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Nicolas Savy
University of Toulouse III, Toulouse Institute of Mathematics UMR C5583, Toulouse, France, nicolas.savy@math.univ-toulouse.fr

Josep Vives
Universitat de Barcelona, josep.vives@ub.edu

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ANTICIPATIVE INTEGRALS WITH RESPECT TO A FILTERED LÉVY PROCESS AND LÉVY–ITÔ DECOMPOSITION

NICOLAS SAVY AND JOSEP VIVES*

Abstract. A filtered process X^k is defined as an integral of a deterministic kernel k with respect to a stochastic process X. One of the main problems to deal with such processes is to define a stochastic integral with respect to them. When X is a Brownian motion one can use the Gaussian properties of X^k to define an integral intrinsically. When X is a jump process or a Lévy process, this is not possible. Alternatively, we can use the integrals defined by means of the so called $\mathcal{S}$-transform or by means of the integral with respect to the process X and a linear operator $\mathcal{K}$ constructed from k. The usual fact that even for predictable Y, $\mathcal{K}^*(Y)$ may not be predictable forces us to consider only anticipative integrals. The aim of this paper is, on the one hand, to clarify the links between these integrals for a given X and on the other hand, to investigate how the Lévy–Itô decomposition of a Lévy process L, roughly speaking $L = B + J$, where $B$is a Brownian motion and J is a pure jump Lévy process, behaves with respect to these integrals.

1. Introduction.

A filtered process or Volterra process X^k is a stochastic process defined as the integral of a deterministic kernel $(s,t) \to k(t,s)$ with respect to an underlying process X. This class of processes is wide and includes, for instance, the fractional Brownian motion, where $X = B$ is a Brownian motion and k a particular kernel (defined by (2.3) in Section 2.4) and the shot noise process, where $X = N$ is a Poisson process and $k(t,s) = g(t-s)$, where g is a deterministic function. Both examples are very important for applications in many domains.

Fractional Brownian motion, a type of Gaussian process introduced in [19], is today a well-known object and several books that treat stochastic calculus with respect to this process are available. We mention [6], [20], and [21]. Applications to finance can be found in [28].

On the other hand, a shot noise process was introduced in [18] and applied to computer failure times. In [15], an application to risk theory is presented and in [25] applications to finance can be found.

In this paper, X will denote a Brownian motion, a pure jump Lévy process or a general Lévy process. In order to make X^k relevant for applications, we have

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to define a stochastic integral with respect to it, but even for a simple function k, X^k is not a martingale nor a Markov process, thus classical techniques developed for defining integrals do not work for these processes.

Many investigations have been made to define such integrals. In the Brownian setting, B^k remains a Gaussian process, thus we can define a stochastic integral, denoted by $\delta_C^{B,k}$, by the use of the chaos decomposition (see [2]). This construction is not possible for pure jump and Lévy filtered processes because these processes are no more Lévy processes.

The $\mathcal{S}$-transform allows us to define directly an integral for filtered processes. Classical references for the $\mathcal{S}$-transform in the White Noise Analysis setting are [16] and [23]. In the following, these integrals will be denoted by $\delta_S^{B,k}$ in the Brownian motion case, by $\delta_S^{J,k}$ in the pure jump case and by $\delta_S^{L,k}$ in the general Lévy case. The Brownian case is studied in [4] and the pure jump case in [5]. The general case will be treated in Section 3.2 of this paper.

Finally, a more versatile idea is to construct, from the kernel k, a linear operator denoted by $\mathcal{K}^*$, and define a stochastic integral with respect to X^k, that will be denoted by $\delta^{X,\mathcal{K}^*}$, from the one with respect to X denoted δ^X by:

$$\delta^{X,\mathcal{K}^*}(Y) = \delta^X(\mathcal{K}^*(Y)).$$

Even for predictable processes Y, $\mathcal{K}^*(Y)$ may be not predictable, thus δ^X has to be defined in a anticipative way, see [9]. So, we have to browse the definitions of anticipative integrals with respect to X. Three main types of constructions have been investigated:

- First, we consider δ_C^X defined by the use of chaos decomposition. This construction has been widely investigated. Let us give some examples of references: for Brownian motion [21], for the standard Poisson process [22] and for Lévy processes [13] and [27].
- Second, we consider δ_S^X defined by the use of the $\mathcal{S}$-transform. In this case, some references are: for Brownian motion [4], for pure jump Lévy processes [5] and for general Lévy processes we will give the definition in this paper.
- Finally, we evoke δ_G^X the integral defined as the adjoint of a stochastic gradient. Some references about this integral are: for Brownian motion, [21, 29], for the standard Poisson process, [7] and for a marked Poisson process, [11]. As far as we know, no version of δ_G^X for a general Lévy process is studied in the literature, whereas a direct definition as a dual operator could be introduced from the gradient operator defined in [17]. It is well known that in the Brownian case $\delta_G^B = \delta_C^B$, meanwhile, even in the simple cases developed in [7] and [11], we have $\delta_G^J(Y) \neq \delta_C^J(Y)$. So we will not consider this kind of integral for the comparisons we investigate here.

The main topic of this paper is, first of all, to highlight the links among these integrals and to deal with what we will call the Lévy–Itô problem. Lévy–Itô decomposition tells us that a Lévy process can be decomposed in two independent components, a Brownian one and a pure jump one. Thus, it is natural to wonder if this decomposition is still true for the integrals considered in this paper. One says that the Lévy–Itô problem is true if for any Y in a suitable domain, we have, roughly speaking,

$$\delta^L(Y) = \delta^B(Y) + \delta^J(Y),$$

and one says that the Lévy–Itô problem is complete if moreover $\delta^B(Y)$ and $\delta^J(Y)$ are independent.

The paper is organized as follows. Section 2 is devoted to the construction of filtered processes. The notations are highlighted and the assumptions discussed. Section 3 browses the different notions of anticipative integrals with respect to the underlying processes. The links between these integrals are studied and the Lévy–Itô problem is investigated. Finally, in Section 4 we develop the construction of anticipative integrals with respect to filtered processes. We compare the different definitions and give some results about the Lévy–Itô problem.

2. Definition of Filtered Lévy Processes.

In the whole paper ℓ denotes the Lebesgue measure on $[0,T]$, $\mathcal{L}^2([0,T])$ denotes the space of square integrable deterministic functions defined on $[0,T]$ equipped with ℓ, and δ_a denotes the Dirac measure concentrated at $\{a\}$.

2.1. The general definition of filtered processes.

Definition 2.1. Let $X = \{X_t,\ t \in [0,T]\}$ be a stochastic process. Let $k : [0,T]^2 \to \mathbb{R}$ be a deterministic function. The *filtered process* X^k of underlying process X is defined by

$$X_t^k := \int_0^t k(t,s)\,\mathrm{d}X_s, \quad t \in [0,T].$$

2.2. The underlying process. Let $L = \{L_t,\ t \in [0,T]\}$ be a Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\{\mathcal{F}_t,\ t \in [0,T]\}$ its completed natural filtration. We refer the reader to [26] for a general theory of Lévy processes. One of the main properties of Lévy processes is the Lévy–Itô decomposition:

Theorem 2.2 (Lévy–Itô decomposition)**.** *There exists a triplet (γ, σ^2, ν), where $\gamma \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$ and ν is a Lévy measure, such that L can be represented as,*

$$L_t = \gamma t + \sigma B_t + J_t, \tag{2.1}$$

where

$$J_t := \int_0^t \int_{|z|>1} z\,\mathrm{d}N(s,z) + \lim_{\epsilon\to 0} \int_0^t \int_{\epsilon<|z|\le 1} z\,\mathrm{d}\tilde{N}(s,z), \quad t \in [0,T], \tag{2.2}$$

and

- *B is a standard Brownian motion,*
- *N is the jump measure associated to L:*

  $$N(E) = \operatorname{card}\{t : (t, \Delta L_t) \in E\} \quad \textit{for any } E \in \mathcal{B}([0,T]\times\mathbb{R}_0),$$

  where $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $\Delta L_t = L_t - L_{t-}$, and $\operatorname{card}\{A\}$ denotes the cardinal of the set A.

- *$\tilde{N}$ is the compensated jump measure associated to L:*

$$\mathrm{d}\tilde{N}(s,z) = \mathrm{d}N(s,z) - \mathrm{d}s \ \ \mathrm{d}\nu(z).$$

The limit in (2.2) *is a.s. uniform on every bounded interval.*

Hypothesis 2.3. The Lévy measure satisfies $\int_{\mathbb{R}} z^2 \,\mathrm{d}\nu(z) < \infty$.

This hypothesis will be useful later to define an integral related to filtered Lévy process. In the following, we will assume that this assumption is fulfilled.

Remark 2.4. Under Hypothesis 2.3, L_t can still be written by (2.1) with a modified γ and considering

$$J_t := \lim_{\epsilon\to 0} \int_0^t \int_{|z|>\epsilon} z \,\mathrm{d}\tilde{N}(s,z), \quad t \in [0,T].$$

Remark 2.5. It is well known that processes B and J are independent, and J is determined only by the measure ν (see for instance [27]).

Remark 2.6. In the case $\nu = 0$, the process L is a Brownian motion with drift γt and volatility σ. In the case $\sigma = 0$, we have a pure jump Lévy process. If, moreover, ν is a finite measure we can write $\nu = \lambda Q$, with Q a probability distribution on $\mathbb{R}$ and $\lambda = \nu(\mathbb{R})$. In this case, the process is a compound Poisson process.

Remark 2.7. For simplicity, we will assume from now on that $\gamma = 0$. In fact, integrals with respect to the "γt" component are nothing but deterministic ones.

2.3. The kernel.

Hypotheses 2.8.

(1) For any $t \in [0,T]$, the function $\begin{array}{cccc} k(t,\cdot): & [0,t] & \longrightarrow & \mathbb{R} \\ & s & \longrightarrow & k(t,s), \end{array}$ is càdlàg and belongs to $\mathcal{L}^2([0,t])$.

(2) For any $s \in [0,T]$ the function $\begin{array}{cccc} k(\cdot,s): & [s,T] & \longrightarrow & \mathbb{R} \\ & t & \longrightarrow & k(t,s), \end{array}$ has bounded variation.

(3) k does not explode on the diagonal, that is, $k(t,t) < \infty$ for all $t \in [0,T]$.

These hypotheses on the kernel k are the one stated in [11] and are reasonable to insure the process to have valuable stochastic properties.

2.4. Examples. The class of filtered processes introduced above covers, among others, the following examples:

- The shot noise process which corresponds to the kernel $k(t,s) = g(t-s)$ for a certain function g defined on $[0,\infty)$. This process has already been shown to be of much interest in a few applications as mentioned in the Introduction. A particular case is the Ornstein-Uhlenbeck process, defined by the function
$$g(u) = e^{\alpha u} 1\!\!1_{\{u \geq 0\}},$$
where α is a positive constant.

- The fractional Brownian motion, that admits the representation

$$B_t^{(H)} = \int_0^t k^{(H)}(t,s)\, dB_s,$$

  with $k^{(H)}$ of the form

$$(s,t) \to k^{(H)}(t,s) := l^{(H)}(t,s)(t-s)^{H-1/2} s^{-|H-1/2|} 1\!\!1_{[0,t]}(s), \tag{2.3}$$

  where $l^{(H)}$ is a bi-continuous function (see [8]).
- The *fractional Lévy motion* defined in [3], constructed by considering the kernel $k^{(H)}$ and a Lévy underlying process. The hypothesis of non explosion on the diagonal excludes the case $H < \frac{1}{2}$.

3. Stochastic Integrals With Respect to the Underlying Process.

3.1. Integrals based on the chaos decomposition. It is well known, since [14], that Lévy processes enjoy the chaotic representation property in a slightly generalized form. For the convenience of the reader, we recall the main ideas of this approach.

For any Borel set E on $[0,T]\times\mathbb{R}$, we define the sets $E^* = \{t \in [0,T] : (t,0) \in E\}$ and $E_0 = E - E^*$, and the measure

$$\mu(E) := \int_{E^*} \sigma^2\, \mathrm{d}\ell(t) + \int_{E_0} z^2\, \mathrm{d}(\ell \otimes \nu)(t,z).$$

Remark 3.1. The measure μ can be written as $\mathrm{d}\mu(t,z) = \mathrm{d}\ell(t) \otimes \mathrm{d}\rho(z)$, with $\mathrm{d}\rho(z) = \sigma^2\, \mathrm{d}\delta_0(z) + z^2\, \mathrm{d}\nu(z)$. In the sequel, we denote $\mathrm{d}\ell(t)$ by $\mathrm{d}t$.

Then for any set E, such that $\mu(E) < \infty$, we can introduce the independent random measure

$$\bar{L}(E) := \int_{E^*} \sigma\, \mathrm{d}B_t + \lim_{m\to\infty} \int_{E_m} z\, \mathrm{d}\widetilde{N}(t,z),$$

where $E_m = \{(t,z) \in E : \frac{1}{m} < |z| < m\}$ and the limit is in the $\mathbb{L}^2(\Omega)$ sense. In short, we can write

$$\mathrm{d}\bar{L}(t,z) = \mathrm{d}\bar{B}(t,z) + \mathrm{d}\bar{J}(t,z),$$

where $\mathrm{d}\bar{B}(t,z) := \sigma\, \mathrm{d}B_t \otimes \mathrm{d}\delta_0(z)$ and $\mathrm{d}\bar{J}(t,z) := z\, \mathrm{d}\tilde{N}(t,z)$ are also independent random measures on $[0,T]\times\mathbb{R}$.

For any collection of disjoint sets E_i, with finite measure μ, we define the multiple stochastic integral $I_n^{\bar{L}}(1\!\!1_{E_1\times\cdots\times E_n})$ of order n with respect to $\bar{L}$ by

$$I_n^{\bar{L}}(1\!\!1_{E_1\times\cdots\times E_n}) = \bar{L}(E_1)\cdots\bar{L}(E_n). \tag{3.1}$$

These multiple stochastic integrals satisfy, for $m \neq n$,

$$\mathbb{E}[I_n^{\bar{L}}(1\!\!1_{E_1\times\cdots\times E_n}) I_m^{\bar{L}}(1\!\!1_{F_1\times\cdots\times F_m})] = 0,$$

and

$$\mathbb{E}[I_n^{\bar{L}}(1\!\!1_{E_1\times\cdots\times E_n}) I_n^{\bar{L}}(1\!\!1_{F_1\times\cdots\times F_n})] = n! \int_{([0,T]\times\mathbb{R})^n} \widetilde{1\!\!1}_{E_1\times\cdots\times E_n} \widetilde{1\!\!1}_{F_1\times\cdots\times F_n}\, \mathrm{d}\mu^{\otimes n},$$

where $\widetilde{f}$ is the symmetrization of the function f.

Then, by linearity and density arguments, the definition of $I_n^{\bar{L}}$ is extended to functions in

$$\mathcal{L}_n^2 := \mathcal{L}^2(([0,T]\times\mathbb{R})^n; \mathcal{B}(([0,T]\times\mathbb{R})^n); \mu^{\otimes n}).$$

Remark 3.2. Notice that if $E = [0,t]\times\mathbb{R}$, for $t \leq T$, then

$$\bar{B}([0,t]\times\mathbb{R}) = \sigma B_t, \quad \bar{J}([0,t]\times\mathbb{R}) = J_t, \quad \bar{L}([0,t]\times\mathbb{R}) = \sigma B_t + J_t = L_t.$$

It is well–known that if F is a square–integrable random variable, measurable with respect to the filtration generated by L, then F has the unique representation usually called chaotic representation property for Lévy processes:

$$F = \sum_{n=0}^{\infty} I_n^{\bar{L}}(f_n), \tag{3.2}$$

where $I_0^{\bar{L}}(f_0) = f_0 = \mathbb{E}(F)$ and f_n is a symmetric function in $\mathcal{L}_n^2$, for any $n \geq 1$. Given this result we can introduce the gradient and divergence operators. We will follow here the abstract point of view presented in [22].

We say that a square-integrable random variable F, given by (3.2), belongs to the domain of the gradient operator $D^{\bar{L}}$, denoted by $\mathbb{D}_{\bar{L}}^{1,2}$, if and only if

$$\sum_{n=1}^{\infty} n n! ||f_n||^2_{\mathcal{L}_n^2} < \infty. \tag{3.3}$$

In this case, we define the random field $D^{\bar{L}}F = \{D_{t,z}^{\bar{L}}F : (t,z) \in [0,T]\times\mathbb{R}\}$ as

$$D_{t,z}^{\bar{L}}F = \sum_{n=1}^{\infty} n I_{n-1}^{\bar{L}}(f_n((t,z),\cdot)).$$

It is well known that $D^{\bar{L}}$ defines a linear and closed operator from $\mathbb{L}^2(\Omega,\mathbb{P})$ into $\mathbb{L}^2(\Omega\times[0,T]\times\mathbb{R}; \mathbb{P}\otimes\mu)$, with dense domain $\mathbb{D}_{\bar{L}}^{1,2}$.

On the other hand we define the divergence operator $\delta_C^{\bar{L}}$ in the following way. If Y has the chaos decomposition

$$Y(t,z) = \sum_{n=0}^{\infty} I_n^{\bar{L}}(y_n(t,z,\cdot)), \quad (t,z) \in [0,T]\times\mathbb{R},$$

where $y_n \in \mathcal{L}_{n+1}^2$ is a symmetric function in the last n variables, then $\delta_C^{\bar{L}}(Y)$ is defined as

$$\delta_C^{\bar{L}}(Y) = \sum_{n=0}^{\infty} I_{n+1}^{\bar{L}}(y_n),$$

provided Y belongs to Dom $\left(\delta_C^{\bar{L}}\right)$, that is,

$$\sum_{n=0}^{\infty} (n+1)! ||y_n||^2_{\mathcal{L}_{n+1}^2} < \infty.$$

It is well known that there is a duality relation between operators $D^{\bar{L}}$ and $\delta_C^{\bar{L}}$ in the sense that if $F \in \mathbb{D}_{\bar{L}}^{1,2}$ and $Y \in \mathrm{Dom}\left(\delta_C^{\bar{L}}\right)$ we have

$$\mathbb{E}\left[\int_{[0,T]}\int_{\mathbb{R}} Y(t,z) D_{t,z}^{\bar{L}} F \,\mathrm{d}\mu(t,z)\right] = \mathbb{E}[\delta_C^{\bar{L}}(Y)F]. \tag{3.4}$$

So, it can be deduced that $\delta_C^{\bar{L}}$ is also a linear and closed operator from $\mathbb{L}^2(\Omega \times [0,T] \times \mathbb{R}; \mathbb{P} \otimes \mu)$ into $\mathbb{L}^2(\Omega, \mathbb{P})$, with dense domain $\mathrm{Dom}\left(\delta_C^{\bar{L}}\right)$.

In order to go deeper in the Lévy problem we need to place our analysis on the canonical space of Lévy processes. This will allow us to obtain probabilistic interpretations of our gradient and divergence operators. We follow the construction of the canonical Lévy space developed in [27].

We denote by $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$ the canonical space of the standard Brownian motion, that is, Ω_B is the space of continuous functions on $[0,T]$, null at the origin, with the topology of the uniform convergence on $[0,T]$, $\mathcal{F}_B$ is the Borel $\sigma-$algebra on Ω_B and $\mathbb{P}_B$ is the standard Wiener measure.

On the other hand, we denote by $(\Omega_{\bar{J}}, \mathcal{F}_{\bar{J}}, \mathbb{P}_{\bar{J}})$, the canonical space of a pure jump Lévy process, with Lévy measure ν, as constructed in [27]. The space $\Omega_{\bar{J}}$ is the space of finite or infinite sequences of pairs $(t,z) \in [0,T] \times \mathbb{R}_0$ such that for any $\epsilon > 0$, only a finite number of pairs satisfy $|z| > \epsilon$.

Finally, we define the general canonical Lévy space on $[0,T]$ as

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_B \otimes \Omega_{\bar{J}}, \mathcal{F}_B \otimes \mathcal{F}_{\bar{J}}, \mathbb{P}_B \otimes \mathbb{P}_{\bar{J}}),$$

where, for $\omega = (\omega', \omega'') \in \Omega_B \otimes \Omega_{\bar{J}}$, the process

$$X_t(\omega) = \gamma t + \sigma B_t(\omega') + J_t(\omega''), \tag{3.5}$$

is a Lévy process with triplet (γ, σ^2, ν).

Analogously, we can consider the following operators:

- Operators D^B and δ_C^B for functionals in $\mathbb{L}^2(\Omega_B, \mathbb{L}^2(\Omega_{\bar{J}}))$ with measure $\mathrm{d}B_t$ on $[0,T]$, as introduced for example in [21]. Divergence operator can easily be extended to process $Y \in \mathbb{L}^2(\Omega \times [0,T] \times \mathbb{R}, \mathbb{P} \otimes \mu)$ writing
$$\delta_C^{\bar{B}}(Y) := \delta_C^B(Y(\cdot, 0)). \tag{3.6}$$
- Operators $D^{\bar{J}}$ and $\delta_C^{\bar{J}}$ for functionals on $\mathbb{L}^2(\Omega_{\bar{J}}, \mathbb{L}^2(\Omega_B))$ with measure $\mathrm{d}\bar{J}(t,z) = z\,\mathrm{d}\tilde{N}(t,z)$ on $[0,T] \times \mathbb{R}$, as introduced in [27]. In fact in [27], $\bar{J}$ is defined on $[0,T] \times \mathbb{R}_0$, but thanks to the factor z in the definition, it is immediate to extend it to $[0,T] \times \mathbb{R}$, by writing
$$\delta_C^{\bar{J}}(Y) := \delta_C^{\bar{J}}(Y \mathbb{1}_{\mathbb{R}_0}). \tag{3.7}$$

Now, we introduce the probabilistic interpretations of operators $D^{\bar{L}}$ and $\delta_C^{\bar{L}}$. The following results are also due to [27], and they establish how we can figure out the random field $D^{\bar{L}}F$ without using the chaos decomposition (3.2).

Let $\mathbb{D}_B^{1,2}(\mathbb{L}^2(\Omega_{\bar{J}}))$ be the family of $\mathbb{L}^2(\Omega_{\bar{J}}, \mathcal{F}_{\bar{J}}, \mathbb{P}_{\bar{J}})$–valued random variables that are in the domain of the classical Malliavin derivative for Hilbert space valued Gaussian random variables $\mathcal{D}^B$. The reader can consult [21] for the basic definitions and properties of this operator. Let us recall now the construction of this space. We say that a random variable F is an $\mathbb{L}^2(\Omega_{\bar{J}})$-valued smooth random variable if it has the form

$$F = f(B_{t_1}, \dots, B_{t_n})Z,$$

with $t_i \in [0,T]$ for any $i = 1, \dots n$ and $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded), and $Z \in \mathbb{L}^2(\Omega_{\bar{J}}, \mathcal{F}_{\bar{J}}, \mathbb{P}_{\bar{J}})$. The derivative of F with respect to B, in the Malliavin calculus sense, is defined as

$$\mathcal{D}^B F = \sum_{i=1}^n \frac{\partial f}{\partial z_i}(B_{t_1}, \dots, B_{t_n}) Z 1_{[0,t_i]}.$$

It is easy to see that $\mathcal{D}^B$ is a closable operator from $\mathbb{L}^2(\Omega_B; \mathbb{L}^2(\Omega_{\bar{J}}))$ into $\mathbb{L}^2(\Omega_B \times [0,T]; \mathbb{L}^2(\Omega_{\bar{J}}))$. Thus we can introduce the space $\mathbb{D}_B^{1,2}(\mathbb{L}^2(\Omega_{\bar{J}}))$ as the completion of the $\mathbb{L}^2(\Omega_{\bar{J}})$-valued smooth random variables with respect to the semi-norm

$$||F||_{1,2,B}^2 = \mathbb{E}\left[||F||^2_{\mathbb{L}^2(\Omega_{\bar{J}})} + ||\mathcal{D}^B F||^2_{\mathbb{L}^2([0,T]\times\Omega_{\bar{J}}))}\right]. \tag{3.8}$$

Moreover, for $\omega = (\omega', \omega'')$, $F \in \mathbb{L}^2(\Omega)$ and $(t,z) \in [0,T] \times \mathbb{R}_0$, we define

$$(\Psi_{t,z}F)(\omega) := \frac{F(\omega_z) - F(\omega)}{z},$$

with $\omega_z = (\omega', \omega_z'')$ and $\omega_z'' = ((t,z), \omega'')$.

The following two lemmas, proved in [1], will be helpful for our purposes:

Lemma 3.3. *Given F and G in $\mathbb{L}^2(\Omega)$ such that $FG \in \mathbb{L}^2(\Omega)$, and $z \neq 0$, we have*

$$\Psi_{t,z}(FG) = \Psi_{t,z}F \cdot G + F \cdot \Psi_{t,z}G + z\, \Psi_{t,z}F \cdot \Psi_{t.z}G.$$

Lemma 3.4. *Let F be a random variable in $\mathbb{L}^2(\Omega)$. Then $F \in \mathbb{D}_{\bar{L}}^{1,2}$ if and only if $F \in \mathbb{D}_B^{1,2}(\mathbb{L}^2(\Omega_{\bar{J}}))$ and $\Psi F \in \mathbb{L}^2(\Omega \times [0,T] \times \mathbb{R}_0; \mathbb{P} \otimes \mu)$. In this case,*

$$D_{t,z}^{\bar{L}} F = 1\!\!1_{\{0\}}(z) \sigma^{-1} \mathcal{D}_t^B F + 1\!\!1_{\mathbb{R}_0}(z) \Psi_{t,z} F.$$

Remark 3.5. Let $F \in \mathbb{D}_{\bar{L}}^{1,2}$.

- For $z = 0$, we have,
$$D_{t,0}^{\bar{L}} F = \sigma^{-1} \mathcal{D}_t^B F = \sigma^{-1} D_t^B F, \tag{3.9}$$
where the second equality comes from [21].
- For $z \neq 0$, we have
$$D_{t,z}^{\bar{L}} F = \Psi_{t,z} F = D_{t,z}^{\bar{J}} F, \tag{3.10}$$
where the second equality comes from [27].

Finally, we present the main result of this section:

Theorem 3.6. *If $Y \in \mathrm{Dom}\left(\delta_C^{\bar{B}}\right) \cap \mathrm{Dom}\left(\delta_C^{\bar{J}}\right)$, then $Y \in \mathrm{Dom}\left(\delta_C^{\bar{L}}\right) \subseteq \mathbb{L}^2(\Omega \times [0,T] \times \mathbb{R}, \mathbb{P} \otimes \mu)$ and*

$$\delta_C^{\bar{L}}(Y) = \sigma \delta_C^{\bar{B}}(Y) + \delta_C^{\bar{J}}(Y).$$

So, the Lévy problem is solved and true.

Proof. Using the duality between operators D and δ we have, for any $F \in \mathbb{D}^{1,2}_{\bar{L}}$,

$$\begin{aligned}
&\mathbb{E}\left[F\delta^{\bar{L}}_C(Y)\right]\\
&= \mathbb{E}\left[\int_0^T\int_{\mathbb{R}} D^{\bar{L}}_{t,z}\,FY(t,z)\,\mathrm{d}\mu(t,z)\right]\\
&= \mathbb{E}\left[\int_0^T\int_{\mathbb{R}} D^{\bar{L}}_{t,z}FY(t,z)\sigma^2\,\mathrm{d}\delta_0(z)\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}} D^{\bar{L}}_{t,z}FY(t,z)z^2\,\mathrm{d}\nu(z)\,\mathrm{d}t\right]\\
&= \mathbb{E}\left[\int_0^T D^B_t F\,Y(t,0)\sigma\,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}} D^{\bar{J}}_{t,z}F\,Y(t,z)z^2\,\mathrm{d}\nu(z)\,\mathrm{d}t\right]\\
&= \mathbb{E}\left[F\sigma\delta^B_C(Y(\cdot,0))\right] + \mathbb{E}\left[F\delta^{\bar{J}}_C(Y)\right]\\
&= \mathbb{E}\left[F\sigma\delta^{\bar{B}}_C(Y)\right] + \mathbb{E}\left[F\delta^{\bar{J}}_C(Y)\right].
\end{aligned}$$

□

In order to deal with processes Y defined on $[0,T]$, we have to restrict the domain of the divergences $\delta^{\bar{J}}_C$ and $\delta^{\bar{L}}_C$. For this, consider the operators $C^i,\ i=1,2,3$ defined for any $f \in \mathcal{L}^2([0,T]\times\mathbb{R})$ by:

$$\begin{aligned}
C^1(f) &= ((t,z) \to \delta_0(z)\,f(t,z)),\\
C^2(f) &= ((t,z) \to z\,f(t,z)),\\
C^3(f) &= ((t,z) \to C^1(f)(t,z) + C^2(f)(t,z)).
\end{aligned}$$

For $f \in \mathcal{L}^2([0,T])$, we denote, for a sake of notational simplicity, $C^1(f)(t,z) = \delta_0(z)\,f(t)$, $C^2(f)(t,z) = z\,f(t)$ and $C^3(f) = C^1(f) + C^2(f)$.

Definition 3.7. Let Y be a random process on $[0,T]$.

- $Y \in \mathrm{Dom}\left(\delta^B_C\right)$ if and only if $C^1(Y) \in \mathrm{Dom}\left(\delta^{\bar{B}}_C\right)$ and then we define
$$\delta^B_C(Y) := \delta^{\bar{B}}_C(C^1(Y)).$$
- $Y \in \mathrm{Dom}\left(\delta^J_C\right)$ if and only if $C^2(Y) \in \mathrm{Dom}\left(\delta^{\bar{J}}_C\right)$ and then we define
$$\delta^J_C(Y) := \delta^{\bar{J}}_C(C^2(Y)).$$
- $Y \in \mathrm{Dom}\left(\delta^L_C\right)$ if and only if $C^3(Y) \in \mathrm{Dom}\left(\delta^{\bar{L}}_C\right)$ and then we define
$$\delta^L_C(Y) := \delta^{\bar{L}}_C(C^3(Y)).$$

Theorem 3.8. *If* $Y \in \mathrm{Dom}\left(\delta^B_C\right) \cap \mathrm{Dom}\left(\delta^J_C\right)$, *then* $Y \in \mathrm{Dom}\left(\delta^L_C\right) \subseteq \mathbb{L}^2(\Omega \times [0,T], \mathbb{P}\otimes\ell)$ *and*

$$\delta^L_C(Y) = \sigma\delta^B_C(Y) + \delta^J_C(Y).$$

So, the Lévy problem is solved and true.

Proof. By Theorem 3.6,

$$\begin{aligned}\delta_C^L(Y) = \delta_C^{\bar{L}}(C^3(Y)) &= \sigma\delta_C^{\bar{B}}(C^3(Y)) + \delta_C^{\bar{J}}(C^3(Y))\\ &= \sigma\delta_C^{B}(C^3(Y)(\cdot,0)) + \delta_C^{\bar{J}}(C^3(Y)).\end{aligned}$$

Notice that $C^3(Y)(t,z)\,\mathbb{1}_{\mathbb{R}_0}(z) = C^2(Y)(t,z)\,\mathbb{1}_{\mathbb{R}_0}(z)$. Using relationships (3.6) and (3.7) and Definition 3.7, we have:

$$\begin{aligned}\delta_C^L(u) &= \sigma\delta_C^{B}(C^3(Y)(\cdot,0)) + \delta_C^{\bar{J}}(C^2(Y))\\ \delta_C^L(u) &= \sigma\delta_C^{B}(Y) + \delta_C^{J}(Y),\end{aligned}$$

which ends the proof. □

3.2. Integrals based on the $\mathcal{S}$-transform. A stochastic integral with respect to X can be defined by the use of the so-called $\mathcal{S}$-transform. This integral will be denoted by δ_S^X. When X is a Brownian motion we refer to [4], when X is a pure jumps process, we refer to [5]. Let us briefly explain the construction of such an integral. For this, we make use of the notations of Section 2.

The main idea of this construction is to consider random processes in a weak sense as an action on test functions. In order to reach a sufficiently wide class of processes, we have to consider a relevant class of test functions. To do this, we have to construct from the process $\{\bar{X}(t,z),\, t\geq 0,\, z\in\mathbb{R}\}$ an auxiliary two-sided process $\{\hat{\bar{X}}(t,z),\, t\in\mathbb{R},\, z\in\mathbb{R}\}$ as follows

$$\hat{\bar{X}}(t,\cdot) = \begin{cases}\bar{X}^1(t,\cdot), & \text{if } t\geq 0\\ \bar{X}^2(-t_-,\cdot), & \text{if } t<0\end{cases}$$

where $\bar{X}^1$ and $\bar{X}^2$ are two independent copies of $\bar{X}$. The restriction to $[0,T]$ is explained in Section 3.3.

In each setting, let Ξ be a subset of $\mathcal{L}^2(\mathbb{R}^2)$. For any $\eta\in\Xi$, consider $I_1^{\hat{\bar{X}}}(\eta)$, the multiple stochastic integral of order 1 with respect to $\hat{\bar{X}}$, defined in (3.1). It is well known that its Wick exponential is given by

$$\exp^{\diamond,\hat{\bar{X}}}\left(I_1^{\hat{\bar{X}}}(\eta)\right) = \sum_{n\geq 0}\frac{1}{n!}\,I_n^{\hat{\bar{X}}}(\eta^{\otimes n}).$$

Thus we can introduce the following transform:

Definition 3.9. For $Y\in\mathbb{L}^2(\Omega,\mathbb{P})$, the $\mathcal{S}$-*transform* of Y associated to $\hat{\bar{X}}$, denoted by $\mathcal{S}^{\hat{\bar{X}}}(Y)$, is the integral transform defined for any $\eta\in\Xi$ by

$$\mathcal{S}^{\hat{\bar{X}}}(Y)(\eta) = \mathbb{E}_{\mathbb{Q}_\eta^{\hat{\bar{X}}}}[Y],$$

where

$$\mathrm{d}\mathbb{Q}_\eta^{\hat{\bar{X}}} = \exp^{\diamond,\hat{\bar{X}}}\left(I_1^{\hat{\bar{X}}}(\eta)\right)\,\mathrm{d}\mathbb{P}.$$

Remark 3.10. The Wick exponential of $I_1^{\hat{\bar{X}}}(\eta)$ coincides with the Doléans-Dade exponential of $I_1^{\hat{\bar{X}}}(\eta)$.

Definition 3.11 (*Wick exponential* in the Brownian setting [4])**.** In the Brownian setting, we have $\Xi = \mathcal{S}(\mathbb{R}^2)$, the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}^2$, and

$$\exp^{\diamond,\hat{\hat{B}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right) = \exp\left(I_1^{\hat{\hat{B}}}(\eta) - \frac{||\eta(\cdot,0)||^2_{\mathcal{L}^2(\mathbb{R})}}{2}\right).$$

Definition 3.12 (*Wick exponential* in the pure jumps setting [5])**.** In the pure jump setting, we have

$$\Xi = \left\{\eta \in \mathcal{S}(\mathbb{R}^2) :\ \forall (t,z) \in \mathbb{R}^2, \eta(0,z) = 0, \eta(t,z) > -1, \left.\frac{\partial}{\partial z}\eta(t,z)\right|_{z=0} = 0\right\},$$

and

$$\begin{aligned}&\exp^{\diamond,\hat{\hat{J}}}\left(I_1^{\hat{\hat{J}}}(\eta)\right)\\ &\quad = \exp\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \log(1+\eta(t,z))\,\mathrm{d}\hat{\hat{J}}(t,z) - \int_{\mathbb{R}}\int_{\mathbb{R}} \eta(t,z)\,z^2\,\mathrm{d}\nu(z)\,\mathrm{d}t\right].\end{aligned}$$

We are also able to define the Wick exponential for Lévy processes:

Definition 3.13 (*Wick exponential* in the Lévy setting)**.** In the Lévy setting we consider:

$$\Xi = \left\{\eta \in \mathcal{S}(\mathbb{R}^2) :\ \forall (t,z) \in \mathbb{R}^2, \eta(0,t) = 0, \eta(t,z) > -1, \left.\frac{\partial}{\partial z}\eta(t,z)\right|_{z=0} = 0\right\},$$

and for $\eta \in \Xi$ we define

$$\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta)\right) = \sum_{n\geq 0}\frac{1}{n!}\,I_n^{\hat{\hat{L}}}(\eta^{\otimes n}).$$

Theorem 3.14. *For $\eta \in \Xi$, we have*

$$\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta)\right) = \exp^{\diamond,\hat{\hat{B}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right)\cdot \exp^{\diamond,\hat{\hat{J}}}\left(I_1^{\hat{\hat{J}}}(\eta)\right). \tag{3.11}$$

Proof. Indeed,

$$I_1^{\hat{\hat{L}}}(\eta) = I_1^{\hat{\hat{B}}}(\eta) + I_1^{\hat{\hat{J}}}(\eta).$$

And in virtue of the property of additivity of Doléans exponential (see for instance [24, Theorem 38], we have,

$$\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta)\right) = \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right)\,\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{J}}}(\eta)\right).$$

Now, it is easily seen, since $I_1^{\hat{\hat{B}}}(\eta) = I_1^{\hat{\hat{L}}}(C^1(\eta)) = I_1^{\hat{\hat{L}}}(\eta\,\delta_0)$, that

$$\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right) = \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta\,\delta_0)\right) = \sum_{n\geq 0}\frac{1}{n!}\,I_n^{\hat{\hat{L}}}((\eta\,\delta_0)^{\otimes n}) \tag{3.12}$$

$$= \sum_{n\geq 0}\frac{1}{n!}\,I_n^{\hat{\hat{L}}}(\eta^{\otimes n}\,(\delta_0)^{\otimes n}) = \sum_{n\geq 0}\frac{1}{n!}\,I_n^{\hat{\hat{B}}}(\eta^{\otimes n}) = \exp^{\diamond,\hat{\hat{B}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right). \tag{3.13}$$

By the same lines, it is easily seen, since $I_1^{\hat{\hat{J}}}(\eta) = I_1^{\hat{\hat{L}}}(C^2(\eta))$, that

$$\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{J}}}(\eta)\right) = \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(C^2(\eta))\right) = \exp^{\diamond,\hat{\hat{J}}}\left(I_1^{\hat{\hat{J}}}(\eta)\right),$$

which ends the proof of (3.11). □

The following theorem is the result which makes the machinery relevant to define an integral. In fact it tells us that a process is perfectly described by its $\mathcal{S}$-transform.

Theorem 3.15. *Let $X = B, J, L$. If $\mathcal{S}^{\hat{\hat{X}}}(Y_1)(\eta) = \mathcal{S}^{\hat{\hat{X}}}(Y_2)(\eta)$ for all $\eta \in \Xi$, then $Y_1 = Y_2$.*

Proof. For the Brownian setting the proof can be found in [4, Theorem 2.2, page 958], for the pure jump setting, the proof is in [5, Proposition 3.4, page 504]. Finally, in the Lévy setting, the proof is strictly the same as in the pure jump case. □

In the following we denote $\mathrm{d}\pi^{\hat{\hat{B}}} = \mathrm{d}\delta_0$ and $\mathrm{d}\pi^{\hat{\hat{J}}}(z) = z^2\,\mathrm{d}\nu(z)$.

Definition 3.16. Consider $X = B, J$ and Y a random field. The *Hitsuda-Skorohod integral* of Y with respect to $\hat{\hat{X}}$ exists if there exists $\Phi \in \mathbb{L}^2(\Omega)$ such that for all $\eta \in \Xi$,

$$\mathcal{S}^{\hat{\hat{X}}}(\Phi)(\eta) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{X}}}(Y(t,z))(\eta)\;\eta(t,z)\;\mathrm{d}\pi^{\hat{\hat{X}}}(z)\,\mathrm{d}t.$$

In this case $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{X}}}\right)$ and thanks to Theorem 3.15, Φ is unique and is denoted by $\delta_S^{\hat{\hat{X}}}(Y)$.

This is Definition 3.1 together with the remark at the beginning of Section 3.2 of [4] for $X = B$ and Definition 3.7 together with Remark 3.9 of [5] for $X = J$.

Theorem 3.17. *Let $X = B, J$ and Y a predictable random field satisfying*

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |Y(t,z)|^2\;\mathrm{d}\pi^{\hat{\hat{X}}}(z)\,\mathrm{d}t\right] < \infty. \tag{3.14}$$

Then $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{X}}}\right)$.

Proof. See Theorem 3.1 in [4] for B and Theorem 3.5 in [5] for J. □

Denote $\mathrm{d}\pi^{\hat{\hat{L}}} = \mathrm{d}\rho$.

Definition 3.18. Consider Y a random field. The *Hitsuda-Skorohod integral* of Y with respect to $\hat{\hat{L}}$ exists if there exists $\Phi \in \mathbb{L}^2(\Omega)$ such that for all $\eta \in \Xi$,

$$\mathcal{S}^{\hat{\hat{L}}}(\Phi)(\eta) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{L}}}(Y(t,z))(\eta)\;\eta(t,z)\;\mathrm{d}\pi^{\hat{\hat{L}}}(z)\,\mathrm{d}t.$$

In this case $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{L}}}\right)$ and thanks to Theorem 3.15, Φ is unique and is denoted by $\delta_S^{\hat{\hat{L}}}(Y)$.

Theorem 3.19. *Let Y be a predictable random field satisfying*

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} |Y(t,z)|^2 \, \mathrm{d}\pi^{\hat{\hat{L}}}(z)\,\mathrm{d}t\right] < \infty. \tag{3.15}$$

Then $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{L}}}\right)$.

Proof. Recall that $\mathrm{d}\rho(z) = \sigma^2\,\mathrm{d}\delta_0(z) + z^2\,\mathrm{d}\nu(z)$. Thus (3.15) implies (3.14) with $X = B$ and $X = J$. Moreover Y being predictable, this implies, by Theorem 3.17, that $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{B}}}\right)$ and $Y \in \mathrm{Dom}\left(\delta_S^{\hat{\hat{J}}}\right)$. □

Theorem 3.20. *Let Y be a predictable random field satisfying* (3.15). *Then*

$$\mathcal{S}^{\hat{\hat{L}}}\left(\delta_S^{\hat{\hat{L}}}(Y)\right)(\eta) = \mathcal{S}^{\hat{\hat{B}}}\left(\delta_S^{\hat{\hat{B}}}(Y)\right)(\eta) + \mathcal{S}^{\hat{\hat{J}}}\left(\delta_S^{\hat{\hat{J}}}(Y)\right)(\eta). \tag{3.16}$$

Proof. Let $\eta \in \Xi$. As $\mathrm{d}\pi^{\hat{\hat{L}}} = \mathrm{d}\pi^{\hat{\hat{B}}} + \mathrm{d}\pi^{\hat{\hat{J}}}$, to prove the result it is enough to show that, for any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{L}}}\left(Y(t,z)\right)(\eta)\;\eta(t,z)\;\mathrm{d}\pi^{\hat{\hat{B}}}(z) = \int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{B}}}\left(Y(t,z)\right)(\eta)\;\eta(t,z)\;\mathrm{d}\pi^{\hat{\hat{B}}}(z), \tag{3.17}$$

$$= \int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{J}}}\left(Y(t,z)\right)(\eta)\;\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{J}}}(z). \tag{3.18}$$

Let us prove (3.17). Note that the reasoning is the same for (3.18). Using Fubini Theorem repeatedly, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{L}}}\left(Y(t,z)\right)(\eta)\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{B}}}(z) &= \int_{\mathbb{R}} \mathbb{E}\left[\exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta)\right)Y(t,z)\right]\eta(t,z)\;\mathrm{d}\pi^{\hat{\hat{B}}}(z)\\ &= \mathbb{E}\left[\int_{\mathbb{R}} \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta)\right)Y(t,z)\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{B}}}(z)\right]\\ &= \mathbb{E}\left[\int_{\mathbb{R}} \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta\delta_0)\right)Y(t,z)\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{B}}}(z)\right].\end{aligned}$$

And in virtue of (3.13), we have

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathbb{R}} \exp^{\diamond,\hat{\hat{L}}}\left(I_1^{\hat{\hat{L}}}(\eta\delta_0)\right)\;Y(t,z)\;\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{B}}}(z)\right]\\ &\qquad= \mathbb{E}\left[\int_{\mathbb{R}} \exp^{\diamond,\hat{\hat{B}}}\left(I_1^{\hat{\hat{B}}}(\eta)\right)\;Y(t,z)\;\eta(t,z)\,\mathrm{d}\pi^{\hat{\hat{B}}}(z)\right],\end{aligned}$$

and the result follows. □

3.3. Links between these two approaches. First of all, notice that, thanks to a well known property of $\mathcal{S}$-transform (see [4] and [5]), we can restrict ourselves to integrals on $[0,T]$ noticing that

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \mathcal{S}^{\hat{\hat{X}}}\left(Y(t,z)\right)(\eta)\;\frac{\partial}{\partial t}\mathcal{S}^{\hat{\hat{X}}}\left(\hat{\hat{X}}(t,z)\right)(\eta)\;\mathrm{d}\pi^{\hat{\hat{X}}}(z)\,\mathrm{d}t\\ &\qquad= \int_0^T \int_{\mathbb{R}} \mathcal{S}^{\bar{X}}\left(Y(t,z)\right)(\eta)\;\frac{\partial}{\partial t}\mathcal{S}^{\bar{X}}\left(\bar{X}(t,z)\right)(\eta)\;\mathrm{d}\pi^{\bar{X}}(z)\,\mathrm{d}t.\end{aligned} \tag{3.19}$$

In the sequel, we focus on this integral on $[0,T]$ and denote it by $\delta_S^{\bar{X}}(Y)$. The domain of $\delta_S^{\bar{X}}$, denoted by $\mathrm{Dom}\left(\delta_S^{\bar{X}}\right)$, is the set of random processes satisfying

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |Y(t,z)|^2 \,\mathrm{d}\pi^{\bar{X}}(z)\,\mathrm{d}t\right] < \infty.$$

Theorem 3.21.

- *If Y belongs to* $\mathrm{Dom}\left(\delta_C^{\bar{B}}\right)$, *then* $Y \in \mathrm{Dom}\left(\delta_S^{\bar{B}}\right)$ *and* $\delta_C^{\bar{B}}(Y) = \delta_S^{\bar{B}}(Y)$.
- *If Y belongs to* $\mathrm{Dom}\left(\delta_C^{\bar{J}}\right)$, *then* $Y \in \mathrm{Dom}\left(\delta_S^{\bar{J}}\right)$ *and* $\delta_C^{\bar{J}}(Y) = \delta_S^{\bar{J}}(Y)$.
- *If Y belongs to* $\mathrm{Dom}\left(\delta_C^{\bar{L}}\right)$, *then* $Y \in \mathrm{Dom}\left(\delta_S^{\bar{L}}\right)$ *and* $\delta_C^{\bar{L}}(Y) = \delta_S^{\bar{L}}(Y)$.

Proof. Let us prove the third point. Consider $Y \in \mathrm{Dom}\left(\delta_C^{\bar{L}}\right)$. Then there exist functions y_n, such that

$$Y(t,z) = \sum_{n\geq 0} I_n^{\bar{L}}(y_n(\cdot,(t,z))).$$

By linearity, we can restrict our attention to the case $Y(t,z) = I_n^{\bar{L}}(y_n(\cdot,(t,z)))$ for fixed n. Thus

$$\delta_C^{\bar{L}}(Y) = I_{n+1}^{\bar{L}}(\widetilde{y_n}).$$

Now, we have for any $\eta \in \Xi$,

$$\begin{aligned}\mathcal{S}^{\bar{L}}\left(\delta_C^{\bar{L}}(Y)\right)(\eta) &= \mathbb{E}\left[I_{n+1}^{\bar{L}}(\widetilde{y_n}) \sum_{j\geq 0}\frac{1}{j!} I_j^{\bar{L}}(\eta^{\otimes j})\right] \\ &= \frac{1}{(n+1)!}\mathbb{E}\left[I_{n+1}^{\bar{L}}(\widetilde{y_n})\ I_{n+1}^{\bar{L}}(\eta^{\otimes(n+1)})\right] \\ &= \left\langle \widetilde{y_n}, \eta^{\otimes(n+1)}\right\rangle_{\mathcal{L}_{n+1}^2}.\end{aligned} \tag{3.20}$$

Now, by the very definition of $\delta_S^{\bar{L}}(Y)$ we have:

$$\begin{aligned}\mathcal{S}^{\bar{L}}\left(\delta_S^{\bar{L}}(Y)\right)(\eta) &= \int_0^T \int_{\mathbb{R}} \mathcal{S}^{\bar{L}}\left(I_n^{\bar{L}}(y_n(\cdot,(t,z)))\right)(\eta)\ \eta(t,z)\,\mathrm{d}\mu(t,z) \\ &= \int_0^T \int_{\mathbb{R}} \frac{1}{n!}\ \mathbb{E}\left[I_n^{\bar{L}}(y_n(\cdot,(t,z)))\ I_n^{\bar{L}}(\eta^{\otimes n})\right]\ \eta(t,z)\,d\mu(t,z) \\ &= \int_0^T \int_{\mathbb{R}} \left\langle y_n(\cdot,(t,z)), \eta^{\otimes n}\right\rangle_{\mathcal{L}_n^2}\ \eta(t,z)\,d\mu(t,z) \\ &= \left\langle y_n, \eta^{\otimes(n+1)}\right\rangle_{\mathcal{L}_{n+1}^2}.\end{aligned} \tag{3.21}$$

Noticing that $\eta^{\otimes(n+1)}$ is a symmetric function, we have

$$\mathcal{S}^{\bar{L}}\left(\delta_C^{\bar{L}}(Y)\right)(\eta) = \mathcal{S}^{\bar{L}}\left(\delta_S^{\bar{L}}(Y)\right)(\eta),$$

for any $\eta \in \Xi$. So, the result is shown in virtue of Theorem 3.15. Moreover equality (3.20) and (3.21) prove that $Y \in \mathrm{Dom}\left(\delta_C^{\bar{L}}\right)$ implies $Y \in \mathrm{Dom}\left(\delta_S^{\bar{L}}\right)$. The proofs of the other points are similar, replacing $\mathrm{d}\mu(t,z)$ by $\mathrm{d}\ell(t) \otimes \mathrm{d}\delta_0(z)$ in the Brownian setting, and by $\mathrm{d}\ell(t) \otimes z^2 \,\mathrm{d}\nu(z)$ in the pure jump setting. □

Definition 3.22. Let Y a random process on $[0,T]$.

- $Y \in \mathrm{Dom}\left(\delta_S^B\right)$ if $C^1(Y) \in \mathrm{Dom}\left(\delta_S^{\bar{B}}\right)$ and $\delta_S^B(Y) = \delta_S^{\bar{B}}(C^1(Y))$.
- $Y \in \mathrm{Dom}\left(\delta_S^J\right)$ if $C^2(Y) \in \mathrm{Dom}\left(\delta_S^{\bar{J}}\right)$ and $\delta_S^J(Y) = \delta_S^{\bar{J}}(C^2(Y))$.
- $Y \in \mathrm{Dom}\left(\delta_S^L\right)$ if $C^3(Y) \in \mathrm{Dom}\left(\delta_S^{\bar{L}}\right)$ and $\delta_S^L(Y) = \delta_S^{\bar{L}}(C^3(Y))$.

Theorem 3.23.
- *If Y belongs to $\mathrm{Dom}\left(\delta_C^B\right)$, then $Y \in \mathrm{Dom}\left(\delta_S^B\right)$ and $\delta_C^B(Y) = \delta_S^B(Y)$.*
- *If Y belongs to $\mathrm{Dom}\left(\delta_C^J\right)$, then $Y \in \mathrm{Dom}\left(\delta_S^J\right)$ and $\delta_C^J(Y) = \delta_S^J(Y)$.*
- *If Y belongs to $\mathrm{Dom}\left(\delta_C^L\right)$, then $Y \in \mathrm{Dom}\left(\delta_S^L\right)$ and $\delta_C^L(Y) = \delta_S^L(Y)$.*

Proof. It is an immediate consequence of the previous theorem and definitions. □

3.4. The Lévy–Itô problem.

Theorem 3.24. *The Lévy problem is solved and true. Indeed,*

- *for any process $Y \in \mathrm{Dom}\left(\delta_C^{\bar{B}}\right) \cap \mathrm{Dom}\left(\delta_C^{\bar{J}}\right)$, we have $Y \in \mathrm{Dom}\left(\delta_C^{\bar{L}}\right) \subseteq \mathrm{Dom}\left(\delta_S^{\bar{L}}\right) \subseteq \mathbb{L}^2(\Omega \times [0,T] \times \mathbb{R})$ and*
$$\delta_S^{\bar{L}}(Y) = \delta_S^{\bar{B}}(Y) + \delta_S^{\bar{J}}(Y), \tag{3.22}$$
- *for any process $Y \in \mathrm{Dom}\left(\delta_C^B\right) \cap \mathrm{Dom}\left(\delta_C^J\right)$, we have $Y \in \mathrm{Dom}\left(\delta_C^L\right) \subseteq \mathrm{Dom}\left(\delta_S^L\right) \subseteq \mathbb{L}^2(\Omega \times [0,T])$ and*
$$\delta_S^L(Y) = \sigma\delta_S^B(Y) + \delta_S^J(Y). \tag{3.23}$$

Proof. Theorem 3.6 tells us that
$$\delta_C^{\bar{L}}(Y) = \delta_C^{\bar{B}}(Y) + \delta_C^{\bar{J}}(Y),$$
and Theorem 3.21 insures that we can replace in each integral the subscript C by subscript S. So, the first result follows. The second one follows exactly the same lines by means of Theorem 3.8. □

3.5. The complete Lévy–Itô problem. It is easily seen that, whatever the setting (Brownian, pure jump and Lévy), the complete Lévy–Itô problem is true if and only if u is deterministic.

4. Stochastic Integrals with Respect to Filtered Processes.

4.1. Intrinsic definitions for the filtered Brownian motion. The filtered Brownian motion introduced in Section 2.4, is an isonormal Gaussian process. In fact, for any $(s,t) \in [0,T]^2$, we have
$$\mathbb{E}\left[B_t^k\, B_s^k\right] = \int_0^{t\wedge s} k(s,u)\; k(t,u)\;\, \mathrm{d}u = \langle k(s,\cdot), k(t,\cdot)\rangle_{\mathcal{L}^2([0,T])}\,.$$

Thus, following a similar construction to the construction in Section 3.1, it is possible to define an intrinsic integral by means of chaos decomposition, denoted by $\delta_C^{B,k}$. We refer to [21] for details.

4.2. By the use of the $\mathcal{S}$-transform. In this section, X denotes B, J or L and $\bar{X}$ denotes $\bar{B}$, $\bar{J}$ or $\bar{L}$. For details on the construction of the integral by the use of $\mathcal{S}$-transform, we refer to [4], for $X = B$, or [5], for $\bar{X} = \bar{J}$. Consider $Y \in \mathcal{L}^2([0,T]\times\mathbb{R})$, then

$$\mathcal{S}^{\bar{X}}\left(\int_0^T \int_{\mathbb{R}} Y(s,z)\,\mathrm{d}\bar{X}(s,z)\right)(\eta) = \int_0^T \int_{\mathbb{R}} \mathcal{S}^{\bar{X}}\left(Y(s,z)\right)(\eta)\ \eta(s,z)\ \mathrm{d}\pi^{\bar{X}}(z)\,\mathrm{d}s.$$

Now taking $Y(s,z) = 1\!\!1_{[0,t]}(s)1\!\!1_A(z)$, we have

$$\begin{aligned}\mathcal{S}^{\bar{X}}\left(\bar{X}([0,t]\times A)\right)(\eta) &= \int_0^T \int_{\mathbb{R}} 1\!\!1_{[0,t]}(s)1\!\!1_A(z)\ \eta(s,z)\ \mathrm{d}\pi^{\bar{X}}(z)\,\mathrm{d}s.\\ &= \int_0^t \int_A \eta(s,z)\ \mathrm{d}\pi^{\bar{X}}(z)\,\mathrm{d}s.\end{aligned}$$

Hence, we can write, in a suggestive notation,

$$\mathcal{S}^{\bar{X}}\left(\bar{X}(\mathrm{d}t,\mathrm{d}z)\right)(\eta) = \eta(t,z)\ \mathrm{d}\pi^{\bar{X}}(z),$$

and thus for any $Y \in \mathrm{Dom}\left(\delta_S^{\bar{X}}\right)$ predictable, we have

$$\mathcal{S}^{\bar{X}}\left(\delta_S^{\bar{X}}(Y)\right)(\eta) = \int_0^T \int_{\mathbb{R}} \mathcal{S}^{\bar{X}}\left(Y(s,z)\right)(\eta)\ \mathcal{S}^{\bar{X}}\left(\bar{X}(\mathrm{d}t,\mathrm{d}z)\right)(\eta)\ \ \mathrm{d}s.$$

Definition 4.1. For $\bar{X} = \bar{B},\bar{J},\bar{L}$, assume Hypothesis 4.2 is fulfilled. Let Y a measurable random field in $\mathbb{L}^2(\Omega\times[0,T]\times\mathbb{R})$. Y is said to have a *Hitsuda-Skorohod integral* with respect to $\bar{X}^k$ if

- for any $\eta \in \Xi$:
$$t \to \int_{\mathbb{R}} \mathcal{S}^{\bar{X}}\left(Y(t,z)\right)(\eta)\ \mathcal{S}^{\bar{X}}\left(\bar{X}^k(\mathrm{d}t,\mathrm{d}z)\right)(\eta) \in \mathcal{L}^1([0,T]),$$
- and there is a $\Phi \in \mathbb{L}^2(\Omega)$ such that for any $\eta \in \Xi$:
$$\mathcal{S}^{\bar{X}}\left(\Phi\right)(\eta) = \int_0^T \int_{\mathbb{R}} \mathcal{S}^{\bar{X}}\left(Y(t,z)\right)(\eta)\frac{\partial}{\partial t}\mathcal{S}^{\bar{X}}\left(\bar{X}^k(\mathrm{d}t,\mathrm{d}z)\right)(\eta)\,\mathrm{d}t. \tag{4.1}$$

By Theorem 3.15, Φ is unique and is denoted $\delta_S^{\bar{X},k}(Y)$, and its domain is denoted by $\mathrm{Dom}\left(\delta_S^{\bar{X},k}\right)$.

Now, restrict ourselves to stochastic processes. To do so, the following assumption has to be fulfilled. In [5], assumptions on the kernel, that guarantees that hypothesis is fulfilled, are given.

Hypothesis 4.2. The mapping

$$t \to \frac{\mathrm{d}}{\mathrm{d}t}\ \mathcal{S}^{\bar{X}}\left(X_t^k\right)(\eta),$$

exists for every $\eta \in \Xi$.

Definition 4.3. For $X = B, J, L$, assume Hypothesis 4.2 fulfilled. Let Y a measurable stochastic process in $\mathbb{L}^2(\Omega \times [0,T])$. Y is said to have a *Hitsuda-Skorohod integral* with respect to X^k if

- $C^i(Y) \in \mathrm{Dom}\left(\delta_S^{\bar{X},k}\right)$,
- for any $Y \in \mathrm{Dom}\left(\delta_S^{X,k}\right)$, $\delta_S^{X,k}(Y) = \delta_S^{\bar{X},k}(C^i(Y))$,

where $i = 1$ for $X = B$, $i = 2$ for $X = J$ and $i = 3$ for $X = L$.

4.3. By the use of an operator. Another more direct approach is to follow [11], [12] or [10] and to use a linear operator $\mathcal{K}^*$ which allows us to define an integral with respect the filtered process by the mean of an integral with respect to the underlying process. For this approach, the following hypothesis has to be fulfilled.

Hypothesis 4.4. The kernel $k : [0,T]^2 \to \mathbb{R}$ is a triangular deterministic function, that is, $k(t,s) = 0$ for all $0 \le t < s \le T$.

Theorem 4.5. *There exists an operator $\mathcal{K}^* : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ linear and continuous, satisfying for any $t \in [0,T]$,*

$$\mathcal{K}^*(\mathbb{1}_{[0,t]}) = k(t,\cdot). \tag{4.2}$$

Remark 4.6. Assumption 4.4 ensures, by equation (4.2) that $\mathcal{K}^*(\mathbb{1}_{[0,t]})$ is null for all $0 \le t < s \le T$ and thus ensures that the processes constructed by means of the operator $\mathcal{K}^*$ are càdlàg.

Proof. Let us sketch the proof. Details are given in [11] and [12]. Introduce the following operator K:

$$\begin{array}{cccc} K: & \mathcal{L}^2([0,T]) & \to & \mathcal{L}^2([0,T]) \\ & f & \to & \int_0^T k(\cdot,s) f(s)\,\mathrm{d}s. \end{array}$$

It is known that K is a continuous operator from $\mathcal{L}^2([0,T])$ into $\mathcal{L}^2([0,T])$. Its adjoint is given by

$$\begin{array}{cccc} K^* & : & \mathcal{L}^2([0,T]) & \to & \mathcal{L}^2([0,T]) \\ & & f & \to & \int_0^T k(s,\cdot) f(s)\,\mathrm{d}s. \end{array}$$

Consider now the operator:

$$\begin{array}{cccc} I_{T-}^T: & \mathcal{L}^2([0,T]) & \to & \mathcal{L}^2([0,T]) \\ & f & \to & \int_\cdot^T f(s)\,\mathrm{d}s. \end{array}$$

I_{T-}^T is continuous from $\mathcal{L}^2([0,T])$ into $\mathcal{L}^2([0,T])$, moreover, its adjoint is given by:

$$\begin{array}{ccccc} I_{0+}^T & : & \mathcal{L}^2([0,T]) & \to & \mathcal{L}^2([0,T]) \\ & & f & \to & \int_0^\cdot f(s)\,\mathrm{d}s. \end{array}$$

Finally, we define the operator:

$$\begin{array}{cccc} \mathcal{K}^*: & \mathcal{L}^2([0,T]) & \to & \mathcal{L}^2([0,T]) \\ & f & \to & K^* \circ [I_{T-}^T]^{-1}(f). \end{array}$$

$\mathcal{K}^*$ is linear and continuous from $\mathcal{L}^2([0,T])$ to $\mathcal{L}^2([0,T])$.
Formally, we have, for all $f \in \mathcal{L}^2([0,T])$:

$$\int_0^T K^*(\delta_t)(s) f(s)\,\mathrm{d}s = K(f)(t) = \int_0^T k(t,s) f(s)\,\mathrm{d}s,$$

which shows that

$$K^*(\delta_t) = k(t,\cdot).$$

On the other hand, we have:

$$I^T_{T-}(\delta_t) = \mathbb{1}_{[0,t]},$$

and (4.2) follows. □

Theorem 4.7. *For any $f \in \mathcal{L}^2([0,T])$, locally bounded, we have*

$$\int_0^t f(s)\ \mathrm{d}J^k_s = \int_0^t \int_{\mathbb{R}_0} \mathcal{K}^*(f)(s)\ \mathrm{d}J_s, \qquad \forall t \in [0,T], \tag{4.3}$$

$$\int_0^t f(s)\ \mathrm{d}B^k_s = \int_0^t \mathcal{K}^*(f)(s)\ \mathrm{d}B_s, \qquad \forall t \in [0,T], \tag{4.4}$$

where the integral in (4.3) has to be understood in the Stieltjes's sense and the integral (4.4) in the Wiener's way.

Let the vector space $\mathcal{I}^k$, that is, the closure of $\mathcal{I}$, be the vector space generated by functions $\{\mathbb{1}_{[0,t]} : t \in [0,T]\}$, with respect to the following inner product:

$$\left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \right\rangle_{\mathcal{I}^k} = \langle k(t,\cdot), k(s,\cdot) \rangle_{\mathcal{L}^2([0,T])}\,.$$

Then $\mathcal{K}^*$ is an isometry from $\mathcal{I}^k$ into $\mathcal{L}^2([0,T])$, and for any $f \in \mathcal{I}^k$, $g \in \mathcal{I}^k$ we have:

$$\mathbb{E}\left[\int_0^T f(s)\ \mathrm{d}J^k_s . \int_0^T g(s)\ \mathrm{d}J^k_s\right] = \langle f, g \rangle_{\mathcal{I}^k}\,, \tag{4.5}$$

$$\mathbb{E}\left[\int_0^T f(s)\ \mathrm{d}B^k_s . \int_0^T g(s)\ \mathrm{d}B^k_s\right] = \langle f, g \rangle_{\mathcal{I}^k}\,. \tag{4.6}$$

Proof. Obviously, we have, for any $t \in [0,T]$,

$$J^k_t = \int_0^T \mathbb{1}_{[0,t]}(s)\,\mathrm{d}J^k_s,$$

and by the very definition of J^k, for any $t \in [0,T]$,

$$J^k_t = \int_0^T k(t,s)\ \mathrm{d}J_s = \int_0^T \mathcal{K}^*(\mathbb{1}_{[0,t]})(s)\ \mathrm{d}J_s,$$

thus, for any $t \in [0,T]$,

$$\int_0^T \mathbb{1}_{[0,t]}(s)\,\mathrm{d}J^k_s = \int_0^T \mathcal{K}^*(\mathbb{1}_{[0,t]})(s)\ \mathrm{d}J_s.$$

Relation (4.3) is true for any function of $\mathcal{I}$ thus, by a continuity argument, it is true for any deterministic function of $\mathcal{L}^2([0,T])$.

Let $1\!\!1_{[0,t]} \in \mathcal{I}$, we have:

$$||1\!\!1_{[0,t]}||_{\mathcal{I}^k} \;=\; ||k(t,\cdot)||_{\mathcal{L}^2([0,T])} \;=\; ||\mathcal{K}^*(1\!\!1_{[0,t]})||_{\mathcal{L}^2([0,T])}.$$

The isometry is verified for the elements of $\mathcal{I}$ and by a limit procedure, using that $\mathcal{K}^*$ is continuous, the result holds. The same arguments hold replacing J by B. Finally,

$$\begin{aligned}\mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}J_s^k.\int_0^T g(s)\,\mathrm{d}J_s^k\right] &= \mathbb{E}\left[\int_0^T \mathcal{K}^*(f)(s)\,\mathrm{d}J_s.\int_0^T\int_{\mathbb{R}_0} \mathcal{K}^*(g)(s)\,\mathrm{d}J_s\right]\\ &= \langle \mathcal{K}^*(f),\mathcal{K}^*(g)\rangle_{\mathcal{L}^2([0,T])} \qquad (4.7)\\ &= \langle f,g\rangle_{\mathcal{I}^k}\,,\end{aligned}$$

where (4.7) comes from the Itô isometry for deterministic integrands and jumps processes. The same arguments hold replacing $\mathrm{d}J$ by $\mathrm{d}B$ evoking in (4.7) the Itô isometry for Wiener integrals. □

Theorem 4.8. *The operator $\mathcal{K}^*$ verifies, for any locally bounded $f \in \mathcal{L}^2([0,T])$,*

$$\int_0^t f(s)\;\mathrm{d}L_s^k = \int_0^t \mathcal{K}^*(f)(s)\;\mathrm{d}L_s, \qquad \forall t \in [0,T]. \tag{4.8}$$

For any f and g in $\mathcal{I}^k$, define the inner product

$$\langle f,g\rangle_{\mathcal{I}^{k,L}} := (1+\sigma^2)\;\langle f,g\rangle_{\mathcal{I}^k}\,,$$

and let $\mathcal{I}^{k,L}$ be the closure of $\mathcal{I}$ with respect to this inner product. Then $\mathcal{K}^*$ is an isometry from $\mathcal{I}^{k,L}$ into $\mathcal{L}^2([0,T])$ and for any $f \in \mathcal{I}^{k,L}$, $g \in \mathcal{I}^{k,L}$ we have:

$$\mathbb{E}\left[\int_0^T f(s)\;\mathrm{d}L_s^k.\int_0^T g(s)\;\mathrm{d}L_s^k\right] = \langle f,g\rangle_{\mathcal{I}^{k,L}}\,.$$

Proof. For any $t \in [0,T]$,

$$L_t^k = \int_0^T 1\!\!1_{[0,t]}(s)\,\mathrm{d}L_s^k,$$

The very definition of $\mathcal{K}^*$ applied to $f = \mathbb{I}_{[0,t]}$ and the definition of L^k, yield that, for any $t \in [0,T]$, we have

$$L_t^k = \int_0^T k(t,s)\,\mathrm{d}L_s = \int_0^T \mathcal{K}^*(1\!\!1_{[0,t]})(s)\,\mathrm{d}L_s.$$

Thus, for any $t \in [0,T]$,

$$\int_0^T 1\!\!1_{[0,t]}(s)\,\mathrm{d}L_s^k = \int_0^T \mathcal{K}^*(1\!\!1_{[0,t]})(s)\,\mathrm{d}L_s.$$

Relation (4.8) is true for any function of $\mathcal{I}$, thus, by a continuity argument, it is true for any deterministic function of $\mathcal{L}^2([0,T])$. Finally, for $f \in \mathcal{I}^{k,L}$ and $g \in \mathcal{I}^{k,L}$,

$$\mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}L_s^k.\int_0^T g(s)\,\mathrm{d}L_s^k\right] = \mathbb{E}\left[\int_0^T \mathcal{K}^*(f)(s)\,\mathrm{d}L_s.\int_0^T \mathcal{K}^*(g)(s)\,\mathrm{d}L_s\right].$$

Now, by independence of B and J, this writes:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}L_s^k . \int_0^T g(s)\,\mathrm{d}L_s^k\right] \\
&= \sigma^2 \mathbb{E}\left[\int_0^T \mathcal{K}^*(f)(s)\,\mathrm{d}B_s . \int_0^T \mathcal{K}^*(g)(s)\,\mathrm{d}B_s\right] \\
&\quad + \mathbb{E}\left[\int_0^T \mathcal{K}^*(f)(s)\,\mathrm{d}J_s . \int_0^T \mathcal{K}^*(g)(s)\,\mathrm{d}J_s\right] \\
&= \sigma^2 \mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}B_s^k . \int_0^T g(s)\,\mathrm{d}B_s^k\right] + \mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}J_s^k . \int_0^T g(s)\,\mathrm{d}J_s^k\right].
\end{aligned}$$

Finally, relation (4.5) and (4.6) enable us to write:

$$\mathbb{E}\left[\int_0^T f(s)\,\mathrm{d}L_s^k . \int_0^T g(s)\,\mathrm{d}L_s^k\right] = (1+\sigma^2)\ \langle f, g\rangle_{\mathcal{I}^k} = \langle f, g\rangle_{\mathcal{I}^{k,L}}\,.$$

□

Definition 4.9. For $I = C, S$, to define the integral with respect to the filtered process associated to $X = B, J, L$, it is enough to say that $Y \in \mathrm{Dom}\left(\delta_I^{X,\mathcal{K}^*}\right)$ if and only if $\mathcal{K}^*(Y) \in \mathrm{Dom}\left(\delta_I^X\right)$ and $\delta_I^{X,\mathcal{K}^*}(Y) := \delta_I^X(\mathcal{K}^*(Y))$.

4.4. A relationship between these integrals.

Theorem 4.10.

- $\mathrm{Dom}\left(\delta_C^{B,k}\right) = \mathrm{Dom}\left(\delta_C^{B,\mathcal{K}^*}\right)$,
- *For all* $Y \in \mathrm{Dom}\left(\delta_C^{B,\mathcal{K}^*}\right)$, $\delta_C^{B,k}(Y) = \delta_C^{B,\mathcal{K}^*}(Y)$.

Proof. See [21] Proposition 5.2.2, page 288. □

Theorem 4.11. *For* $X = B, J, L$, *and for all* $Y \in \mathrm{Dom}\left(\delta_C^{X,\mathcal{K}^*}\right)$,

$$\delta_C^{X,\mathcal{K}^*}(Y) = \delta_S^{X,\mathcal{K}^*}(Y).$$

Proof. By definition, for $X = B, J, L$

$$\delta_I^{X,\mathcal{K}^*}(Y) = \delta_I^X(\mathcal{K}^*(Y)), \quad I = C, S,$$

the proof is thus an obvious consequence of Theorem 3.23. □

Finally, it remains to prove the following result:

Theorem 4.12. *For* $X = B, J, L$, $\mathrm{Dom}\left(\delta_S^{X,\mathcal{K}^*}\right) = \mathrm{Dom}\left(\delta_S^{X,k}\right)$ *and for all* $Y \in \mathrm{Dom}\left(\delta_S^{X,k}\right)$, $\delta_S^{X,\mathcal{K}^*}(Y) = \delta_S^{X,k}(Y)$.

Proof. We give the proof for $X = L$. It is exactly the same in the other cases. Let $Y = Fh$ with $F \in \mathbb{L}^2(\Omega)$ and $h \in \mathcal{L}^2([0,T])$. To prove the theorem, it is enough to show that, for any $\eta \in \Xi$,

$$\mathcal{S}^{\bar{L}}\left(\delta_S^{L,\mathcal{K}^*}(Y)\right)(\eta) = \mathcal{S}^{\bar{L}}\left(\delta_S^{L,k}(Y)\right)(\eta).$$

By the very definition of integrals by means of the $\mathcal{S}$-transform (relations (3.19) and (4.1)), we have :

$$\mathcal{S}^{\bar{L}}\left(\delta_S^L(\mathcal{K}^*(Y))\right)(\eta) = \int_0^T \mathcal{S}^{\bar{L}}\left(\mathcal{K}^*(Y)(t)\right)(\eta)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta)\,\mathrm{d}t,$$
$$\mathcal{S}^{\bar{L}}\left(\delta_S^{L,k}(Y)\right)(\eta) = \int_0^T \mathcal{S}^{\bar{L}}\left(Y(t)\right)(\eta)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta)\,\mathrm{d}t.$$

The particular form of u allows us to write:

$$\mathcal{S}^{\bar{L}}\left(Y(t)\right)(\eta) = \mathbb{E}_{\mathbb{Q}_\eta^{\bar{L}}}\left[Y(t)\right] = h(t)\,\mathbb{E}_{\mathbb{Q}_\eta^{\bar{L}}}\left[F\right], \tag{4.9}$$
$$\mathcal{S}^{\bar{L}}\left(\mathcal{K}^*(Y)(t)\right)(\eta) = \mathbb{E}_{\mathbb{Q}_\eta^{\bar{L}}}\left[\mathcal{K}^*(Y)(t)\right] = \mathcal{K}^*(h)(t)\,\mathbb{E}_{\mathbb{Q}_\eta^{\bar{L}}}\left[F\right],$$

thus, it remains to show that, for any $\eta \in \Xi$,

$$\int_0^T h(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta)\,\mathrm{d}t = \int_0^T \mathcal{K}^*(h)(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta)\,\mathrm{d}t. \tag{4.10}$$

Now, we have

$$\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta) = \mathcal{S}^{\bar{L}}\left(\delta_S^L(k(t,\cdot))\right) = \int_0^T \mathcal{S}^{\bar{L}}\left(k(t,s)\right)(\eta)\,\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S}^{\bar{L}}\left(L(s)\right)(\eta)\,\mathrm{d}s,$$

but for deterministic k, it comes from (4.9) that $\mathcal{S}^{\bar{L}}\left(k(t,s)\right)(\eta) = k(t,s)$, which yields to

$$\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta) = \mathcal{S}^{\bar{L}}\left(\delta_S^L(k(t,\cdot))\right) = \int_0^T k(t,s)\,\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S}^{\bar{L}}\left(L(s)\right)(\eta)\,\mathrm{d}s.$$

Let us point out that, for a differentiable function f such that $f(0) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}f = \left[I_{0^+}^T\right]^{-1}(f).$$

All the processes X, defined in this work, are assumed to verify $X(0) = 0$. Then we can write, for any $t \in [0,T]$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta) = \left[I_{0^+}^T\right]^{-1}(\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta)),$$

thus

$$\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta) = K \circ \left[I_{0^+}^T\right]^{-1}(\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta)),$$

finally,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{S}^{\bar{L}}\left(L^k(t)\right)(\eta) = \left[I_{0^+}^T\right]^{-1} \circ K \circ \left[I_{0^+}^T\right]^{-1}(\mathcal{S}^{\bar{L}}\left(L(t)\right)(\eta)).$$

Thus, noticing that the dual of I_{0+}^T is $I_{T^-}^T$, we have

$$\begin{aligned}\int_0^T h(t)\frac{\mathrm{d}}{\mathrm{d}s}\,\mathcal{S}^{\bar{L}}\left(L^K(t)\right)(\eta)\,\mathrm{d}t &= \left\langle h, \frac{\mathrm{d}}{\mathrm{d}s}\,\mathcal{S}^{\bar{L}}\left(L^K(\cdot)\right)(\eta)\right\rangle_{\mathcal{L}^2([0,T])} \\ &= \left\langle h, \left[I_{0+}^T\right]^{-1}\circ K\circ\left[I_{0+}^T\right]^{-1}\left(\mathcal{S}^{\bar{L}}\left(L(\cdot)\right)(\eta)\right)\right\rangle_{\mathcal{L}^2([0,T])} \\ &= \left\langle K^*\circ\left[I_{T^-}^T\right]^{-1}h, \left[I_{0+}^T\right]^{-1}\left(\mathcal{S}^{\bar{L}}\left(L(\cdot)\right)(\eta)\right)\right\rangle_{\mathcal{L}^2([0,T])} \\ &= \left\langle \mathcal{K}^*(h), \left[I_{0+}^T\right]^{-1}\left(\mathcal{S}^{\bar{L}}\left(L(\cdot)\right)(\eta)\right)\right\rangle_{\mathcal{L}^2([0,T])},\end{aligned}$$

which is exactly the relation (4.10). This ends the proof for simple processes. The theorem remains true for any process by a limiting procedure. □

4.5. Lévy–Itô problem for filtered processes.

Theorem 4.13.
- *For any* $Y \in \mathrm{Dom}\left(\delta_C^{L,\mathcal{K}^*}\right)$, $\delta_C^{L,\mathcal{K}^*}(u) = \delta_C^{B,\mathcal{K}^*}(Y) + \delta_C^{J,\mathcal{K}^*}(Y)$.
- *For any* $Y \in \mathrm{Dom}\left(\delta_S^{L,\mathcal{K}^*}\right)$, $\delta_S^{L,\mathcal{K}^*}(u) = \delta_S^{B,\mathcal{K}^*}(Y) + \delta_S^{J,\mathcal{K}^*}(Y)$.
- *For any* $Y \in \mathrm{Dom}\left(\delta_S^{L,k}\right)$, $\delta_S^{L,k}(u) = \delta_S^{B,k}(Y) + \delta_S^{J,k}(Y)$.

Proof. The proof of the first two statements is nothing but an application of Theorems 3.6 and 3.24 to $\mathcal{K}^*(u)$. The third statement is a consequence of the second statement and Theorem 4.12. □

4.6. The complete Lévy–Itô problem. The filtered process and its underlying process have the same filtration. See [12] for Brownian motion and [11] for Poisson process. The proof can be easily extended to the general Lévy case. Thus the remark of Section 3.5 extends to filtered processes.

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Nicolas Savy: Université Paul Sabatier, Institut de Mathématiques de Toulouse UMR 5219, 31062 Toulouse Cedex 09, France

E-mail address: nicolas.savy@math.univ-toulouse.fr

Josep Vives: Universitat de Barcelona, Dpt. de Matemàtiques i Informàtica, Gran Via 585, 08007 Barcelona, Catalunya, Spain

E-mail address: josep.vives@ub.edu