{2} \right] \left[ \frac{2\Omega - 2(n_1 + n_{-1})}{2} + 1 \right]^{-\frac{1}{2n}}
\]

\[+ \frac{M_{0,0,0}^{T}}{2} + \left[ \frac{1}{\Omega} \right] [n_1 - n_{-1}]^2\frac{q^{-1}}{2n} \rightarrow \Omega + 3. \] (2.151)

The second-order operator (2.149) is a Casimir invariant only in the non-deformed limit of the theory. We saw that in that limit its eigenvalue ($\Omega + 3$) labels the Sp(4) representations. While an explicit form for the second-order Casimir operator of $\text{sp}_q(4)$ for other $q$-deformed schemes can be given [24], this is not the case here because the suitable for nuclear physics applications scheme includes, by construction (2.99), a dependence on the shell structure. Nevertheless, the importance of the second-order operator (2.149) in the $q$-deformed case is obvious. It is an operator that consists of number preserving products of all ten $q$-deformed generators, and the $q$-deformed pair basis states (2.77) are its eigenvectors. Its zeroth-order approximation commutes with the generators of the $q$-deformed symplectic symmetry. It also gives a direct relation between the expectation values of the second-order products of the operators that build $O_2(\text{sp}_q(4))$.

As presented in this Chapter, the mathematical apparatus related to the $\text{sp}(4)$ algebra, the fully-paired representation of Sp(4) and the corresponding $q$-deformed picture, is built and a new tool for investigation of nuclear structure phenomena is ready. The next step is to apply the theoretical approach to nuclear many-body systems and in doing so we make an attempt to reveal the beauty of group theory and algebraic methods to describe real world physics.
Chapter 3

An Algebraic Pairing Model with Sp(4) Dynamical Symmetry and Its Non-Linear ($q$-Deformed) Extension

The pairing problem, which was suggested by Racah [6] in atomic physics as a seniority scheme to describe coupling of identical electrons, was introduced first to nuclear structure by Jahn and Flowers [79, 7] to completely classify the states of the $j^n$ nuclear configurations. It triumphed in its first physical application, the theory of superconductivity [80], which suggested quasi-bound states of electron pairs (Cooper pairs) with equal and opposite momenta near the Fermi surface to explain the correlation between electrons in superconductors as they arise from the interaction with lattice vibrations. This leads to the appearance of a superconducting gap in the originally continuous energy spectrum of the system. Similar type of correlation effects were suggested by Bohr, Mottelson and Pines [81] to explain the energy gap observed in the spectra of even-even nuclei and the concept was soon after applied by Belyaev in the first detailed (mean-field) study of pairing in nuclei in terms of independent quasi-particles [82]. Pairing theory was then adopted in nuclear physics as a fundamental concept for describing binding energies of nuclei and their low-lying vibrational spectra [83, 84, 85, 86, 87]. Recently, there has been renewed interest in this problem because of new experimental studies of exotic nuclei with relatively large proton excess or with $N \approx Z$.

This revival of interest in pairing follows from the recent development of radioactive beam facilities [19] and attempts to bridge from nuclear structure considerations to astrophysical phenomena [17, 18].

Along with approximate mean field solutions (for a review see [88]), the pairing problem can be solved exactly by means of various group theoretical methods, which allow one to explore the underlying symmetries. The SU(2) seniority model [16, 77, 89] provides for a good description of semi-magic nuclei and nuclei with large proton or neutron excess, where the like-particle pairing plays a dominant role. The generalization to the SO(5) model [8, 9, 10, 11, 12, 14] introduces a relation between identical-particle and proton-neutron ($pn$) isovector (isospin $\tau = 1$) pairing modes. The addition of an isoscalar ($\tau = 0$) $pn$ pairing
channel\(^1\) is described within the framework of the SO(8) model\(^2\) [91, 92, 93], the Interacting Boson Model (IBM) [94] and the pseudo-SU(4) shell-model [95].

In the limit of dominant isovector \(pn\) pairing correlations, a simple SO(5) seniority model [15, 45] is suitable. Light and medium mass nuclei along with unstable nuclei on the proton-rich side of the valley of stability are expected to exhibit larger \(pn\) pairing effects than do heavy stable nuclei, in which the valence protons and neutrons lie in different shells and their Fermi surfaces are relatively displaced. Our goal is to investigate properties of the isovector pairing interaction within the context of a fermion realization of the symplectic \(sp(4)\) algebra [96], which is isomorphic to \(so(5)\).

An additional degree of freedom can be introduced through the quantum (\(q\)-) deformation of the classical \(sp(4)\) Lie algebra (Chapter 2). While this preserves the underlying symplectic symmetry, it introduces non-linear terms into the theory leading also, as in the “classical” case, to an exact (but \(q\)-deformed) solution of the problem and its limiting cases. With a view towards applications, the additional parameter of the deformation leads to the possibility of greater flexibility and richer structures within the framework of algebraic descriptions. In a Hamiltonian theory this implies a dependence of the matrix elements of the interaction on the deformation parameter while preserving the integrals of motion of the system. A property with physics impact for the \(q\)-deformed content of the states and interaction matrix elements is the dependence of the deformed anticommutation relations (2.64) on the shell dimension \(\Omega\) and the operators that count the number of particles, \(N_{\pm 1}\). Existing applications of the \(q\)-deformed algebraic structures to the pairing problem are restricted mainly to the like-particle \(su_q(2)\) limit [31, 32, 33] of the dynamical symmetry approach presented here for \(sp_q(4)\).

### 3.1 Theoretical Model with \(Sp(4)\) Dynamical Symmetry

The problem to be solved, in a many-body Schrödinger equation formulation, is

\[
H\Psi(1, 2, \ldots, A) = (\hat{T}(1, 2, \ldots, A) + \hat{V}(1, 2, \ldots, A))\Psi(1, 2, \ldots, A) = E\Psi(1, 2, \ldots, A),
\]

where \(\hat{T}\) is the operator of the total kinetic energy of the system of \(A\) nucleons, \(\hat{V}\) is the many-body interaction operator and \(\Psi(1, 2, \ldots, A)\) is the wave function describing a state of

\(^1\)Protons (neutrons) can form pairs only with total isospin one (isovector channel), while a proton and a neutron can be coupled also in an isoscalar (\(\tau = 0\)) mode with total spin one and total isospin zero. Such a state is, in fact, the ground state of the deuteron \(^2\)D. The isovector \(0^+\) state (a \(\tau = 1 \, pn\) pair) turns out to be unbound (as well as the dineutron \(nn\) and the diproton \(pp\)) revealing a stronger bare \(\tau = 0 \, pn\) interaction than the \(\tau = 1 \, pn\) pairing.

\(^2\)A limited applicability of the SO(8) model follows from the assumption of an \(LS\) classification and no spin-orbit coupling. Once a spin-orbit splitting is added to the model Hamiltonian, the subspace constructed out of \(L = 0\) pairs is no longer decoupled and hence one is forced to solve the eigenproblem in the full shell-model space [90].

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energy \( E \) in the energy spectrum of the nuclear system. While it may be easy to formulate the problem in this way, it is very difficult to find solutions: the comparatively large number of the interacting nucleons, and yet a number not big enough to enforce statistical laws, makes the problem impossible to solve exactly. There is a need for models.

For a model to be successful, it should be simple enough to be accessible to analysis and yet detailed enough to provide for a good depiction of the relevant physical situation. Models are typically useful for describing only certain groups of phenomena as they isolate the most important degrees of freedom and bring them explicitly into the theory as an effective parametrized interaction. Other degrees of freedom can contribute only implicitly and their effect is limited to renormalization of the parameters of the model [27].

In the nuclear shell-model, the motion of \( A \) nucleons is assumed to be independent of each other in same average field expressed by a central potential \( U(r) \) and can be described by the Hamiltonian

\[
H_0 = \sum_{i=1}^{A} (T_i + U(r_i)). \tag{3.2}
\]

The spherical potential \( U(r) \), which arises from the mutual interaction between the nucleons, is not explicitly known and can be approximated to a sufficiently good degree by a simple three-dimensional harmonic oscillator well corrected with a spin-orbit \( s \cdot l \) splitting [1] and an orbit-orbit \( l^2 \) force (that shifts higher-\( l \) levels downward). In a limit of a strong spin-orbit term the \( jj \)-coupling scheme is obtained, which we employ. The correlation effects of internucleon forces neglected in \( H_0 \) typically give rise to a deviation of \( H_0 \) from the general \( A \)-particle Hamiltonian \( H \) (3.1). When \( H \) is restricted to two-body interactions the residual interaction, \( H - H_0 \), between nucleons \( i \) and \( j \) is

\[
H_{\text{res}} = \frac{1}{2} \sum_{i,j=1}^{A} V_{ij} - \sum_{i=1}^{A} U(r_i), \tag{3.3}
\]

which although relatively weak plays an important role in various nuclear properties, especially when this interaction has a correlated coherent character. The residual interaction then accounts for strong short-range two-particle correlations (pairing) and long-range multipole forces responsible for the collective dynamics of a nuclear system. Most important among the latter is of a quadrupole type, which is related to nuclear shape deformations and in fact can be hidden in a deformed mean-field (Nilsson potential [97]), in which independent particles move (deformed shell-model).

In nuclear shell-models two principal limitations are encountered: (i) the number of configurations necessary to provide an adequate representation of the nuclear state under investigation is typically very large, and (ii) uncertainties exist in the effective residual interaction. An approach that minimizes uncertainties of type (ii) invokes group symmetries and is related to the fact that the wave functions of a quantum mechanical system can be characterized by their invariance properties under certain symmetry transformations. In
addition, if one can find near invariant operators, the associated symmetries can be used to help reduce the dimensionality of a model space to tractable size (the complete many-body space is thus decomposed to subspaces that are only weakly coupled and introduces small uncertainties of type (i)). Interaction matrix elements are determined by only a few parameters, which in turn can be determined from experiment. This approach constitutes a major class of group theoretical fermion models [98].

A significant simplification (without compromising realism) is achieved if one considers shell closures with closed shells as part of an inert core that is spherical and do not affect directly the single-particle motion of the valence nucleons in the last unfilled shell. In this case, the Sp(4) model describes the motion of \( N_+ \) valence protons and \( N_- \) valence neutrons above a doubly-magic inert nuclear core, which corresponds to the vacuum state, \(|0\rangle\), in the model. In the \( \text{su}^+(2) \) limit (Table 2.1), the generators \( \tau_{0,\pm} \) are associated with the components of the isospin of the valence particles. The \( \text{su}^0(2) \) limit describes proton and neutron pairs \((pn)\), while the \( \text{su}^+(2) \oplus \text{su}^-(2) \) limit is related to coupling between identical particles, proton-proton \((pp)\) and neutron-neutron \((nn)\) pairs. The pairing residual interaction couples valence particles or holes distributed in the last not completely filled major shell.

3.1.1 Why Do We Need the Sp(4) Approach if the Sp\((2j+1)\) Model Is out There?

Since the early days of nuclear structure physics, there has been much effort devoted to understanding the nature of the proton-neutron interaction and to implementing its effect on the nucleonic motion in various models [99, 100, 101, 102, 103, 104, 105]. One can assume, as a zero-order approximation of no \( pn \)-interaction, independent groups of protons and neutrons, each of which gives rise to characteristic set of levels. Then, the effect of the interaction between the protons and the neutrons leads to a coupling of these two and yields different low-energy states with the same total angular momentum resulting, for example, in a “repulsion” of the levels [106, 107]. However, McCullen showed that for odd-odd nuclei in \( 1f_{7/2} \) this separation does not work, and following Talmi’s group theoretical method based on \( \text{Sp}(2j+1) \) [26, 83] in the conventional seniority scheme of Racah and Flowers [6, 7], proposed a detailed study of the \( 1f_{7/2} \) nuclear shell [108]. The exactly solvable approach provides for a general but very complicated way to account for the \( pn \) component of the nuclear interaction. It is the “quasi-spin” formalism introduced by Helmers [8] that allows for a simple expansion of Kerman’s identical pairing \( \text{SU}(2) \) group to \( \text{SO}(5) \sim \text{Sp}(4) \) that includes protons and neutrons, while retaining the detailed representation of the nuclear interaction and the exact solution of the eigenproblem. Along with the group theoretical concept, early achievements aimed at including a \( pn \) component have been accomplished by different formulations of mean-field approximations [109, 110, 111, 112, 113] (for a review see [88]) – although many of them do not include important \( pn \tau = 1 \) pairs and also lack isospin invariance (the third isospin projection looses its normal definition and physical meaning due to introduction of independent quasi-particles).

In the nuclear shell-model a basis set of wave functions for a system with \( n \) (valence)
nucleons is constructed by taking a Slater determinant (which is fully antisymmetrized in accordance with the Pauli principle) of \( n \) single-particle wave functions represented in the second quantization by the fermion operators \( c_{jm}^{(\tau=1/2)\sigma} \). For a given \( j \), there are \((2\tau+1)(2j+1) = 4\Omega \) possible single-particle orthonormal wave functions, which constitute an orthonormal basis of a \( 2(2j+1) \)-dimensional vector space and form an irreducible unitary representation of \( U(2(2j+1)) \). In the conventional seniority scheme of Racah and Flowers [6, 7], states of a simple configuration \( j^n \) comprised of both protons and neutrons are classified according to the reduction chain (with labels shown in parenthesis)

\[
U(2(2j+1))_{(j^n)} \supset U(2j+1)_{(\tau)} \supset \text{Sp}(2j+1)_{(w,t)} \supset \text{SO}(3)_{(J)}, \quad (3.4)
\]

where an \( IUR \) of \( U(2(2j+1)) \) is formed by the \( n \)-particle antisymmetric wave functions with total isospin \( \tau \), each of the \( \tau \) values labeling an \( IUR \) of \( U(2j+1) \). The label \( b \) specifies the degenerate states for which a symplectic symmetry \((w,t)\) occurs more than once for a given isospin \( \tau \), and \( a \) provides for a complete labeling classifying degenerate states of same \( J \) in a given \((w,t)\)-irrep [6, 7, 8, 10, 11].

The “quasi-spin” approach of Helmers [8], on the other hand, yields a classification scheme with the same quantum numbers (3.4) based on two parallel group chains starting with a different and ingenious group decomposition of \( U(2(2j+1)) \), namely

\[
U(2(2j+1))_{(j^n)} \supset \text{Sp}(2j+1)_{(j^{\nu})} \otimes \text{SO}(5)_{(w\leftarrow u,t;n,b,\tau)} \cup_{(a)} \text{SO}(3)_{(J)}, \quad (3.5)
\]

where the dependence on \( n, b \) and \( \tau \) is transferred solely to \( \text{SO}(5) \) (locally isomorphic to \( \text{Sp}(4) \)). The group chain of \( \text{Sp}(2j+1) \) is the one associated with conventional seniority but now is completely specified by the simple configuration \( j^{\nu} \), where \( \nu \) is the total seniority number that counts particles not coupled in a \( J = 0 \) pair and is related to the maximum number \( w \) as \( w = 4\Omega - \nu \) [8, 10, 11]. In the specific representation of \( t = 0 \) (hence, \( \nu = 0 \)) \( J \) is simply zero, and \( t \) and \( b \) are redundant in the labeling scheme (as discussed in Chapter 2). In this way, the simple “quasi-spin” approach (which we also follow) not only offers an elegant way to understand the results from the conventional seniority scheme of Racah and Flowers [6, 7, 8, 10, 11] but also allows for a straightforward expansion of the like-particle \( \text{SU}(2) \sim \text{Sp}(2) \) group [16] to \( \text{SO}(5) \sim \text{Sp}(4) \) to include interacting protons and neutrons, which otherwise has proven to be too complicated to work with. According to Hecht, this method “constitutes a valuable tool in deriving” the otherwise very complicated nuclear matrix elements when both protons and neutrons are considered ([10], Nucl. Phys. A102, 11 (1967), p. 11). A detailed comparison that reveals the power of the \( \text{Sp}(4) \) method versus the conventional seniority spectroscopy can be found in the literature [8, 9, 10, 11]. As an aside, we mention that for the \( \text{Sp}(2j+1) \) approach “a different symmetry group is necessary as a starting point of the group chain for each \( j^{\nu} \), which is not the case for \( \text{Sp}(4) \) ([10], p. 12). Hence, a \( q \)-deformed version can be accomplished only for one algebra, \( \mathfrak{sp}(4) \). Also, \( \text{Sp}(4) \) is
a smaller group and the method is simple and not unnecessarily complicated [11]: the \( \text{Sp}(4) \) group has 10 generators compared to \( (j + 1)(2j + 1) \) generators of \( \text{Sp}(2j + 1) \). Because of this, the use of \( \text{Sp}(4) \), even if only to reproduce some of the results of the \( \text{Sp}(2j + 1) \) scheme, is justified. At the same time, \( \text{Sp}(4) \) is the simplest possible group to investigate in details the isovector (\( \tau = 1 \)) pairing correlations and symmetry energy.

### 3.1.2 Model Space

Within the \( q \)-deformed as well as the non-deformed algebraic approach, the basis states \( |n_1, n_0, n_{-1}(q)\rangle \), (2.19) or (2.77), model \( 0^+ \) states with dominant isovector pair correlations in a nucleus with \( N_+ = 2n_1 + n_0 \) valence protons and \( N_- = 2n_{-1} + n_0 \) valence neutrons. Linear combinations of the basis, (2.19) or (2.77), give the model interaction eigenvectors, which are expected to be almost equivalent to the isospin eigenstates (e.g., Table 2.5 and Table 2.6): due to the possibility of very weak isospin mixing, \( \tau \) is an almost good quantum number for most applications. We refer to the nuclear states that can be represented realistically enough by the model interaction eigenvectors and to which the \( \text{Sp}(4) \) model is applied as isovector-paired states. The isovector-paired states consist of \( 0^+ \) isobaric analog states of all even-\( A \) nuclei. These corresponds to the \( 0^+ \) ground state of the even-even nucleus of maximum (or minimum) weight \( i = \pm \tau \) in a \( \tau \)-multiplet\(^3\). Hence, the lowest-lying isovector-paired states include:

1. \( 0^+ \) ground states of even-\( A \) nuclei (of all even-even nuclei and only of those of the odd-odd nuclei with a \( J = 0 \) ground state);

2. the lowest isobaric analog \( 0^+ \) excited state of an odd-odd nuclide (with \( A \) and \( i \)) with a \( J \neq 0 \) ground state, which corresponds to the \( 0^+ \) ground state of the even-even neighbor with the same mass \( A \) and absolute value of the isospin projection \( |i| + 1 \).

The importance of the isovector pairing for binding energies is suggested by experimental data, namely a \( 0^+ \) ground state with \( \tau = 1 \) for most \( N = Z \) odd-odd nuclei with mass number \( A > 40 \) [114, 115, 116, 117], and by the results of various theoretical studies [118, 119, 120, 121, 122, 123, 124, 125, 126].

As we explained in detail (in Chapter 2), the classification scheme of the model basis states according to \( \text{sp}(4) \) reduction chains yields a systematics of nuclei as mapped to a vector with fixed quantum numbers \( (n, i) \), or alternatively \( (N_+, N_-) \). In this way the \( \text{Sp}(4) \) symmetry provides for a simultaneous natural classification of nuclei (as belonging to a single-\( j \) level or a major shell (multi-\( j \))) and of their corresponding ground and excited states including their isovector-paired \( 0^+ \) states. The classification scheme is illustrated for the simple case of \( 1d_{3/2} \) with \( \Omega_{j=3/2} = 2 \) in Table 3.1, and for \( 1f_{7/2} \) with \( \Omega_{j=7/2} = 4 \) in Table 3.2. The total number of the valence particles, \( n = N_+ + N_- \), enumerates the rows and the eigenvalue \( i \) of the third projection of the valence isospin \( \tau_0 \) enumerates the columns. Isotopes of an

\[^3\] Charge independence of the nuclear force implies that isobaric analog nuclei will exhibit similar energy spectra for the states as belonging to the \( \tau \)-multiplets.
Table 3.1: Classification scheme of even-\(A\) nuclei in the 1\(d_{3/2}\) shell, \(\Omega_{3/2} = 2\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(i = 2)</th>
<th>(i = 1)</th>
<th>(i = 0)</th>
<th>(i = -1)</th>
<th>(i = -2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(^{32})S(_{16})</td>
<td>(^{34})S(_{16})</td>
<td>(^{34})S(_{18})</td>
<td>(^{34})S(_{18})</td>
<td>(^{34})S(_{20})</td>
</tr>
<tr>
<td>2</td>
<td>(^{36})Ar(_{16})</td>
<td>(^{36})Cl(_{17})</td>
<td>(^{36})Cl(_{19})</td>
<td>(^{36})Cl(_{19})</td>
<td>(^{36})Cl(_{20})</td>
</tr>
<tr>
<td>4</td>
<td>(^{40})Ca(_{16})</td>
<td>(^{40})Ca(_{18})</td>
<td>(^{40})Ca(_{18})</td>
<td>(^{40})Ca(_{20})</td>
<td>(^{40})Ca(_{20})</td>
</tr>
<tr>
<td>6</td>
<td>(^{44})Ca(_{16})</td>
<td>(^{44})Ca(_{18})</td>
<td>(^{44})Ca(_{18})</td>
<td>(^{44})Ca(_{20})</td>
<td>(^{44})Ca(_{20})</td>
</tr>
<tr>
<td>8</td>
<td>(^{48})Ca(_{16})</td>
<td>(^{48})Ca(_{18})</td>
<td>(^{48})Ca(_{18})</td>
<td>(^{48})Ca(_{20})</td>
<td>(^{48})Ca(_{20})</td>
</tr>
</tbody>
</table>

The element are situated along the right-and-down diagonals, and isotones along the left-and-up diagonals, and the rows consist of isobars of a given mass number. The shape of the table is symmetric with respect to \(i\) (with the exchange \(n_1 \leftrightarrow n_{-1}\)), as well as with respect to \(n - 2\Omega\) (middle of the shell). This is a consequence of the charge independent nature of the interaction and the Pauli principle, respectively. The \(q\)-deformed states, while classified by the “classical” quantum numbers (Table 3.1 and Table 3.2), are in general different from the classical ones and coincide with them in the limit \(q \to 1\).

In working in a subspace spanned by the pair basis states (2.19), the Sp(4) model reduces the full shell-model space tremendously and yields an exact solution of the eigenvalue problem for describing isovector-paired \(0^+\) states in nuclei. However, this simplification ignores mixing of states with other configurations that involve the core or higher-lying major shells — both being negligible so the theory retains sufficient for single-\(j\) levels such as the \(1d_{3/2}\) and \(1f_{7/2}\) orbits and within major shells like \(1f_{5/2} 2p_{1/2} 2p_{3/2} 1g_{7/2}\). Another possible limitation relates to isoscalar (\(\tau = 0\)) \(pn\) mixing since the present level of experimental results do not allow one to answer sharply the question of whether or not one can really ignore true isoscalar mixing; no exact study to date, nor does the available data, allows one to sort out these effects.

However, while coupling of protons and neutrons in the isoscalar channel may be important and dominant in some cases [127, 128] (also, refer to the example of the \(\tau = 0\) ground state of the deuteron), we exclude — to the best of our ability — the ground states of nuclei that show fingerprints of such isoscalar mode. In fact, in odd-odd nuclei there is a clear signature if an isoscalar mode dominates in the ground state: the angular momentum of the state is not zero and the (almost) good isospin quantum number is \(\tau = |i|\). In their ground states, even-even nuclei are dominated by isovector pair correlations [120] responsible for the empirically observed pairing gap in these nuclei [81]. In short, the Sp(4) model space is limited to the isovector-paired \(0^+\) states that reveal the dominance of a correlated formation of isovector pairs.

The long-standing debate on the competition of isovector and isoscalar modes in ground states continues, waiting for new experimental results in the unexplored area of the isotopic chart with the help of the promising radioactive beam facilities [19]. Unfortunately, the theory is confusing due to the different languages employed: from quasi-particles through symmetries to empirical phenomena; from pure isovector pairing through isoscalar-isovector mixing to pure \(pn\) isoscalar coupling. To introduce a common terminology, we will briefly
Table 3.2: Classification scheme of even-$A$ nuclei and their *isovector-paired* states in the $1f_{7/2}$ shell, $\Omega_{7/2} = 4$. The shape of the table is symmetric with respect to the sign of $i$ and $n - 2\Omega$. The basis states are labeled by the numbers of particle pairs $(n_1, n_0, n_{-2})$ (2.19). The subsequent action of the SU$^\mu(2)$ generators (shown in parenthesis) constructs the constituents in a given SU$^\mu(2)$ multiplet ($\mu = \tau, 0, \pm$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$i = 0$</th>
<th>$i = -1$</th>
<th>$i = -2$</th>
<th>$i = -3$</th>
<th>$i = -4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(0, 0, 0)<em>{40}^{20}$Ca$</em>{20}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\sqrt[4]{21}$Sc$_{21}$</td>
<td>$(0, 0, 1)<em>{44}^{42}$Ca$</em>{22}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$(1, 0, 1)<em>{44}^{44}$Sc$</em>{23}$</td>
<td>$(0, 1, 1)<em>{44}^{44}$Ca$</em>{24}$</td>
<td>$(0, 0, 2)<em>{44}^{44}$Ca$</em>{24}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$(0, 3, 0)<em>{46}^{46}$V$</em>{23}$</td>
<td>$(1, 0, 2)<em>{46}^{46}$Ti$</em>{24}$</td>
<td>$(0, 2, 1)<em>{46}^{46}$Sc$</em>{25}$</td>
<td>$(0, 1, 2)<em>{46}^{46}$Ca$</em>{26}$</td>
<td>$(0, 0, 3)<em>{46}^{46}$Ca$</em>{26}$</td>
</tr>
<tr>
<td>8</td>
<td>$(2, 0, 2)<em>{48}^{48}$Cr$</em>{24}$</td>
<td>$(1, 1, 2)<em>{48}^{48}$V$</em>{25}$</td>
<td>$(0, 3, 1)<em>{48}^{48}$Ti$</em>{26}$</td>
<td>$(0, 2, 2)<em>{48}^{48}$Sc$</em>{27}$</td>
<td>$(0, 1, 3)<em>{48}^{48}$Ca$</em>{28}$</td>
</tr>
<tr>
<td>$(A^+_{-1})$</td>
<td>$(A^+_{0})$</td>
<td></td>
<td></td>
<td></td>
<td>$(A^+<em>{+1})</em>{48}^{48}$Ca$_{28}$</td>
</tr>
</tbody>
</table>

review the nature of the isovector and isoscalar interactions. A nucleon (with two states, proton or neutron [129]) carries $s = \frac{1}{2}$ spin and $\tau = \frac{1}{2}$ isospin. A pair of two nucleons may be found in either an isovector spin-singlet state (total isospin $\tau = 1$ and total spin zero) or an isoscalar spin-triplet state (total isospin $\tau = 0$ and total spin one). The first corresponds to *isovector pairing* that couples particles in time-reversed spatial orbits (antiparallel spins) with $J = 0$ and isospin projection 1 (a $pp$ pair), 0 ($pn$) and $-1$ ($nn$). It favors states with good isospin. For a $j^n$ configuration there are sufficiently many such $J = 0$ states and as protons and neutrons are added the coherent character of the isovector pairing interaction can gradually build up correlations that favor sphericity. The second case, an isoscalar spin-triplet, corresponds to *isospin pairing* of a proton and a neutron in a $\tau = 0$ pair and it favors states with good spin. In the $jj$-coupling scheme this implies that the proton and the neutron can occupy the same $j$-level (which is not forbidden by the Pauli principle as they have parallel spins but they differ by the isotopic spin) as well as spin-orbit “partner” orbitals, such as $l_{j+\frac{1}{2}}$ and $l_{j-\frac{1}{2}}$, when their spin third projection is zero. The isoscalar $pn$ mode introduces non-axial deformations in the intrinsic state. Once the spatial $pn$ correlations in spin-orbit partners become dominant, the system deforms. In such a case, the emerging microscopic picture is one in which deformation is produced by the isoscalar $\tau = 0$ part of the $pn$ residual interaction, rather than by a long-range quadrupole-quadrupole interaction.
This is most likely to happen in neutron-rich light nuclei in the $(2s, 1d)$ shell \[130\] and in medium nuclei with protons and neutrons in spin-orbit partners belonging to different major shells \[105\]. Both cases lie beyond the scope of our Sp(4) model; such effects are not expected to be large for the nuclei under consideration.

While strong proton-neutron interactions definitely exist in both the $\tau = 0$ and $\tau = 1$ channels with the bare $\tau = 0$ interaction being stronger than the $\tau = 1$ one (as seen from the binding of the deuteron in the $\tau = 0$, but not $\tau = 1$, state), only the $\tau = 1$ pairing correlations as a coherent superposition of configurations leads to a pairing gap \[124, 125, 126\]. A possible explanation is that stronger spin-orbit coupling with increasing nuclear mass results in large energy splitting between the spin-orbit partner orbitals favored by $\tau = 0$ pairs, while it leaves those involved in $\tau = 1$ pairing degenerate \[131\]. Thus, the $\tau = 1$ collective pairs may become greatly favored and hence an almost pure isovector pair structure results \[90\].

At the same time the isoscalar $pn$ interaction must not be neglected \[121\] because it constitutes a strong force that favors the lowest possible isospin states. In this context, as we will show in the next section, the isoscalar $pn$ force (if $J$-independence is assumed) is directly related to a symmetry $\tau(\tau+1)$ term (with its optimum energy value at $\tau = 0$). Hence, which state is the ground state is determined by the competition between this symmetry energy (dominated by $\tau = 0$ interactions) and the $\tau = 1$ pairing energy \[121, 125, 126\]. In even-even $N = Z$ nuclei, the $\tau = 0$ ground state ($k$ isovector pairs are coupled to total isospin $\tau = 0, 2, \ldots, k$) is favored by both the $\tau = 1$ pairing and symmetry energies [a state in which there are only isoscalar $pn$ pairs looses the $\tau = 1$ pairing energy]. In the odd-odd $N = Z$ nuclei, the $\tau = 1$ state gains the pairing energy, but loses the symmetry energy relatively to the $\tau = 0$ state. While the symmetry energy (as a volume term) decreases with nuclear mass approximately as $1/A$, the pairing term drops off as $A^{-1/2}$, and hence, for the light odd-odd $N = Z$ nuclei the symmetry energy is bigger than the pairing energy resulting in a $\tau = 0$ $J > 0$ ground state (from the perspective of a theory with explicit isoscalar ‘pairing’ term this implies a very strong $\tau = 0$ ‘pairing’ component and an isoscalar pair structure). On the other hand, for heavy ($A > 40$) odd-odd $N = Z$ nuclei the pairing energy becomes larger than the symmetry energy and the ground state becomes a $\tau = 1$ configuration. There are fluctuations from the smooth trend due to the influence of shell effects on the pairing such as the exceptions $^{34}$Cl and $^{58}$Cu \[131, 125, 126\]. Nuclei with $N \neq Z$ are then easily explained as the proton (neutron) excess coupled to an isovector mode of $pp$ ($nn$) pairs in addition to a system of protons and neutrons with $N = Z$, which follows a behavior that is similar to that described above.

The interplay between like-particle pairing and the $pn$ interaction (both isovector pairing and isoscalar force [symmetry energy]) can be illustrated by the following simple example of an even-even system of two protons and two neutrons (Figure 3.1). The eigenstates of the isovector pairing interaction are linear combinations of a state with like-particle pairs, a $pp$ pair ($x_p \leftrightarrow y_p$) and a $nn$ pair ($x_n \leftrightarrow y_n$) and a state with two $pn$ isovector pairs, ($x_p \leftrightarrow y_n$) and ($x_n \leftrightarrow y_p$). In general, the two like-particle pairs and the two $pn$ pairs can occupy any two $(m, -m)$ levels of different $m$, but the most energetically favorable configuration will be when all are found in the same $(m, -m)$ levels ($\alpha$ cluster), which in addition leads to a
strong $pn$ interaction in the isoscalar $\tau = 0$ channel ($(x_p \leftrightarrow x_n)$ and $(y_n \leftrightarrow y_p)$) favoring the lowest isospin state, $\tau = 0$, and thus, to a large increase of the binding energy of the ground state. In this context, a theory may imply that isoscalar $pn$ pair structure and a non-negligible isoscalar $pn$ ‘pairing’ force are essential for the energy of the ground state of even-even nuclei. This is a matter of terminology – in the viewpoint of our Sp(4) model the isovector $\tau = 1$ channel is regarded as the only pairing interaction of a coherent character, the $pn$ isoscalar force is related to the symmetry energy of the nuclear system and the Sp(4) pair basis vectors (2.19) as states of correlated $\tau = 1$ isovector pairs describe the isovector-paired nuclear states with a very good degree of approximation.

Figure 3.1: A schematic illustration of two protons (red sphere) and two neutrons (blue sphere) in the $jj$-coupling scheme.

In short, in our investigation we consider in detail the isovector-paired states of the even-even and odd-odd nuclei with nuclear masses, $32 \leq A \leq 100$, and nucleons in the $1d_{3/2}$ and $1f_{7/2}$ single levels and $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell. Their description will be attempted in the framework of microscopic models based on the $sp_{(q)}(4)$ algebras in both “classical” and $q$-deformed pictures.

### 3.1.3 Model Hamiltonian

In our effort to search for an effective force that is realistic enough to explain essential features of pairing correlations in a nuclear system, we first make use of an algebraic construction of the model interaction based on the concept of a dynamical symmetry. How realistic this is will be discussed further as it is related to a general microscopic two-body interaction that preserves particle number and angular momentum.

The concept of a dynamical symmetry of a many-body (nuclear) system (see [90] for a review of dynamical symmetry models) is based on the assumption of a ‘primary’ symmetry
with an associated dynamical algebra, which has the property that the Hamiltonian of the system can be expressed in terms of its basic operators (the symmetry generators). The Hamiltonian must, of course, respect the true symmetry of the system (embedded in the ‘primary’ symmetry). An important aspect of this reduction is that it can sometimes be achieved analytically, while preserving the solvable character of the many-body problem.

A Hamiltonian \( H \) is said to have a symmetry \( G \) (alternatively, to be invariant under \( G \)) if it is invariant under a set of infinitesimal transformations generated by the \( \hat{g}_i \) operators that close a Lie algebra \( g \), that is,

\[
[H, \hat{g}_i] = 0 \quad \text{for all} \quad \hat{g}_i \in g.
\]  

(3.6)

In this way, the Hamiltonian can be constructed via the Casimir invariant of \( g \) since it commutes with all the group generators. In order not to be too abstract, let us consider the \( su^\tau(2) \) subalgebra of \( sp(4) \). A Hamiltonian that is isospin invariant can be written as

\[
H^\tau = \eta_1 \tau^2,
\]

where \( \eta_1 \) is a parameter. The eigenvectors of the Hamiltonian operator are the isospin eigenstates \( |\tau, i\rangle \), which are degenerate, \( H^\tau |\tau, i\rangle = \epsilon_\tau |\tau, i\rangle \), since \( H^\tau \) commutes with all three generators of \( SU^\tau(2) \), \( \tau_{0,\pm} \), and the states \( |\tau, i\rangle \) and \( \tau_{0,\pm} |\tau, i\rangle \) should have the same energy.

Now, consider the reduction \( su^\tau(2) \supset u^\tau(1) \). A Hamiltonian with a symmetry \( su^\tau(2) \) must necessarily have the \( u^\tau(1) \) symmetry (\([H, \tau_0] = 0 \) and they are simultaneously diagonalizable). However, if the condition of \( su^\tau(2) \) isospin symmetry is too strong, a possible breaking of the \( su^\tau(2) \) symmetry can be imposed as

\[
H'_\tau = \eta_1 \tau^2 + \eta_2 \tau_0^2,
\]

(3.7)

which in general corresponds to the linear combination of the two second-order Casimir operators of the algebra \( g \) and its subalgebra. The Hamiltonian \( H'_\tau \) is invariant under \( U^\tau(1) \) but does not commute with the rest of the generators of \( SU^\tau(2) \) and hence the \( SU^\tau(2) \) symmetry is broken. Since both Casimir invariants in (3.7) commute, \( H'_\tau \) eigenstates have good (not invariant) isospin (they are the same as for \( H^\tau \)) with eigenvalues \( \eta_1 \epsilon_\tau + \eta_2 \epsilon_i \). The Hamiltonian \( H'_\tau \) is said to have \( SU^\tau(2) \) as a dynamical symmetry. This is the essential feature of the dynamical symmetry concept, namely, although the eigenvalues of \( H'_\tau \) depend on both \( \tau \) and \( i \) (and hence \( SU^\tau(2) \) is not an exact symmetry), the eigenstates do not change during the breaking of the \( SU^\tau(2) \) symmetry: the dynamical symmetry breaking splits but does not admix the eigenstates [90].

In analogy to the example shown above for the isospin limit, one can build Hamiltonian operators for the other limiting cases related to isovector pairing as linear combinations of the two Casimir invariants of \( su^{\mu=0,\pm}(2) \) and \( u^{\mu=0,\pm}(1) \) algebras.

Within this algebraic framework, the most general Hamiltonian [96] (restricted to two-body interaction between the nucleons) of a system with \( Sp(4) \) dynamical symmetry can be expressed through the first-order invariant of \( sp(4) \) and the second-order Casimir operators of \( sp(4), u^{\mu}(2), su^{\mu}(2) (C^\tau_{2,0,\pm} \text{ (Table 2.2)}), \) and \( u^{\mu}(1) \) according to the reduction chains,
(2.13) or (2.14),

\[ H = -\eta_1 C_2^\tau - \eta_2 r_0^2 - \eta_3 C_2^0 - \eta_4 (N^0)^2 - \eta_5 (C_2^+ + C_2^-) - \eta_6 (N^+ (N^+ - 1) + N^- (N^- - 1)) - \eta_7 \hat{N} + \eta_8, \]

where \( \eta_i, i = 1, 2, \ldots, 8 \), are real coefficients and the contribution due to the Casimir operator of \( \mathfrak{sp}(4) \) scales the Hamiltonian within a constant. The operators \( N^{0,\pm} \) are related to the number operators \( \hat{N}, \hat{N}_\pm \) (Table 2.1). When some of the \( \eta_i \)-coefficients vanish such that (3.8) consists of Casimir operators of subalgebras belonging to a single reduction of \( \mathfrak{sp}(4) \), one obtains the corresponding limiting case and the eigenvalue problem can be solved analytically. When all of the coefficients are zero except \( \{\eta_1, \eta_3, \eta_7, \eta_8\} \neq 0 \) the Hamiltonian has a true \( U^\tau(1) \) symmetry (preserves the third isospin projection) and it breaks \( \text{Sp}(4), U^\tau(2) \) and \( SU^\tau(2) \) but has them as dynamical symmetries. For \( \{\eta_2, \eta_3, \eta_4, (\eta_7, \eta_8)\} \neq 0 \) the dynamical symmetries are determined according to the \( \mu = 0 \) \( pn \)-pairing group reduction limit of \( \text{Sp}(4) \supset U^0(2) \), and for \( \{\eta_5, \eta_6, (\eta_7, \eta_8)\} \neq 0 \) both \( SU^\tau(2) \) are dynamical symmetries in the like-particle pairing limit according to the reduction (2.14). The ratios \( \eta_2/\eta_1, \eta_4/\eta_3, \eta_6/\eta_5 \) determine the extent to which the \( SU^{\mu=0, \pm}(2) \) symmetry in each limit is broken.

Within the microscopic picture, the general one-body and two-body Hamiltonian in the second quantized representation \([85, 86, 87]\) is

\[ H = -\sum_{\mu_1, \mu_2} T_{\mu_1, \mu_2} c_{\mu_1}^\dagger c_{\mu_2} - \frac{1}{4} \sum_{\mu_1, \mu_2, \mu_3, \mu_4} V_{\mu_1 \mu_2, \mu_3 \mu_4} c_{\mu_1}^\dagger c_{\mu_2} c_{\mu_3} c_{\mu_4}, \]

with interaction matrix elements \( V_{\mu_1 \mu_2, \mu_3 \mu_4} \) that are antisymmetric under the exchange of \( \mu_1 \) and \( \mu_2 \), as well under the exchange of \( \mu_3 \) and \( \mu_4 \). The two-body interaction between \( n \) nucleons in the \( j^n \) configuration that: (1) commutes with the angular momentum operator \( J \) (invariant under three-dimensional rotations), and (2) is isospin invariant, can be written in analogy to the interaction used by Kerman [16] for one kind of particle,

\[ H_{\text{int}}^J = -\sum_{J, M, \tau, \tau_0} E_{J\tau} C_{JM\tau\tau_0}^\dagger C_{JM\tau\tau_0}, \]

where

\[ C_{JM\tau\tau_0}^\dagger = \sum_{m, \sigma_1, \sigma_2 = \tau_0 - \sigma_1} \langle jmJM - m|JM\rangle \left\langle \sigma_1 1 2 | \sigma_2 3 | \tau \tau_0 \right\rangle c_{jM\sigma_1}^\dagger c_{jM\sigma_2}, \]

is the proton-neutron version of Kerman’s generalized pair creation operator\(^4\) for coupling particles to angular momentum \( J \) and isospin \( \tau \) with an energy \( E_{J\tau} \) (3.10), where \( \langle jmJM - m|JM\rangle \) and \( \left\langle \sigma_1 1 2 | \sigma_2 3 | \tau \tau_0 \right\rangle \) are the Clebsch-Gordan coupling coefficients, and \( J \) and \( \tau \) may

\(^{4}\)The term ‘generalized’ was used by Kerman in the sense that the particles are coupled to any angular momentum \( J \) not only to \( J = 0 \) and should not be confused with the generalization to multi-shell dimension introduced in Chapter 2.
take simultaneously the values \((\tau = 1, \ J = 0, 2, \ldots, 2j - 1)\) and \((\tau = 0, \ J = 1, 3, \ldots, 2j)\), in accordance with the Pauli principle. Following this discussion we write

\[
H_{\text{int}}^j = -\frac{1}{2} \sum_{\{\sigma\}} \sum_{M,m,m'} \langle V \rangle c_{j,m+1,\sigma_1}^{\dagger} c_{j,m-1,\sigma_2}^{\dagger} c_{j,M-m',\sigma_3} c_{j,M-m',\sigma_4}
\]

\[
= -\frac{1}{4} \sum_{\{\sigma\}} V_{\sigma_1,\sigma_2;\sigma_4,\sigma_3}^{(J=0)} \left( \sum_{m} (-)^{j-m} c_{j,m,\sigma_1}^{\dagger} c_{j,-m,\sigma_2} \right)_{J,M=0} \times \left( \sum_{m'} (-)^{j-m'} c_{j,-m',\sigma_3} c_{j,m'\sigma_4} \right)_{J,M=0}
\]

\[
-\frac{1}{2} \sum_{\{\sigma\}} V_{\sigma_1,\sigma_2;\sigma_4,\sigma_3}^{(J)} \sum_{M,m} c_{j,m,\sigma_1}^{\dagger} c_{j,M-m,\sigma_2} c_{j,M-m',\sigma_3} c_{j,m'\sigma_4} \left|_{m=m'} \right._{J \neq 0} \left. \right. \]  

\[
- \sum_{\{\sigma\}} V_{\sigma_1,\sigma_2;\sigma_4,\sigma_3}^{(J)} \sum_{M,m,m'} c_{j,m,\sigma_1}^{\dagger} c_{j,M-m,\sigma_2} c_{j,M-m',\sigma_3} c_{j,m'\sigma_4} \left|_{m \neq m'} \right. \right. , \quad (3.12)
\]

where \(\{\sigma\} = \{ (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \} = \{ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \} \). The coefficients

\[
\langle V \rangle = \langle jm\sigma_1, j(M-m)\sigma_2 | V | jm'\sigma_4, j(M-m')\sigma_3 \rangle = V_{j\sigma_1,j(M-m)\sigma_2;jm'\sigma_4,j(M-m')\sigma_3}^{(J)}
\]

are the matrix elements of the two-body interaction potential that scatters two particles in single-particle states of quantum numbers \((j, m', \sigma_1)\) and \((j, M - m', \sigma_3)\) into states with \((j, m, \sigma_1)\) and \((j, M - m, \sigma_2)\). In the second part of (3.12) we take into account the degeneracy of the single \(j\)-level (no energy dependence on the magnetic quantum number \(m\) of the interaction matrix elements).

The microscopic interaction (3.12) can be generalized straightforwardly to multi-shell configurations and related to the generators of the generalized symplectic \(\text{Sp}(4)\) group. The first term of the second expression in (3.12) corresponds to \(J = 0\) \((M = 0)\) pairing and is given by the product of a pair creation operator and a pair annihilation operator \(A_k^\dagger A_k\) \((k = 0, \pm)\), the second term includes high-\(J\) \((J \neq 0, \ m = m')\) components of the interaction and can be represented by \{\(\tau_+, \tau_-\) \(\sim \hat{N}/(2\Omega)\), \(\hat{N}(\hat{N} - 1)\) \} and \(\tau_0\tau_0 - \hat{N}/4\). The rest of the sum is the residual interaction that is neglected. Assuming that the \(J\)-dependence of the interaction matrix elements distinguishes only the \(J = 0\) coupling from the rest of the high-\(J\) interaction strengths, we introduce the parameters

\[
J = 0 \quad \text{pairing} \quad \text{interaction} \quad \left\{ \begin{array}{l}
\langle ++ | V_{F_0} | ++ \rangle = \langle -- | V_{F_0} | -- \rangle = F/\Omega, \\
\langle -- | V_{F_0} | ++ \rangle = \langle ++ | V_{F_0} | -- \rangle = G/\Omega,
\end{array} \right.
\]

\[
\langle ++ | V | ++ \rangle = \langle -- | V | -- \rangle = C + D/2, \\
\langle -- | V | ++ \rangle = 2C - D, \quad \langle ++ | V | -- \rangle = E/\Omega.
\]

\[(3.13)\]
In this way, a natural microscopic approach yields a suitable effective interaction, which preserves the total number of particles, \( \hat{N}, \) and the third isospin projection, \( \tau_0, \) consists of one- and two-body terms, and is expressed through group generators with Sp(4) dynamical symmetry \([46, 96]\)

\[
H = -\epsilon \hat{N} - GA_0^\dagger A_0 - F(A_{+1}^\dagger A_{-1} + A_{-1}^\dagger A_{+1}) - \frac{E}{2\Omega}(\{\tau_+, \tau_-\} - \frac{\hat{N}}{2\Omega}) - C\frac{\hat{N}(\hat{N} - 1)}{2} - D(\tau_0^2 - \frac{\hat{N}^2}{4})
\]

\[
= -\epsilon \hat{N} - GA_0^\dagger A_0 - F(A_{+1}^\dagger A_{-1} + A_{-1}^\dagger A_{+1}) - \frac{E}{2\Omega}(\tau^2 - \frac{\hat{N}}{2}) - D(\tau_0^2 - \frac{\hat{N}^2}{4})
\]

\[
-(C + \frac{E}{2\Omega})\frac{\hat{N}(\hat{N} - 1)}{2} - (D - \frac{E}{2\Omega})\tau_0^2, \quad \tau_0 \leq \frac{\hat{N}}{2}
\]

(3.14)

where \( G, F, E, C \) and \( D \) are phenomenological constant interaction strength parameters \((G \geq 0, F \geq 0 \text{ for attraction})\) and \( \epsilon > 0 \) is a Fermi level energy. The connection (3.13) with the interaction matrix elements gives a real physical meaning to the phenomenological strength parameters, and, therefore, their estimation can lead to a microscopic description of the nuclear interaction.

Specific features of the phenomenological Hamiltonian (3.14) are that it breaks the isospin symmetry \((D \neq \frac{E}{2\Omega})\) [which actually appears as a dynamical symmetry if \( F = G \)] and it mixes states with definite isospin values \((F \neq G)\). This is different from other applications of non-deformed and \(q\)-deformed \(sp(4)\) or \(o(5)\) algebras with isospin-invariant Hamiltonians \([10, 15, 34]\). Although the degree of isospin mixing is expected to be very small \((G \text{ should be very close to } F)\) it may still add an interesting contribution to the study of the isospin mixing. The isospin breaking \((D \neq \frac{E}{2\Omega})\), on the other hand, introduces an additional degree of freedom that allows a parabolic energy dependence on \(i\) preserving at the same time the proton-neutron symmetry of the nuclear force. For light nuclei, where isospin is known to be a good symmetry, we expect \( F \cong G \) and \( D \cong \frac{E}{2\Omega} \).

The Hamiltonian (3.14) expressed in terms of the Sp(4) generators corresponds to the one introduced as linear combinations of the invariant operators of \(sp(4)\) and its subalgebras (3.8). The \( \eta_i\)-coefficients \((i = 1, 2, \ldots, 8)\) in (3.8) are not necessarily linearly independent and are related to the phenomenological parameters of the model (3.14) in the following way:

\[
\eta_1 = \frac{E}{2\Omega} \quad \eta_2 = (D - \frac{1}{2\Omega}E) \quad \eta_3 = \frac{G}{2\Omega} \quad \eta_4 = -\frac{(G - 2C)}{2\Omega} \quad \eta_5 = -\frac{F}{\Omega} \quad \eta_6 = \frac{E}{2\Omega} \quad \eta_7 = \begin{cases} \epsilon - C(1 - 4\Omega)/2 - D/4 - (E - G)/(4\Omega), & n \leq 2\Omega \\ \epsilon - C(1 - 4\Omega)/2 - D/4 + (E - G)/(4\Omega), & n > 2\Omega \end{cases} \\
\eta_8 = \begin{cases} 2C\Omega^2 + G/2, & n \leq 2\Omega \\ 2C\Omega^2 + G/2 + (E - G), & n > 2\Omega. \end{cases}
\]

(3.15)

The difference in \( \eta_7 \) and \( \eta_8 \) before and after mid-shell \((2\Omega)\) reflects the implementation of the particle-hole concept in the pairing terms, which is explained next.

Possible applications of the Hamiltonian to real nuclei can be determined through a detailed investigation of the various terms introduced in (3.14). The first two terms \((G, F)\) of the Hamiltonian (3.14) account for \( J = 0 \) isovector \((\tau = 1)\) pairing between non-identical
and identical particles, respectively. To reflect the assumption that a zero pairing energy corresponds to a state with no possible breaking of a pair [87], a particle-hole concept is incorporated in these two terms. For one type of particles, hole pair-creation (annihilation) operators are introduced via the hole creation and annihilation single-particle operators [87]

\[ b_{jm\sigma}^\dagger = c_{j,-m,\sigma}, \quad b_{jm\sigma} = c_{j,-m,\sigma}^\dagger, \]  

(3.16)

and the pairing interaction between the holes is equivalent to a change from the particle to the hole number operator, \( N_{\pm 1} \rightarrow 2\Omega - N_{\pm 1} \) for \( N_{\pm} > \Omega \). In analogy to the like-particle pairing, a hole pair-creation (annihilation) operator can be also introduced for \( pn \) pairs. This corresponds to \( \hat{N} \rightarrow 4\Omega - \hat{N} \) for \( n > 2\Omega \). In this way the pairing terms in the Hamiltonian are particle-hole conjugate. It is important to note that the rest of the terms are not particle-hole conjugates, which allows the single-particle term to account correctly for the increase in the volume of the nucleus (proportional to the mass number \( \mathcal{A} \)) when particles are added, as well the third projection of the isospin to remain well defined as half the difference between the protons and neutrons (as particles).

The next term \( (E) \) can be related to the symmetry energy [10, 27] as its expectation value in states with definite isospin is

\[ \langle n, \tau, i | -\frac{E}{2\Omega}\tau^2 | n, \tau, i \rangle = -\frac{E}{2\Omega}\tau (\tau + 1) \approx \varepsilon_{\text{sym}}, \]  

(3.17)

which enters as a symmetry term in many nuclear mass relationships [132, 133]. The second-order Casimir invariant of \( \mathfrak{sp}(4), C_2(\mathfrak{sp}(4)) \) (2.12) with eigenvalue \( \Omega + 3 \), establishes a linear dependence among the terms in the Hamiltonian (3.14),

\[ \frac{\tau^2}{\Omega} = C_2(\mathfrak{sp}(4)) - 2 \left( A_1^\dagger A_{-1} + A_{-1}^\dagger A_1 + A_0^\dagger A_0 \right) + \frac{N_{-2\Omega}}{2\Omega} + \frac{N_1 + N_{-1} - 2\Omega}{\Omega} - \frac{(\hat{N} - 2\Omega)^2}{4\Omega}, \]  

(3.18)

which yields to a direct relation between the symmetry and pairing contributions, a fact that has been already pointed out in a phenomenological analysis based on the experimental nuclear masses and excitation energies [126]. The existing relation, due to \( C_2(\mathfrak{sp}(4)) \), holds even when the particle-hole concept is implemented in the pairing terms (on the right-hand side of the relation (3.18)). The dependence among terms in the Hamiltonian could have lead to a reduction of the number of the phenomenological parameters if terms linear in particle-hole conjugate number operators did not appear. Particularly, \( N_1 + N_{-1} \) in (3.18) changes according to the particle-hole concept for \( N_\pm > \Omega \) and cannot be added to the linear in \( \hat{N} \) single-particle term in the Hamiltonian (3.14). This is why the parameter \( E \) is not redundant in the analysis and cannot be set to zero without a loss of generality. We refer to the \( E \)-term as a symmetry term, although it is common to address the symmetry energy in a slightly different way: the \( \tau (\tau + 1) \)-term together with the isospin dependence of the isovector pairing term (that follows from (3.18)) yield both symmetry \( \sim \tau^2 \sim (Z - N)^2 \) and Wigner \( \sim \tau \) energies [134]. The first one was originally included in the Bethe-Weizsäcker semi-empirical mass formula [135, 136] and implies that the nuclear symmetry energy has the tendency toward stability for \( N = Z \). The \( (Z - N)^2 \) dependence emerges in the kinetic
energy of the nuclear Fermi gas model, where this energy is optimal for $N = Z$, and as well due to a specific feature of the nuclear force, which implies that the proton-neutron interaction is on average stronger than that between like particles. The Wigner energy is associated with proton-neutron exchange interactions and is responsible for a sharp energy cusp at $N = Z$ leading to an additional binding of self-conjugate nuclei [85].

The last two terms in (3.14), $C$ and $D$, as we have already pointed out, arise in the dynamical-symmetry formalism. Together with the $E$-term, they are related to the microscopic nature of the $pn$ isoscalar ($\tau = 0$) interaction,

$$H_{pn}^{\tau=0} = -\frac{E}{2\Omega}(\tau^2 - \frac{\hat{N}}{2} - \frac{\hat{N}^2}{4}), \quad (3.19)$$

shown in the second expression for the Hamiltonian (3.14). As can be seen clearly from this expression (3.14) and from relation (3.13), for $D = E/2\Omega$ and $C + D/2 = 0$ we obtain the $J$-independent $pn$ isoscalar force. It is closely related to the symmetry energy ($E$), is diagonal in the isospin basis, and can be compared to [30, 137]. Therefore, the Sp(4) model interaction consists of isovector ($pp$, $nn$, $pn$) pairing and an isoscalar ($pn$) force in addition to a possible isospin-breaking term and high-$J$ identical-particle interactions.

In this way, the phenomenological Hamiltonian (3.14) can be used to describe general properties of the nuclear interaction, which serves as a motivation to fit the theoretical expectation values of (3.14) to the energies of the corresponding isovector-paired $0^+$ states of nuclei in a broad region of the nuclear chart. Still, possible model limitations are expected. First, the Hamiltonian (3.14) lacks the $Q \cdot Q$ quadrupole-quadrupole interaction responsible for shape deformation and for strong enhanced quadrupole transitions in collective rotational bands. Its effect is expected to be non-negligible in the region of medium nuclei especially toward the middle of the $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell. Within the framework of the conventional deformed shell-model we assume that this degree of freedom is hidden in the parametrization of the mean-field potential. However, an isospin invariant $Q \cdot Q$ force can be expressed as a linear combination with $J$-dependent coefficients of $\sum_{M\tau_0} C_{J\tau \tau_0}^{\dagger} C_{J\tau \tau_0}$ (in terms of the generalized pair operators (3.11)) and its $J = 0$, $J$–odd and $J$–even components overlap with the isovector pairing, the isoscalar $pn$ interaction [30] and the high-$J$ identical-particle interaction in the model (3.12). In this context the $Q \cdot Q$ interaction is not independent of the model Hamiltonian (3.14) and may be implicitly taken into account through a parameter renormalization. Second, we should note that when applied to multi-$j$ levels the Sp(4) model treats the $j$ orbits as degenerate. Hence, the parameters of the effective model interaction are likely to be influenced by the non-degeneracy of the orbits. Nevertheless, as the dynamical symmetry properties of the two-body interaction in nuclei from this region are not lost, the model remains a good multi-$j$ approximation and the extent to which it provides for a realistic description is further tested in comparison to experiment.

As for the microscopic “classical” approach, the most general Hamiltonian of a system with a $q$-deformed symplectic dynamical symmetry and conserved proton and neutron particle numbers can be constructed in terms of the $\mathfrak{sp}_q(4)$ basis operators in a way that is analogous to (3.8) and is chosen to coincide with the non-deformed one (3.14) in the limit
where $\epsilon^q = \epsilon^q + \left(\frac{1}{2} - 2\Omega\right) C_q + \frac{D_q}{4} > 0$ is the Fermi level of the nuclear system, $G_q$, $F_q$, $E_q$, $C_q$ and $D_q$ are constant interaction strength parameters. In principle, the $q$-deformed set of phenomenological parameters,

$$\gamma_q = \{\epsilon_q, G_q, F_q, E_q, C_q, D_q\},$$

may be different than the non-deformed counterparts,

$$\gamma = \{\epsilon, G, F, E, C, D\}.$$  \hspace{0.5cm} (3.21)

From a “classical” perspective, the deformation introduces higher-order, many-body terms into a theory that starts with only one-body and two-body interactions (3.14). The way in which the higher-order effects enter into the theory is governed by the [X] form (2.34). Since $\varkappa$ and $q$ are related to one another, $q = e^\varkappa$, everything is tied to the deformation with

$$[X] = \frac{\sinh(\varkappa X)}{\sinh(\varkappa)} = X \left(1 + \varkappa^2 \frac{X^2 - 1}{6} + \varkappa^4 \frac{3X^4 - 10X^2 + 7}{360} + \ldots \right) \xrightarrow{\varkappa \rightarrow 0(q \rightarrow 1)} X.$$  \hspace{0.5cm} (3.23)

Clearly, an operator like $[\hat{N}]$ introduces non-linear (non-negligible if comparatively large $\varkappa$) terms in the Hamiltonian such as $\hat{N}^3$, $\hat{N}^5$, etc., which correspond to many-body interactions of the kind\(^5\)

$$\begin{align*}
&c_{jm\sigma}^{\dagger}c_{jm'\sigma'}^{\dagger}c_{jm\sigma'}c_{jm'\sigma'} \\
&c_{jm\sigma}^{\dagger}c_{jm'\sigma'}^{\dagger}c_{jm'\sigma''}^{\dagger}c_{jm\sigma}c_{jm'\sigma'}c_{jm'\sigma''} \\
&c_{jm\sigma}^{\dagger}c_{jm'\sigma'}^{\dagger}c_{jm'\sigma''}^{\dagger}c_{jm''\sigma'''}^{\dagger}c_{jm\sigma}c_{jm'\sigma'}c_{jm'\sigma''}c_{jm''\sigma'''} \\
&\vdots
\end{align*}$$

Thus, regarding the non-deformed Cartan subalgebra ($\hat{N}$ and $T_0 \equiv \tau_0$), $q$-brackets of such operators appear in the Hamiltonian and their meaning and use is clear. On the other hand, the effect of the $q$-deformation on the interaction that enters through the $q$-deformed $\text{sp}_q(4)$ operators (such as $B_0^0 B_0^0$ or $T^2$, (3.20)) is best revealed through the eigenvalues of the model Hamiltonian, which will be illustrated in the next section.

In this way, with the use of a single parameter, $q$, a simple Hamiltonian like (3.20) is used to account, in a prescribed fashion, for many-body interactions in an $n$ nucleon system, while preserving the symplectic $\text{Sp}_q(4)$ dynamical symmetry and leaving the constants of motions of the system unaffected.

\(^5\)More precisely, sum over repeated indices is to be understood (Einstein’s convention). As well, note that even though there are only odd powers of $\hat{N}$, interactions such as the four-body term (3.26) will also appear (due to $\hat{N}^3$). A final remark regards the fact that the many-body interactions are in terms of the “classical” (but not the $q$-deformed) fermion operators, which are the one representing the real nucleons.
3.2 Matrix Elements of the “Classical” and $q$-Deformed Model Interactions

In the SU$^0(2)$ limit ($pn$-coupling) the energy eigenvalue of the non-deformed $pn$ pairing interaction $GA_0^\dagger A_0$ is

$$\varepsilon_{pn} = \frac{G}{\Omega} n_0 \frac{2\Omega - n + n_0 + 1}{2} = \frac{G}{8\Omega} (n - 2\nu_0)(4\Omega - n - 2\nu_0 + 2)$$ (3.27)

and in the SU$^\pm(2)$ limit (like-particle coupling) the energy of the non-deformed $pp$ and $nn$ pairing interaction $FA_1^\dagger A_\mp$ is

$$\varepsilon_{pp(nn)} = \frac{F}{\Omega} n_\pm(\Omega + n_\pm - N_\pm + 1) = \frac{F}{4\Omega} (N_\pm - \nu_1)(2\Omega - N_\pm - \nu_1 + 2).$$ (3.28)

In each limit, as described in Chapter 2, $2\nu_0 = 2n_{-1} + 2n_{-1}$ and $\nu_1 = n_0$ are the seniority quantum numbers that counts the particles not coupled in $J = 0$ $pn$ pairs, and the protons (neutrons) not coupled to $J = 0$ $pp$ ($nn$) pairs, respectively. In the fully-paired basis (2.19), $\nu_0$ and $\nu_1$ give the number of remaining pairs that can be formed after coupling the fermions in the primary pairing mode and they vary by $\Delta \nu_{0,1} = 2$. The eigenstates for both $pn$ and $pp + nn$ pairing interactions are $|i, s^0, n\rangle$ and $|s^+ = s^-, N_+, N_-\rangle$ (Table 2.4), respectively, which coincide with the pair basis state (2.19) for the maximum eigenvalue of the corresponding quasi-spin.

In general, the Hamiltonian (3.14) is not diagonal in the pair basis set (Table 3.2). The linear combinations of the basis states describe the spectrum of the isovector-paired $0^+$ states for a given nucleus. The pairing Hamiltonian $H_{pair}$, ((3.14) with $E = C = D = 0$ and $\epsilon = 0$),

$$H_{pair} = -GA_0^\dagger A_0 - F(A_1^\dagger A_{-1} + A_{-1}^\dagger A_{+1}) = -G\mathfrak{N}_{pn} - F\mathfrak{N}_{pp + nn},$$ (3.29)

gives a transition between the states with different kinds of pairing while preserving the total number of pairs, $n$, and the isospin projection, $i$, that is, two $pn$ pairs scatter into a $pp$ and a $nn$ pair, and vice versa

$$|H_{pair}| |n_1, n_0, n_{-1}\rangle = (\varepsilon_{pn} + \varepsilon_{pp} + \varepsilon_{nn}) |n_1, n_0, n_{-1}\rangle - \frac{G}{\Omega} n_1 n_{-1} |n_1 - 1, n_0 + 2, n_{-1} - 1\rangle$$

$$- \frac{F}{\Omega} n_0 (n_0 - 1) |n_1 + 1, n_0 - 2, n_{-1} + 1\rangle,$$ (3.30)

where $\varepsilon_{pn,pp,nn}$ are given in (3.27) and (3.28), the matrix representation of $\mathfrak{N}_{pn,pp + nn}$ is shown in (2.128) and (2.129) and $n_1, n_0, n_{-1}$ are particle or hole pairs.

The $q$-analog of the pairing Hamiltonian (3.29) is given in terms of the $q$-deformed operators (3.20) and its absolute value is

$$|H_{q, pair}| = G_q B_0^\dagger B_0 + F_q (B_{+1}^\dagger B_{-1} + B_{-1}^\dagger B_{+1}) = G_q \mathfrak{N}_{qn}^q + F_q \mathfrak{N}_{pp + nn}^q,$$ (3.31)
The matrix representation of the $q$-deformed pairing Hamiltonian (3.31) are also derived in an analytical form with the help of (2.130) and (2.131) and the $q$-analog of (3.30) is

$$|H_{q,pair}| n_1, n_0, n_{-1} = \left(\varepsilon_{pn}^q + \varepsilon_{pp}^q + \varepsilon_{nn}^q\right)|n_1, n_0, n_{-1} - \frac{G}{\Omega} \tilde{n}_{-1}| n_1 - 1, n_0 + 2, n_{-1} - 1 - \frac{F}{\sqrt{\rho_+ \rho_-}} \sum_{k=1}^{n_0-1} S_q(k) |n_1 + 1, n_0 - 2, n_{-1} + 1), \quad (3.32)$$

where $\tilde{n}_{-1}$ and $S_q(k)$ are defined in (2.134) and (2.126). In (3.32) the eigenvalue, $\varepsilon_{pn}^q$, of the $q$-deformed $pn$ pairing interaction ($G_q B_0^q B_0$) in the $su_q^0(2)$ limit and the eigenvalues, $\varepsilon_{pp,nn}^q$, of the $q$-deformed identical pairing ($F_q B_{\pm 1}^q B_{\pm 1}$) in the $su_q^\pm(2)$ limits are

$$\varepsilon_{pn}^q = G_q \left[\frac{1}{2\Omega}\right] \left[\frac{n - 2n_0}{2}\right] \left[\frac{4\Omega - n - 2n_0 + 2}{2}\right] = G_q \left[\frac{1}{2\Omega}\right] [n_0] \frac{\sqrt{n_0 + 1 - n + n_0}}{n_0}, \quad (3.33)$$

$$\varepsilon_{pp,nn}^q = F_q \hat{\rho} \left[\frac{1}{\Omega}\right] \left[\frac{n_\pm - \nu_1}{2}\right] \left[\frac{2\Omega - n_\pm - \nu_1 + 2}{2}\right] = F_q \hat{\rho} \left[\frac{1}{\Omega}\right] [n_\pm] \frac{n_\pm + 1 - N_\pm + n_\pm}{n_\pm}. \quad (3.34)$$

The higher-order terms, which correspond to many-body interactions, can be recognized through the expansion of the eigenvalues of the $q$-deformed Hamiltonian (3.20). In the pairing limits, the influence of the deformation on the pairing interaction is revealed through the eigenvalues of $|H_{q,pair}|$ (3.31) in each limit, (3.33) and (3.34), which are expanded in orders of $\kappa$ ($q = e^{\kappa}$) as

$$su_q^0(2) : \varepsilon_{pn}^q = \frac{G_q}{G} \varepsilon_{pn}\{1 + \kappa^2 \frac{(n_0^2 - 4\Omega^2 - 1) + \left(\frac{2\Omega \varepsilon_{pm}}{n_0 G}\right)^2}{24\Omega^2} + O(\kappa^4)\}, \quad (3.35)$$

$$su_q^\pm(2) : \varepsilon_{pp,nn}^q = \frac{F_q}{F} \varepsilon_{pp,nn}\{1 + \kappa^2 \frac{(n_0^2 - 4\Omega^2 - 1) + \left(\frac{2\Omega \varepsilon_{pm}}{n_0 G}\right)^2}{24\Omega^2} + O(\kappa^4)\}, \quad (3.36)$$

where the expansions include higher-order terms that may not be negligible and the non-deformed energies (3.27) and (3.28) are the zeroth-order approximation of the corresponding deformed pairing energies. While the proton-neutron interaction is even with respect to the deformation parameter $\kappa$, the identical particle pairing may also include odd terms through the coefficients $\rho_\pm$ (2.69) when $\tilde{\rho} = \rho_\pm$ (non-symmetric commutation relations in $sp_q(4)$). When symmetric commutations are used, $\tilde{\rho} = \rho$, the like-particle interaction is even with respect to $\kappa$ and can be compared with earlier studies [31, 76]. While the quadratic coefficient in the expansion of (3.36) is positive, the one in the expansion of (3.35) is negative.
This leads to a decrease of the binding energy of the \( pn \) pairs as \(|\kappa|\) increases from zero. As the deformation parameter increases from the "classical" limit, the like-particle pairing is strengthened, yielding a larger pairing gap (when \( \tilde{\rho} = \rho \)). The expansions in the pairing limits ((3.35) and (3.36)) introduce non-linear terms with respect to the pair numbers, space dimension and the non-deformed pairing energies, \( \varepsilon_{pn} \) and \( \varepsilon_{pp(\nu)} \), indicating the higher-order nature of the isovector pairing interactions between nucleons.

Similarly, in the isospin limit, the \( q \)-deformed analog \( (-E_q^{\tau}T^2, (3.20)) \) of the symmetry energy \( \varepsilon_{sym} \) (3.17), related to the eigenvalue (Table 2.10) of the second-order Casimir invariant of \( su_q^*(2) \), \( T^2 \) (Table 2.9), is expanded in orders of \( \kappa \) \((q = e^{\kappa})\)

\[
\varepsilon_q^{\tau} = -\frac{E_q^{\tau}}{2\Omega}(\tau + 1)\left\{1 + \frac{\kappa^2(\tau + 1)^2 + \tau^2 - 1 - 4\Omega^2}{24\Omega^2} + O(\kappa^4)\right\}. \quad (3.37)
\]

In this limit, the \( q \)-deformation leads to a decrease (larger for greater \( \tau \) values) of the magnitude of the non-deformed symmetry energy. The appearance of higher-order terms in the \( \tau \) isospin in the interaction is not a mere mathematical consequence without a precedent in physics. In particular, Hecht discovered a \( \tau^2(\tau + 1)^2 \) dependence in his SO(5) model that results from admixtures of higher \( \nu \) seniority numbers [10]. Indeed, even though the basis states of our model are fully-paired \((\nu = 0\) as a total seniority\), the seniority quantum numbers introduced in the different pairing limits \( \nu_1 \) \((\nu_0) \) (Chapter 2) are not in general good quantum numbers in the isospin limit (isospin eigenstates) and the admixture of the various values of \( \nu_1 \) \((\nu_0) \) may provide an explanation to the \( \tau \) dependence in (3.37) when many-body effects take place.

The expansions of the \( q \)-deformed energies in the limiting cases, (3.35)-(3.37), serve as a simple example of the contribution of the \( q \)-deformation compared to the non-deformed model, which is a straightforward result of the quantum definition, (2.34). They clearly point out the nature of the \( q \)-deformed Hamiltonian related to the many-body interactions.

In both the "classical" and \( q \)-deformed cases, the matrix elements of the total Hamiltonian, (3.14) and (3.20), can be obtained with the help of (3.30) and (3.32) for the isovector pairing interactions and by using the analytical formulae, (2.144) and (2.145), multiplied by \( E/2 \) for the symmetry term. The rest of the terms in the Hamiltonians are diagonal in the pair basis (2.19). The energy eigenvalues are generally obtained through a numerical diagonalization of the Hamiltonian matrix for a given nucleus \((N_+ \) fixed\), which is typically of a small dimension \([^\min(N_+, N_-)/2 + 1\), where \( N_\pm \) is the eigenvalue of the particle-hole conjugate number operator (Table 2.4)).

To move from the abstract to applications, we need to penetrate through the mathematical complexity and turn it into a tool for understanding real nuclei and interactions. In this Chapter, we constructed a model with Sp(4) dynamical symmetry (and its \( q \)-deformation) and its boundaries of applicability were explored; specifically we looked at the model space
and the associated effective interaction. In the non-deformed limit as well as in the $q$-deformed generalization, a phenomenological Hamiltonian was written in terms of the generators of the group and this in turn was related to a general microscopic pairing Hamiltonian. Next, the $\text{Sp}(4)$ model and its $q$-deformed extension will be applied to the isovector-paired $0^+$ state energies in nuclei for single-$j$ levels, namely $1d_{\frac{3}{2}}$ and $1f_{\frac{7}{2}}$, and for a multi-$j$ $1f_{\frac{7}{2}}2p_{\frac{3}{2}}2p_{\frac{1}{2}}1g_{\frac{9}{2}}$ major shell. Whether the models yield a good description of the relevant real-world nuclear systems is the topic of the next chapter.
Chapter 4

Applications to Nuclear Structure

Understanding the structure of nuclei has, and continues, to challenge nuclear physicists, driving forward novel theoretical ideas and experimental techniques, including specialized resources and computer power: from the surprising gleam of radium observed by Madam Curie to huge modern-day accelerator facilities; from Rutherford’s discovery of the nuclear concept with most of the mass of an atom concentrated at its center and several order of magnitude smaller than the atom itself to the quark substructure of individual nucleons; from Fermi-scale nuclear structure exploration to a possible quark-gluon plasma in gigantic stars; and from current experimental techniques to futuristic methods to achieve extreme conditions in a laboratory environment that simulate the Big Bang environment.

All of these efforts provide us with a large collection of nuclear data and information. Based on this work, we are able to compare experimental results [138, 139, 140] with observables described by the Sp(4) model and its non-linear extension (q-deformed model). For a given nucleus (fixed total particle number \(n\) and isospin projection \(i\), or alternatively proton number \(N^+\) and neutron number \(N^-\)), the eigenvalues \(E_0\) of the model Hamiltonian of the “classical” limit (3.14) and of the q-deformed generalization (3.20) give estimates for the energy of the isovector-paired \(0^+\) states of the nuclear system. For even-even nuclei and for the odd-odd nuclei with \(J = 0\) ground states \((N \approx Z)\), the lowest isovector-paired \(0^+\) state is the nuclear ground state and the positive value of its energy is defined as the binding energy, \(|BE|\). The binding energy of a nucleus is an important quantity because it is related to the nuclear mass and lifetime.

If the model achieves good agreement with experiment it can be used to provide a microscopic explanation of the interactions that are involved and an interpretation of the related phenomena. However, the model Hamiltonian (3.14), which accounts for the strong interaction between nucleons, leads to a striking deviation of the theory from experiment – not surprisingly, since the Coulomb repulsion between the protons of a nucleus needs to be taken into consideration. Even though the Coulomb interaction is rather weak compared to the strong nuclear interaction (and does not fundamentally influence the nuclear energy spectrum [relatively to the ground state]) it significantly affects the binding energy of the nuclei and cannot be neglected. This is why, before we continue with our investigation of the pair-
ing correlations in atomic nuclei we need to discuss the effects of the Coulomb interaction on the nuclear system.

4.1 Isospin-Violating Coulomb and Nuclear Interactions

A fundamental feature of nuclear structure is a basic symmetry between neutrons and protons, namely the *charge independence* of the nuclear force. This implies that the \( pp \) interaction and the \( nn \) interaction are equal to the \( \tau = 1 \) \( pn \) interaction and leads to ‘rotational’ invariance in isotopic space. Clear evidence for this can be found in the striking similarity in the energy spectra of different isobars [85]. However, the isospin invariance is violated by the electromagnetic interaction. For the lightest nuclei, the symmetry breaking effects, mainly associated with the Coulomb force (and magnetic forces) between nucleons, are relatively small and can be rather accurately treated as perturbations. In heavy nuclei, the effects are comparatively larger, yet the validity of the isospin quantum number is not totally lost and \( \tau \) is an almost-good quantum number [85, 141].

The primary effect of the Coulomb force is to introduce into the theory a dependence on the third isospin projection, \( \tau_0 \), resulting in splitting of the energies of the isobaric analog nuclei (a \( \tau \)-multiplet) without coupling different isospin multiplets. At the same time, the isospin-violating part of the Coulomb interaction leads to small isospin mixing in nuclear ground states increasing with \( Z \) and largest for \( N = Z \). An interesting observation made by Dobaczewski and Hamamoto is that the \( pn \) interaction, which is known to be a driving force towards deformed shapes is the one working to restore the isospin symmetry [39]. The ground state isospin impurity is theoretically estimated to be as small as a percent for nuclei in the \( 1f_7^2 \) [85], and up to \( 4 - 5\% \) (for \( ^{100}\text{Sn} \)) toward the \( 1f_7^22p_1^22p_3^11g_7^2 \) shell closure\(^1\) [39]. The mixing probability coming from other sources than the Coulomb interaction is expected to be smaller. Such source is the isospin non-conserving part of the nuclear Hamiltonian, which includes effects due to the proton-neutron mass difference \( (\Delta m/m = 1.4 \times 10^{-3}) \) and small charge dependent components in the strong nucleonic interaction [141]. However, the weaker isospin mixing can be revealed if a proper account of the Coulomb energy that contributes to the total nuclear energy is taken into consideration.

In our investigation, we adopt a phenomenological Coulomb correction to the experimental energies such that a nuclear system can be regarded as if there is no Coulomb interaction between its constituents. The *Coulomb corrected* experimental energy, \( E_{0,\text{exp}} \), of an *isovector-paired* \( 0^+ \) state is adjusted to be

\[
E_{0,\text{exp}}(N_+, N_-) = E_{0,\text{exp}}^C(N_+, N_-) - E_{0,\text{exp}}^C(0, 0) + V_{\text{Coul}}(N_+, N_-),
\]

\(^1\)As far as the higher-energy part of the spectra is concerned, even a small isospin-violating coupling (due to either Coulomb or nuclear interaction) can produce large admixtures of the almost degenerate states with different isospins. These states are not in the main scope of our study.
where\(^2\) \(E_{0,\text{exp}}^C\) is the total energy measured including the Coulomb energy. In order to focus only on the contribution from the valence shell, the binding energy of the core \(E_{0,\text{exp}}^C(0,0)\) is subtracted in (4.1) and the \(V_{\text{Coul}}(N_+, N_-)\) Coulomb correction for a nucleus with mass \(A\) and \(Z\) protons is taken relative to the core \(V_{\text{Coul}}(N_+, N_-) = V_{\text{Coul}}(A, Z) - V_{\text{Coul}}(A_{\text{core}}, Z_{\text{core}})\). The recursion formula for the \(V_{\text{Coul}}(A, Z)\) Coulomb energy is derived in [142] with the use of the Pape and Antony formula [143]

\[
V_{\text{Coul}}(A, Z) = \begin{cases} V_{\text{Coul}}(A, Z - 1) + 1.44 \frac{(Z - 1/2)}{A^{1/3}} - 1.02 & Z > Z_s \\ V_{\text{Coul}}(A, Z + 1) - 1.44 \frac{(Z + 1/2)}{A^{1/3}} + 1.02 & Z < Z_s, \end{cases} \tag{4.2}
\]

where \(Z_s = A/2\) for \(A\) even or \(Z_s = (A + 1)/2\) for \(A\) odd. When \(Z = Z_s\) the Coulomb potential is given by

\[
V_{\text{Coul}}(A, Z_s) = \begin{cases} 0.162Z_s^2 + 0.95Z_s - 18.25 & Z_s \leq 20 \\ 0.125Z_s^2 + 2.35Z_s - 31.53 & Z_s > 20. \end{cases} \tag{4.3}
\]

The (4.2) Coulomb correction agrees with the one proposed in [144] for the \(1f_{7/2}\) level,

\[
V_{\text{Coul}}(N_+, N_-) = 0.300 \frac{N_+(N_+ - 1)}{2} - 0.065 N_+ N_- + 7.229 N_+ + 0.215i^2 + (7.129 + 0.15n)i + (3.5645 + 0.021n)n, \tag{4.4}
\]

where we substitute \(N_\pm = n/2 \pm \frac{i}{2}\) in the second expression. Although we chose not to use the correction (4.4) as it is restricted only to the \(1f_{7/2}\) orbit, its analytical form is a straightforward illustration of the fact that the asymmetry between protons and neutrons due to the Coulomb effect cannot be simulated solely by the model Hamiltonian (3.14), which is symmetric under proton-neutron exchange. The \(i^2\) dependence of the Coulomb energy (as in (4.4)) can be understood in the light of the empirical mass formula where the Coulomb energy\(^3\) may be taken proportional to \(Z^2/\sqrt{A} = (A/2)^2/\sqrt{A}\). Hence, in general, the bulk effect of the Coulomb interaction can be represented by \(a + bi + ci^2\), where the three components are isoscalar, isovector and isotensor (of rank 2) in nature (with respect to the isospin \(su^\tau(2)\) ‘rotations’), respectively, and \(a, b, c\) are functions of the nuclear characteristics.

The Coulomb corrected energies (4.1) should be the one to reflect solely the nuclear properties of the many-nucleon systems. Assuming charge independence of the nuclear force, their description can be provided by an isoscalar Hamiltonian as (3.14) with \(G = F\) and \(D = E/(2\Omega)\). However, the violation of the charge independence is well established – the purely nuclear parts of the pp force and the \(\tau = 1\) pn force differ from each other – which appears to be associated with the electromagnetic structure of the nucleons [141]. An analysis of the \(1S\) (\(L = 0\), spin zero (singlet)) scattering in the pn system and the low-energy

\(^2\)To avoid confusions we mention that in (4.1) the energies are assumed positive for bound states; \(V_{\text{Coul}}\) is also defined positive.

\(^3\)From E\&M classes we know that the electrostatic energy of a sphere of radius \(R\) and charge \(Z\) is \(\frac{Ze^2}{2R}\), where \(e\) is the charge of the electron, and take the radius of a nucleus \(R \approx 1.2A^{1/3}\text{fm}\).
$pp$ scattering lead to the estimate that the interaction between protons and neutrons ($V_{pn}^{\tau=1}$) in $\tau = 1$ states are more attractive than the force between the protons ($V_{pp}$) by 2% [145],

$$|V_{pn}^{\tau=1} - V_{pp}| / V_{pp} \sim 2\%.$$  \hspace{1cm} (4.5)

In addition, evidence on the $^1S$ scattering length in the $nn$ system obtained from an analysis of reactions involving two neutrons in the final state leads to the conclusion of charge symmetry, $V_{pp} = V_{nn}$, namely there is no difference (within 1%) between $pp$ and $nn$ interactions [146]. Furthermore, after the Coulomb energy is taken into account the discrepancy in the isobaric multiplet energies is bigger for the seniority zero ($\nu = 0$) levels as compared to $\nu > 0$ states indicating the presence of a short range charge dependent interaction [147].

"The problem of broken symmetry is one of general significance in nuclear and elementary particle physics" [85] (Vol. I, p.37), which needs special attention and may be associated with novel and interesting physics. Charge dependent but charge symmetric nucleon-nucleon interaction brings into the nuclear Hamiltonian a small isotensor (of rank two) component (with zero third isospin projection so that the Hamiltonian commutes with $\tau_0$). In our Sp(4) model this is achieved by introducing the two additional terms,

$$H_{IM} = (G - F)A^+_0 A_0,$$
$$H_{INC} = (D - \frac{E}{2\Omega})(\tau^2_0 - \frac{\hat{N}}{4}),$$  \hspace{1cm} (4.6)

to the isoscalar (isospin conserving $(IC)$) part (with $G = F$ and $D = \frac{E}{2\Omega}$) in the model Hamiltonian (3.14). As we have already mentioned, the first correction, in contrast to the second (isospin non-conserving $(INC)$) one, introduces small isospin mixing (IM).

Another source of isospin mixing is the component of the nuclear interaction that scatters a $pn$ isovector ($\tau = 1$) pair into a $pn$ isoscalar ($\tau = 0$) pair, which leads to an additional (much smaller than the one induced by the Coulomb interaction) isovector-isoscalar mixing of the near-lying $\tau = 0$ and $\tau = 1$ states in odd-odd nuclei (as mentioned before, for even-even nuclei both isovector and isoscalar $pn$ couplings yield a $\tau = 0$ state). As the diagonal component of this interaction is already included in the Sp(4) model Hamiltonian (isoscalar force proportional to $\tau^2$), the effect of its off-diagonal part on the energy when treated as a perturbation will be of second order, while the isospin mixing due to $H_{IM}$ brings in the energy a first-order correction. Hence, even though both the strength of the perturbation interaction, $H_{IM}$ in (3.14), and the corresponding mixing of isospin eigenstates may be smaller than the ones in the case of isovector-isoscalar mixing, the nuclear energies are affected by the $H_{IM}$ correction, which cannot be simply attributed to the much smaller (by more than an order of magnitude) second-order energy correction due to $(\tau = 0) - (\tau = 1)$ mixing.

At the same time, one should be aware that if the Coulomb energy correction is not well-determined, the $H_{INC}$ isospin symmetry breaking term proportional to $\tau^2_0$ in the nuclear Hamiltonian (3.14) will actually detect the $i^2$ trend of the additionally needed fine adjustments due to the Coulomb interaction, which will be absorbed in the $(D - \frac{E}{2\Omega})$ param-
eter. In this way, the estimate of the pure nuclear isospin non-conserving (but not $\tau$ mixing) interaction strength may not be free of this type of uncertainties.

There is also a possible scenario where the $H_{IM}$ correction (4.6) only (or partly) simulates an additional energy splitting of the isobaric analogs that has the energy functional dependence close to the one produced by $H_{IM}$, but does not induce isospin mixing effects. Therefore, such an interaction is expected to give a (binding) energy increase primarily around $i = 0$ as $H_{IM}$ is weakened after $Z - N$ becomes non-zero reflecting the characteristic feature of the $pn$ component of the isovector pairing (as it will be illustrated in Section 4.4). The nature of the simulated isospin non-mixing interaction can be understood in the following way. In the $su^0(2)$ limit, the energies of $H_{IM}$ for $n$ and $i$ fixed (given nucleus) are expressed as 

$$
\varepsilon_{pn} = \frac{G - F}{8\Omega} (n - 2(|i| + 2k))(4\Omega - n - 2(|i| + 2k) + 2), \quad k = 0, 1, 2, ..., \frac{1}{2}(\tilde{n} - |i|), \quad (4.7)
$$

where we use the fact that the minimum value of the $\nu_0$ seniority number is $|i|$ and it varies by 2 (Table 2.4). This implies that $H_{IM}$ brings into the total energy of the isospin eigenstates (2.25) (which are also eigenstates of the model Hamiltonian (3.14) with $G = F$) a dependence on $i^2$ and $|i|$. Therefore, as the $i^2$ dependence has already been given a degree of freedom via the $D$ parameter, an isospin breaking nuclear interaction proportional to $|i|$ may be the one needed to describe the nuclear energies instead of $H_{IM}$. It is clear that such an interaction does not mix isospin (and hence the model Hamiltonian commutes with $\tau$) but has properties similar to the $pn$ isovector interaction, namely its energy magnitude increases toward $N = Z$ (for a negative coupling strength parameter). However, as pointed out in the previous Chapter, such an interaction has its origin in the so-called Wigner energy, which is already included in the original Hamiltonian as the term that is linear in $\tau$ in the $\tau(\tau + 1)$ symmetry energy (due to the $\tau^2$ interaction). Hence, any additional $\tau$ dependence will lead to the energy correction $\Delta \varepsilon_W$,

$$
-\frac{E}{2\Omega} \tau(x + \tau) = -\frac{E}{2\Omega} \tau^2 - \frac{E}{2\Omega} x\tau = \varepsilon_{sym} - \frac{E}{2\Omega} \tau(x - 1),
$$

where $-\tilde{W}_\tau$ together with the $\tau$ dependent part of the isovector pairing energy is the Wigner energy [134, 148, 29] and $x$ is the additional degree of freedom. The Wigner energy is associated with an extra binding energy at $N = Z$ and is usually parametrized as $-W(A)|N - Z| = -W(A)2|i|$. The $\tau(\tau + x)$ dependence (4.8) is investigated in many mass formulae, where $x$ is allowed to take different values. Originally, Wigner [134] proposed $x = 4$ based on the SU(4) spin-isospin symmetry of nuclear forces, which however is severely broken throughout the nuclear chart except probably in the lightest nuclei. Yet, the observed energy systematics of isobaric analog states is best explained by $x = 1$ [149]. Recent analysis of binding energies for nuclei with masses $10 \leq A \leq 64$ illustrates that the data is distributed somewhat between $x = 0$ and $x = 4$ with $x = 1$ being the optimum value to give a good
account of the overall experimental data [125]. Values slightly greater than one \((x > 1)\) are probably the most feasible [126]. At the same time the Wigner energy includes an additional term for \(N = Z\) odd-odd nuclei, \(-W(A)|N - Z| - d(A)\delta_{NZ}\delta_{Z,odd}\), where \(d(A)\) is a parameter in a close relation to the \(W(A)\) strength\(^4\). In the framework of the Sp(4) model, these nuclei also have the additional \((d-)\)term in the Wigner energy, \(-\tilde{W}\tau\), since for them \(\tau = |i| + 1\), but it is possible for the absolute value of its strength to be reduced with respect to \(|\tilde{W}|\) (as both \(\tilde{W}\) and \(d\) are negative, a smaller \(|d|\) value implies an increase in the (binding) energy). Even now, since such a correction is needed for the \(N = Z\) odd-odd nuclei only, its overall effect is probably suppressed to some extent due to the good description of the even-even nuclei. If the corrections in the Wigner energy \((d\)-term and/or \((4.8)\)) were a reality in nuclei in place of \(H_{IM}\), any isospin mixing produced by the interaction \((3.14)\) (with \(G \neq F\)) would be artificial. However, as \(x\) is very close to unity and \(d\) does not bring a substantial change, it is more likely that the isospin mixing \(H_{IM}\) interaction really exists although may be overestimated if a term of the form \(\Delta\varepsilon_{W} = -\tilde{W}\tau\frac{\gamma - 1}{2}\) \((4.8)\) in addition to a constant term for \(N = Z\) odd-odd nuclei are neglected. This scenario may take place for medium nuclei like those with nucleons filling the \(1f_{\frac{3}{2}}2p_{\frac{1}{2}}2p_{\frac{3}{2}}1g_{\frac{3}{2}}\) major shell. A further discussion on the isospin mixing and its effects will be carried throughout the rest of this Chapter.

In what follows the solutions to the eigenvalue problem \(H|0^+; n, i\rangle = E_0|0^+; n, i\rangle\) and to its \(q\)-deformed analog \(H_q|0^+; n, i\rangle_q = E_0^q|0^+; n, i\rangle_q\) are linked to experiment. The Coulomb corrected energies of the nuclear \textit{isovector-paired} states are estimated by the model Hamiltonian eigenvalues. The wave functions of these \(0^+\) states are represented to a good approximation by the relevant Hamiltonian eigenvectors \([\text{with an (almost) good isospin quantum number}]\), which, in general, are linear combinations of the pair basis \((2.19)\) for \(n\) and \(i\) fixed

\[
|0^+; n, i\rangle = \frac{1}{\mathcal{N}} \sum_{\nu_1 = n_0} C_{\nu_1} |n+1, n_0, n-1\rangle
\]

with a normalization coefficient \(\mathcal{N}\) and weight coefficients \(C_{\nu_1}\). The latter depend on the effective interaction parameters, \(\gamma\) \((3.22)\) \((\gamma_q (3.21))\).

---

\(^4\)Most of the studies that include an investigation of the Wigner term consider binding energies, for which practically all odd-odd nuclei (and of course, all even-even) have \(\tau = |i|\) isospin and almost all \((A > 40)\) odd-odd \(N = Z\) nuclei have \(\tau = |i| + 1\) isospin in their ground state. In the sophisticated semi-empirical analysis [29] of nuclear masses, which makes use of a wide-ranging data compilation, the Wigner energy is defined as \(-W(A)(|N - Z| + \delta_{NZ}\delta_{Z,odd})\), \(W = -30/A\) MeV. The Wigner energy can be also recognized as a part of a \textit{congruence} energy introduced in the Thomas-Fermi model as an exponential form \(-10e^{-0.2|N - Z|/A}\) MeV (to prevent the term of becoming positive) with an expansion for small \(i\), \(\sim -10 + 42|N - Z|/A\), and phenomenological constants \((W = -42/A\) MeV) determined from a large-scale experimental data [148], where the value of \(d = -30/A\) MeV has been used. A similar result, \(W \approx d\), but including a significantly smaller sample of \(\tau = 0\) states in \(N = Z\) odd-odd is reached in [128].
4.2 Estimate for the Model Interaction Parameter: Lowest Isovector-Paired $0^+$ State Energy for Even-$A$ Nuclei

In our investigation, four groups of even-$A$ nuclei in the mass range $32 < A < 164$ are considered: (I) $1d_{5/2} (\Omega = 2)$ with a core $^{32}{\text{S}}$ (Table 3.1); (II) $1f_{7/2} (\Omega = 4)$ with a core $^{40}{\text{Ca}}$ (Table 3.2); (III) $1f_{5/2}2p_{3/2}2p_{1/2}1g_{7/2}$ major shell ($\Omega = 11$) with a core $^{56}{\text{Ni}}$; and (IV) $1g_{7/2}2d_{5/2}2d_{3/2}3s_{1/2}1h_{11/2}$ major shell ($\Omega = 16$) with a core $^{100}{\text{Sn}}$. In each group, the number of the valence protons (neutrons) varies in the range $N_\pm = 0, \ldots, 2\Omega$ and the total number of nuclei that enter into the Sp(4) systematics is $2\Omega(\Omega + 1) + 1$ (13 for (I), 41 for (II), 265 for (III) and 545 for (IV)).

The phenomenological model parameters $\gamma$ (3.22) and $\gamma_q$ (3.21) are determined by a non-linear least-squares fit of the $E_0$ theoretical energies (maximum eigenvalues of $|H|$ (3.14) or $|H_q|$ (3.20)) to the $E_{0,\exp}$ Coulomb corrected experimental energies of the lowest isovector-paired $0^+$ states. In this procedure, the first three groups of nuclei are considered, (I)-(III). As an optimization problem, the fitting procedure minimizes the residual sum of squares (over the statistics data),

$$S = \sum_{N_+,N_-} (E_0(N_+,N_-) - E_{0,\exp}(N_+,N_-))^2,$$

where $E_0(N_+,N_-)$ is a non-linear function of the $\gamma_q$ parameters, which we assume to be constant for all nuclei within a major shell. Specifically, the non-linearity is in $G_q$, $F_q$ and $E_q$ involved in the diagonalization of the model interaction, and in the $q$ parameter in the case of the deformed model. The chi-statistics,

$$\chi = \sqrt{\frac{S}{N_d - n_p}},$$

defines the goodness of the fit, where $n_p$ is the number of the fitting parameters and $N_d$ is the number of data cases. In each group investigated, the number of nuclei with available data is $N_d = 13$ in (I), $N_d = 36$ in (II) and $N_d = 100$ in (III). The uncertainties of the experimental energies are not included as their measurement is very precise in the cases considered. For the nuclei that enter the statistics, the experimental energy errors are on average $\approx 4$ keV, 50 keV, 100 keV (for the three regions, respectively), which is in powers of ten less than the magnitude of the energies (several hundreds MeV). Moreover, the greatest uncertainty in $E_{0,\exp}$ among the data sets of each of the three groups of nuclei corresponds to a relative error of 0.01% (I), 0.11% (II) and 0.17% (III).

Table 4.1 shows the parameters and statistics, obtained from the fitting procedure in both the non-deformed (“non-def” column) and deformed cases (“$q$-def” column). The quantities marked with the symbol * are not varied in a given fit and their number is not included in the $n_p$ number of parameters: $n_p = 6$ in all the non-deformed fits and $n_p = 1$ in the
Table 4.1: Fit parameters and statistics. \( G, F, C, D, \epsilon \) and \( \chi \) are in \( \text{MeV} \), \( S \) is in \( \text{MeV}^2 \). Quantities marked with the symbol * are fixed for a given fit.

<table>
<thead>
<tr>
<th></th>
<th>(I) ( 1d_{\frac{3}{2}} )</th>
<th>(II) ( 1f_{\frac{7}{2}} )</th>
<th>(III) ( 1f_{\frac{7}{2}} 2p_{\frac{1}{2}} 2p_{\frac{3}{2}} 1g_{\frac{7}{2}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>non-def</td>
<td>q-def</td>
<td>non-def</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0*</td>
<td>-0.015</td>
<td>0*</td>
</tr>
<tr>
<td>( q = e^\kappa )</td>
<td>1*</td>
<td>0.985</td>
<td>1*</td>
</tr>
<tr>
<td>( G/\Omega )</td>
<td>0.709</td>
<td>0.709*</td>
<td>0.525</td>
</tr>
<tr>
<td>( F/\Omega )</td>
<td>0.702</td>
<td>0.702*</td>
<td>0.453</td>
</tr>
<tr>
<td>( C )</td>
<td>0.815</td>
<td>0.815*</td>
<td>0.473</td>
</tr>
<tr>
<td>( D )</td>
<td>-1.282</td>
<td>-1.282*</td>
<td>-0.971</td>
</tr>
<tr>
<td>( E/(2\Omega) )</td>
<td>-1.409</td>
<td>-1.409*</td>
<td>-1.120</td>
</tr>
</tbody>
</table>

\( q \)-deformed fits. First, we focus our attention only on the “classical” results, which will be followed by a detailed discussion on the \( q \)-deformed outcome.

### 4.2.1 Analysis in the Non-Deformed Limit

In all three cases of the “classical” model, there is a good agreement with experiment (small \( \chi \)-statistics), as can be seen in Table 4.1, as well as in Figure 4.1 for the isobars \( A = 40 \div 56 \) in the \( 1f_{\frac{7}{2}} \) level (region (II)). The theory predicts the lowest 0\(^+\) isovector-paired state energy of nuclei with a deviation \( \chi/\Delta E_{0,\text{exp}} \times 100[\%] \) of 0.7\% for (I) and 0.5\% for (II) and (III) in the corresponding energy range considered, \( \Delta E_{0,\text{exp}} \).

The estimate for the parameters (Table 4.1) reveals the properties of the nuclear interaction as interpreted by the connection (3.13). The \( J = 0 \) pairing interaction \( \langle V_{\text{pair}} \rangle \) is always attractive \( (G, F > 0) \), while the overall high-\( J \) component of identical-nucleon coupling \( \langle \pm |V| \pm \rangle \) might be repulsive \( (C + D/2) \). The \( J > 0 \) proton-neutron “direct” interaction \( \langle - |V| + \rangle \) is attractive, but not the “exchange” part of it \( \langle - |V| - \rangle \) \( (E < 0) \).

There are many different values for the like-particle pairing strength used in literature. The most common value is taken to be proportional to \( 1/A \), typically, by a constant factor \( 22/A \div 27/A \) \([77, 103, 84, 85, 150]\) or by a functional dependence on \( Z \) and \( N \) \((F_{p(n)} \) for proton (neutron) pairing strength) \([151]\),

\[
F_p = \frac{17.90 + 0.176(N - Z)}{A}, \quad F_n = \frac{18.95 + 0.078(N - Z)}{A}, \quad Z < 88, \quad (4.12)
\]

and is consistent with the experimental pairing gaps derived from the odd-even mass differences \([152, 86]\). The values of \( F \), obtained by our Sp(4) model, fall within the limits of their
Figure 4.1: Coulomb corrected $0^+$ state energy, $E_0$, in MeV versus the isospin projection $i$ for the isobars with $A = 40$ to $A = 56$ in the $1f_{7/2}$ level, $\Omega^+ = 4$. The experimental binding energies $E_{\text{BE,exp}}$ (symbol “×”) are distinguished from the experimental energies of the isobaric analog $0^+$ excited states $E_{0,\text{exp}}$ (symbol “◦”). Each line connects theoretically predicted energies of an isobar sequence.

Isospin Invariance Breaking $D \neq E/(2\Omega)$ and Isospin Mixing $G \neq F$

Based on our estimate for the parameters (Table 4.1) and the correlations (3.15), the extent to which the symmetry in each limit ($\mu = \tau, 0, \pm$) is broken can be evaluated. In the $\text{SU}^+(2)$ limit, the breaking of the isospin invariance $\eta_2/\eta_1$ is in general small for light nuclei ($\eta_2/\eta_1 = 0.090$ for $1d_{3/2}$ and $\eta_2/\eta_1 = 0.133$ for $1f_{7/2}$). For medium nuclei in the $1f_{7/2} 2p_{1/2} 2p_{3/2} 1g_{9/2}$ major shell the isospin breaking is significantly greater, $\eta_2/\eta_1 = 0.628$. 

estimation. In this way, they are expected to reproduce the low-lying vibrational spectra of spherical nuclei in the $\text{SU}^+(2)$ limit of the model.

Since the Wigner energy ($-W^2\tau$) is implicitly included in the $\text{Sp}(4)$ energy estimation as the linear in $\tau$ term in the $pn$ isoscalar force (proportional to the symmetry term) and in the isovector pairing through the second-order Casimir invariant of $\text{sp}(4)$ (2.12), its interaction strength parameter is expressed through the model parameter $W = E - F_{4\Omega}$. Its estimated values from the three regions (I), (II) and (III) are found to lie on a curve $W = -(31 \pm 2)/A$ with a very good correlation coefficient $R^2 = 0.96$ and a remarkably close value to most other estimates: $W = -30/A$ [29], $W = -37/A$ [133], $W = -37.4/A$ [137], $W = -42/A$ [148] and $W = -47/A$ [126].
Analysis of the results (Table 4.1) shows that for $1d_{5/2}$ the pairing parameters are almost equal ($G \approx F$) as it is expected for light nuclei (good isospin symmetry), and they differ for $1f_{7/2}$ by 0.07 and for $(III)$ by 0.06, $pm$ isovector strength being more attractive. The small difference between both parameters establishes a rather good approximation if they are considered equal (enforcing charge independence), which leads to a great simplification of the pairing problem. That is why most of isovector ($pp$, $pn$, $nn$)-coupling study has been done assuming good isospin, that is $G = F$. An investigation along isospin violating $G > F$ effects has been approached in other studies [118, 111] mainly using phenomenological arguments from comparison to experimental energies, as well as an isospin non-conserving $\tau_0^2$-like nuclear interaction has been originally suggested in [147].

The questions regarding how strong the isovector $pn$ strength really is and to what extent the isospin symmetry is broken by a pure $\tau_0^2$ nuclear interaction (4.6) remain still open – there are no sharp answers at the present level of experimental results and microscopic theoretical interpretations. The only related experimental fact presently available emerges from the free nucleon-nucleon data [145], which establishes the charge dependence of the nuclear force (Section 4.1), that is, the interaction in the $pn \tau = 1$ system is slightly (by 2%) more attractive than the one in the $pp$ system (4.5). A consistency check for such charge dependence (4.5) in the Sp(4) model can be found in the comparison of the $i = 0$ two-body model interaction, $H(n, \tau = 1, i = 0; \varepsilon = 0)$ (3.14), to $H(n, \tau = 1, i = 1(-1); \varepsilon = 0)$ in the $\tau = 1$ multiplets,

$$
\frac{H(n, \tau = 1, i = 0; \varepsilon = 0) - H(n, \tau = 1, i = 1; \varepsilon = 0)}{H(n, \tau = 1, i = 1; \varepsilon = 0)} \times 100\% ,
$$

(4.13)

which, for example in the $1f_{7/2}$ shell, is on average $5 \sim 2.5\%$. At the same time, the charge dependence experimental result cannot shed a light on the individual strengths of the $H_{IM}$ and $H_{INC}$ interactions (4.6) because their simultaneous contributions (along with the rest components of the nuclear interaction) determine the overall interaction between both nucleons.

While the $(II)$ case of $1f_{7/2}$ yields an excellent and consistent estimate for the interaction strengths and energy prediction, it is possible that the isospin-violating strengths, $G - F$ and $D - E/(2\Omega)$, for the $1f_{7/2}2p_{3/2}2p_{1/2}1g_{9/2}$ major shell are slightly overestimated – due to small correction to the $\tau$-dependent nuclear interaction (4.8) in addition to the isospin mixing $H_{IM}$ (4.6) and due to fine Coulomb interaction adjustments, respectively (as we explained in detail in Section 4.1). This is combined with other effects due to the non-degeneracy of the orbitals and non-negligible shape deformation in this major shell. The latter may be debated as being responsible for an artificial non-zero ($G - F$) difference in $H_{IM}$ due to an increase in (binding) energies of the deformed nuclei since a $Q \cdot Q$ interaction ($Q$ is the quadrupole moment operator) is missing from the Hamiltonian (3.14). Such shape deformations are typically bigger towards half-filled major shells, where the number of nucleons is large enough to

\footnote{This estimation does not aim to confirm the charge-dependence, which is very difficult at this level of accuracy. It only reflects the fingerprints of the experimental data in the properties of the model interaction.}
develop collectivity. In order to investigate the qualitative way this collective mode influences our model, we examine the energy differences, \( E_0 - E_{0,\text{exp}} \) (model predictions compared to the available experimental values), for the nuclei in the \( 1f_2^2 2p_1^2 2p_3^1 1g_2^2 \) shell (Figure 4.2). In the case when \( G = F \) (a), it turns out that the region of deformed nuclei (around \( N_\pm = \Omega \)) is explained well by the model. The reason may be hidden in the fact that the overall additional energy due to shape deformations is absorbed in the mean-field parameters of the model. This may be the explanation why the semi-magic nickel (Ni) isotopes \((N_+ = 0)\) are overestimated by the Sp(4) model, namely the neutrons in these nuclei are likely to feel a weaker average potential than the one estimated by the model. Since the effects due to shape deformation are somehow hidden in the mean-field then the nuclear energies for all the nuclei in the mid-shell region, in the case of \( G = F \) (Figure 4.2(a)), should be affected in a continuous way (such a smooth pattern is empirically observed in the behavior of the energies of the first \( 2^+_1 \) states). This is why the sudden change in \( E_0 - E_{0,\text{exp}} < 0 \) observed near \( N = Z \) cannot be explained due to the fact that the model lacks the \( Q \cdot Q \) interaction and if it was included the bigger \( E_0 - E_{0,\text{exp}} < 0 \) difference in the \( N = Z \) nuclei would disappear or decrease. At the same time, when the \( H_{IM} \) interaction is ‘turned on’ (Figure 4.2(b)), an increase of the energies along the \( N = Z \) line yields a better agreement to experiment leaving everything else almost unchanged, as it is expected (Section 4.1). Although the lack of \( Q \cdot Q \) in the Hamiltonian is a limitation of the model (yet a reasonable approximation) the slight overestimate for \( G = F \) is more likely, as suggested in Section 4.1, to be due to small corrections in the Wigner energy \( (\sim \tau) \) that does not mix isospin values.

Figure 4.2: The difference, \( E_0 - E_{0,\text{exp}} \), relative to the maximum range considered \( \Delta E_{\text{max}} = E_{Z=50,N=50} - E_{Z=28,N=28} \), between the theoretically predicted and experimental energies for the even-even nuclei with available data in the \( 1f_2^2 2p_1^2 2p_3^1 1g_2^2 \) major shell \((^{56}\text{Ni core})\) in the cases of (a) \( G = F \), and (b) \( G \neq F \). The other parameters are kept fixed with values given in Table 4.1 (II).

The reasons stated suggest that we should not pursue a sharp estimate for the isospin mixing in the states for the \( 1f_2^2 2p_1^2 2p_3^1 1g_2^2 \) shell. Nevertheless, the very good agreement of
the energy function with experiment in this region (Table 4.1, (III)) and the fundamental ideas behind the construction of the model interaction suggests that we can use the model Hamiltonian (3.14) for the energy estimation and basic interpretation of phenomena observed in the region for medium nuclei. Further tests will confirm this suggestion.

The outcome of the fits presented in Table 4.1 should not to be overlooked – the freedom allowed by introducing additional parameters (as $G$ and $D$) reflects the symmetries observed in light nuclei (good isospin) and the comparatively larger symmetry-breaking as expected in medium-mass nuclei. Hence, the charge dependence of the nuclear force, being a very challenging problem, yields results qualitatively consistent with the physical reality.

**Smooth Functional Dependence on Nuclear Mass**

When the results from all the three non-deformed fits of a small uncertainty are considered (Table 4.1), the values for the pairing strengths are found to lie on a curve that decreases with nuclear mass (Figure 4.3)

\[
\begin{align*}
\frac{G}{\Omega} &= \frac{25.7 \pm 0.5}{A}, \quad R^2 = 0.99, \\
\frac{F}{\Omega} &= \frac{23.9 \pm 1.1}{A}, \quad R^2 = 0.96,
\end{align*}
\]

where $R^2$ is a coefficient of correlation and represents the proportion of variation in the strength parameter accounted for by the analytical curve. This allows for their further prediction for the region of the $1g_{\frac{3}{2}}2d_{\frac{5}{2}}2d_{\frac{3}{2}}3s_{\frac{1}{2}}1h_{\frac{11}{2}}$ major shell

(IV) : \[
\frac{G}{\Omega} = 0.194 \text{ MeV and } \frac{F}{\Omega} = 0.181 \text{ MeV},
\]

where we use an average nuclear mass of $\bar{A} = 132$ (determined as the mean of the mass numbers of the nuclei at both closed shells, $^{100}$Sn and $^{164}$Pb). For this group of nuclei the energy spectrum of $Z \approx N$ and proton-rich nuclei is not yet measured and the available data is not sufficient to determine such parameters as $G$ that decreases rapidly with proton (neutron) excess. However, once $G$ and $F$ are estimated (4.16) the available data in this region is suitable to be used in a fit to determine the rest of the parameters

(IV) : \[
C = 0.142 \text{ MeV}, \quad D = -0.484 \text{ MeV}, \quad \frac{E}{2\Omega} = -0.702 \text{ MeV},
\]

\[
\varepsilon = 10.886 \text{ MeV} \quad (S = 392.505 \text{ MeV}^2, \quad \chi = 1.924 \text{ MeV}),
\]

which together with the values obtained for the other three regions, (I), (II) and (III) (Table 4.1), show also an overall dependence on the mass $A$

\[
\begin{align*}
\frac{E}{2\Omega} &= \frac{-52 \pm 5}{A} \frac{A}{(R^2 = 0.75)}, \\
D &= \frac{-37 \pm 5}{A} + (-0.24 \pm 0.09) \frac{A}{(R^2 = 0.97)}, \\
C &= \left( \frac{32 \pm 1}{A} \right)^{1.7 \pm 0.2} \frac{A}{(R^2 = 0.99)}.
\end{align*}
\]
The existence of such a smooth functional dependence of the interaction strength parameters on the nuclear mass $A$ reveals their _global_ behavior, namely the interactions in the model Hamiltonian (3.14) are related to global behavior common to all nuclei.

### 4.2.2 Comparisons to Other Theoretical Models

The model with a symplectic Sp(4) dynamical symmetry gives a good estimate for the relevant experimental $0^+$ state energies with an indication for this being the small value of $\chi$-statistics (Table 4.1 and Figure 4.1 for region (II)). Part of our results, namely for the binding energies (but not for the excited $0^+$ state energies), can be compared to other theories. A direct comparison of the chi-statistics is impossible because of the different data sets and energy levels determined by the various theories. However, if we select only the data subsets that are equivalent for the nuclei in the $1d_\frac{3}{2}$ and/or $1f_\frac{7}{2}$ orbits, our results are much closer to the experimental numbers than those for the Hartree-Fock-Bogoliubov ($HFB$) model [28] and the semi-empirical model [29] and comparable with those of the $jj$-coupling shell models of [27, 26], based on the conventional seniority method, and of [30], the so-called isovector and isoscalar pairing plus quadrupole model. Comparing the strength...
parameters of the common components of the model interaction in our Sp(4) model and in the \( P + Q \cdot Q \) model [30], the values of the parameters lie close to each other for the single \( j = 7/2 \) level, which is a further test of the simple Sp(4) approach. In short, the Sp(4) model stands in a good position among other models when applied to light nuclei in single-\( j \) levels. In this region, many symmetries are conserved (as seen also in Table 4.1, (I) and (II)) allowing for a possible reduction of the number of fitting parameters and yet increasing the goodness-of-the-fit measure, \( \chi \), of our model. Here once again the asset of the simple group theoretical concept based on the Sp(4) symmetry is revealed in easily reproducing results of more complicated approaches, such that involve higher-rank groups or extensive shell-model calculation in a large model space. The simplicity allows for novel investigations and multi-\( j \) generalizations.

Based on the conventional seniority method (related to Sp(2\( j + 1 \))), Talmi has derived a binding energy formula (for the lowest seniority, \( \nu = 0 \) \( t = 0, J = 0 \) and \( \nu = 1 \) \( t = 1/2, J = j \)) [26, 83, 27]

\[
Cn + \alpha \frac{n(n-1)}{2} + \beta (\tau(\tau+1) - \frac{3}{4}n) + \gamma \lfloor \frac{n}{2} \rfloor,
\]

(4.21)

which includes a symmetry term (\( \sim \tau(\tau+1) \)) and a very simple pairing term proportional to the number of pairs (\( \lfloor \frac{n}{2} \rfloor \) gives the smaller of the integer numbers that bracket \( \frac{n}{2} \)). The form of (4.21) is a very good illustration of the complementarity of both methods based on Sp(2\( j + 1 \)) (4.21) and Sp(4) (3.14), while the simplicity of the latter gives a more detailed insight into the isovector pairing correlations, in accordance with the discussion in Section 3.1.1. When the Coulomb energy repulsion is taken away, (4.21) has been applied to the \( 1d_{3/2} \) and \( 1f_{7/2} \) orbits. Talmi himself warns about the values of the parameters obtained in their fit: “the value of \( \gamma \) yields the expected energy spacing to be equal to 0.8\( \gamma = 2.83 \) MeV. This is considerably higher than the measured values.” ([27], p. 562). In comparison, the approach based on the Sp(4) quasi-spin formalism yields an estimate for the pairing strength parameters that falls in the limits of the usually used pairing parameters consistent with the experimental pairing gaps. In addition, the simple two-body Sp(4) (“classical”) pairing model gives a very reasonable estimate for the rest of the two-body nuclear interaction strengths (Table 4.1). Besides, the challenging problem of the isospin symmetry breaking follows naturally from the development of the dynamical symmetry concept for the Sp(4) group.

4.2.3 Deformed Non-Linear Model

To explore the physics of \( q \)-deformation, we again perform the fitting procedures for the same regions ((I), (II), (III) and (IV), Table 4.1) but using the deformed Hamiltonian (3.20) and the asymmetry \( \tilde{\rho} = \rho_+ \) factor (2.69). This was carried out in two steps. First, for each group of nuclei, a fit was made with the \( \gamma_q \) (3.21) and \( q \) parameters allowed to vary. The \( \gamma_q \) set that was found differed very little from that of the non-deformed case, \( \gamma \) (3.22), shown in Table 4.1. In short, varying the deformation parameter affects the interaction strengths very little.
This means that, in the “classical” picture, the two-body nuclear interaction strengths, $\gamma$, can be assigned the best-fit global values for the model space under consideration without compromising overall quality of the theory. The same values, by virtue of $\gamma_q$ being very close to $\gamma$, are assigned to the interaction strength parameters in the $q$-deformed model. In this way, the corresponding $q$-deformed Hamiltonian possesses a precious asset, namely it contains in itself exactly the two-body “classical” Hamiltonian (3.14). This is because the zeroth order of each of the interactions in $H_q$ (3.20) is the corresponding two-body interaction and the strength of the latter is the same as in $H_{cl}$ as we assign $\gamma_q = \gamma$ (due to the observed decoupling of $q$ from $\gamma$). In a word, the observation that the $q$-deformation does not vary the two-body interaction strength underscores the fact that the deformation represents something fundamentally different, a feature that cannot be “mocked up” by allowing the strengths of the non-deformed interaction to absorb its effect.

Based on this result, we considered the deformation to be independent of the other parameters and as a second step, we varied only $q$ in the fit (the rest of the parameters were kept fixed with values obtained from the non-deformed fit). The results are shown in the “$q$-def” columns in Table 4.1. The fits with and without a deformation can be compared by using the residual sum of squares ($S$), which is always smaller in the deformed case (Table 4.1). The overall $1/A$ dependence of the $q$-parameter also holds as for the interaction strength parameters (Figure 4.3),

$$(-9 \pm 5)/A + (1.3 \pm 0.1) \quad (R^2 = 0.6),$$

although its dependence on $A$ is likely to be more complicated (comparatively small correlation coefficient, $R^2$). The value of $q$ used for the (IV) region is obtained in a fitting procedure, $q = 1.137$ ($\kappa = 0.128$), with $S = 355.998$ MeV$^2$ and $\chi = 1.807$ MeV.

Although it stands in contrast with other $q$-deformed applications [31, 33], the decoupling of the $q$-deformation from the interaction strengths is not an assumption but results from comparisons to experimental data over total of 149 nuclei. It implies that while leaving the strength of the two-body interactions unchanged, the $q$-deformation allows one to take into account, in a prescribed way, complicated higher-order dependence of the energy eigenvalues on the number of nucleons/pairs that cannot be reproduced by any two-body interaction (for example, see (3.35) and (3.36)). Moreover, similar terms are expected to arise from many-body interactions between the particles. In this way the $q$-parameter introduces some non-linear residual interaction not present in the two-body Hamiltonian (3.14).

The observed independence of the pairing strengths on the $q$-parameter suggests that while the deformation does not change the strength to couple two particles, it can model many-pair effects and can influence the energy spectrum [153]. As an illustration, in each of the dynamical limits we investigate the quantities $R_{pn} = \varepsilon_{pn}^q/\varepsilon_{pn}$ and $R_{pp+nn} = (\varepsilon_{pp}^q + \varepsilon_{nn}^q)/(\varepsilon_{pp} + \varepsilon_{nn})$ that give an additional contribution to the pairing energy in the deformed case$^6$ (Figure 4.4) (compare to the analytical expansion with respect to $\kappa$ of the energies, (3.35) and (3.36)). In the limit of $pn$-pairing, $R_{pn}$ does not significantly change when $q$ is close

$^6$In the case of a symmetric $\tilde{\rho} = \rho$ factor the dependence of $R_{pn}$ ($R_{pp+nn}$) on $\kappa$ decreases (increases) with increasing $|\kappa|$. 

81
to unity and it decreases for all $q \neq 1$. The ratio $R_{pp+nn}$ increases (decreases) monotonically with $q$ only for nuclei with a primary $pp$ ($nn$) coupling. Regarding the pairing interaction only, in the limit of identical-particle coupling, when $q$ increases from unity ($q > 1$) neutron pairs are less bound and proton pairs give a larger pairing gap, and vice versa for $q < 1$.

In this way, the deformation parameter can determine the degree to which the $pp$ isovector coupling differs from the $nn$ one.

Figure 4.4: Non-symmetric ratios $R_{pn}$ and $R_{pp+nn}$ as functions of the $q$-parameter for several nuclei with a typical behavior in the $1f_2$ level.

Despite the close similarity between the $\mathfrak{su}_+^q(2)$ and $\mathfrak{su}_-^q(2)$ algebras, the different behavior of the multiplication asymmetric constants $\rho_\pm$ (Table 2.9) is responsible for different impact of the deformation in various isotopes. This accounts for the differences in the experimental data between mirror nuclei even after the Coulomb energy correction is applied – its origin may be due to additionally needed fine Coulomb corrections or due to a charge asymmetry in the nuclear force. From analysis of the $pp$ and $nn$ scattering, the latter was suggested to be less than a percent, namely $(V_{nn} - V_{pp})/V_{pp} \leq 1\%$ [141], and it is probable its effect to be slightly more feasible when many-body interactions are involved. Indeed, one finds, for example, that in the $1f_2$ level the proton-rich Ni isotopes yield on average the $q$-deformed energy difference (in contrast to the charge-symmetric “classical” Hamiltonian (3.14), where $H(N_-, N_+) - H(N_+, N_-) = 0$)

$$
\frac{H(N_-, N_+ = 8; \varepsilon = 0) - H(N_+ = 8, N_-; \varepsilon = 0)}{H(N_+ = 8, N_-; \varepsilon = 0)} \times 100\% \sim 1\%,
$$

which is not a proof of existing charge asymmetry in the nuclear interaction, as any confirmation is hindered by the high level of accuracy and the presence of charge asymmetric
Coulomb potential, but rather a suggestion for a plausible explanation. Yet, the very small charge asymmetry resulted from the experimental scattering analysis allows one usually to consider the proton-neutron symmetry as a fundamental one and the nuclear force in almost all models is constructed charge-symmetric. That is why, in our further investigation on the role of the $q$-parameter we make use of the symmetric $\tilde{\rho}$ factor (2.69) and retain the charge symmetry in the $q$-deformed case as in the “classical” picture. In this situation, a fit that varies only the $q$-parameter over all nuclei in a shell results in a value of $q$ very close to unity and much smaller than the $q$ value obtained for each region of nuclei with the asymmetry $\tilde{\rho}$ factor shown in Table 4.1. This comes to say that the fully symmetric $q$-deformation cannot add significant improvements as far as the global properties of the nuclear structure is considered.

The significance of the higher-order terms that enter through the $q$-deformed theory can be estimated through a comparison with experiment. In general, the fitting procedures determine values for $\kappa$ (Table 4.1) that are small. The reason may be that while higher-order effects may be significant in an individual nucleus (as related to the local nuclear properties) they probably cancel on average when the $q$-parameter is kept one and the same for all nuclei. This suggests the need for a more elaborate investigation of the role of the $q$-deformation in each nucleus and the relation of the $q$-parameter to the underlying nuclear structure. However, in two of the cases, (II) and (III), it is of an order of magnitude greater than the estimate for other physical applications ([32] and references there) and for the $1f_{7/2}$ shell, our value ($q = 1.132$) is comparable to the values obtained in a $q$-deformed like-particle seniority model [31]: $q = 1.1585$ for the neutron pairs and $q = 1.1924$ for protons. For the nuclei in the multi-$j$ shell our model yields a bigger $q$-parameter than for the lighter nuclei in single-$j$ shell (Table 4.1), where the small number of valence nucleons is not sufficient to build strong non-linear correlations. This suggests that the $q$-deformation is more significant for masses $A > 56$.

4.3 Energy Spectra for the Isovector-Paired $0^+$ States: Predicted Ground and Excited $0^+$ States

The fitting procedure not only estimates the magnitude of the interaction strength and determines how well the model Hamiltonian “explains” the experimental data, it also can be used to predict nuclear energies that have not been measured. From the fit for the $1f_{7/2}$ case the binding energy of the proton-rich $^{48}\text{Ni}$ nucleus is estimated to be 348.19 MeV, which is by 0.07% greater than the sophisticated semi-empirical estimate of [29]. Likewise, for the odd-odd nuclei that do not have measured energy spectra the theory can predict the energy of their lowest $0^+$ isobaric analog state: 358.75 MeV ($^{44}\text{V}$), 359.49 MeV ($^{46}\text{Mn}$), 357.56 MeV ($^{48}\text{Co}$), 394.16 MeV ($^{50}\text{Co}$). The predicted energies are calculated for $q = 1.132$ (Table 4.1 (II)) as the fit with deformation has a smaller uncertainty compared to the non-deformed one. The $\text{Sp}_q(4)$ model predicts the relevant $0^+$ state energies for additional 165 even-$A$ nuclei in the medium mass region (III) plotted in Figure 4.5 (in the “classical” case).
binding energies for 25 of them are also calculated in [29]. For these even-even nuclei, we predict binding energies that on average are by 0.05% (non-deformed case) and by 0.008% (for $q = 1.240$) less than the semi-empirical approximation [29].

Figure 4.5: Theoretical energies including the Coulomb energy contribution of the lowest isovector-paired $0^+$ states for isobars (marked in different colors) with mass number $A = 56, 58, \ldots, 100$ in the $1f_{5/2}^22p_{1/2}^22p_{3/2}^21g_{9/2}$ major shell ($^{56}$Ni core) in comparison to experimental values (black ‘×’) and semi-empirical estimate in [29] (blue ‘×’).

Without varying the values of the interaction strength parameters obtained in the fits of comparatively small residual mean square ($\chi^2$) (Table 4.1), the energy of the higher-lying isovector-paired $0^+$ states can be theoretically calculated and they agree remarkably well with the available experimental values$^7$ (Figure 4.6). This agreement, which is observed not only in single cases but throughout the shells, represents an astonishing result. Since the higher-lying isovector-paired $0^+$ states constitute an experimental set independent of the data that enters the statistics to determine the $\gamma$ model parameters, such an result is, first, an independent test of the physical validity of the strength parameters, and, second, an indication that the

$^7$The energy spectra of nuclei in the (III) and (IV) region with nuclear masses $56 < A < 164$ is not yet completely measured, especially the higher-lying $0^+$ states.
interactions interpreted by the model Hamiltonian are the main driving force that defines the properties of these states. In this way, the simple $\text{Sp}(4)$ model provides for a reasonable prediction of the *isovector-paired* (ground and/or excited) $0^+$ states in proton-rich nuclei with energy spectra not yet experimentally fully explored. The $q$-deformed extension of the model does not change the energy spectra significantly (relatively to the difference between the theoretical and experimental energies) due to the small value of the $q$-parameter, that is kept the same throughout a shell.

![Energy spectra](image)

**Figure 4.6:** Theoretical and experimental (black lines) energy spectra of the higher-lying *isovector-paired* $0^+$ states for isotopes in the $1f_{7/2}$ shell ($^{40}$Ca core). Insert: First excited *isovector-paired* $0^+$ state energy in $^{36}$Ar in the $1d_{3/2}$ shell ($^{32}$S core) in comparison to its experimental value.

Another interesting investigation that reveals the features of the model nuclear interaction follows. It involves discrete approximation of derivatives of the energy function and conveys the idea that a remarkable reproduction of the nuclear energies does not straightaway guarantee agreement of the theoretical energy differences to the experimental ones. This is because the energy differences reflect the fine properties of the nuclear structure filtering out the strong mean-field influence. This study is exclusively focused on the “classical” model since the $q$-deformation considered as a global characteristic (same value for all the nuclei in a shell) adds very little to the analysis. The role of the $q$-parameter for a local improvement within an individual nuclear system will be discussed afterwards.
4.4 Staggering Behavior of the $0^+$ State Energies

The observed staggering of energy levels in atomic nuclei requires a theory that goes beyond mean-field considerations [85]. Staggering data contains detailed information about the properties of the nucleonic interaction and suggests the existence of high-order correlations in the collective dynamics. Most studies of staggering focus on two aspects of the phenomena. There are discrete angular momentum dependent oscillations of physical observables; namely, of $M1$ transitions in nuclei [154] or of the energy levels themselves (e.g., in octupole [155, 156, 157], superdeformed [158, 159, 160], ground and $\gamma$ [85, 161, 162] bands in atomic nuclei, as well as in molecular rotational bands [163]). And then there are sawtooth patterns of different physical quantities (most commonly binding energies) that track with changes in the number of particles in a system (both in nuclei [164] and in metallic clusters [165, 166]).

In nuclear structure physics, staggering behavior of the second type is observed when one changes in a systematic way the usual nuclear characteristics such as proton ($Z$), neutron ($N$), mass ($A$) or isospin projection ($|Z - N|/2$) numbers. Examples of these nuclear phenomena include odd-even mass staggering (OEMS) [85, 167, 168, 169, 170, 171, 172, 173], odd-even staggering in isotope/isotone shifts [174, 175], and zig-zag patterns of the first excited $2^+_1$ state energies in even-even nuclei [176]. The staggering behavior of a nuclear observable is most easily seen when discrete derivatives of second- or higher-order in its variable(s) are considered. The aim of this approach is to filter out the strong mean-field (global) effects and in so doing reveal weaker specific features. In this way, for example, the OEMS, which is usually attributed to the nuclear pairing correlations, manifests itself in certain finite differences of the binding energies that can provide for a measure of the empirical pairing gap [152, 85]. Likewise, various discrete approximation of derivatives (filters) of the binding energies can be considered to investigate detailed properties of the nuclear structure [177, 178, 179, 180, 137, 181, 125, 126].

In this section, we consider the lowest isovector-paired $0^+$ states of even-$A$ nuclei in the mass range $40 \leq A \leq 100$. This includes the binding energies of the $0^+$ ground states of even-even $A$ nuclei and odd-odd nuclei with a $(J^p = 0^+)$ ground state. Our aim is to investigate how various, comparatively small but not insignificant, parts of the interaction between nucleons influence these states when we consider higher-order discrete derivatives of their energies within the framework of the convenient Sp(4) systematics [182, 183, 184].

The symplectic Sp(4) scheme not only allows for a systematic investigation of staggering patterns in the experimental energies of the even-$A$ nuclei, it also offers a simple algebraic model for interpreting the results. Moreover, this detailed investigation serves as a test for the validity and reliability of the Sp(4) model and the interactions it includes.

4.4.1 Properties of the Isovector Pairing Interaction: Staggering Behavior of the Pairing Energies

Before we consider certain finite energy differences that can take away the strong mean-field influence, we first examine in details specific parts of the nuclear interaction by studying
the behavior of the terms included in the model Hamiltonian, particularly the pairing and symmetry term. We have already shown in the previous sections that the Sp(4) model leads to a good reproduction of the experimental energies of the lowest isovector-paired $0^+$ state for even-$A$ nuclei with nuclear masses $32 \leq A \leq 164$, and of the higher-lying isovector-paired $0^+$ states for the $32 \leq A \leq 56$ mass range, where experimental data is available. The good agreement with experiment allows us to relate the various terms that build the model Hamiltonian to the corresponding components of the nuclear interaction that shape the overall dynamics of the nuclear system in the isovector-paired $0^+$ states and hence to investigate the properties of these constituents of the nuclear interaction.

Figure 4.7: Identical particle ($G = 0$, $F = 1.81$ (a)) and non-identical particle ($G = 2.10$, $F = 0$ (b)) maximum pairing energies in the SU$^\pm$ (2) and SU$^0$ (2) limits versus the isospin projection, $i$, for isobars with $A = 40$ to $A = 56$ in the $1f_{7/2}$ level.

The model with Sp(4) dynamical symmetry permits an independent investigation of the $pn$ and like-particle isovector pairing interactions in the limiting cases of the non-deformed, (3.27) and (3.28), [as well as the deformed, (3.35) and (3.36)] version of the theory. In the SU$^\pm$ (2) limit, the symplectic model reproduces the properties of the identical-nucleon pairing ($\epsilon_{pp} + \epsilon_{nn}$) (3.28), for which the usual parabolic dependence of $\epsilon_{pp(nn)}$ on $N_\pm$ holds [16, 77, 27, 87]. For a given nucleus, the maximum value of $\epsilon_{pp} + \epsilon_{nn}$ is the energy of the nuclear ground state in this limit with maximum number of like-particle pairs. The dependence of the maximum like-particle pairing energy on the isospin projection $i$ (Figure 4.7(a)) reveals a distinct $\Delta i = 1$ staggering pattern as one goes from an odd-odd nucleus to its even-even isobaric neighbor. In contrast with this, the $pn$ limit (maximum $\epsilon_{pn}$, corresponding to maximum number of $pn$ pairs in a nucleus) shows a smooth behavior except an $i = 0$ discontinuity (Figure 4.7 (b)). The SU$^0$ (2) limiting case yields a proton-neutron coupling
that has its maximum when $N_+ = N_-$ ($i = 0$), which is consistent with $\alpha$-clustering theories [178, 186, 181, 185] and the increasing role of the $pn$ pairing interaction toward self-conjugate light and medium nuclei where protons and neutrons fill the same shell. In both limits (SU$^\pm(2)$ and SU$^0(2)$), the pairing energy decreases when the difference between proton and neutron numbers increases. The limiting cases correspond only to those of the nuclei in a shell with $i = \tilde{n}/2, \tilde{n}/2 - 1$ ($\tilde{n}$ is a particle-hole conjugate total number of particles, Table 2.4), that is, the last two diagonals that surround the diamond shape of Table 3.2 (Table 3.1).

In most nuclei the $pn$ and like-particle isovector pairing interactions coexist and their mutual influence is expected to alter the limiting behaviors that are described by SU$^0(2)$ and SU$^\pm(2)$ (Figure 4.7). The contribution of each of the pairing modes in the total pairing energy (3.29),

$$\langle H_{\text{pair}} \rangle = \langle H_{pn} \rangle + \langle H_{pp} + H_{nn} \rangle,$$

is given as

$$\langle H_{pp} + H_{nn} \rangle = F \left\langle A_{+1} \right| A_{-1} + A_{-1} \left| A_{+1} \right\rangle$$

$$\langle H_{pn} \rangle = G \left\langle A_{0} \right| A_{0} \right\rangle,$$

where $\langle \ldots \rangle$ denotes the expectation value of an interaction in the eigenstates of the model Hamiltonian (3.14). Specifically, in the present investigation (section) we take these to be
the eigenstates of maximum total energy (as corresponding to the lowest \textit{isovector-paired} \(0^+\) states). When compared to the limiting cases (Figure 4.7), the main features of the concurrent like-particle and \(pn\) isovector pairing interactions are not changed dramatically (Figure 4.8): one can still observe the \(\Delta i = 1\) staggering pattern of the \(\langle H_{pp} + H_{nn}\rangle\) identical-nucleon pairing energies in an isobaric sequence and the increasing \(pn\) pairing energy, \(\langle H_{pn}\rangle\), toward \(N = Z\) nuclei. However, the \(pn\) pairing energy is likely to decrease faster away

![Figure 4.9: Energy contributions to the total lowest \textit{isovector-paired} \(0^+\) state energies versus \(N_\text{val}\) valence neutrons for the \(N = Z\) nuclei in the 1\(f_{7/2}\) level: (a) \(pn\), \(pp + nn\) and total pairing energies; (b) symmetry energy (\(E/2\) term in (3.14)) and pairing + symmetry energy.](image)

from \(i = 0\) as compared to the case when the like-particle pairing is switched off. The difference between proton and neutron numbers \(|Z - N|\) at which the \(pn\) pairing energy decreases almost twice relatively to its maximum value can be estimated by the width, \(w_{pn}\), of a simple Gaussian approximation of the \(\langle H_{pn}\rangle\) energies in an isobaric sequence (Figure 4.8(b)),

\[
E_{pn} e^{-\frac{\chi^2}{2w_{pn}^2}},
\]

where \(E_{pn}\) is the height of the Gaussian function in MeV. For isobars in the 1\(f_{7/2}\) shell, the \(pn\) pairing energies yield the following parameters

\[
\begin{align*}
w_{pn} & = 1.0 \pm 0.2, & E_{pn} & = 2.0 \pm 0.4 \ (R^2 = 0.84), & A & = 44, \\
&w_{pn} & = 0.6 \pm 0.1, & E_{pn} & = 3.9 \pm 0.7 \ (R^2 = 0.80), & A & = 46, \\
&w_{pn} & = 1.37 \pm 0.09, & E_{pn} & = 2.9 \pm 0.2 \ (R^2 = 0.94), & A & = 48.
\end{align*}
\]
The results yield a width $|Z - N|/2 \approx 1$ indicating that in the $1f_{7/2}$ shell the $pn$ pairing energy decreases rapidly after $Z \sim N \pm 2$. A similar rapid decrease of the $pn$ pairing contribution is expected also for the nuclei in the $1f_{7/2} 2p_{3/2} 2p_{1/2} 1g_{9/2}$ major shell.

Figure 4.10: Non-identical ($pn$) and identical ($pp$ and $nn$) pairing “numbers”, $N_{pn}$ and $N_{pp+nn}$, versus $N_-$ valence neutrons (a) for the $N_+ = N_-$ nuclei with $Z = 20$ to $Z = 28$ in the $1f_{7/2}$ level; (b) for the Ti isotopes in the $1f_{7/2}$ level.

Due to the fact that the $pn$ interaction weakens quickly with $i$ the competition between the $pn$ and like-particle pairing interaction will have the largest impact in the $N = Z$ nuclei. For these self-conjugate nuclei, a $\Delta n = 2$ staggering is observed for both pairing interactions (Figure 4.9(a)) [15, 187, 123]. For $N_+ = N_-$ odd-odd nuclei the $\tau = 1$ $pn$ pairs give the dominant contribution, while for the even-even $N_+ = N_-$ nuclei both pairing modes contribute almost equally with a slightly greater like-particle contribution. The $\tau = 1$ like-particle pairing energy and the $\tau = 1$ $pn$ pairing energy yield $\Delta n = 2$ staggering patterns that are of opposite phases and the total isovector pairing energy has a smooth behavior. It is the contribution from the symmetry term ($E$ term in (3.14)) [114, 125, 126] that makes an accurate theoretical prediction of the regular zig-zag pattern of the experimental energies in isobaric sequences possible, as it decreases the energy of the odd-odd nuclei with respect to their even-even neighbors (Figure 4.9(b)).
Rough measures for the number of $pn$ and like-particle pairs are the quantities defined in Chapter 2 (see (2.128) and (2.129))

$$N_{pn} = \frac{1}{G} \langle H_{pn} \rangle, \quad N_{pp+nn} = \frac{1}{F} \langle H_{pp} + H_{nn} \rangle,$$

(4.31)

respectively, which are related to the pairing gaps [15, 45]. The “number” of $pn$ pairs (Figure 4.10(a)) is bigger than the “number” of $pp(nn)$ pairs for odd-odd $N = Z$ nuclei, and is of the same order as for the even-even $N = Z$ nuclei [15, 187, 123] consistent with the charge independence (within few percents) of the nuclear force [124, 126]. In an isotopic chain (Figure 4.10(b)), the “number” of $pn$ pairs peaks at $N = Z$ and the “number” of $nn$ pairs is maximum when the neutrons fill half of their available space ($\Omega$). Although the number of protons does not change within an isotopic sequence, the “number” of $pp$ pairs varies due to the close relationship to the rest two components of the isovector pairing interaction. Such existence of a strong connection between the three components of the isovector pairing makes a treatment of the $pn$ $\tau = 1$ mode separated from the $pp$ and $nn$ modes unrealistic [15].

### 4.4.2 Fine Structure Effects and Discrete Derivatives Based on the Sp(4) Classification Scheme

The symplectic Sp(4) model (namely, the $E_0$ maximal eigenvalues of $|H|$ (3.14)) reproduces the Coulomb corrected $E_{0,\text{exp}}$ energies of the isovector-paired $0^+$ states quite well. A more detailed investigation and a significant test for the theory is achieved through the discrete derivatives of the $E_0$ energy function

$$g^{(m)}_\delta(x) = \frac{g^{(m-1)}_\delta(x-\frac{\delta}{2}) - g^{(m-1)}_\delta(x+\frac{\delta}{2})}{\delta}, \quad m \geq 2$$

$$g^{(1)}_\delta(x) = \begin{cases} E_0(x+\frac{\delta}{2}) - E_0(x-\frac{\delta}{2}), & m - \text{even} \\ E_0(x+\delta) - E_0(x), & m - \text{odd} \end{cases},$$

(4.32)

where the variable is $x = \{n, i, N_+, N_-\}$ according to the Sp(4) classification. Recall that the dynamical Sp(4) symmetry furnishes in a natural way a simultaneous classification scheme of nuclei (Table 3.1) and of their corresponding ground and excited states including the isovector-paired $0^+$ states (Table 3.2) that are mapped to the algebraic multiplets according to the $sp(4)$ reduction chains. In each multiplet, the nuclear characteristics vary in the following steps, $\Delta n = 2$ in each $i$-multiplet (columns), $\Delta i = 1$ in each $n$-multiplet (rows), $\Delta N_\pm = 2$ in each $N_\pm$-multiplet (diagonals).

The filters (4.32) are $(m + 1)$-point expressions and account for the mutual behavior of neighboring nuclei. The first $(m = 1)$ discrete derivative (4.32) is related to the $\delta$-particle

---

The discrete-derivative formulae are valid for any energy function, e.g. $E_{0,\text{exp}}$ (4.1), $\varepsilon_{pn(pp+nn)}$ ((3.27) and (3.28)), $\varepsilon_{\text{sym}}$ (3.17) etc., and hence their definitions in terms of $E_0$ should be considered in a general sense.

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separation energy. When \( m \geq 3 \) the \( Stg_{\delta}^{(m)}(x) \) discrete derivative is independent of mean-field effects and only provides for a description of higher-order terms in the variable \( x \), as well as for discontinuities in the energy function.

The mixed derivatives also provide useful information about the nuclear fine structure effects and are defined as

\[
Stg_{\delta_1,\delta_2}^{(2)}(x,y) = \frac{E_0(x + \delta_1, y + \delta_2) - E_0(x + \delta_1, y) - E_0(x, y + \delta_2) + E_0(x, y)}{\delta_1\delta_2},
\]  

(4.33)

where the variables represent quantities among the set \( (x, y) = \{ n, i, N_+, N_-, \} \).

Different types of discrete derivatives of the theoretical (3.14) and experimental energies according to the Sp(4) classification are considered and various staggering patterns are investigated in the following sections. The parameters in the energy operator (3.14) are not varied and their fixed values are given in Table 4.1 (“non-def” column). The corresponding components of the interaction isolated through the energy difference filters can be explained in analogous ways as in [179, 180], in addition to the advantage that because they are free of Coulomb effect they reflect phenomena related only to nuclear forces.

4.4.3 Discrete Derivatives with Respect to \( N_+ \) and \( N_- \): \( N = Z \)

Irregularities

For even-even nuclei, the discrete approximation of the \( \partial E_0^C / \partial N_\pm \) first derivative of the binding energies (including the Coulomb repulsion energy (4.2)) is related to the well known two-proton (two-neutron) separation energy, which is usually defined as

\[
S_{2p(2n)}(N_\pm) = E_0^C(N_\pm) - E_0^C(N_\pm - 2).
\]  

(4.34)

The Sp(4) theory reproduces very well the available experimental data, especially the irregularity at \( N_+ = N_- \) (see Figure 4.11(a) for a relation to proton number and Figure 4.11(b) for the difference of the Coulomb corrected energies, \( E_0 \), versus neutron number). The zero point of \( S_{2p} \) determines the two-proton-drip line, which according to the Sp(4) model for the \( 1f_{7/2}2p_{1/2}1g_{9/2} \) major shell lies near the following nuclei: \( ^{60}\text{Ge}, ^{64}\text{Se}, ^{68}\text{Kr}, ^{72}\text{Sr}, ^{76}\text{Zr}, ^{78}\text{Zr}, ^{82}\text{Mo}, ^{86}\text{Ru}, ^{90}\text{Pd}, ^{94}\text{Cd}, \) and can be compared to [188, 189, 29]. For odd-odd nuclei the zero point of \( S_{2p} \) can be also determined \( (^{60}\text{Ga}, ^{64}\text{As}, ^{68}\text{Br}, ^{72}\text{Rb}, ^{76}\text{Y}, ^{84}\text{Y}, ^{82}\text{Nb}, ^{86}\text{Tc}, ^{90}\text{Rh}, ^{94}\text{Ag}) \) although it does not define the drip line, as \( S_{2p} \) is a relation of the lowest isovector-paired \( 0^+ \) state energies \( E_0 \) rather than of the binding energies for most odd-odd nuclei.

As a whole, the higher-order derivatives with respect to proton (neutron) number have a smooth behavior. This is because these derivatives reflect changes only within a sequence of either even-even or odd-odd nuclei. The discretization of the \( \partial^2 E_0 / \partial N_\pm^2 \) second-order derivative (4.32),

\[
4\delta I_{pp(nn)}(N_\pm) = E_0(N_\pm + 2) - 2E_0(N_\pm) + E_0(N_\pm - 2) = 4Stg_2^{(2)}(N_\pm),
\]  

(4.35)
accounts for the interaction between the last two \( pp \) \((nn)\) pairs in the \((N_\pm + 2)\) nucleus (Figure 4.12(a)). The average interaction, \( \delta I_{pp(nn)} \), may be used as an alternative way to the defining equation used by Zamfir and Casten \[180\]

\[
\delta V_{pp(nn)}(N_\pm) = \frac{1}{2} \left\{ E_0(N_\pm + 1) - E_0(N_\pm - 1) - [E_0(N_\pm) - E_0(N_\pm - 2)] \right\},
\]

(4.36)

to approximate the non-pairing like-particle interaction\(^9\) (of the last two protons (neutrons)). It shows no outlined staggering pattern but a repulsive peak around the \( N = Z \) nuclei in very good agreement with experiment and with the results and discussions of \[180\]. Another smaller peak is observed around mid-shell (Figure 4.12(a)), which is due to the particle-hole discontinuity introduced in the pairing theory. The analysis yields that as a whole the \(\text{Sp}(4)\) model reproduces the fine structure effects in interactions isolated via the \(Stg_2^{(2)}(N_\pm)\) filters.

Another aspect of the nuclear interaction is revealed by the second-order discrete mixed derivative (4.33) of the energy \[132\],

\[
\delta V_{pn}(N_+, N_-) = \frac{E_0(N_+ + 2, N_- + 2) - E_0(N_+ + 2, N_-) - E_0(N_+, N_- + 2) + E_0(N_+, N_-)}{4}.
\]

(4.37)

For even-even nuclei it was found to represent the residual interaction between the last proton and the last neutron \[179, 190\] and it was empirically approximated by \(40/A\) \[137\]. The theoretical discrete derivative (Figure 4.12(b)) agrees remarkably well with the experiment, especially in reproducing the typical behavior at \(N_+ = N_-\), and is consistent with the

\(^9\)The meaning of “non-pairing” relates to \(J \neq 0\) and \(\tau \neq 1\) interaction or any interaction that is different than the isovector pairing. Also, here the approximation is of \(O(1/\Omega)\).
Figure 4.12: Second discrete derivatives of the $E_0$ energy ($1\frac{3}{2}^+ 2p_\frac{1}{2} 2p_\frac{3}{2} 1g_\frac{9}{2}$ shell): (a) with respect to $N\pm$, $\delta I_{pp(nn)}(N\pm)$, as an estimate for the non-pairing like-particle nuclear interaction in MeV for the $N(Z) = 34, 36, 38$-multiplets; (b) with respect to $N_+$ and $N_-$, $\delta V_{pn}(N_+, N_-)$, as an estimate for the residual interaction between the last proton and the last neutron, for Zn, Ge, Sr isotopes.

empirical trend: excluding the $N = Z$ irregularity the Sp(4) model yields an estimate of $\delta V_{pn}$ on average $\sim 0.71$ MeV for $1\frac{3}{2}^+ (^{40}Ca$ core) and $\sim 0.52$ MeV for the $1\frac{3}{2}^+ 2p_\frac{1}{2} 2p_\frac{3}{2} 1g_\frac{9}{2}$ major shell ($^{56}Ni$ core). It is well-known that the attractive peak in the self-conjugate nuclei cannot be described by a model with an isovector interaction only [190] and in this respect our model achieves this result due to the additional terms included in the Hamiltonian (3.14), mainly the symmetry term (Figure 4.13). As the latter is proportional to the $pn$ isoscalar force (3.19) it is clear that if the $pn$ $\tau = 0$ interaction was neglected the magnitude of the $N = Z$ peak would be greatly reduced. Also, it is not sufficient to include only the $J = 1$ (and not higher-$J$) component of the $pn$ isoscalar interaction [137] (as it is usually done when explicit constructions of isoscalar $pn$ ‘pairs’ is attempted [121]). While the $pn$ interaction spikes, $\delta V_{pn}$, for $N = Z$ are attractive, the like-nucleon $\delta I_{pp(nn)}$ anomalies are repulsive (Figure 4.12). An explanation for that [180] is found in the nature of the attractive $N = Z$ singularities in $\delta V_{pn}$ as arisen from the enhanced spatial symmetry of the wave functions and the basically attractive nature of the $pn$ interaction. In a similar way, it is reasonable that, for a basically repulsive non-pairing like-particle interaction, enhanced spatial symmetry should lead to repulsive spikes in $\delta I_{pp(nn)}$. The $\delta V_{pn}$ energy difference provides for a powerful test for the symplectic model: the theory not only gives a thorough description of the isovector $pn$ and like-particle pairing but additionally accounts for $J > 0$ components of the $pn$ interaction in a consistent way with the experiment. As a result the model can be used to provide for a reasonable prediction of $\delta V_{pn}$ of proton-rich exotic nuclei as well as odd-odd nuclei.

The discussion on the discrete derivatives analyzed in this section is certainly valid for the nuclei in the $1\frac{3}{2}^-$ level (Figure 4.13) although it is illustrated mainly for the major
Figure 4.13: $\delta V_{pn}$ in MeV for Ti-isotopes in the $1f_{7/2}$ shell: (a) of the total binding energy; (b) of the $\tau = 1$ pairing energy. The isovector pairing interaction is not enough to reproduce the experimental peak at $N = Z$.

shell, $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ (III) (Figure 4.11 and Figure 4.12). The reason is that the latter ($\Omega = 11$) shell has the advantage that the data for the nuclei in this region is richer and the observed effects of various phenomena are more developed compared to the $1f_{7/2}$ ($\Omega = 4$) level. However, as it was mentioned, the model interaction for the (III) region carries a few additional approximations and any conclusions made are based on the fact that same patterns are observed also in the $1f_{7/2}$ level. The model Hamiltonian (3.14) reproduces the experimental trend in both $1f_{7/2}$ and $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ shells, whereas for the latter there may be a non-negligible influence of an additional $|i|$-dependent correction$^{10}$ (4.8) to the $pn$ exchange interaction. However, it will only reduce slightly the not at all major contribution of the isovector pairing to the peaks in Figure 4.12 in accordance to the discussion in Section 4.1 and Figure 4.13. Hence, the present analysis retain their validity, which is in addition to the fact that we do not aim to explain the exact mechanism of the phenomena observed – it is beyond the goals of our model and probably requires a sophisticated microscopic theory as large-scale as detailed it can be. In short, in both regions the good theoretical reproduction of the experimental values for $\delta I_{pp(n\bar{n})}$ and $\delta V_{pn}$ cannot lessen the significance and the fundamental nature of the interactions the model Hamiltonian includes.

### 4.4.4 Discrete Derivatives with Respect to $n$ and $i$: Staggering Behavior and Pairing Gaps

The Sp(4) classification scheme can also be used to investigate energy differences with respect to the total number of particles ($n$) and their isospin projection ($i$). Indeed, in contrast with the typical smooth behavior observed for discrete derivatives with respect to $N_+$ and $N_-$ that was highlighted in the previous section, the derivatives with respect to $n$ and $i$ are the ones that reveal distinct staggering effects. They give a relation between even-even (ee) and odd-odd (oo) nuclei and the patterns can be referred as an “ee − oo” staggering.

$^{10}$An $|i|$ energy dependence is definitely discontinuous at $i = 0$.  

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Second and Higher-order Derivatives in One Variable

The discrete derivatives, $Stg_1^{(m)}(i)$, $m = 1, 2, \ldots$, show a prominent $\Delta i = 1$ staggering of the experimental energies of the lowest $0^+$ isovector-paired states for different isobaric multiplets (see Figure 4.14 for the $1f_{5/2}$ shell and Figure 4.15(a) for nuclei above the $^{56}Ni$ core). The theory reproduces this staggering very well. For each of the $i$-multiplets ($i$ fixed), a $\Delta n = 2$

Figure 4.14: The $Stg_1^{(1,2)}(i)$ discrete derivatives for different isobaric multiplets for even-$A$ nuclei with valence nucleons in the $1f_{5/2}$ shell with a core $^{40}Ca$.

Figure 4.15: Discrete derivatives $Stg_8^{(m)}(i)$ ($1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell, a $^{56}Ni$ core): (a) $\delta = 1$, $m = 2, 3, 4$ for $A = 76$ isobars; (b) $\delta = 2$, $m = 2, 4$ for $(i = -1)$-multiplet $[N = Z + 2]$

The staggering effect is also observed for the experimental values via the energy filters $Stg_2^{(m)}(n)$, $m = 1, 2, \ldots$, and successfully predicted by the symplectic model (Figure 4.16 ($1f_{7/2}$) and Figure 4.15(b) ($1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$)).

The staggering amplitudes of both $Stg_1^{(m)}(i)$ and $Stg_2^{(m)}(n)$, while almost independent of the total number of particles $n$, increase with increasing difference in proton and neutron
numbers, \(i\), and hence the “\(ee-oo\)” staggering effect is greater for the proton- (neutron-) rich nuclei than around \(N \approx Z\). Also, the amplitude of \(Stg_1^{(m)}(i)\) increases in higher-order derivatives. This analysis shows a more complicated dependence of the energy function on the isospin projection \(i\) than on the mass number \(A\).

The first, \(m = 1\), discrete derivative,

\[
S_{\text{pn}} = 2Stg_2^{(1)}(n) = E_0(n + 2) - E_0(n),
\]

where \(i\) is fixed, corresponds to the energy gained when a \(\tau = 1\) \(pn\) pair is added (Figure 4.16(a) (\(1f_{\frac{7}{2}}\)) and Figure 4.17 (a \(^{56}\)Ni core)). \(S_{\text{pn}}\) is the true \(pn\) separation energy only when \(E_0\) is the binding energy of the odd-odd nucleus involved in its calculation. The experimental data, where available, is also shown in Figure 4.17 and the \(\text{Sp}(4)\) model follows the distinctive zig-zag pattern very well. A \(\Delta n = 4\) bifurcation separates the nuclei into two groups: one of even-even nuclei \([(n/2 + i)-\text{even}]\) and another of odd-odd nuclei \([(n/2 + i)-\text{odd}]\). The \(S_{\text{pn}}\) energy difference has a smooth behavior within each group. The magnitude of \(S_{\text{pn}}\) is proportional to the total number of particles and increases (decreases) with \(i\) for odd-odd (even-even) nuclei\(^{11}\) (Figure 4.17). Furthermore, the \(Stg_1^{(1)}(n) = \frac{Stg_2^{(1)}(n+2)+Stg_2^{(1)}(n)}{2}\) energy difference shows no \(\Delta n = 4\) staggering (average values of two consecutive data points in Figure 4.17). This indicates that the addition of an \(\alpha\)-like cluster has almost the same effect for both even-even and odd-odd nuclei. This statement does not contradict the stronger binding of even-pairs nuclei as compared to odd-pairs ones, which is detected via \(S_{\text{pn}}\) and via the filter investigated by Gambhir et al. [178]

\[
BE(Z + 2, N + 2) - \frac{BE(Z + 2, N) + BE(Z, N + 2)}{2}.
\]

\(^{11}\)When \((n/2 + i)\) corresponds to an odd-odd nucleus \(S_{\text{pn}}\) is related to the properties of the even-even \((n + 2)\) nucleus.
• Pairing Gaps

The $Stg_1^{(m)}(i)$ and $Stg_2^{(m)}(n)$ energy differences, $m = 1, 2, ...$, described above, isolate effects related to the various types of pairing in addition to non-monopole interactions resulting in changes in energy due to the different isospin values (symmetry term). As noted in [179, 180], the significance of the various energy filters can be understood using phenomenological arguments that can be given a simple and useful graphical representation. Specifically, each nucleus can be represented by an inactive core, schematically illustrated by a box, □, in which the interaction between the constituent particles does not change. Active particles beyond this core can be represented by solid or empty dots, for protons or neutrons, above the box.

The second-order filter,

$$Stg_1^{(2)}(i) = E_0(i + 1) - 2E_0(i) + E_0(i - 1), \quad n = \text{const},$$

$$= E_0(N_+ + 1, N_- - 1) - 2E_0(N_+, N_-) + E_0(N_+, N_-), \quad (4.40)$$

when centered at an odd-odd [(n/2 + i)-odd] self-conjugate (i = 0) nucleus, represents the pairing gap relation $2\Delta$

$$Stg_1^{(2)}(i = 0) \quad (\frac{n}{2}-\text{odd}) = \begin{array}{c}
\square + \bigcirc - 2 \bigcirc \\
\end{array}

\approx 2\Delta \equiv 2\Delta_{pp} + 2\Delta_{nn} - 4\Delta_{pn}. \quad (4.41)$$

The result (4.41) follows from the well-known definition of the empirical like-particle pairing gap [85]

$$\Delta_{pp(nn)} \equiv \frac{BE(N_+\pm1, N_-\mp1) - BE(N_+, N_- - 1) - 2\{BE(N_+, N_- - 1) - BE(N_+, N_- - 1)\}}{2} \quad (4.42)$$

$$= \frac{1}{2} \left(\begin{array}{c}
\bigcirc - \square - 2 \left\{ \begin{array}{c}
\square - \square \\
\end{array} \right\} \\
\end{array} \right),$$

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which isolates the isovector pairing interaction of the \((N_+)^{th}\) and \((N_\pm + 1)^{th}\) protons (neutrons) for an even-even \((N_+ - 1, N_- - 1)\)-core (marked by a square) \[180\]. We also define the \(pn\) isovector pairing gap

\[
\Delta_{pn} \equiv \frac{1}{2} \left( E_0(N_+, N_-) - BE(N_+, N_- - 1) - \{BE(N_+, N_- - 1) - BE(N_--1, N_- - 1)\} \right)
\]

\[
= \frac{1}{2} \left( \bigcirc - \bullet - \{\bigcirc - \bigcirc\} \right),
\]

as the pairing interaction of the \((N_+)^{th}\) proton and the \((N_-)^{th}\) neutron. In order to account correctly for the \(\tau = 1\) mode of the \(pn\) pairing one should consider in (4.43) the \(E_0\) energy of the odd-odd \((N_+, N_-)\) nucleus (that is, the energy of the isobaric analog state rather than its ground state energy, \(BE\)). For the remaining even-even nuclei in (4.40), replacing the symbol \(E_0\) with \(BE\) is justified. In the computation of \(\Delta\), all odd-\(A\) binding energies in (4.42) and (4.43) cancel so their theoretical calculation is not required.

The \(\Delta\) relation of the gaps is a measure of the difference in the isovector pairing energy between even-even and odd-odd nuclei. For odd-odd \(N = Z\) nuclei information about \(\Delta\) is extracted via the \(Stg_1^{(2)}(i)\) energy filter (4.40). Both experimental and model estimations yield \(\Delta \approx 0\) for all the odd-odd \(i = 0\) nuclei in the \(1f_{\frac{5}{2}}\) shell (for example, see solid (purple) line with empty squares in Figure 4.18 for \(A = 46, i = 0\)). The result reflects the fact that in this case all three isovector pairing gaps, \(\Delta_{pp}, \Delta_{nn}\) and \(\Delta_{pn}\), are equal \[126, 125\].

![Figure 4.18: Theoretical staggering amplitudes for the total energy in comparison to experiment, for the isovector pairing energy, the \(pn\) and the like-particle pairing energies, and for the symmetry energy for \(A = 48, A = 46\) and \(A = 44\) nuclei in the \(1f_{\frac{5}{2}}\) shell (a \(^{40}Ca\) core).](image)

A different scenario regarding two aspects is encountered when one considers the \(Stg_1^{(2)}(i)\) discrete derivative centered at an even-(\(n/2 + i\)) \(N = Z\) nucleus (relative to a \((N_+ - 2, N_- - 2)\)-
core)

\[
Stg_1^{(2)}(i = 0) = \begin{array}{c}
\square + \square - 2 \square, \\
(n/2 + i) - \text{even}
\end{array} \\
\approx -\frac{2}{3} \tilde{\Delta} + I_2^{I\neq 0, \tau\neq 1},
\]

(4.44)

where an additional non-pairing two-body interaction, \(I_2^{I\neq 0, \tau\neq 1}\), is not filtered out in this case. Here, for example, \(I_2^{I\neq 0, \tau\neq 1}\) is related to the non-pairing interaction of the three protons and of the three neutrons in the odd-odd nuclei (4.44). Another new feature of (4.44) is that \(Stg_1^{(2)}(i = 0)\) does not simply account for the energy gained when two pm pairs are created (in the first two odd-odd nuclei) and the energy lost to destroy a pp pair and a nn \(\tau = 1\) pairs co-exist. A good approximation that serves well in estimating the pairing gaps is to assume that a \(2p - 2n\) formation above the inactive core (\(\square\)) consists of \(n_0 = 2/3\) \(pm\) pairs, \(n_1 = 2/3\) \(pp\) pairs and \(n_{-1} = 2/3\) \(nn\) pairs (rather than a proton pair \(n_1 = 1\) and a neutron pair \(n_{-1} = 1\)). This is in analogy to an even-even \(n = 4\) nucleus where the \(pp\), \(nn\) and \(pn\) “numbers of pairs” are the same and equal to one third the total number of pairs, \(n/2\) [15, 45, 96]. Additionally, the relations like (4.41) - (4.44) are based on the assumptions that the interaction of a particle with the core is independent of the type of added/removed particles and is the same for all protons (neutrons) above the core. Finally, all the approximations are of an order \(O(1/\Omega)\).

The additional non-monopole two-body residual interaction \(I_2^{I\neq 0, \tau\neq 1}\) should be also taken into account for the rest \(i \neq 0\) of the (even-even, \(ee\), and odd-odd, \(oo\)) nuclei

\[
Stg_1^{(2)}(i \neq 0) \approx \begin{cases} 
-\frac{4}{3} \tilde{\Delta} + I_2^{I\neq 0, \tau\neq 1}, & \text{ee} \\
\frac{4}{3} \tilde{\Delta} + I_2^{I\neq 0, \tau\neq 1}, & \text{oo}.
\end{cases}
\]

(4.45)

The main contribution to the \(I_2^{I\neq 0, \tau\neq 1}\) interaction is due to the symmetry energy as is apparent from the Sp(4) model.

The very close theoretical reproduction of the experimental staggering allows us to use the symplectic model as a microscopic explanation of the observed effects in the \(1f_{7/2}\) level through the investigation of the different terms in the Hamiltonian (3.14) (Figure 4.18). According to the Sp(4) model (Figure 4.8 and Figure 4.9), the “ee-oo” staggering behavior is recognized in the pairing energies and appear due to the discontinuous change of the seniority number(s) driven by the \(\tau = 1\) pairing interaction\(^{12}\) (Figure 4.9). Even values of the

\(^{12}\)Recall that within the fully-paired \((t = 0)\)-representation of Sp(4) (Chapter 2) in the like-particle pairing limit the \(n_0\) number of \(pn\) pairs gives the number of protons (neutrons) not coupled to \(J = 0\) \(pp\) (\(nn\)) pairs and hence defines the usual seniority quantum number \([6, 16]\), \(\nu_1 = n_0\). On the other hand, in the \(pn\) pairing limit \(2n_0 = 2n_{+1} + 2n_{-1}\) counts the particles not coupled in \(J = 0\) \(pn\) pairs and we regard it as another seniority number. However, the dependence of \(n_0\) on \(\nu_1\) within a given nucleus allows one to consider only \(\nu_1\) in the analysis; specifically, for a system of \(n\) valence particles with isospin projection \(i = (Z - N)/2\), the fully paired states (2.19) differ in their coupling mode as the seniority quantum number \(\nu_1\) \((n_0 = n/2 - \nu_1)\) changes by \(\pm 2\) (\(\mp 2\)).
seniority quantum number \( (\nu_1) \) in even-even nuclei and odd values for odd-odd nuclei lead to a change in \( pn \) and like-particle pairing energies in opposite directions. After the contribution from the isovector pairing energy is taken away, the theoretical staggering amplitude, \( (-)^{\nu_1+1} Stg_2^{(2)}(i) \), has still a (typically large) component from the remaining \( (J \neq 0, \tau \neq 1) \) interactions in the Hamiltonian (3.14), mainly the symmetry \( (\tau^2) \) term (Figure 4.18, long-dashed (purple) line with solid squares). This is the same non-monopole nuclear interaction, \( I_2^{J \neq 0, \tau \neq 1} \), that was suggested in (4.44) and (4.45) using phenomenological arguments. Indeed, the symmetry energy contribution is significant and non-zero in all nuclei but the odd-odd \( N = Z \) (Figure 4.18), which is consistent with the discussion above [(4.41), (4.44), (4.45)].

Also, an estimation of the pairing gaps is possible based on the examination of the model Hamiltonian but the theoretical staggering amplitudes of the \( \tau = 1 \) pairing energies (shown in Figure 4.18) need to be rescaled in accordance with (4.41), (4.44) and (4.45).

In a way that is analogous to that used in (4.45), the second-order discrete derivative with respect to \( n \) (can be compared to the filter used in [125])

\[
Stg_2^{(2)}(n) = \frac{E_0(n+2) - 2E_0(n) + E_0(n-2)}{4}, \quad i = \text{const},
\]

is related to the pairing gap relation

\[
Stg_2^{(2)}(n) \approx \begin{cases} 
-\frac{\Delta}{3} + I_2^{J \neq 0, \tau \neq 1}, & \text{ee} \\
\Delta/3 + I_2^{J \neq 0, \tau \neq 1}, & \text{oo},
\end{cases}
\]

(4.47)

where in the odd-odd case, for example, \( I_2^{J \neq 0, \tau \neq 1} \) is the non-pair interaction of the last two protons with the last two neutrons in the \( (n+2) \) nucleus. The effects due to \( \Delta \) cannot be isolated via (4.47) because of the additional non-zero contribution due to the symmetry energy. However, the staggering amplitude of the discrete derivative (4.46), \(-3(-)^{\nu_1+i} Stg_2^{(2)}(n)\), of the theoretical total, \( pp \) \( (nn) \) and \( pn \) pairing energies can provide for estimation of the pairing gaps, \( \tilde{\Delta} \), \( \Delta_{pp(nn)} \) and \(-2\Delta_{pn} \), respectively (Figure 4.19(a)). The like-particle pairing gap can be compared to the empirical value of \( \Delta_{pp} + \Delta_{nn} = 24/A^{1/2} \) [85] (solid (purple) line). The gap is smaller in odd-odd nuclei as compared to their even-even neighbors. This is a consequence of a decrease in the like-particle pairing energy in the odd-odd nuclei due to the blocking effect while there is an increase in energy due to the \( pn \) pairing. The \( pn \) isovector pairing gap increases toward \( i = 0 \) and eventually gets almost equal to \( \Delta_{pp(nn)} \) for odd-odd nuclei around the \( N = Z \) region, which is in agreement with the discussion of [125, 126].

Furthermore, an average of the additional non-pair interaction is achieved by the fourth-order derivatives both in \( n \) \( (Stg_2^{(4)}(n)) \) and \( i \) \( (Stg_1^{(4)}(i)) \)

\[
\tilde{\Delta}_{i|\neq 0,1} \approx \frac{3}{16} (-)^{n/2+i} (Stg_2^{(4)}(i) - I_2^{J \neq 0, \tau \neq 1})
\]

(4.48)

\[
\approx 3(-)^{n/2+i} (Stg_2^{(4)}(n) - I_2^{J \neq 0, \tau \neq 1}).
\]

(4.49)

Assuming that the \( pn \) pairing gap is negligible for high-\( i \) nuclei in large shells like the \( 1f_{7/2}^2p_{1/2}^2p_{3/2}^21g_{9/2} \) major shell, the gap relation (4.48) or (4.49) provides for a rough estimation of the like-particle pairing gaps. With the use of the model Hamiltonian (3.14) we can
Figure 4.19: Estimate for the pairing gaps: (a) total isovector pairing gap $\tilde{\Delta}$, $2\Delta_{pn}$ and $\Delta_{pp} + \Delta_{nn}$, as well as the empirical like-particle pairing gap $\Delta_{pp} + \Delta_{nn} = 24/A^{1/2}$ shown for comparison, for $A = 48$ and $A = 46$ nuclei versus the isospin projection, $i$ ($1f_{7/2}$ shell); (b) like-particle pairing gap (according to (4.48)) versus the mass number, $A$, for $i = \pm 6, \pm 7, \pm 8$ multiplets in the $1f_{5/2}^2p_{1/2}^2p_{3/2}^2g_{9/2}$ major shell.

estimate the additional $I_2^{J\neq 0, \tau \neq 1}$ interaction with the major input being the symmetry energy. Although the existence of a very small mixing of isospin values complicates the computation of the symmetry energy for nuclear systems with very large interaction matrices, as a very good approximation one may use $|\varepsilon_{sym}|$ (3.17) with isospin values $\tau = |i|$ for even-even nuclei and $\tau = |i| + 1$ for odd-odd nuclei (refer to the isospin eigenstate definition (2.25) and the text after). Once the fourth-order discrete derivative (4.32) of the approximated symmetry energy is removed from $Stg_1^{(4)}(i)$ (4.48), the like-particle pairing gaps $\Delta_{pp} + \Delta_{nn}$ are found to be in a very good agreement with the experimental approximation of $24/\sqrt{A}$ for the $(i = \pm 6, \pm 7, \pm 8)$-multiplets in the $1f_{5/2}^2p_{1/2}^2p_{3/2}^2g_{9/2}$ major shell (Figure 4.19(b)). For lower $|i|$ values the difference increases due to an increase in the $pn$ pairing gap as mentioned above. As a whole, the agreement would not be possible if the significant energy contribution due to the symmetry energy was not taken into account.

When $i = 0$, the absence of the large effect of the symmetry energy in odd-odd nuclei (4.41) permits an investigation of the $\tilde{\Delta}$ total isovector pairing gap for the challenging region (III) of nuclei in the $1f_{5/2}^2p_{1/2}^2p_{3/2}^2g_{9/2}$ major shell. The experimental values of $\tilde{\Delta}$ for the odd-odd $N = Z$ nuclei in (III) (for which energy data is available) confirm the results observed for the odd-odd self-conjugate nuclei in the $1f_{7/2}$ level (II) (Figure 4.18), namely that $\tilde{\Delta}$ is negative and close to zero. While in (II) the empirical $\Delta$ gap is very close to zero, in
(III) it deviates more from the zero point (e.g., $\bar{\Delta} \sim -0.5$ MeV in $^{84}$Mo). A negative (and small) $\bar{\Delta}$ indicates that the $pn$ isovector pairing gap, $\Delta_{pn}$, tends indeed to be slightly bigger than the like-particle ones, $\Delta_{pp(nn)}$, resulting in a small redistribution of the $pn$ and like-particle isovector pairing energies, which from a theoretical point of view is achieved by letting $G \gtrsim F$ in the model Hamiltonian. This is another result to show the validity of such assumption. However, while the theoretical estimate of $\Delta$ for odd-odd $N=Z$ nuclei succeeds remarkably well in reproducing the experimental results in the $1f\frac{7}{2}$ level (Figure 4.18), in the $1f\frac{5}{2}2p\frac{3}{2}2p\frac{1}{2}1g\frac{9}{2}$ shell it is on average $\sim 1$ MeV lower than the experimental one, which may be a signal that the strength parameters difference, $G-F$ (Table 4.1), is slightly overestimated in this region. Here again, a plausible explanation (that was suggested in Section 4.1) is the influence on $(G-F)$ of a small additional correction to the Wigner energy around $N=Z$ nuclei (refer to (4.8) and text), which if considered can reduce the $|\Delta|$ gap while leaving the total energy almost unchanged. Notwithstanding, the qualitative description pictured by the simple model Hamiltonian is by no means changed and the role of the underlying interactions is clearly revealed.

- **Second-Order Mixed Derivatives**

Next we consider the second-order discrete mixed derivative of the relevant energies with respect to the total number $n$ and the isospin projection $i$

$$Stg_{2,1}^{(2)}(n, i) = \frac{E_0(n+2,i+1) - E_0(n+2,i) - E_0(n,i+1) + E_0(n,i)}{2}$$  \hspace{1cm} (4.50)

$$\approx \begin{cases} 
\frac{2}{3} \bar{\Delta} + I_2^{I \neq 0, \tau \neq 1}, & ee \\
-\frac{2}{3} \bar{\Delta} + I_2^{I \neq 0, \tau \neq 1}, & oo,
\end{cases}$$  \hspace{1cm} (4.51)

where in addition to the pairing gaps relation, $\bar{\Delta}$, there is the contribution due to the non-pairing interaction, $I_2^{I \neq 0, \tau \neq 1}$. For example, for the odd-odd [even-even] case it is the positive [negative] non-pairing average interaction between the last three protons [neutrons] in the $(n+2)[n], i+1)$ nucleus with a $(n-2)[n-4], i)$ core. Within the Sp(4) framework the additional non-pairing contribution corresponds to the staggering of the symmetry energy approximation, $-\varepsilon_{sym} (3.17)$, of $(-)^{n/2+i+1} E_2(2|i| + 3)$.

The filter (4.50) isolates fine structure effects between two $i$-multiplets (Figure 4.20(a)) and two consecutive isobaric sequences (Figure 4.20(b)). Clearly, it reveals a $\{\Delta n, \Delta i\} = \{2, 1\}$ symmetric oscillating pattern as it is observed in the experiment. Its positive (negative) value is centered at even-even (odd-odd) nuclei and its amplitude increases (decreases) with $|i|$. This mixed discrete derivative (4.50) serves as another test for the Sp(4) model and allows for a detailed investigation of the non-pairing, like-particle interactions involved.

To isolate the effect of non-pairing interactions (again, it is understood to order $1/\Omega$), an energy difference with respect to both $N_\pm$ and $i$ can be considered. The second discrete derivative of the energy

$$Stg_{1,1}^{(2)}(N_\pm, i) = \frac{E_0(N_\pm+1,i+1) - E_0(N_\pm+1,i) - E_0(N_\pm,i+1) + E_0(N_\pm,i)}{2}$$  \hspace{1cm} (4.52)
Figure 4.20: Second-order energy filter $Stg^{(2)}_{2,1}(n, i)$ for nuclei above the $^{56}Ni$ core with respect to the nuclear mass, $A$ (a) and the isospin projection, $i$ (b).

represents the negative [positive] non-pairing two-body interaction of the last two neutrons [protons] with a proton and a neutron in the $(N_{\pm} + 1, i[+1])$ nucleus. It shows prominent $\Delta i = 1$ staggering patterns for different $i$-multiplets (Figure 4.21). While in the framework of the Sp(4) model its amplitude does not depend on $N_{\pm}$ and $i$ except for irregularities around the mid-shell, the magnitude of the few experimental values (where data exists) tends to be slightly lower away from the closed shell. As a whole, the results show that the staggering behavior of this interaction is due to the fine structure features in the relationship between the like-particle and $pn$ non-pairing interactions and differs between proton-rich and neutron-rich nuclei.

Regarding (4.52) and the other discrete approximations of the derivatives in section 4.4.4, it is clear that the oscillating patterns that exist and their regular appearance throughout the nuclear chart cannot be a simple artifact due to errors in the experimental or theoretical energies. Even more, the staggering amplitudes are usually (very) large compared to the energy uncertainties.

For all the discrete derivatives that we have investigated above and that show “ee-oo” staggering behavior, the discontinuity of the symmetry term (due to discrete changes in the isospin value) plays an important role. In contrast, when these discrete derivatives include states of odd-odd nuclei with a dominant $\tau = 0$ pn coupling there is a constant or no contribution due to the symmetry energy and hence yield patterns of different shapes and interpretations. Our investigation does not aim to account for such effects. It is focused on the “ee-oo” staggering behavior of the $E_0$ energies of the lowest isovector-paired states as observed from the experimental data and reproduced remarkably well by the Sp(4) model without any parameter variation.

If reality were only a mean-field theory, none of the finite energy differences would reveal regular or irregular staggering effects. Indeed, the results obtained show that this is not the
The theoretical discrete derivatives investigated not only followed the experimental patterns but their magnitude was found to be in a remarkable agreement with the data. The specific parts of the nuclear interaction that were isolated through such filters were identified using phenomenological arguments and the Sp(4) model interpretation. The present study brings forward a very useful result – a finite energy difference (4.41) was found that, for a specific case, can be interpreted as an isovector pairing gap, \( \tilde{\Delta} = \Delta_{pp} + \Delta_{nn} - 2\Delta_{pn} \), which is related to the like-particle and \( pn \) isovector pairing gaps. Indeed, they correspond to the \( \tau = 1 \) pairing mode because we do not consider the binding energies for all the nuclei but the respective isobaric analog \( 0^+ \) states for the odd-odd nuclei with a \( J \neq 0^+ \) ground state. This investigation is the first of its kind. Moreover, the relevant energies are corrected for the Coulomb interaction and therefore the isolated effects reflect solely the nature of the nuclear interaction. Small deviations from the experimental data are attributed to other two-body interactions or higher-order correlations that are not included in the theoretical model.

In short, we explored independent finite energy differences based on a simple \( \mathbf{sp}(4) \) algebraic classification scheme. The results suggest that this theoretical framework can be used to reproduce various experimental results including observed staggering behavior in fine structure effects of nuclear collective motion.
4.5 Isospin Mixing and Non-Analog $\beta$ Decay Transitions

We are back to the question on isospin mixing but this time qualitative results are sought. As we have already said, since the model allows $G$ not to be equal to $F$, yet very close to each other, the eigenvectors of the Hamiltonian (3.14) do not have definite isospin (or have an almost good isospin). Both empirical evidence (such as scattering analysis and finite energy differences) and the comparison of the model to experimental data (Table 4.1) does not yield equal pairing strengths resulting in a coupling of isospin eigenstates from different isospin multiplets with a degree of mixing expected to be very small (Section 4.1).

In order to estimate the magnitude of the isospin admixture (Table 4.2), we evaluate the percent overlap of the $0^+; n, i$ state (4.9) with the $|n, \tau, i\rangle$ isospin eigenvector,

$$\delta_{\bar{\tau}, \tau} = \left| \langle n, \tau, i | 0^+; n, \tilde{\tau}; n, i \rangle \right|^2 \times 100\%.$$

The $0^+$ states with almost good isospin $\tilde{\tau}$ quantum number are obtained in the diagonalization of the Hamiltonian (3.14) and the isospin states are derived as eigenvectors of $C_2(\mathfrak{su}(2))$ (Table 2.2).

Table 4.2: Sp(4) model estimate for the overlap [%] of the lowest isovector-paired $0^+$ state with the states of definite isospin for the nuclei in $1d_{5/2}$ (I) and $1f_{7/2}$ (II). The table is symmetric with respect to the exchange of proton and neutron ($N^+ \leftrightarrow N^-$) and to the sign of $n - 2\Omega$.

| $^A_X$ | $(N^+, N^-)$ | $\tau = 0$ | $\tau = 1$ | $\tau = 2$ | $\tau = 3$ | $\tau = 4$
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(I) $^{36}$Ar</td>
<td>(2, 2)</td>
<td>99.9999</td>
<td>-</td>
<td>0.0001</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(II) $^{44}$Ti</td>
<td>(2, 2)</td>
<td>99.90</td>
<td>-</td>
<td>0.10</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$^{46}$Ti</td>
<td>(2, 4)</td>
<td>-</td>
<td>99.98</td>
<td>-</td>
<td>0.02</td>
<td>-</td>
</tr>
<tr>
<td>$^{48}$Ti</td>
<td>(2, 6)</td>
<td>-</td>
<td>-</td>
<td>99.997</td>
<td>-</td>
<td>0.003</td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>(3, 3)</td>
<td>-</td>
<td>99.98</td>
<td>-</td>
<td>0.02</td>
<td>-</td>
</tr>
<tr>
<td>$^{48}$V</td>
<td>(3, 5)</td>
<td>-</td>
<td>-</td>
<td>99.994</td>
<td>-</td>
<td>0.006</td>
</tr>
<tr>
<td>$^{58}$Cr</td>
<td>(4, 4)</td>
<td>99.83534</td>
<td>-</td>
<td>0.16465</td>
<td>-</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

The overlap percentages given in Table 4.2 show that the nuclear isovector-paired $0^+$ states have primarily isospin $\tau = |i|$ for even-even, and $\tau = |i| + 1$ for odd-odd nuclei, with a very small mixture of the higher possible isospin values. The $\delta$ isospin mixing increases as $Z$ and $N$ approach one another and approach the middle of the shell. For nuclei occupying a single-$j$ shell, the mixing of the isospin states is less than 0.17%. Although the isospin mixing is negligible for light nuclei in the $j = 3/2$ (I) shell, it is clearly bigger for the $j = 7/2$ (II) shell. The mixing is expected to be even stronger in multi-shell configurations, which is not considered due to the possible slight overestimate for the $(G - F)$ difference in this region as discussed above.
A rough estimate of the δ_{\tilde{\tau},\tau} mixing induced by $H_{IM}$ (4.6) in the Sp(4) model can be obtained in comparison to the (τ = 0) – (τ = 1) (isoscalar-isovector) isospin mixing, δ_{0,1}, in the nuclear ground state due to the Coulomb interaction and a smaller isospin non-conserving nuclear interaction. For $1f_{7/2}$ isoscalar-isovector isospin mixing is of order of 1% [85, 39, 41] and in perturbation theory the τ = 2 isovector-paired 0+ level for $^{48}$Cr (with maximum admixture in the $1f_{7/2}$ level) is expected to be mixed in the ground state of the order of

$$\delta_{0,2} \sim \left( \frac{E_{\tau=1} - E_{\tau=0}}{E_{\tau=2} - E_{\tau=0}} \right)^2 \delta_{0,1} \sim 0.4\%,$$

(4.54)

where we use the experimental energies for $^{48}$Cr ($E_{\tau=1} - E_{\tau=0} \sim 5.8$ MeV, $E_{\tau=2} - E_{\tau=0} \sim 8.8$ MeV) as a rough estimate for the unperturbed energy differences. Hence, if equal coupling strengths are assumed the mixing in the ground state of $^{48}$Cr from the (τ = 2) level is estimated to be around 0.4%. The pure nuclear isospin mixing in the Sp(4) model (0.16%) is within this limit and it is expected to be smaller than the one related to the Coulomb interaction. The ∆τ = 2 mixing in the Sp(4) model is smaller (of an order of magnitude) than the ∆τ = 1 mixing (mainly due to Coulomb interaction) and hence harder to be detected.

Laboratory studies of non-analog Fermi (∆J = 0) β-decay transitions $0^+ \rightarrow 0^+$ provide for an excellent test of the isospin admixture. If there was no isospin mixing, any β transition between nuclear states belonging to different τ multiplets would be forbidden. The experimental results clearly reveal the existence of isospin mixing [35, 36, 37, 38].

For a pure Fermi transition the $f_t$ value (in seconds) that is sometimes called comparative lifetime and is inversely proportional to the decay rate is nucleus-independent according to the conserved-vector-current (CVC) hypothesis and given by (see for example [40])

$$f_t = \frac{K}{G_V^2 |M_F|^2}, \quad K = 2\pi^3 \hbar \ln 2 \frac{(\hbar c)^6}{(m_e c^2)^5},$$

(4.55)

where $K/(\hbar c)^6 = 8.120270(12) \times 10^{-7}$ GeV$^{-4}$s ($m_e$ is the mass of the electron), $G_V$ is the vector coupling constant for nuclear β decay ($K/G_V^2 = 6200s$), and $M_F$ is the Fermi matrix element between the final (F) and initial (I) states in the decay generated by the raising (for β$^-$ decay) and lowering (β$^+$) isospin transition operator τ$_\pm$, $\langle F|\tau_\pm|I\rangle$, which in respect to our model is given as

$$|M_F|^2 = |\langle 0_\tau^+; n, i \pm 1|\tau_\pm|0_\tau^+; n, i \rangle|^2.$$

(4.56)

It is common the isospin impurity caused by isospin non-conserving forces in nuclei to be estimated as a correction to the Fermi matrix element $|M_F|^2$ of the superallowed analog $0^+ \rightarrow 0^+$ transition [35, 40] rather than as a δ_{\tilde{\tau},\tau} overlap (4.53) of the nuclear wave function to isospin eigenstates. Such correction for τ analogs is defined as $\delta_C = 1 - |M_F|^2/\{\tau(\tau + 1) - \tau_0^F \tau_0^I\}$, and should not be directly compared to the degree of mixing evaluated by the Sp(4) model by (4.53) (Table 4.2).

A very interesting result follows from the estimate for the nuclear Fermi β decay rate. When compared to the decay rate for purely leptonic muon decay, it determines a value
for the Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix element [191] between $u$ and $d$ quarks ($v_{ud}$). This in turn furnishes a precise test of the unitary condition of the CKM matrix under the assumption of the three-generation standard particle model.

Table 4.3: $\beta$-Decay transitions and their $ft$ values for nuclei in the $1f_2^2$ level.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta^+$ decay</th>
<th>$\beta^-$ decay</th>
<th>$\lg_{10}ft$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$^{44}<em>{23}$V $\rightarrow ^{44}</em>{22}$Ti</td>
<td>$^{44}<em>{21}$Sc $\rightarrow ^{44}</em>{22}$Ti</td>
<td>6.06</td>
</tr>
<tr>
<td>6</td>
<td>$^{46}<em>{25}$Mn $\rightarrow ^{46}</em>{24}$Cr</td>
<td>$^{46}<em>{21}$Sc $\rightarrow ^{46}</em>{22}$Ti</td>
<td>6.60</td>
</tr>
<tr>
<td>8</td>
<td>$^{48}<em>{24}$Cr $\rightarrow ^{48}</em>{23}$V</td>
<td>$^{46}<em>{22}$Ti $\rightarrow ^{46}</em>{23}$V</td>
<td>6.74</td>
</tr>
<tr>
<td>8</td>
<td>$^{27}<em>{27}$Co $\rightarrow ^{28}</em>{26}$Fe</td>
<td>$^{21}<em>{21}$Sc $\rightarrow ^{22}</em>{22}$Ti</td>
<td>7.17</td>
</tr>
<tr>
<td>8</td>
<td>$^{26}<em>{26}$Fe $\rightarrow ^{28}</em>{25}$Mn</td>
<td>$^{22}<em>{22}$Ti $\rightarrow ^{23}</em>{23}$V</td>
<td>7.13</td>
</tr>
<tr>
<td>8</td>
<td>$^{25}<em>{25}$Mn $\rightarrow ^{26}</em>{24}$Cr</td>
<td>$^{23}<em>{23}$V $\rightarrow ^{24}</em>{24}$Cr</td>
<td>5.90</td>
</tr>
<tr>
<td>10</td>
<td>$^{27}<em>{27}$Co $\rightarrow ^{29}</em>{26}$Fe</td>
<td>$^{23}<em>{23}$V $\rightarrow ^{24}</em>{24}$Cr</td>
<td>6.60</td>
</tr>
<tr>
<td>10</td>
<td>$^{26}<em>{26}$Fe $\rightarrow ^{28}</em>{25}$Mn</td>
<td>$^{24}<em>{24}$Cr $\rightarrow ^{25}</em>{25}$Mn</td>
<td>6.74</td>
</tr>
<tr>
<td>12</td>
<td>$^{27}<em>{27}$Co $\rightarrow ^{29}</em>{26}$Fe</td>
<td>$^{25}<em>{25}$Mn $\rightarrow ^{26}</em>{26}$Fe</td>
<td>6.06</td>
</tr>
</tbody>
</table>

The small mixing of the $0^+$ isospin eigenstates from different isospin multiplets yields very small but non-zero $|M_F|^2$ for non-analog $\beta^\pm$ decay transitions between the isovector-paired $0^+$ states. For the nuclei in the $1f_{7/2}$ shell, such transitions to the $0^+$ ground state of the daughter nucleus (or the lowest isobaric analog $0^+$ state for $^{48}\text{Mn}$ and $^{48}\text{V}$) are shown in Table 4.3 along with the $ft$ values. Four of the transitions are classified as forbidden ($\lg_{10}ft \geq 7$), other eight are also suppressed ($\lg_{10}ft \approx 7$) and the six to an even-even $N = Z$ nucleus appear to have comparatively larger decay rate ($\lg_{10}ft \approx 6$). Although the detection of the $\Delta\tau = 2$ non-analog transitions is hindered by the higher isospin-mixing governed by the Coulomb interaction, the theoretical Sp(4) model suggests the existence of $\beta$-decay branches to these non-analog states.

In the last three sections we focused on the “classical” Sp(4) model and its description of the properties of the isovector-paired states in nuclear systems. Next, we discuss the role of $q$-deformation in the nuclear dynamics.

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4.6 On the Physical Significance of $q$-Deformation in Atomic Nuclei

Although a feature of a quantum theory is that in the $q \to 1$ limit one recovers the non-deformed results it is under a question if the $q$-deformed generalization of the "classical" model will survive the test of the fundamental physical laws.

One of the essential features of the $q$-deformation concept involving non-commutative coordinates has lead to a major class of studies related to the idea of non-commutative geometry. The latter is realized as one replaces the space-time coordinates by non-commuting operators while the deformation (non-commutativity) is introduced in a rather natural and mathematically rigorous way [192]. In such an approach, it has been suggested that when a new fundamental constant is imposed by experiments, the passage from one level of physical theory to another (such as the transition from the classical mechanics to quantum mechanics) can be understood using deformation theory [193]. However such a space-time geometry deformation leads to a $q$-deformed quantum mechanics, which effects are expected to manifest mainly at very short distances (much) smaller than $10^{-19}$ m (at the present level of experimental accuracy quantum electrodynamics is tested to be correct at least down to $10^{-19}$ m) [194]. It is suggested that the description of space-time geometry involves non-commutative structure but at the Planck scale. Hence, it is clear that such an approach leads to effects much beyond the quantum mechanics, which in its standard concept has proven to be a firm ground for nuclear structure descriptions.

But the situation can be very different if one considers a $q$-deformation of the building blocks of a two-body Hamiltonian without compromising fundamental symmetries inherent to the quantum mechanical theory. In such scenarios, the $q$-deformation accounts for non-linear contributions of higher-order (many-body) interactions without affecting physical observables. Such is the essence of the $q$-deformed $\mathfrak{sp}_q(4)$ algebra, introduced in Chapter 2, where the operators of proton and neutron numbers $N_{\pm1}$ and their linear combinations, the total number of particles operator $\hat{N}$ and the third isospin projection $\tau_0$, remain non-deformed and retain the physical meaning of the observables associated with them. In addition, the $q$-deformation of $\mathfrak{sp}(4)$ preserves the "classical" (fundamental) properties of the angular momentum operator, $\mathbf{J}$, as the symplectic algebra is decoupled from the $\mathfrak{so}(3)$ algebra of three-dimensional rotations (3.5) and all the transformations in the $q$-deformed fermion occupation vector space generated by the basis operators of $\mathfrak{sp}_q(4)$ conserve angular momentum. Hence, the model Hamiltonian transforms as a scalar under three-dimensional rotations in coordinate space. It also conserves the number of particles and the third isospin projection but it is not necessary to be isospin invariant (under "rotations" in an abstract isotopic space) as we encountered this situation many times even in regard to the "classical" nuclear force.

The group theoretical $q$-deformed approach allows one to construct $q$-deformed nuclear Hamiltonians with exact solutions. By considering a $q$-deformed generalization of some nuclear structure models, the role of the deformation can be explored by comparing the "classical" and $q$-deformed results with the experimental data [31, 76].
4.6.1 Novel Properties of \(q\)-Deformation

In the previous study where the \(q\)-deformation was treated as nucleus-independent (global) (Section 4.2) the \(q\)-parameter was found as high as 1.240 in the charge-asymmetric model (non-symmetric \(\rho_±\) factor (2.69)) and very close to unity, \(q \approx 1\), when charge symmetry was imposed (via the symmetric \(\rho\) factor). The value of \(q\) was suggested to be greater if its influence is not averaged over all nuclei in a major shell. Such study may reveal the local properties of the \(q\)-deformation as related to each individual nucleus \([195, 196]\).

We focus on two groups of nuclei, namely in the \(1f_{7/2}\) single-\(j\) orbit (II) and in the \(1f_{5/2}2p_{1/2}1g_{9/2}\) major shell (III), where the “classical” model has already demonstrated to agree quite well with experiment for the \(0^+\) states under consideration. The limitations encountered in the larger region (III) require the conclusions to be drawn in a consistency with the results from (II). Yet the limited size of the region (II) that somehow hinders the detection of distinct patterns is surpassed by examinations of the broader area (III), where the observed effects are typically more fully developed. Since the \(q\)-deformed model is applied to the same regions of nuclei where the “classical” model has already proven to provide for a reasonable description of the isovector-paired \(0^+\) states, the \(q\)-deformation does not remedy the non-deformed model but complementarily can improve it.

The idea, that Bahri came up with, is as follows. Within an individual nucleus any deviation between the “classical” model prediction and the experiment is to be attributed to a presence of effects governed by the \(q\)-deformation. A smooth behavior, if found, of its characteristic \(\kappa\) parameter \((q = e^\kappa)\) as one goes from one nucleus to another is indicative of physical significance of the deformation, extending to the very nature of the nuclear interaction itself \([195]\). This is complementary to the observed decoupling of the \(q\)-deformation parameter from the two-body interaction strength, which implies that the \(q\)-deformed Hamiltonian contains the exact “classical” two-body interaction in itself and suggests a link between the \(q\)-deformation and the many-body interactions (Section 4.2).

We consider a \(q\)-deformed model with a charge symmetric \((\tilde{\rho} = \rho)\) nuclear interaction (3.20) with fixed strength parameters \(\gamma_q = \gamma\) (Table 4.1) that reverts to the “classical” model (3.14) when \(q \to 1\), or alternatively \(\kappa \to 0\). Within a nucleus, the expectation value of the energy operator \(E_0^q = \langle H_q \rangle\) in the lowest isovector-paired \(0^+\) nuclear state (as represented by \(|0^+; n, i\rangle\) (4.9)) depends only on the deformation parameter, \(q\) \((\kappa)\), and gives the corresponding predicted energy by the “classical” model in the limit \(\langle H_q \rangle \underset{\kappa \to 0}{\longrightarrow} \langle H_{cl} \rangle\) (Figure 4.22). The small deviations, \(E_0 - E_{0,exp}\), between the non-deformed and experimental energies (see also Figure 4.2), are expected to decrease, even get zero, by varying the \(q\)-parameter, that is, by introducing the higher-order interactions in nuclei.

The deviation of the predicted \(0^+\) state energy, \(E_0^q\), from the corresponding experimental number, \(E_{0,exp}\),

\[
|E_0^q - E_{0,exp}|^2,
\]

is minimized with respect to \(\kappa\) \((= \ln q)\) for each individual nucleus. This procedure yields values for \(\kappa\) as two types of cases are encountered (Figure 4.22).
The difference between the $q$-deformed theoretical and the corresponding experimental energies as a function of the deformation parameter $\kappa$ for a typical near-closed shell nucleus (red solid line) and for a mid-shell nucleus (blue dashed line).

1. The first case of $E_0(= E_{\kappa=0}^q) < E_{0,\exp}$ leads to two symmetric solutions for $\kappa$, that is one physical solution $|\kappa|$. They correspond to the roots of the equation $E_{\kappa}^q = E_{0,\exp}$, by virtue of which the $q$-deformed theoretical energy reproduces exactly the experimental energy at the value of $|\kappa|$ obtained. This is typical for near-closed shell nuclei.

2. The second case of $E_0 > E_{0,\exp}$ determines one value of $\kappa$ at the minimum of the $q$-deformed energy $E_{\kappa}^q$. The minimum occurs at the “classical” energy, $E_0$, with $\kappa \approx 0$ and its difference from the experimental value can be attributed to the presence of other types of interactions that are not included explicitly but rather approximated implicitly in the model.

In physical nuclear structure applications, the values for the deformation parameter, $\kappa$, found in this procedure represent a measure of the extent to which the higher-order many-body interactions are significant within a given nucleus. The contribution of the higher-order terms above the two-body interaction is revealed through the energy difference between the $q$-deformed and non-deformed eigenvalues of the model Hamiltonian, $E_{\kappa}^q - E_0$ (Figure 4.23). For nuclei with non-zero energy differences, the deformed model with the local $q$-parameter improves the prediction of the energies compared to the “classical” global model and reproduces the experimental values exactly.

The solutions for the deformation parameter $|\kappa|$ are found to fall on a smooth curve that tracks with the energy of the lowest $2_1^+$ states (Figure 4.24). This outcome is significant in two aspects. First, the smooth behavior of the deformation parameter with respect to a change in the proton or neutron number suggests that the nature of the $q$-deformation in
nuclear systems is deeply rooted in the basic internucleon interactions. The second aspect is in relation to the lowest $2^+_1$ states\textsuperscript{13}. These energies are largest near closed shells where the pairing effect is essential for determining the low-lying spectrum and decrease with increasing collectivity and shape deformation. Similar behavior is suggested for the $q$-deformation in the sense that dominant pairing correlations are accompanied by non-negligible many-body interactions as prescribed by $H_q$ (3.20), while long-range collectivity suppresses their overall contribution. As an illustrative example, let us recall the system of two protons and two neutrons, $2p - 2n$ (Figure 3.1), where the strong two-body isovector pairing interaction couples the particles in $pp$, $pn$ and $nn$ pairs, which are most favored by the strong isoscalar $pn$ force if all four particles occupy only two orbits (with projections $(m, -m)$). Such configuration is in turn greatly favored by three- and four-body forces, clustering all four particles together in an $\alpha$-like system. If higher-order interactions were not considered the overall picture would be the one of a two-body interaction with larger strength, which needs however to vary between different nuclei. When more particles come to play it is possible that the

\textsuperscript{13}Practically without exception the first excited state of an even-even nucleus is a $2^+$ state, which supports the assumption that this state is an excited state of the ground state configuration (the nucleons occupy the same single-particle $j$-state as in the ground state but the relative orientation of the $j$'s of the different nucleons is changed). The largest number of possibilities of creating a $2^+_1$ state occurs in the middle between closed shells with a corresponding minimum in the $0^+ - 2^+$ separation. The existence of large admixtures of configurations around mid-shell (probably responsible for a “smearing out” of most of the sub-shell effects) may have an averaging effect equivalent to the assumption of a deformable core which is the starting point in the collective model [106].

Figure 4.23: The energy difference between $q$-deformed and non-deformed ("classical") total energies for isotopes in the $1f_{7/2}$ level with a $^{40}$Ca core. The global parameters, $\gamma$, have the values given in Table 4.1, (II).
Figure 4.24: Deformation parameter κ (symbol ■) as a function of valence neutron numbers (N−) for various isotopes in the 1f52p12p31g92 major shell with 56Ni as a core. The solid line is the excitation energies of the 21+ level measured in MeV. The arrows indicate N = Z with the value of q = eκ. The global parameters, γ, have the values given in Table 4.1, (III).

Various higher-order interactions accounted by the q-deformation work against each other and their contribution becomes negligible. Although quite simple this example can give the general idea suggested by the empirical trend of the q-deformation. In a word, the observed smooth behavior of the deformation parameter, even though a qualitative result, gives some insight into the understanding of the nature of the q-deformation and reveals its functional dependence on the model quantum numbers.

The analysis yields that the many-body nature of the interaction is most important around closed shells and the regions with N+ ≈ N− (Figure 4.25). For these nuclei the q-parameter has significant values and the experimental energies can be reproduced exactly. An interesting point is that q tends to peak for even-even nuclei when N+ = N− where strong pairing correlations are expected (Figure 4.24 and Figure 4.25). The behavior of the q-parameter is persistent in both regions under consideration, namely 1f72 and 1f72p12p31g92.
shells, which is best seen for the $N = Z$ nuclei as the $i = 0$ multiplet evolves continuously from the first orbit to the next shell (with $^{56}$Ni being the last nucleus in the multiplet in $1f_{7/2}$ and the first for the $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell) (Figure 4.26).

“Classical” values of the $q$-deformation parameter ($q \approx 1$) are found in nuclei with only one or two particle/hole pairs from a closed shell. This is an expected result since the number of particles is insufficient to sample the effect of higher-order terms in a $q$-deformed interaction. For these nuclei the non-deformed limit gives a good description.

Around mid-shell ($N \approx 2\Omega$) the $q$-deformation adds little improvement to the theory with the experimental values remaining close to the “classical” limit. This suggests that for these nuclei the many-body interactions as prescribed by the $q$-deformation are negligible. Besides there may be other parts of the many-body interaction that are present in the mid-shell region but they are more likely to be absorbed in the mean-field potential of the model,
Figure 4.26: The $\kappa$ deformation parameter as a function of the nuclear mass $A$ for the even-even $N = Z$ nuclei in the $1f_{7/2}$ level and $1f_{5/2} 2p_{3/2} 2p_{1/2} 1g_{9/2}$ major shell (separated by a dotted line).

Similarly to the two-body shape deformation force in the non-deformed case. The results imply that even though the $q$-parameter gives additional freedom for all the nuclei, it only improves the model around regions of dominant pairing correlations.

As observed in Figure 4.24 the $q$-deformation, and hence the development of non-linear effects, is related in a non-trivial way to the underlying nuclear structure. In general, the many-body interactions yield very complicated matrix elements and the analytical modeling of some of them is made possible due to the quantum extension of $sp(4)$. For even-even nuclei a functional dependence of $\kappa$ on the total number of particles $n$ and the isospin projection $i$ is found in the form

$$\kappa(n, i) = A(n - 1)(\frac{n}{2\Omega} + B - 2\Theta(n - 2\Omega))e^{-0.5(x^2)} + D\Theta(n - 2\Omega)|i|\sqrt{\frac{n}{2\Omega} - 1}, \quad (4.58)$$

where $\Theta(x)$ is the step-function defined as

$$\Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}. \quad (4.59)$$

A fit ($\chi = 0.13$) to the values of $\kappa$ for the even-even nuclei in the $1f_{5/2} 2p_{3/2} 2p_{1/2} 1g_{9/2}$ shell (Figure 4.27) estimates the parameters of the function to be

$$A = -2.86, \quad B = 0.21, \quad C = 2.46, \quad D = 0.12. \quad (4.60)$$

An interesting observation is that the exponential dependence on $i^2$ in (4.58) yields a width of $C/2 = 1.23$, which is an indication that the first component of the $\kappa(n, i)$ function decreases rapidly after $|Z - N| > 2$, a tendency that has been observed for the $pn$ isovector pairing interaction for the case of $1f_{7/2}$ (Section 4.4).

The functional dependence found for the deformation parameter, $\kappa$, suggests an approach to the role of $\kappa$ from a different perspective. Instead of focusing on each individual nucleus,
one can consider again all the nuclei in a major shell and perform a non-linear fitting procedure of the relevant $q$-deformed nuclear energies, $E_{q0}^n$, to the experimental data as done in Section 4.2. But this time, the $q$-parameter is given a functional dependence on the nuclear characteristics, say $n$ and $i$, with parameters estimated in the optimization procedure. If the functional dependence is chosen to be as (similar to) (4.58) then the estimate for the parameters will be close to the values (4.60) and the energy prediction for many of the nuclei will reproduce almost exactly the experimental values. However, such an approach is not easy in two aspects, namely the highly non-linear fitting procedure and the fact that we do not know \textit{a priori} the functional dependence of $\kappa$, nor a bit of the way it influences the real nuclear states from one nucleus to another.

### 4.6.2 $q$-Deformed Parameter and ‘Phase Transition’

As described above, higher-order interactions in nuclei such as the ones investigated via the use of the local $q$-parameter are found to develop smoothly with mass number and be nucleus-dependent in their nature. Throughout a major shell for even-even nuclei, they outline two regions, one of negligible higher-order correlations (I) and another where the
latter are significant (II). While in the first ‘phase’ (I) the deformation parameter $\kappa$ is zero, it is considerable in (II) where non-linear many-body fields are required. In analogy with statistical mechanics, the $q$-parameter appears as an order parameter, $\kappa (q = e^\kappa)$, for a ‘phase transition’ between the ‘phases’ (I) and (II)\(^{14}\) (green boundaries in Figure 4.25).

![Figure 4.28: The $\kappa$ order parameter versus the $E_{4^+}/E_{2^+}$ ratio for isotopes of $Z = 30 - 38$ nuclei in the $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell. The ‘phase transition’ where $\kappa$ zeros is shown in yellow in Figure 4.25.](image)

The behavior of the $\kappa$ order parameter can be traced as one varies the proton or neutron mass number (Figure 4.24). Such a control parameter is related to the identity of the nuclei rather than to the properties of the underlying nuclear interaction. As indicative for the latter, the $E_{4^+}/E_{2^+}$ ratio furnishes a good choice, where $E_{J^+}$ is the experimental energy of the first $J = 2$ and $J = 4$ excited levels (Figure 4.28). The physical meaning behind the $E_{4^+}/E_{2^+}$ quantity has been revealed by Casten et al. in a phenomenological analysis of the energy systematics in medium and heavy even-even nuclei where the ratio was found to be $E_{4^+}/E_{2^+} \approx 1$ for nuclei dominated by a seniority structure (seniority regime), $E_{4^+}/E_{2^+} \approx 2$ for (anharmonic) vibrational nuclei (vibrator regime) and $E_{4^+}/E_{2^+} \approx 3.33$ for well-deformed rotational nuclei (rotator regime)\(^{203}\). Zero value of $\kappa$ is observed for comparatively higher

\(^{14}\) A criticism may be raised here as we talk about phase transitions in small finite systems such as nuclei. Small systems do not exhibit true phase transitions (as defined within the framework of statistical mechanics). But it is reasonable to say that they can have an order parameter: such that represents the symmetry (order) in a quantum mechanical system and reflects any changes in the properties of the system, which are governed by this symmetry. As a very fundamental issue, the nature of phase transitions in small systems is a subject of many investigations (in nuclear structure physics it was first investigated in details\(^{197, 198}\) in the view of the Interacting Boson Model (IBM)\(^{199}\)). A way to approach a precise definition of this concept is to require a transition in a finite system to be a precursor of a phase transition in the corresponding thermodynamical limit of infinite particle number\(^{200}\). As in solid state physics, in large (infinite) systems the phase transition to a superconducting phase below a critical temperature is well established also in nuclear physics with neutron stars being such infinite systems\(^{201}\). It has been recently shown that, in turn, finite fermionic systems exhibit clear features reminiscent of the pairing phase transition in infinite fermionic systems\(^{202}\).
$E_{4+}/E_{2+}$ ratios. Strong non-linear effects ($\kappa > 0$) are found for lower $E_{4+}/E_{2+}$ ratios. Complementarily to the analysis of the $E_{2+}$ energies this result shows that for shape deformed even-even nuclei the many-body effects introduced by the specific $q$-deformation do not have an overall significant effect in their ground state, while spherical shapes give rise to higher-order interactions such as described by $H_q$ (3.20). Typically, for the $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell the order parameter $\kappa$, tends to zero around $E_{4+}/E_{2+} \approx 2 - 2.5$, where the ‘phase transition’ occurs (see Figure 4.24 for a ‘phase transition’ when $n < 2\Omega$). This implies that the ‘phase transitions’ from non-negligible (II) to negligible (I) many-body effects as detected by the $\kappa$ parameter are observed soon after the appearance of a collective structure. An additional study of the odd-odd nuclei in the $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell and of other major shells where the model is applicable is expected to reveal more of the hidden role of the $q$-deformation. Such a study is presently on its way.

In summary, the $q$-deformed extension of the $\text{Sp}(4)$ model is compared to experimental data resulting in a smooth functional dependence of the deformation parameter $q$ on the proton and neutron numbers, which resembles the behavior of the lowest $2^+$ state energies. In addition, the $q \neq 1$ results are uniformly superior to those of the non-deformed limit. The results suggest that the deformation has physical significance over-and-above the simple pairing gap concept, extending to the very nature of the nuclear interaction itself and beyond what can be achieved by simply tweaking the parameters of a two-body interaction. Since the $q$-deformation of $\text{Sp}_q(4)$ introduces into the nuclear Hamiltonian higher-order, non-linear terms in the $pn$, $pp$ and $nn$ pairing interaction, as well as in $pn$ isoscalar interaction (hence, non-linear dependence on the isospin $\tau$), the outcome suggests the presence and importance of higher-order many-body interactions accompanying dominant pairing correlations in nuclei, especially for nuclei just beyond closed shells and with $N \approx Z$. Hence, the specific features of the nuclear structure can be investigated through the use of a local $q$ value that varies smoothly with nuclear mass number. This is in addition to the good description of the global and common properties of the nuclear dynamics provided by the “classical” two-body interaction. The results also underscore the need for additional studies to achieve a more comprehensive understanding of $q$-deformation in nuclear physics.

### 4.7 The $\text{sp}_{(q)}(4)$ Algebraic Approach: the Closing Stage

The finale of this work is a prelude to what can be further achieved in terms of both the theoretical and practical aspects within the framework of the simple and beautiful $\text{sp}(4)$ model and its $q$-deformed non-linear generalization. We briefly summarize the main ideas without attempting a detailed presentation and discussion.

Before we focus on possible developments of the present investigation, we start with two other studies of physical interest (Figure 2.1), namely the monopole-plus-pairing model (2) and the two-dimensional vibration-rotation model (3). They are based on the same $\text{Sp}(4)$ algebraic structure and differ only in the interpretation of the $\sigma$ quantum number. Hence,
both $\mathfrak{sp}_q(4)$ algebraic approaches discussed in Chapter 2 and Chapter 3 can be applied to these model spaces without any change of the theory. Certainly, individual objects will have different physical meanings but the mathematical construction is “ready-to-go”. The $q$-deformed version of $\mathfrak{sp}(4)$ will provide both models, (2) and (3), with the capacity to incorporate non-linear many-body interactions in their two-body descriptions.

- **Understanding the $q$-Deformation**

As we have already emphasized, the physical role of the $q$-deformation certainly needs further exploration. The outcome of our study relates the deformation to higher-order interactions in nuclei and reveals the close connection of the $q$-deformation to the underlying nuclear structure, yet leaves several questions open. What is the exact mechanism for developing many-body interactions in different nuclei and can it be explained by a $q$-deformed theory? What is then the exact relation of the $q$-deformation, on one hand, to the nuclear structure and, on the other hand, to higher-order effects? An attempt on the first part of the question is made via the functional dependence on nuclear characteristics found for the deformation parameter (4.58). Here again, what is the fundamental meaning behind this dependence? The second part of the question is still another puzzle. We may think of the $q$-deformation as it collects certain many-body interactions and presents them as a $q$-deformed two-body interaction. We may regard it as nothing more but a non-linear transformation to a “quasi-particle” description where the “quasi-particles” are subjective only to a two-body interaction, in close similarity to the Bogolyubov transformation [204] to independent quasi-particles in the BCS superconductivity approach. Even though the parameter of such a transformation, $q$, can be viewed as a ‘detector’ of the many-body interactions, the exact connection is still hidden behind the fact that there is no known simple function that transforms the “classical” fermions into the $q$-deformed (“quasi”-) particles (Appendix C).

A deeper understanding of the role of the $q$-deformation can be achieved if one employs more general models with different Hamiltonians and even different model spaces but restricted to a deformation representative of many-body interactions (it is obvious that models where the $q$-parameter is related to a space-time deformation are not relative to such investigations). If the main features of the $q$-deformation consistently develop from one model to another then we have a clear indication of their fundamental origin. In addition, the exploration of other models can provide further answers to questions like, are there other kinds of higher-order many-body interactions different from the one introduced by $\mathfrak{sp}_q(4)$ with a comparatively significant contribution? If yes, do the kinds compete or do they complement each other in a way that one of them appear where the other kind is negligible?

And finally, it may be interesting to understand how the $\mathfrak{sp}_q(4)$ algebraic structure appears as a fundamental dynamical symmetry of the nuclear interaction in regions of dominant pairing correlations and what the $q$-parameter represents as a characteristic of such a symmetry.
• Description of Odd-$A$ Nuclei in the Sp$(4)$ Framework

In our investigation in the realm of pairing, we focus on the isovector-paired states since pairing correlations manifest themselves most distinctly in fully-paired $0^+$ states. However, such a space (2.19) is not a restriction of the $\text{sp}(q)(4)$ algebraic approach and we can further extend both “classical” and $q$-deformed models if we enlarge the model space to include odd-$A$ nuclei. The dynamics of these nuclear systems is typically more complicated than for even-even nuclides. However, for odd-$A$ nuclei with proton and neutron numbers close to the magic numbers the single-particle shell-model proves successful in describing the nuclear properties when one considers all the nucleons but the odd particle as an even-even inert core. Hence, the states in odd-$A$ nuclei that are influenced the most by isovector pairing can be represented as an odd fermion and completely-coupled nucleons (even in number). This is achieved by a slight modification in the fully-paired states (2.19) \cite{16}

$$|\Omega; n_1, n_0, n_{-1}, i_v\rangle = \left(A^\dagger_1\right)^{n_1} \left(A^\dagger_0\right)^{n_0} \left(A^\dagger_{-1}\right)^{n_{-1}} |i_v\rangle,$$  

(4.61)

where $i_v$ is the isospin value of the odd particle, $i_v = \frac{1}{2}$ if it is a proton and $i_v = -\frac{1}{2}$ for a neutron. The state $|i_v\rangle$ is constructed above the vacuum by creation fermion operators, $c^\dagger_{jm,\sigma = i_v}$ (hence it commutes with $A^\dagger_{0,\pm 1}$). Since the odd particle is not coupled then $A^\dagger_{0,\pm 1} |i_v\rangle = 0$, while the isospin operators change its isospin projection. The states (4.61) span now a subspace $E^-_{J=j}$ in the odd $E^-$ space and have total seniority one and total angular momentum $J = j$ determined by the $j$ angular momentum of the odd particle.

Now, in the bigger model space, $E^+_{J=0} \oplus E^-_{J=j}$, the seniority quantum numbers introduced in the limits of $pn$ and like-particle pairing should take proper account of the odd nucleon and can be redefined as, $2\nu_0 = 2(n_1 + n_{-1}) + N_{\text{odd},1/2} + N_{\text{odd},-1/2}$ and $\nu_{\pm 1} = n_0 + N_{\text{odd},\pm 1/2}$, where $N_{\text{odd},\pm 1/2}$ is the number of the fermions (protons or neutrons) not coupled in any $J = 0$ pair ($N_{\text{odd},\pm 1/2} \leq 1$) and is zero in even-$A$ nuclei. With such a substitution, the eigenvalues of the quasi-spin operators, $s^0 \pm$ (2.29) and (2.27), expressed in terms of the seniority numbers are still valid (as well as in the $q$-deformed case, (2.81) and (2.85)). The energies in the pairing limits, (3.27) and (3.28) ((3.33) and (3.34)), give the nuclear energy spectra of the relevant states (with $J = 0$ for $n$ even and $J = j$ for $n$ odd) from the $E^+_{J=0} \oplus E^-_{J=j}$ space when $\nu_{0,\pm 1}$ change by two.

A complete investigation in this direction will allow for a systematic study of nuclear properties of even-even, odd-odd and odd-even nuclei in the framework of only one model and its $q$-deformed extension. The possible development of the Sp$(4)$ model to include odd-$A$ nuclei may lead to interesting results especially in relation to studies on staggering, pairing correlations and significance of the $q$-deformation.

• The Mass of Neutrino

Recently, a very interesting observation has shown that the problem on the mass of neutrino can be linked to algebraic structures, which for the neutrino mass Lorentz invariance constraints is the symplectic $\text{sp}(4)$ algebra \cite{205}.
The mass of neutrino is still a very widely discussed problem – there is no up-to-date deep understanding of the fundamental origin of neutrino mass, nor an exact measurement of the neutrino mass. The significance of the problem follows also from the fact that the neutrino mass is probably the best indication for physics beyond the standard model.

In the framework of the neutrino mass problem the following operators close on the four different realizations of the unitary subalgebra of $\mathfrak{sp}(4)$ (Table 2.1 and Table 2.2). The operators associated with Dirac charges $D_+$, $D_-$ and $D_0$ close on the $\mathfrak{su}^*(2)$ algebra, while right(left)-handed charges $R_+ R_-, R_0 (L_+, L_-, L_0)$ form a basis in the $\mathfrak{su}^{\pm}(2)$ algebra, where $D_0 \doteq (R_0 + L_0)$. Additionally, the operators $A_+, A_-$ and $A_0 \doteq (R_0 - L_0)$ close on the $\mathfrak{su}^0(2)$ algebra, where $A_0$ is proportional to the neutrino number operator. The ten operators $D_+, D_-, L_+, L_-, L_0, R_+, R_-, R_0, A_+$, and $A_-$ close the symplectic $\mathfrak{sp}(4)$ Lie algebra. The physical meaning behind the $\mathfrak{su}^{\tau, \pm}(2)$ algebras is that they generate the neutrino mass matrix. The relation to the generalized pairing problem allows for a general mass term in the Hamiltonian to be expressed as [205]

$$H_m = m_D(D_+ + D_-) + m_L(L_+ + L_-) + m_R(R_+ + R_-). \quad (4.62)$$

A natural scheme for the smallness of the neutrino masses is provided by the see-saw mechanism (see [205] and references there). A see-saw Hamiltonian is generated by a particular Pauli-Gürsey transformation, which in fact has been shown to be an SU$^0(2)$ rotation (generated by $A_+, A_-$ and $A_0$) embedded in the associated Sp(4) Lie group [205].

A further extension of this model to introduce a $q$-deformation in the neutrino mass generators in the framework of the $q$-deformed $\mathfrak{sp}_q(4)$ approach may lead to novel and interesting results regarding the mass of neutrino and its fundamental origin.

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Obviously, there is more to explore using what we have developed as part of this dissertation project. Hopefully, I will have an opportunity to explore this and more as I continue down life’s pathway and my personal scientific journey.
Chapter 5

Conclusion

We constructed an algebraic $\mathfrak{sp}(4)$ model and its extension to many-particle interactions in a $q$-deformed $\mathfrak{sp}_q(4)$ framework. Both models, which are complementary to each other, proved to be useful and consistent with experiment, providing for a microscopic description of nuclear structure, particularly with respect to pairing correlations and proton-neutron interactions.

As a foundation of our investigation, the fermion realization of the symplectic $\mathfrak{sp}(4)$ algebra, isomorphic to $\mathfrak{so}(5)$, was reviewed and presented in detail in relation to the charge independent pairing model \cite{10, 11, 12, 13, 14, 15, 45} for a single-$j$ and multi-$j$ orbits. The reduction chains of $\mathfrak{sp}(4)$ provided for a natural and useful classification scheme of nuclei and their states, which in turn allowed for a wide-ranging systematic study of nuclear properties.

We discovered a $q$-deformation of the fermion realization of the symplectic $\mathfrak{sp}(4)$ algebra that corresponds to the standard Drinfeld-Jimbo construction for $\mathfrak{so}_q(5)$ and leaves the physical observables undeformed. In this way, the $q$-deformed $\mathfrak{sp}_q(4)$ constitutes a novel mathematical result as well as a rich theoretical structure that can be applied to nuclear systems without giving up the fundamental symmetries.

The dependence of the $q$-deformation on the dimensionality of the space makes the generalization of $\mathfrak{sp}_q(4)$ to multi-shell dimensions unique and non-trivial. In both the single-$j$ and multi-$j$ cases, we identified the subalgebraic structure of $\mathfrak{sp}_q(4)$ and constructed the second-order Casimir invariants for each realization of the $q$-deformed $\mathfrak{su}_q(2)$ subalgebra. We derived an analytical form of the matrix representation of the $\mathfrak{sp}_q(4)$ basis operators in a space of fully-paired states. We also found a $q$-deformed second-order operator, which coincides with the Casimir invariant of $\mathfrak{sp}(4)$ in the $q \to 1$ limit, that is diagonal in the $q$-deformed basis set with its zeroth-order approximation commuting with all the $\mathfrak{sp}_q(4)$ operators.

These algebraic concepts lead to the construction of a “classical” Hamiltonian that has the symplectic $\text{Sp}(4)$ group as a dynamical symmetry. We used the theory to describe isovector pairing correlations and high-$J$ interactions (including diagonal isoscalar $pm$ force) in nuclei. We established a relation of the group theoretical approach to a general microscopic pairing Hamiltonian. The limits of applicability of the model were carefully examined.

The theory was first exploited for fitting calculated energies to the relevant experimental
0+ state (binding) energies for even-A nuclei with masses 32 ≤ A ≤ 164 in single-j levels, namely 1d5/2 and 1f7/2, and in major shells, 1f5/2 2p1/2 2p3/2 1g9/2 and 1g7/2 2d5/2 2d3/2 3s1/2 1h11/2. In general, the fitting procedure yielded results that are in good agreement with the experiment providing for a physically valid estimation of the interaction strength parameters. The latter was found to follow an overall smooth 1/A decrease for 32 ≤ A ≤ 100.

Based on these outcomes, the question on breaking of the isospin invariance and isospin mixing was tackled. The degree of \( \Delta \tau = 2 \) isospin mixing was calculated for the nuclei in the 1d5/2 and 1f7/2 orbits resulting in practically no mixing for 1d5/2 and a \( \tau \)-mixing for 1f7/2 much smaller than the \( \Delta \tau = 1 \) mixing induced mainly by the Coulomb interaction, in agreement with experimental analysis. The isospin admixture found for nuclei in the 1f7/2 shell predicted the observation of forbidden 0+ → 0+ non-analog (\( \Delta \tau = 2 \)) \( \beta \) decay transitions.

Throughout the nuclear chart (32 ≤ A ≤ 164), the Sp(4) model was used to provide for an estimate for the lowest isovector-paired 0+ state energies, which include the binding energies for even-A nuclei with a 0+ ground state. They agree rather well with the available experimental values. Moreover, we were able to reproduce the energy spectra of the higher-lying isovector-paired 0+ states for 1d5/2 and 1f7/2 quite well, with no adjustable parameters. The model was then used to obtain a reasonable prediction of the relevant energies, not yet measured, for the proton-rich region and for \( N \approx Z \) nuclei outside the valley of stability.

The theoretical model was further tested through second- and higher-order discrete derivatives of the energies of the lowest isovector-paired 0+ states in the Sp(4) systematics, without any parameter variation. The investigation that considers these 0+ states (rather than ground states for all nuclei) is the first of its kind and brought to light a finite energy difference that, for a specific case, represents the nuclear isovector pairing gap, which is related to the like-particle and \( pn \) isovector pairing gaps. The proposed model was used to successfully interpret: the two-proton (two-neutron) separation energy \( S_{2p(2n)} \) for even-even nuclei (hence determined the two-proton drip line), the \( S_{pn} \) energy difference when a \( pn \) \( \tau = 1 \) pair is added, the observed irregularities around \( N = Z \), the prominent “ee-oo” staggering between even-even and odd-odd nuclides, the like-particle and \( pn \) isovector pairing gaps, and the large contribution to the finite energy differences due to the symmetry term (accounting for \( J = 1 \) and higher-\( J \) \( pn \) interactions). We noted that the oscillating “ee-oo” effects correlate with the alternating of the seniority numbers related to the \( pn \) and like-particle isovector pairing, which is in addition to the larger contribution due to the discontinuous change in isospin values associated with the symmetry energy.

At the same time, the \( q \)-deformation of sp(4) was used to construct a \( q \)-deformed model Hamiltonian, which in turn provided for an exact solution of the \( q \)-deformed eigenvalue problem. The \( q \)-Hamiltonian was found to be an extension of the “classical” two-body interaction by the virtue of the observed decoupling of the \( q \)-parameter from the interaction pairing strengths. In short, while the \( q \)-parameter does not influence the two-body part of the nuclear interaction, it introduces many-body higher-order components into the “classical” two-body Hamiltonian. The non-linear feature due to \( q \) was found to be responsible for a possible change of the like-particle and \( pn \) isovector pairing gaps – more the \( q \)-parameter differs from its “classical” limit bigger the gap change gets.
The $q$-deformed theoretical energies were fit to the relevant experimental data resulting in an evaluation of $q$ as a fitting parameter. When the $q$-parameter is considered global and kept the same within a major shell, the values of $q$ that we obtained were found to differ little from their "classical" limit. Yet, the $q$-deformed case yielded the optimum overall results. In addition to the broken symmetries of the non-deformed model, the asymmetric realization of the $\mathbf{sp}_q(4)$ algebra was found to break slightly the symmetry between protons and neutrons, which again is negligible for light nuclei and consistent with observations. The symmetric realization of the $\mathbf{sp}_q(4)$ algebra, on the other hand, conserves the charge symmetry and yielded much smaller global values for $q$ compared to the asymmetric case.

The outcome suggests that the nature of the $q$-deformation must be of a local character and its influence cancels out when averaged over all nuclei in a major shell. This lead to the first step towards a more elaborate investigation of the role of the $q$-deformation in each individual nucleus. Indeed, we discovered that values of the $q$-parameter as obtained in comparison with experiment have a smooth functional dependence on the proton and neutron numbers which qualitatively reveals the fundamental relation of the $q$-parameter to the underlying nuclear structure. The smooth behavior of $q$ within the Sp(4) systematics was found to resemble the behavior of the lowest $2^+$ state energies ($E_{2^+}$). As the latter are indicative of the dominance of the pairing correlations in the low-lying energy spectrum of a nuclear system, this observation shows that the many-body interactions as prescribed by the $q$-deformation are closely linked to the regions of dominant pairing correlations. Moreover, the role of the deformation parameter $\varkappa (= \ln q)$ was given another perspective, namely to furnish an order parameter between 'phases' of non-negligible and negligible (where $\varkappa = 0$) many-body higher-order interactions introduced by $\mathbf{sp}_q(4)$. The behavior of the order parameter with changing $E_{4^+}/E_{2^+}$ ratio (indicative of the collectivity of the system) shows an abrupt zeroing of $\varkappa$ soon after collectivity develops.

The outcome of the present investigation shows that, in comparison to experiment, the simple $\mathbf{sp}(4)$ algebraic approach reproduces not only global trends of the relevant energies but as well the smaller fine features driven by isovector pairing correlations and higher-$J^p$ $pn$ and like-particle nuclear interactions. In addition to this, the variations within individual nuclei due to higher-order many-body interactions are described by the non-linear $q$-deformed $\mathbf{sp}_q(4)$ theory with a local $q$-parameter fundamentally linked to the very nature of the nuclear interaction.
Bibliography


Appendix A

Lie Algebras and Lie Groups

A comprehensive knowledge of exact definitions, theorems and proofs in group theory is not necessary for applying its essential features and results to physics. Nevertheless, we present basic definitions and concepts from group theory to show the simple and methodological basis of the theory. A more comprehensive review can be found in the following textbooks [206, 207, 208, 209, 210, 211, 212] (and many others).

**Definition** Let $g$ be a finite-dimensional vector space over the field $K$ of real or complex numbers. The vector space $g$ is called a Lie algebra over $K$ if there is a rule of composition $g \times g \to g: (X,Y) \to [X,Y]$ in $g$ which satisfies the following axioms:

1. $[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$ for $\alpha, \beta \in K$ (linearity),
2. $[X, Y] = -[Y, X]$ for all $X, Y \in g$ (antisymmetry),
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in g$ (Jacobi associativity),

(A.1)

where the operation $[ , ]$ is called Lie multiplication and in general it is non-associative. A real Lie algebra is defined over the field $K$ of real numbers, while a complex Lie algebra over the field $K$ of complex numbers.

**Definition** A Lie algebra $g$ is abelian if $[X, Y] = 0$ for all $X, Y \in g$.

**Definition** A Lie algebra $g$ is associative if for every $X, Y \in g$ a product $XY$ in $g$ is defined, which satisfies: $X(Y + Z) = XY + XZ$, $(X + Y)Z = XZ + YZ$ and $(XY)Z = X(YZ) = XYZ$.

**Definitions** A subspace $n$ of the Lie algebra $g$ is a subalgebra if $[n, n] \subset n$ and a proper subalgebra of $g$ if at least one element of $g$ is not contained in $n$. A subalgebra $n$ of the Lie algebra $g$ is an ideal (also called invariant) if $[g, n] \subset n$. A maximal ideal $n$, which satisfies the condition $[g, n] = 0$ is called the center of $g$ and because $[n, n] = 0$ the center is always commutative.

A mapping of one algebraic structure into another similar algebraic structure is called a homomorphism $(h)$ if it preserves all combinatorial operations associated with that structure.
(\mathfrak{g} \rightarrow \mathfrak{g}')$. If the mapping is in addition one-to-one, or faithful, so that an inverse is well defined and exists, it is called an isomorphism \((\mathfrak{g} \sim \mathfrak{g}')\). The isomorphism of \(\mathfrak{g}\) in itself \((\mathfrak{g} \sim \mathfrak{g})\) is called automorphism.

**Definition** A Lie algebra \(\mathfrak{g}\) is simple if it is not Abelian and does not possess a proper invariant Lie subalgebra.

**Definition** A Lie algebra \(\mathfrak{g}\) is semi-simple if it does not possess an Abelian invariant subalgebra.

If \(\mathfrak{g}\) is simple then \(\mathfrak{g}\) is semi-simple, the converse is not true. There are four infinite classes of complex algebras, \(A_{n-1}\) \((n > 1)\), \(B_n\) \((n > 1)\), \(C_n\) \((n > 2)\) and \(D_n\) \((n > 3)\), that together with five finite others, \(G_2\), \(F_4\), \(E_6\), \(E_7\) and \(E_8\), constitute all the non-isomorphic simple complex algebras, where the isomorphism holds \(A_{n-1} \sim \mathfrak{sl}(n, \mathbb{C}) \supset \mathfrak{su}(n)\) \((n > 1)\), \(B_n \sim \mathfrak{so}(2n + 1, \mathbb{C}) \supset \mathfrak{so}(2n + 1)\) \((n \geq 1)\), \(C_n \sim \mathfrak{sp}(2n, \mathbb{C}) \supset \mathfrak{sp}(2n)\) \((n \geq 1)\) and \(D_n \sim \mathfrak{so}(2n, \mathbb{C}) \supset \mathfrak{so}(2n)\) \((n \geq 1)\). The \(\mathfrak{sp}(2n) = \mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{su}(4n)\) with \(n = 2\) is isomorphic to \(C_2\) and it is a simple algebra. An example for a semi-simple Lie algebra is \(\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(3)\).

**Definitions** Consider the linear map \(\text{ad}(X)\) of a Lie algebra \(\mathfrak{g}\) into itself defined as

\[
\text{ad}(X)Y = [X, Y], \quad X, Y \in \mathfrak{g},
\]

then the set \(\mathfrak{g}_n = \{\text{ad}(X), X \in \mathfrak{g}\}\) is called the adjoint algebra. Now, let \(Y_1, Y_2, \ldots, Y_d\) be a basis in \(\mathfrak{g}\) then for any \(X \in \mathfrak{g}\)

\[
[X, Y_j] = \text{ad}(X)_{kj}Y_k, \quad k, j = 1, 2, \ldots, d,
\]

where \(\text{ad}(X)\) is \(d \times d\) matrix (summation over repeated indices is implied) and \(d\) is the dimension of the \(\mathfrak{g}\) Lie algebra. The set of matrices \(\text{ad}(X)\) forms a \(d\)-dimensional representation of \(\mathfrak{g}\) called the adjoint representation of \(\mathfrak{g}\). For every element of the basis, \(Y_p\) \((p = 1, 2, \ldots, d)\), the matrix \(\text{ad}(Y_p)\) is defined as \(\text{ad}(Y_p)_{kj} = c^k_{pj}\) and the set of \(d^3\) numbers \(c^k_{pj}\) is called the structure constants of \(\mathfrak{g}\) with respect to the basis \(Y_1, Y_2, \ldots, Y_d\). According to the axioms (2) and (3) of the algebra definition (A.1) the structure constants satisfy the following conditions

\[
c^k_{pj} = -c^k_{jp},
\]

\[
c^k_{pl}c^l_{rs} + c^k_{rl}c^l_{sp} + c^k_{sl}c^l_{pr} = 0.
\]

**Definitions** Let \(\mathfrak{g}\) be a Lie algebra of a dimension \(d\). A representation of \(\mathfrak{g}\) is said to be reducible if it is equivalent to a representation \(\mathfrak{t}\) of \(\mathfrak{g}\) that can be partitioned for every \(X \in \mathfrak{g}\) in the form

\[
\mathfrak{t}(X) = \begin{pmatrix}
\mathfrak{t}_{11}(X) & \mathfrak{t}_{12}(X) \\
0 & \mathfrak{t}_{22}(X)
\end{pmatrix}_{d \times d},
\]

and a representation is said to be irreducible if it is not reducible. A representation of a Lie algebra \(\mathfrak{g}\) is said to be completely reducible if it is equivalent to a representation \(\mathfrak{t}'\) of \(\mathfrak{g}\) that
for every $X \in g$ has the form

$$t'(X) = \begin{pmatrix}
  t'_{11}(X) & 0 & 0 & \ldots \\
  0 & t'_{22}(X) & 0 & \ldots \\
  0 & 0 & t'_{33}(X) & \ldots \\
  \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}_{d \times d},$$  \hspace{1cm} (A.6)

where $t_{ii}$ are irreducible representations of $g$.

**Definition** Let $g$ be a Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$. The collection of all possible products of $X \in g$ taken in all possible orders ($K \oplus g \oplus (g \otimes g) \oplus (g \otimes g \otimes g) \ldots$) is called the universal enveloping algebra, $U(g)$, of $g$.

The center $\mathfrak{z}$ of the universal enveloping algebra $U(g)$ is the set of all elements $C$ in $U(g)$ which satisfy

$$[C, X] = 0, \quad \text{for all } X \in g. \hspace{1cm} (A.7)$$

The operators $C$ are said to be invariant. The dimension of the center $\mathfrak{z}$ (the number of the invariant operators of $g$) defines the rank $l$ of $g$ (later on we relate the rank of $g$ to the number of the commuting operators in $g$). The invariant operator of $g$ that is second-order in the basis elements is called the second-order Casimir invariant,

$$C_2 = g^{ik}X_iX_k, \hspace{1cm} (A.8)$$

where the symmetric Cartan metric tensor is defined as $g_{ik} = \text{Tr} \text{ad}(X_i)\text{ad}(X_k) = \delta_{ij}\delta_{lk}$. For every semi-simple Lie algebra $g$ of dimension $d$ and of rank $l$ there exists a set of $l$ invariant polynomials of operators $X_i$, $i = 1, 2, \ldots, d$, whose eigenvalues completely characterize the finite dimensional irreducible representations.

The following table A.1 shows the dimension $d$ (the number of basis elements in $g$) and the rank $l$ (the number of invariant operators of $g = \text{number of commuting operators in } g$) of a Lie algebra realized in $n$ dimensions (for example, the $\mathfrak{sp}(4)$ Lie algebra realized in terms of four tensor operators, $(t_{1/2}), (t_{-1/2}), (t_{1/2})^\dagger (t_{-1/2})^\dagger$, is 10-dimensional, of rank 2).

Table A.1: Properties of algebras.

<table>
<thead>
<tr>
<th>algebra</th>
<th>$d$ (dimension of $g$)</th>
<th>$l$ (rank of $g$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{u}(n)$</td>
<td>$n^2$</td>
<td>$n$</td>
</tr>
<tr>
<td>$\mathfrak{su}(n)$</td>
<td>$n^2 - 1$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$\mathfrak{so}(n)$</td>
<td>$\frac{1}{2}n(n - 1)$</td>
<td>$\lceil n/2 \rceil$</td>
</tr>
<tr>
<td>$\mathfrak{sp}(n)$</td>
<td>$\frac{1}{2}n(n + 1)$</td>
<td>$n/2$</td>
</tr>
</tbody>
</table>

Next, we introduce the concept of a group and the correspondence between Lie groups and their Lie algebras.

**Definition** A group $G$ is a set of elements $x_1, x_2, \ldots, x_m \in G$ with an operation called group multiplication such that
1. if \( x_i, x_j \in G \) then \( x_i x_j \in G \) (closure),

2. if \( x_i, x_j, x_k \in G \) then \( x_i(x_j x_k) = (x_i x_j) x_k \) (associativity),

3. there exists an element \( e \in G \) such that \( ex = xe = x \), for all \( x \in G \) (existence of identity),

4. for every \( x \in G \) there exists an element \( x^{-1} \in G \) such that \( xx^{-1} = x^{-1}x = e \) (existence of unique inverse),

where \( m \) is the order of \( G \) (under the assumption that the \( m \) elements are independent in \( G \) such that every \((m + 1)\)th element in \( G \) can be expressed in terms of these \( m \) elements).

**Definition** An abstract group \( G \) is said to be a **Lie group** if \( G \) is an analytic manifold and the mapping \((x, y) \rightarrow xy^{-1}\) of the product manifold \( G \times G \) into \( G \) is analytic. (To every Lie group \( G \) there is a unique (up to isomorphism) Lie algebra with basis elements being the tangent vectors at the identity \( e \) of \( G \). Two Lie groups are **locally isomorphic** if and only if their Lie algebras are isomorphic.)

This is the mathematical aspect of a group — a set of elements that must obey the four axioms. In most physical applications, the elements of a group can be simply viewed as transformations of certain kinds. Such transformations are typically represented as matrices.

**Definition** If there exists a homomorphic mapping \( x \rightarrow T(x) \) of a group \( G \) onto a group of non-singular \( d \times d \) matrices \( T(x) \) with matrix multiplication as the group multiplication operation then the group of matrices \( T(x) \) forms a \( d \)-dimensional (matrix) **representation** \( T \) of \( G \). It satisfies the following conditions: \( T(xy) = T(x)T(y) \) and \( T(e) = 1 \) (generally, the map \( x \rightarrow T(x) \) is a representation of \( G \) if these conditions are satisfied).

Every element of the Lie group \( G \) lying in a small neighborhood of the identity \( e \in G \) can be parameterized by \( d \) parameters \( p_1, p_2, \ldots, p_d \) (no two such sets of parameters corresponding to the same element \( x \) of \( G \)) with the identity \( e \) being parametrized by \( p_1 = p_2 = \cdots = p_d = 0 \). Then the operators \( X^i = (\partial T(x(p_1, \ldots, p_d))/\partial p_i)_{p_1=\cdots=p_d=0} \), \( i = 1, 2, \ldots, d \), form the Lie algebra \( \mathfrak{g} \) associated with the Lie group \( G \) (the number of parameters, \( d \), of \( G \) is the same as the dimension of the Lie algebra \( \mathfrak{g} \) (Table A.1)). Every element \( X^i \) of the Lie algebra \( \mathfrak{g} \) of a finite Lie group \( G \) can be associated with one-parametric subgroup of \( G \) defined by \( T(x(0, \ldots, p_i, \ldots, 0)) = e^{X^i p_i} \), \( i = 1, 2, \ldots, d \), (no summation over the repeated indices), where \( G \) consists of all the one-parametric subgroups. The operators \( X^i \), \( i = 1, 2, \ldots, d \), play the role of **generators** of the transformations \( x \) (with respect to the \( p_i \) parameter) represented by (the matrix) \( T(x) \). They are also called **infinitesimal operators** for the \( G \) group since for \( x \) in an infinitesimally small neighborhood of \( e \) they induce the infinitesimal changes of \( T(x) \), \( X^i = \lim_{p_i \to 0} \frac{T(x)-1}{p_i} \).

As a familiar example, consider the SO(3) group of rotations around a unit vector in the three-dimensional coordinate space \( \mathfrak{n} \), which has the elements \( \mathfrak{D}(\mathfrak{n}, \phi) = e^{-\frac{i}{2} \phi \mathbf{J} \mathfrak{n}} \) with a set of parameters \( \phi = (\phi_x, \phi_y, \phi_z) \). Hence, \( \mathbf{J} \) are the generators of three-dimensional rotations in \( \mathbb{R}^3 \) and \( J_x, J_y, J_z \) close on the \( \mathfrak{so}(3) \) Lie algebra. An example related to matrix representations is the Lie algebra \( \mathfrak{sl}(n, \mathbb{C}) \), which is realized as the set of all \( n \times n \) complex traceless matrices.

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The $\text{SL}(n, \mathbb{C})$ group then consists of all elements $x = e^{(p)X}, \ X \in \mathfrak{sl}(n, \mathbb{C})$, which by virtue of the identity $\det e^{(p)X} = e^{(p)\text{Tr}X}$, is the set of all $n \times n$ unimodular (det = 1) matrices.

Transformations (as elements of the Lie group $G$ with a Lie algebra $\mathfrak{g}$) that do not change the energy of a system ($G H G^{-1} = H \leftrightarrow [H, \mathfrak{g}] = 0$, where $H$ is the Hamiltonian operator) determine the symmetries that the system possesses. Symmetries play a crucial role in finding the equation of motion of a quantum mechanical system since they imply conserved quantities (integrals of motion).

### A.1 Cartan Subalgebra and Root Subspaces

**Definition** A subalgebra $\mathfrak{h}$ of a semi-simple algebra $\mathfrak{g}$ is called a *Cartan subalgebra* if

(i) $\mathfrak{h}$ is a maximal abelian subalgebra in $\mathfrak{g}$, and

(ii) $\text{ad}(H)$ is completely reducible for every $H \in \mathfrak{h}$, where $\text{ad}$ is the adjoint representation of $\mathfrak{h}$.

The dimension of the Cartan subalgebra $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ defines the rank $l$ of $\mathfrak{g}$ (Table A.1).

Let $H_1, H_2, \ldots, H_l$ be a basis of a Cartan subalgebra $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ of rank $l$ and dimension $d$. Then the $d \times d$ matrices $\text{ad}(H_i), i = 1, 2, \ldots, l$, are simultaneously diagonalizable and there exists a basis $H_1, H_2, \ldots, H_l, E_{a_1}, E_{a_2}, \ldots, E_{a_{d-l}}$ of $\mathfrak{g}$, called the *Cartan-Weyl basis*, such that for a linear function $\alpha$ on the complex vector space $\mathfrak{h} \subset \mathfrak{g}$ the linear subspace $\mathfrak{g}^\alpha$ of $\mathfrak{g}$ is defined by the condition

$$\mathfrak{g}^\alpha = \{ E \in \mathfrak{g} : [H, E] = \alpha(H)E \ \text{for all} \ H \in \mathfrak{h} \}.$$  \hspace{1cm} (A.9)

If $\mathfrak{g}^\alpha \neq \{0\}$ then $\alpha$ is called a (non-zero) *root* and $\mathfrak{g}^\alpha$ the root subspace. Let $\Delta$ be the set of non-zero roots then the Lie algebra is a direct sum of the Cartan subalgebra and the root subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$  \hspace{1cm} (A.10)

and $\dim \mathfrak{g}^\alpha = 1$ for every $\alpha \in \Delta$ (that is, roots are non-degenerate except $\alpha = 0$). If $\alpha, \beta \in \Delta$ then (i) $-\alpha \in \Delta$ such that if $E_\alpha \in \mathfrak{g}^\alpha, E_{-\alpha} \in \mathfrak{g}^{-\alpha}$ then $[E_\alpha, E_{-\alpha}] \subset \mathfrak{h}$, $\alpha \neq 0$, and (ii) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha + \beta}$ if $\alpha + \beta \neq 0$.

If we choose as the basis of the Lie algebra the Cartan-Weyl basis, we obtain the so-called *Cartan-Weyl set of commutation relations* for a semi-simple complex Lie algebra, which for all $\alpha, \beta \in \Delta$ hold in the form

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha, \quad H_i \in \mathfrak{h},$$
$$[E_\alpha, E_\beta] \begin{cases} = 0, & \alpha + \beta \neq 0, \alpha + \beta \notin \Delta \\ \sim \alpha \cdot \mathbf{H}, & \alpha + \beta = 0, \\ \sim E_{\alpha + \beta}, & \alpha + \beta \in \Delta, \end{cases}$$  \hspace{1cm} (A.11)
where $\alpha$ and $H$ are vectors in the $\mathfrak{h}$ vector space with components $\alpha(H_i)$ and $H_i$, $i = 1, 2, \ldots, l$, respectively. In the Cartan-Weyl basis of $\mathfrak{g}$ the second-order Casimir invariant (A.8) has the form

$$C_2 = \sum_{i=1}^{l} g^{ik} H_i H_k + \sum_{\alpha \in \Delta} E_\alpha E_{-\alpha},$$

(A.12)

where the Cartan metric tensor $g_{ik}$ is defined after (A.8).

All generators $H_i$ of the Cartan subgroup $H \subset G$ are diagonal in the space of the representations of the $G$ group. In physical applications, if $G$ is a symmetry group of some physical system, the generators $H_i$ are simultaneously diagonalizable, hence observables.

### A.2 Symplectic $\mathfrak{sp}(4)$ Lie Algebra: Root System

Basis operators in the Cartan subalgebra of $\mathfrak{sp}(4)$ are the two commuting $N_{\pm 1}$ operators. The root system for $\mathfrak{sp}(4)$ (roots and root subspaces) can be represented by a root space diagram, which is the one for $C_2$. Let $e_1$ and $e_2$ be orthogonal vectors along “the $\frac{1}{2}(N_+ - \Omega)$ and $\frac{1}{2}(N_- - \Omega)$” in a two-dimensional plane and their length be $1/2$ (Figure A.1). Then for

$$\mathbf{H} = \left( \frac{N_+ - \Omega}{2}, \frac{N_- - \Omega}{2} \right)$$

the root subspaces and the non-zero roots, expressed in terms of the

![Figure A.1: Root space diagram of $\mathfrak{sp}(4)$ ($C_2$).](image-url)
simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2 = (\frac{1}{2}, -\frac{1}{2})$ (short) and $\alpha_2 = 2\mathbf{e}_2 = (0, 1)$ (long), are

$$\sqrt{\Omega} A_{+1}^\dagger, \quad 2\alpha_1 + \alpha_2 \rightarrow (1, 0),$$
$$\sqrt{\Omega} A_{-1}, \quad -2\alpha_1 - \alpha_2 \rightarrow (-1, 0),$$
$$\sqrt{\Omega} A_{-1}^\dagger, \quad \alpha_2 \rightarrow (0, 1),$$
$$\sqrt{\Omega} A_{-1}, \quad -\alpha_2 \rightarrow (0, -1),$$
$$\sqrt{2\Omega} A_0^\dagger, \quad \alpha_1 + \alpha_2 \rightarrow (\frac{1}{2}, \frac{1}{2}),$$
$$\sqrt{2\Omega} A_0, \quad -\alpha_1 - \alpha_2 \rightarrow (-\frac{1}{2}, -\frac{1}{2}),$$
$$\sqrt{2\Omega} \tau_+, \quad \alpha_1 \rightarrow (\frac{1}{2}, -\frac{1}{2}),$$
$$\sqrt{2\Omega} \tau_-, \quad -\alpha_1 \rightarrow (-\frac{1}{2}, \frac{1}{2}). \quad (A.13)$$

Clearly, the dimension of each root subspace is one and $\mathfrak{sp}(4)$ is a direct sum of the Cartan subalgebra with a basis $N_{\pm 1}$ and the eight root subspaces in (A.13). The ten basis operators constitute the Cartan-Weyl basis. The infinitesimal operators which generate $\mathfrak{sp}(4)$ are a subset of those which generate the group $\mathfrak{su}(4)$, where the latter are denoted by $A_{ij}$ with $i, j = 1, \ldots, 4$, $\sum A_{\alpha\alpha} = 0$, and commutation properties $[A_{ij}, A_{kl}] = A_{il}\delta_{jk} - A_{kj}\delta_{il}$. The root space diagram for $\mathfrak{so}(5)$ ($B_2$) can be obtained from the one for $\mathfrak{sp}(4)$ ($C_2$) by a simple $+45^\circ$ rotation, where the basis in the Cartan subalgebra is $\{\hat{N} \text{ and } \tau_0\}$.

Based on the root system (A.13) the general form of the commutation relations of $\mathfrak{sp}(4)$ ($\sim \mathfrak{so}(5)$) can be given as follows ($N_{l=0,\pm}$ are defined in Table 2.1)

$$\begin{aligned}
[\hat{N}, A_{\pm}^k] &\subset A_{\pm}^k, \quad (A.14) \\
[\hat{N}, \tau_l] &= 0, \quad (A.15) \\
[\tau_l, \tau_k] &\subset \tau_{l+k}, \quad (A.16) \\
[A_{\pm}^l, A_{\pm}^k] &= 0, \quad (A.17) \\
[A_{l}^\dagger, A_k] &\subset \tau_l \oplus \tau_{l+k}, \quad (A.18)
\end{aligned}
$$

$$\begin{aligned}
[l + k = 0 \quad l + k \neq 0]
[\tau_l, A_{\pm}^k] &\subset A_{l+k}^\pm, \quad (A.19) \\
l, k = 0, \pm 1.
\end{aligned}$$

For $\hat{N} = N_{+1} + N_{-1}$ and $\tau_0 = (N_{+1} - N_{-1})/2$, the commutation relations can be compared to the ones (A.11) for the Cartan-Weyl basis of $\mathfrak{sp}(4)$. 

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Appendix B

“Second” Second Quantization or Quantum Group (Algebra) Concept

The name “quantum groups” and their precise formulation is given by Drinfeld [21]. In fact, they are not groups but algebras (that is why we refer to them as $q$-deformed algebras) and their notion may not be related to any quantization. Although the concept may be referred as a “second” second quantization, it is the very formalism of second quantization [85, 86, 87] that does not introduce a subsequent quantization of standard quantum mechanics, nor quantum field theory, but it is rather an occupation number representation. Hence, its “second” quantization ($q$-deformation) can be viewed as an occupation number representation with different symmetries imposed on the nucleon wave functions implying different ($q$-deformed) commutation relations between the creation and annihilation operators. (The concept of quantum groups can also enter as a deformation quantization [213] to another (quantization) level of the theory, but this is a feature outside of the scope of this work.)

It takes a chain of definitions to define the concept of a quantum group [21, 51]. The definitions of quantum group, Hopf algebra (introduced by H. Hopf in 1941), bi-algebra and antipode are next presented.

B.1 Definition of Quantum Groups

**Definition** A *quantum group* is defined to be a (not necessary commutative) Hopf algebra.

**Definition** A *Hopf algebra* is a bi-algebra with antipode.

**Definition** Let $a$ be an associative algebra with unity $1$ over a field $K$ of complex numbers ($\mathbb{C}$). Then a *bi-algebra* on $a$ is defined by four morphisms $m, \Delta, \eta$ and $\epsilon$

\[
\begin{align*}
   a \otimes a &\xrightarrow{m} a \\
   a &\xrightarrow{\Delta} a \otimes a, \\
   K &\xrightarrow{\eta} a \\
   a &\xrightarrow{\epsilon} K,
\end{align*}
\]

where

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• the linear mapping, $m$, is the multiplication in $a$: $m(X \otimes Y) = XY$ for all $X, Y \in a$,
• the homomorphism of $a$, $\Delta$, is called the co-multiplication,
• the operation $\eta$ is defined by $\eta(c) = c1$ for all $c \in K$,
• the homomorphism of $a$, $\epsilon$, is called the co-unit and $\epsilon(XY) = \epsilon(X)\epsilon(Y)$ for all $X, Y \in a$,

and which satisfy the following axioms (given also as commutative diagrams):

1. associativity: $m(m \otimes \text{id}) = m(\text{id} \otimes m)$, where $\text{id}$ denotes the identity mapping (of $a$),

\[
\begin{array}{ccc}
\text{id} \otimes m & \Rightarrow & a \\
\downarrow & & \downarrow m \\
\text{id} \otimes m & \Rightarrow & a \otimes a
\end{array}
\]  

\[
\begin{array}{ccc}
m \otimes \text{id} & \Rightarrow & a \\
\downarrow & \Rightarrow & \downarrow m \\
m \otimes \text{id} & \Rightarrow & a \otimes a
\end{array}
\]  

(B.2)

2. co-associativity: $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$,

\[
\begin{array}{ccc}
\Delta & \Rightarrow & \Delta \otimes \text{id} \\
\downarrow & \Rightarrow & \downarrow \Delta \otimes \text{id} \\
\Delta & \Rightarrow & \Delta \otimes \Delta
\end{array}
\]

(B.3)

3. unit: $m(X \otimes 1) = m(1 \otimes X)$,

\[
\begin{array}{ccc}
\text{id} \otimes \eta, \eta \otimes \text{id} & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow m \\
\text{id} \otimes \eta, \eta \otimes \text{id} & \Rightarrow & a \otimes a
\end{array}
\]  

(B.4)

4. co-unit: $(\epsilon \otimes \text{id})\Delta = (\text{id} \otimes \epsilon)\Delta$,

\[
\begin{array}{ccc}
\Delta & \Rightarrow & \epsilon \otimes \text{id}, \text{id} \otimes \epsilon \\
\downarrow & \Rightarrow & \downarrow \epsilon \otimes \text{id}, \text{id} \otimes \epsilon \\
\Delta & \Rightarrow & \epsilon \otimes \text{id}, \text{id} \otimes \epsilon
\end{array}
\]  

(B.5)

5. connecting axiom:

\[
\begin{array}{ccc}
m & \Rightarrow & \Delta \\
\downarrow & \Rightarrow & \downarrow \Delta \\
m & \Rightarrow & \Delta \otimes \Delta
\end{array}
\]

\[
\begin{array}{ccc}
\Delta \otimes \Delta & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow \Delta \otimes \Delta \\
\Delta \otimes \Delta & \Rightarrow & a \otimes a
\end{array}
\]

\[
\begin{array}{ccc}
a \otimes a & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow a \otimes a \\
a \otimes a & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow a \otimes a \\
a \otimes a & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow a \otimes a \\
\Delta & \Rightarrow & \Delta \otimes \Delta
\end{array}
\]

\[
\begin{array}{ccc}
a \otimes a & \Rightarrow & a \otimes a \\
\downarrow & \Rightarrow & \downarrow a \otimes a \\
\Delta & \Rightarrow & \Delta \otimes \Delta
\end{array}
\]

(B.6)

where $S_{(23)}$ is the morphism exchanging the second and third places in the product.
Definition An antipode of a bi-algebra \((a, m, \Delta)\) is a linear map \(\gamma: a \to a\) such that

\[
m(id \otimes \gamma)\Delta(X) = m(\gamma \otimes id)\Delta(X) = \epsilon(X)1,
\]

where \(X \in a\), that is, the following diagram is commutative

\[
\begin{array}{ccc}
\Delta & \gamma \otimes id & \Delta \\
a \otimes a & a \otimes a & a \\
\alpha & \epsilon & \epsilon \\
\Delta & id \otimes \gamma & a \otimes a
\end{array}
\]

(B.7)

The \(\gamma\) antipode is anti-homomorphism: \(\gamma(XY) = \gamma(Y)\gamma(X)\) for all \(X, Y \in a\) and it reverses multiplication and co-multiplication.

B.2 Standard \(U_q(\mathfrak{su}(2))\) Deformation

The standard Drinfeld-Jimbo quantum deformation of the \(\mathfrak{su}(2)\) Lie algebra, \(U_q(\mathfrak{su}(2))\), is characterized by the commutation relations

\[
[J_+, J_-] = [2J_0], \quad [J_0, J_{\pm}] = \pm J_{\pm},
\]

(B.9)

where the basis operators, \(J_{0,\pm}\), are \(q\)-deformed. The algebra, which is over \(\mathbb{C}\), has a unit denoted by \(1\). The \(U_q(\mathfrak{su}(2))\) algebra is a Hopf algebra when is completed by the operations

co-product:

\[
\Delta(J_{\pm}) = J_{\pm} \otimes q^{J_0} + q^{-J_0} \otimes J_{\pm},
\]

\[
\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0,
\]

coop-unit:

\[
\epsilon(1) = 1, \quad \epsilon(J_{0,\pm}) = 0,
\]

antipode:

\[
\gamma(J_{\pm}) = -q^{\pm}J_{\pm}, \quad \gamma(J_0) = -J_0.
\]

(B.10)

The concept of co-multiplication can be understood in physics by the example of the angular momentum operator \(J\) in quantum mechanics, which can be added as \(J = J^{(1)} + J^{(2)}\). More precisely, as the total angular momentum operator acts on product kets, \(|\psi\rangle = |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)}\), the operator addition is \(J = J^{(1)} \otimes 1 + 1 \otimes J^{(2)}\). This action actually defines a co-multiplication \(\Delta: \Delta(J) = J \otimes 1 + 1 \otimes J\). In other words, the vector addition of angular momentum in quantum mechanics defines a commutative co-multiplication in a bi-algebra. In general, the co-multiplication in quantum group symmetry is not commutative, which means that the addition of \(q\)-angular momenta depends on the order \([51]\).
Appendix C

$q$-Deformed $\mathfrak{sp}_q(4)$ Algebra: a Further Discussion

C.1 On the Mapping of the $q$-Deformed Fermion Operators to Their “Classical” Counterparts

The $q$-deformed fermion operators that satisfy the anticommutation relations (2.99) (or (2.64) in the single-$j$ case) and that enter into the fermion realization of $\mathfrak{sp}_q(4)$ do not have a known transformation function to their “classical” counterparts. We have already mentioned in Section 2.3.4 that a different set of anticommutation relations (2.89) leads to an analytic mapping of the $q$-deformed fermion operators to the “classical” ones but such a $q$-deformation turns out to be trivial. Regarding the $q$-deformation that we use for the nuclear structure applications (2.99) we make the following proposition.

**Proposition** Let $\Omega$ be the generalized dimension of fermion occupation space and be a fixed integer parameter. There does not exist an analytic mapping $F$ of $q$-deformed fermion operators, $A = \{(\alpha_{j^m\sigma}^\dagger, \alpha_{j^m\sigma})$ for all $j^m\sigma\}$, which satisfy the anticommutation relations (2.99)

\[
\alpha_{j^m\sigma}\alpha_{j'^{m'}\sigma'}^{\dagger} + q^{\pm 1}\alpha_{j'^{m'}\sigma'}^{\dagger}\alpha_{j^m\sigma} = q^{\pm \frac{Nq^2r}{2\Omega} \delta_{j,j'} \delta_{m,m'}},
\]

\[
\{\alpha_{j^m\sigma}, \alpha_{j'^{m'}\sigma'}^{\dagger}\} = 0 \ (\sigma \neq \sigma'), \hspace{1cm} \{\alpha_{j^m\sigma}^{\dagger}, \alpha_{j'^{m'}\sigma'}\} = 0, \hspace{1cm} \{\alpha_{j^m\sigma}, \alpha_{j'^{m'}\sigma'}\} = 0,
\]

to “classical” fermion operators, $C = \{(c_{j^m\sigma}^{\dagger}, c_{j^m\sigma})$ for all $j^m\sigma\}$, of standard anticommutation relations (2.1), $c_{j^m\sigma}c_{j'^{m'}\sigma'}^{\dagger} + c_{j'^{m'}\sigma'}^{\dagger}c_{j^m\sigma} = \delta_{j,j'} \delta_{m,m'} \delta_{\sigma,\sigma'}$ and $\{c_{j^m\sigma}^{\dagger}, c_{j'^{m'}\sigma'}\} = \{c_{j^m\sigma}, c_{j'^{m'}\sigma'}^{\dagger}\} = 0$. The $q$-deformed objects transform to the “classical” operators only in the limit, $A \xrightarrow{q \to 1} C$, where they coincide.

**Proof** Let assume that there exists an analytic mapping $F$, $A \xrightarrow{F} C$, such that

\[
f_{\alpha_{j^m\sigma}^{\dagger}} = c_{j^m\sigma}^{\dagger}, \hspace{1cm} \alpha_{j^m\sigma}g = c_{j^m\sigma}, \hspace{1cm} (C.1)
\]
where \( f \) and \( g \) are continuous non-commutative \( q \)-functions that represent \( F \) and \( f, g \neq 0 \). If same Hermitian conjugation \(^{(1)}\) is assumed for both \( q \)-deformed and “classical” objects then \( f \) and \( g \) are adjoint \((f = g^\dagger)\).

Starting with the non-trivial “classical” standard anticommutation relation with \( j = j', m = m' \) and \( \sigma = \sigma' \), the \( F \) mapping \((C.1)\) yields

\[
\begin{align*}
    c_{j\sigma} c_{j\sigma}^\dagger + c_{j\sigma}^\dagger c_{j\sigma} &= 1 \Rightarrow \\
    \alpha_{j\sigma} g f \alpha_{j\sigma}^\dagger + f \alpha_{j\sigma}^\dagger \alpha_{j\sigma} g &= 1 \Rightarrow \\
    \alpha_{j\sigma} g f \alpha_{j\sigma}^\dagger + \alpha_{j\sigma}^\dagger \alpha_{j\sigma} f g' &= 1,
\end{align*}
\]

(C.2)

where we introduce the continuous function

\[
\alpha_{j\sigma} f \alpha_{j\sigma}^\dagger = \alpha_{j\sigma}^\dagger \alpha_{j\sigma} f'.
\]

(C.3)

The last equation in \((C.2)\) gives the corresponding anticommutation relation between the \( q \)-deformed fermion operators and can be compared to the explicit form of the latter \((2.99)\) (or refer to the Proposition above) after a few steps

\[
\begin{align*}
    \alpha_{j\sigma} \alpha_{j\sigma}^\dagger + q^{\pm 1} \alpha_{j\sigma}^\dagger \alpha_{j\sigma} &= q^{\pm \frac{N_2}{2\Omega}} \Rightarrow \\
    \alpha_{j\sigma} \alpha_{j\sigma}^\dagger q^{\frac{N_2}{2\Omega}} + q^{\pm 1} \alpha_{j\sigma}^\dagger \alpha_{j\sigma} q^{\frac{N_2}{2\Omega}} &= 1 \Rightarrow \\
    \alpha_{j\sigma} q^{\frac{N_2}{2\Omega} - 1} \alpha_{j\sigma}^\dagger + \alpha_{j\sigma}^\dagger \alpha_{j\sigma} q^{\frac{N_2}{2\Omega} + 1} &= 1,
\end{align*}
\]

(C.4)

where we use \((2.57)\). A comparison of both \((C.2)\) and \((C.4)\) leads to the following assignments

\[
g f = q^{\frac{N_2}{2\Omega} - 1}, \quad f' g = q^{\frac{N_2}{2\Omega} + 1}.
\]

(C.5)

The system of both equations relates \( f \) and \( f' \) as follows

\[
\begin{align*}
    q^{\frac{N_2}{2\Omega} - 1} = g f' &= q^{\frac{N_2}{2\Omega} + 1} g^{-1} \Rightarrow f' q^{\frac{N_2}{2\Omega} - 1} = q^{\frac{N_2}{2\Omega} + 1} g^{-1} gf \Rightarrow f' = q^{\frac{1}{2\Omega} \pm 1} f,
\end{align*}
\]

(C.6)

which by the definition of \( f' \) \((C.3)\) implies

\[
\begin{align*}
    f \alpha_{j\sigma} \alpha_{j\sigma} &= \alpha_{j\sigma}^\dagger \alpha_{j\sigma} q^{-\frac{1}{2\Omega} + 1} f \\
    f \alpha_{j\sigma} \alpha_{j\sigma} &= \alpha_{j\sigma}^\dagger \alpha_{j\sigma} q^{\frac{1}{2\Omega} - 1} f.
\end{align*}
\]

(C.7)

After both equations are subtracted

\[
0 = \alpha_{j\sigma} \alpha_{j\sigma} (q^{\frac{1}{2\Omega} - 1} - q^{-\frac{1}{2\Omega} + 1}) f;
\]

(C.8)

we find out the general solutions

\[
q^{\frac{1}{2\Omega} - 2} = 1 \quad \text{or} \quad f = 0,
\]

(C.9)
where we use that $\alpha_{jm\sigma}^{\dagger} \alpha_{jm\sigma} = [N_{2\sigma}/(2\Omega)]$ is in general non-zero. The result implies a trivial solution $q = 1$, which is expected since $A \rightarrow \mathcal{C}$, and two other solutions, $\Omega = 1/2$ and $f = 0$. Note that the first one, $\Omega = 1/2$, actually gives an analytic mapping, $f = g = q^{\pm N_{2\sigma} - 1/2}$ (with $f = g^{\dagger}$ being a $q$-function of the non-deformed self-adjoint operator $N_{2\sigma}$), but this solution is not physically valid ($\Omega \geq 1$ as $j \geq 1/2$). Hence, we are left with only one solution $f = 0$, but the $f$ transformation by construction cannot be zero resulting in a contradiction to the initial assumption. This proves the proposition. Therefore, there is no analytic transformation that maps the $q$-deformed fermion operators (2.99) to their “classical” counterparts.

C.2 Comparison between the $q$-Deformed $sp_q(4)$ and $so_q(5)$ Algebras

The realization of $sp_q(4)$ after the renormalization procedure (2.76) corresponds to the standard Drinfeld-Jimbo construction for $so_q(5)$ [24] with commuting Cartan generators, $q^{h_1}$ and $q^{h_2}$. The Cartan-Chevalley basis that consist of two triplets $(e_1, f_1, h_1)$ and $(e_2, f_2, h_2)$ corresponding to the simple roots $\alpha_1$ and $\alpha_2$, respectively, can be related to the $su_q(-2)$ operators when their commutation relations are compared to the ones in [24]

\begin{align*}
(e_1, f_1, h_1) &\leftrightarrow \left( \frac{T_{+}}{\sqrt{[\omega]}}, \frac{T_{-}}{\sqrt{[\omega]}}, T_{0} \right), \\
(e_2, f_2, h_2) &\leftrightarrow \left( \frac{B_{-1}^{\dagger}}{\sqrt{\rho_{\pm}[2\omega]}}, \frac{B_{1}}{\sqrt{\rho_{\pm}[2\omega]}}, N^{-} \right).
\end{align*}

The correspondence between the rest operators is

\begin{align*}
(e_\pm^3, f_\pm^3) &\leftrightarrow (B_{0}q_{\pm N_{-1}}, B_{0}q_{\pm N_{-1}}), \\
(e_4, f_4) &\leftrightarrow (B_{+1}, B_{-1}).
\end{align*}

The commutation relations between all the operators are equivalent in both algebras ($so_q(5)$ and $sp_q(4)$) with the exception of the two commutators $[e_l, f_l] = [2h_l]_{k=l}$, $l = 1, 2$, which correspond to their $sp_q(4)$ analog within the parameter $\omega$, since $k = \omega l$. The results prove the isomorphism of the $q$-deformed $sp_q(4)$ algebra and its standard $su_q(2)$ subalgebras to the $so_q(5)$ algebra and its subalgebraic structure.

C.3 Normalization Coefficients of the Basis States in the Limiting $su_q(2) \subset sp_q(4)$ Cases

The pair basis states, (2.19) and (2.77), are not orthonormalized as the isospin eigenvectors are (2.25). Although an orthogonal basis is preferable to work with, the orthogonalization of
(2.19) looses the physical and natural meaning of seniority quantum numbers and coupling schemes. However, the normalization of the pair states (2.19) is needed. Next, we present the normalization coefficients in an analytical form of the basis states in the limiting cases of $\text{sp}(4)$ and $\text{sp}_q(4)$, which are derived using (2.28) and (2.30) (and their $q$-deformed analogs). In general, the normalization coefficients can be computed numerically with the help of a computer code for non-commutative multiplication (Appendix D).

C.3.1 The $\text{su}^{\pm}(2)$ Limit: Identical-Particle Pairing

In the non-deformed case, a normalized pair basis states is given as

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{N}_\mu (n_1, n_0, n_{-1})} |n_1, n_0, n_{-1}\rangle,$$  \hspace{1cm} (C.14)

where the normalization coefficients for the lowest possible seniority number ($\nu_1 = n_0 = 0$ or 1) in this limit, $\mu = \pm$, are derived recursively

$$\mathcal{N}_\pm^2 (n_1, n_0, n_{-1}) = n_1! n_{-1}! \prod_{l=0}^{n_1-1} \left( 1 - \frac{n_0 + l}{\Omega} \right) \prod_{l=0}^{n_{-1}-1} \left( 1 - \frac{n_0 + l}{\Omega} \right).$$  \hspace{1cm} (C.15)

In the $q$-deformed case, the normalization coefficients of the normalized pair basis states,

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{M}_\mu (n_1, n_0, n_{-1})} |n_1, n_0, n_{-1}\rangle,$$  \hspace{1cm} (C.16)

for the case of maximum like-nucleon pairs ($\nu_1 = n_0 = 0$ or 1 and $\mu = \pm$) are calculated to be

$$\mathcal{M}_\pm^2 (n_1, n_0, n_{-1}) = \rho_+ \rho_- [n_1]_\Omega ! [n_{-1}]_\Omega ! \prod_{l=0}^{n_1-1} \left[ 1 - \frac{n_0 + l}{\Omega} \right] \times$$

$$\prod_{l=0}^{n_{-1}-1} \left[ 1 - \frac{n_0 + l}{\Omega} \right],$$  \hspace{1cm} (C.17)

where the $q$-deformed factorial is defined in (2.53).

C.3.2 The $\text{su}^0(2)$ Limit: Isovector $pn$ Pairing

In the “classical” case, the normalization coefficients of the normalized pair states (C.14) for the lowest possible seniority number ($\nu_0 = |i|$) in this reduction limit, $\mu = 0$, are found to be

$$\mathcal{N}_0 (0, n_0, 0) = \mathcal{P}_0 (0, n_0, 0),$$
$$\mathcal{N}_0 (n_1, n_0, 0) = \mathcal{P}_0 (n_1, n_0, 0) \mathcal{N} (n_1),$$
$$\mathcal{N}_0 (0, n_0, n_{-1}) = \mathcal{P}_0 (0, n_0, n_{-1}) \mathcal{N} (n_{-1}),$$  \hspace{1cm} (C.18)

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where a recursive procedure yields

\[ P_0^2 (n_1, n_0, n_{-1}) = n_0! \prod_{k=0}^{n_0-1} \left( 1 - \frac{2(n_{-1} + n_1) + k}{2\Omega} \right), \quad (C.19) \]

\[ N_0^2 (n_{\pm1}) = n_{\pm1}! \prod_{l=0}^{n_{\pm1}-1} \left( 1 - \frac{l}{\Omega} \right). \quad (C.20) \]

The last result (C.20) is consistent with (C.15) for \( n_0 = 0 \).

The \( q \)-deformed normalized pair basis states (C.16) for the case of maximum \( pn \) pairs \( (\nu_0 = |i| \text{ and } \mu = 0) \) have the normalization coefficients

\[ M_0 (0, n_0) = Q_0 (0, n_0) , \]
\[ M_0 (n_1, n_0) = Q_0 (n_1, n_0) M (n_1) , \]
\[ M_0 (0, n_0, n_{-1}) = Q_0 (0, n_0, n_{-1}) M (n_{-1}) , \quad (C.21) \]

where we derive

\[ Q_0^2 (n_1, n_0, n_{-1}) = [n_0]_{\frac{1}{2\Omega}}! \prod_{k=0}^{n_0-1} \left[ 1 - \frac{2(n_{-1} + n_1) + k}{2\Omega} \right]. \quad (C.22) \]

The normalization coefficients \( M (n_{\pm1}) \) of a state with \( n_0 = 0 \) and \( n_{\mp1} = 0 \) is obtained with the help of (C.17),

\[ M^2 (n_{\pm1}) = \rho_{\pm} [n_{\pm1}]_{2\Omega}! \prod_{l=0}^{n_{\pm1}-1} \left( 1 - \frac{l}{\Omega_j} \right). \quad (C.23) \]
Appendix D

The Job of the CPU

In the XXIst century, at the threshold of the third millennium, computers go hand-in-hand with science. They are as fast as user-friendly. Computers offer a great help to scientific projects especially regarding impossible for humans tasks. At the same time science gives its best to help computers develop, a recent example being quantum computers.

Parallel to our project in nuclear structure physics, a MATHEMATICA [214] computational package is developed for non-commutative (NC) calculations in the framework of the fermion realization of the \( \mathfrak{sp}(4) \) algebra and its non-linear extension \( \mathfrak{sp}_q(4) \). The development of such computer programs is driven by the need from the theoretical viewpoint for a great number of symbolic (rather than numerical) calculations, which are especially more complicated in the \( q \)-deformed case.

Although the concept of a non-commutative product (\( \ast \ast a \neq ab \)) is present in MATHEMATICA (as a basic function “NonCommutativeMultiply” or equivalently “\( \ast \ast \)”), even the notion of a commutator is not internally defined there. Presently, add-on packages exist with definitions for some NC-algebraic operations, which are primarily constructed to handle quantum mechanical operators [215, 216]. Gueorguiev has implemented a different and general approach to group theory computations based on the concept of fermion or boson realizations, which in addition has the flexibility to be applied to various algebraic structures only by assigning the respective (anti-) commutation relations between the one-index (fermion) boson operators [70]. It is a very efficient and fast computational technique, which is in the foundation of the NC-algebra programs we develop for fermions with two indices (such as \( \alpha^\dagger_{m\sigma} \) and \( \alpha_{m\sigma} \) in \( \mathfrak{sp}_q(4) \)). In addition, the construction of the \( \mathfrak{sp}_q(4) \) operators (as (2.103)-(2.100)) requires an implementation of a subroutine that defines symbolic rules to handle \( \delta \)-function summations (discrete Dirac function). In doing this, we aim on symbolic algebraic NC-computations rather than numerical results.

The computational package is mainly used to calculate or verify various relations of the basis operators of \( \mathfrak{sp}_q(4) \) and their action on the basis states, as well as the normalization of the basis states and their overlaps.

The eigenvalue pairing problem as well as the estimation of the interaction strength parameter make use of internally defined functions in MATHEMATICA. The non-linear fitting
procedure is accomplished using Levenberg-Marquardt method for minimization of the $\chi^2$ statistics. Given initial values for the parameters the procedure proceeds iteratively and is repeated until $\chi^2$ effectively stops decreasing. Levenberg-Marquardt method performs very well in practice although it is not uncommon, especially for highly non-linear systems, to find the parameters wandering around the minimum in a flat valley of complicated topography. Also, a complete failure by a zero pivot is possible, but very unlikely [217].

Such a method gives no guarantee that the minimum found is the global one. In fact, it is most likely that local minima are detected. This feature actually turns into advantage in regard to physical applications. This is because a reasonable local minimum with physically valid estimate for the fitting parameters is better than a global one with invalid outcome. The existence of such an optimal solution that is not of a physical interest follows from limitations of the model Hamiltonian, restricted model space or limited experimental data. However, a good initial guess of the parameters and a reasonable judgement of the outcome help the computers to help us the most.
Vita

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