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ON CONSTRUCTING SOME MEMBRANES FOR A SYMMETRIC α -STABLE PROCESS

M. M. OSYPCHUK AND M. I. PORTENKO

ABSTRACT. Two kinds of membranes located on a fixed hyperplane S in a Euclidean space are constructed for a symmetric α -stable process with $\alpha \in (1, 2)$. The first one has the property of killing the process at the points of the hyperplane with some given intensity $(r(x))_{x \in S}$. This kind of membranes can be called an *elastic screen* for the process, by analogy to that in the theory of diffusion processes. The second one has the property of delaying the process at the points of S with some given coefficient $(p(x))_{x \in S}$. In other words, the points of S , where $p(x) > 0$, are *sticky* for the process constructed. We show that each one of the membranes is associated with some initial-boundary value problem for pseudo-differential equations related to a symmetric α -stable process.

1. Introduction

Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ be a standard Markov process in a d -dimensional Euclidean space \mathbb{R}^d whose transition probability density g_0 (with respect to the Lebesgue measure on \mathbb{R}^d) is given by the equality

$$g_0(t, x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(x - y, \xi) - ct|\xi|^\alpha\} d\xi, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d, \quad (1.1)$$

where $c > 0$ and $\alpha \in (1, 2)$ are fixed parameters (see [4, Theorem 3.14]). This process is called a symmetric (more precisely, rotationally invariant) α -stable process. The generator of it is denoted by \mathbf{A} and this is a pseudo-differential operator with its symbol given by $(-c|\xi|^\alpha)_{\xi \in \mathbb{R}^d}$.

Let ν be a fixed unit vector in \mathbb{R}^d and S denote the hyperplane in \mathbb{R}^d orthogonal to ν , that is $S = \{x \in \mathbb{R}^d : (x, \nu) = 0\}$. By \mathbf{B}_ν we denote a pseudo-differential operator with the function $(2ic|\xi|^{\alpha-2}(\xi, \nu))_{\xi \in \mathbb{R}^d}$ as its symbol.

We will consider two kinds of transformations of the process $(x(t))_{t \geq 0}$ (this is a short notation for our process). The first one is connected with the Feynman-Kac formula. Let $(r(x))_{x \in S}$ be a given bounded continuous function with non-negative values. We show that there exists a W-functional $(\eta_t(r))_{t \geq 0}$ of the process $(x(t))_{t \geq 0}$

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such that its characteristic is given by

$$\mathbb{E}_x \eta_t(r) = \int_0^t d\tau \int_S g_0(\tau, x, y) r(y) d\sigma_y, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (1.2)$$

where the inner integral is a surface one.

As is well-known (see [4, Chapter 10]), there exists a standard Markov process $(x^*(t), \mathcal{M}_t^*, \mathbb{P}_x^*, \zeta)$ in \mathbb{R}^d (ζ is the life time of the process) such that the equality

$$\mathbb{E}_x^*(\varphi(x^*(t)) \mathbb{1}_{\zeta > t}) = \mathbb{E}_x(\varphi(x(t)) \exp\{-\eta_t(r)\}) \quad (1.3)$$

is valid for $t > 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ (this is the notation for the Banach space of all continuous bounded functions on \mathbb{R}^d with real values and the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^d} |\varphi(x)|$). We show that the function (1.3) (denote it by $u(t, x, \varphi)$, $t \geq 0$, $x \in \mathbb{R}^d$, $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$) is a solution to the following initial-boundary value problem.

Problem A. For a given $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, a continuous function $(u(t, x))_{t > 0, x \in \mathbb{R}^d}$ is being looked for such that it satisfies

- (i) the equation $\frac{\partial u}{\partial t} = \mathbf{A}u$ in the region $t > 0$, $x \notin S$;
- (ii) the initial condition $u(0+, x) = \varphi(x)$ for all $x \in \mathbb{R}^d$;
- (iii) the boundary condition $\frac{1}{2} \mathbf{B}_\nu u(t, \cdot)(x+) - \frac{1}{2} \mathbf{B}_\nu u(t, \cdot)(x-) = r(x)u(t, x)$ for all $t > 0$ and $x \in S$.

The symbol $\mathbf{B}_\nu u(t, \cdot)(x+)$ (respectively, $\mathbf{B}_\nu u(t, \cdot)(x-)$) for $t > 0$ and $x \in S$ means the limit value of the function $\mathbf{B}_\nu u(t, \cdot)(z)$, as z approaches x along any curve lying in a finite closed cone \mathcal{K} in \mathbb{R}^d with vertex at x such that $\mathcal{K} \subset \{z \in \mathbb{R}^d : (z, \nu) > 0\} \cup \{x\}$ (respectively, $\mathcal{K} \subset \{z \in \mathbb{R}^d : (z, \nu) < 0\} \cup \{x\}$).

The second transformation is connected with some random change of time. Let a continuous bounded function $(p(x))_{x \in S}$ with non-negative values be given. For $t \geq 0$, we put

$$\zeta_t = \inf\{s \geq 0 : s + \eta_s(p) \geq t\}, \quad \hat{x}(t) = x(\zeta_t), \quad \hat{\mathcal{M}}_t = \mathcal{M}_{\zeta_t}.$$

It is well-known (see, for example, [4, Chapter 10]) that the process $(\hat{x}(t), \hat{\mathcal{M}}_t, \mathbb{P}_x)$ is also a standard Markov process in \mathbb{R}^d . We show that the function

$$\hat{u}(t, x, \varphi) = \mathbb{E}_x \varphi(\hat{x}(t)), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (1.4)$$

is a solution to the following problem.

Problem B. For a given $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, a continuous function $(u(t, x))_{t > 0, x \in \mathbb{R}^d}$ is being looked for such that it satisfies the condition (i), the initial condition (ii) and the following boundary condition (for $t > 0$ and $x \in S$)

$$(iii') \quad p(x) \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \mathbf{B}_\nu u(t, \cdot)(x+) - \frac{1}{2} \mathbf{B}_\nu u(t, \cdot)(x-).$$

If $\alpha = 2$ (and $c = \frac{1}{2}$), then our process is a standard Brownian motion, and the operator \mathbf{A} coincides with $\frac{1}{2} \Delta$ (Δ is the Laplace operator) and \mathbf{B}_ν coincides with $\frac{\partial}{\partial \nu}$ (the derivative in the direction ν). The facts that in this case the functions (1.3) and (1.4) solve *Problems A* and *B*, respectively, are well-known (some results of the kind can be found in the books [4, 6] and also in [1, 2, 7] and many others).

The article is organized as follows. In Section 2 some auxiliary results are presented. Sections 3 and 4 are devoted to solving the *Problems A* and *B*, respectively.

2. Single-layer Potentials for a Symmetric α -stable Process and the Feynman-Kac Formula.

2.1. The function g_0 defined by (1.1) is continuous in the region $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Moreover, it is uniformly continuous in any region of the form $(t, x, y) \in [\gamma, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for $\gamma > 0$. As follows from [3], it satisfies the inequality

$$g_0(t, x, y) \leq N \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha}}, \quad t > 0, \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d, \quad (2.1)$$

where N is a positive constant. The inequalities of the kind in more general situations including similar inequalities for (fractional) derivatives of g_0 can be found in [5].

2.2. Let $\nu \in \mathbb{R}^d$ be a fixed unit vector and S be the hyperplane in \mathbb{R}^d orthogonal to ν . The following formula

$$\int_S e^{i(\xi, y)} g_0(t, x, y) d\sigma_y = \frac{1}{\pi} \int_0^\infty e^{-ct(|\xi|^2 + \rho^2)^{\alpha/2}} \cos(\rho(x, \nu)) d\rho \quad (2.2)$$

holds true for all $t > 0$, $x \in \mathbb{R}^d$ and $\xi \in S$ (see [8]). Combining (2.1) and (2.2) (for $\xi = 0$), we arrive at the inequality

$$\int_S g_0(t, x, y) d\sigma_y \leq N \frac{t}{(t^{1/\alpha} + |(x, \nu)|)^{1+\alpha}} \quad (2.3)$$

valid for all $t > 0$ and $x \in \mathbb{R}^d$ with some positive constant N .

2.3. In accordance with the definition of \mathbf{B}_ν (see Section 1), the following equality (for fixed $t > 0$ and $y \in \mathbb{R}^d$)

$$\mathbf{B}_\nu g_0(t, \cdot, y)(x) = \frac{2ic}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\{i(x - y, \xi) - ct|\xi|^\alpha\} |\xi|^{\alpha-2}(\xi, \nu) d\xi$$

is fulfilled for all $x \in \mathbb{R}^d$. Integrating by parts leads us to the formula

$$\mathbf{B}_\nu g_0(t, \cdot, y)(x) = \frac{2(y - x, \nu)}{\alpha t} g_0(t, x, y) \quad (2.4)$$

2.4. Let $(\psi(t, x))_{t \geq 0, x \in S}$ be a continuous function with real values satisfying the inequality $|\psi(t, x)| \leq Ct^{-\beta}$ for all $t > 0$ and $x \in S$ with some constants $C > 0$ and $\beta < 1$. We put

$$V_0(t, x) = \int_0^t d\tau \int_S g_0(t - \tau, x, y) \psi(\tau, y) d\sigma_y, \quad t > 0, \quad x \in \mathbb{R}^d. \quad (2.5)$$

This function is well-defined, as the following estimations show

$$\begin{aligned} |V_0(t, x)| &\leq C \int_0^t \tau^{-\beta} d\tau \int_S g_0(t - \tau, x, y) d\sigma_y \\ &\leq CN \int_0^t \tau^{-\beta} (t - \tau)^{-1/\alpha} d\tau \\ &= CN \frac{\Gamma(1 - \beta)\Gamma(1 - 1/\alpha)}{\Gamma(2 - \beta - 1/\alpha)} t^{1 - \beta - 1/\alpha}. \end{aligned}$$

Moreover, this function is continuous in the region $t > 0$ and $x \in \mathbb{R}^d$. It is called a single-layer potential.

The following properties of the function V_0 are proved in [8].

2.4.A. The function V_0 is a solution of the equation $\frac{\partial V_0}{\partial t} = \mathbf{A}V_0$ in the region $t > 0$ and $x \notin S$.

2.4.B. The following relations $\mathbf{B}_\nu V_0(t, \cdot)(x_\pm) = \mp \psi(t, x)$ are held for all $t > 0$ and $x \in S$ (the sense of the left hand side is explained in Section 1).

Remark 2.1. Relation 2.4.B are some analogy to the well-known theorem on the jump of the (co-)normal derivative of a single-layer potential in the classical theory of potentials. The term analogous to the so-called direct value of the derivative vanishes in 2.4.B, since $\mathbf{B}_\nu g_0(t, \cdot, y)(x) = 0$ for $y \in S$ and $x \in S$ (see (2.4)).

2.5. Let $(v(x))_{x \in \mathbb{R}^d}$ be a continuous bounded function with real values. We put for $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, $t > 0$ and $x \in \mathbb{R}^d$

$$Q(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) \exp \left\{ \int_0^t v(x(\tau)) d\tau \right\} \right).$$

The well-known Feynman-Kac formula asserts that Q satisfies the equation

$$\frac{\partial Q}{\partial t} = \mathbf{A}Q + v(x)Q$$

in the region $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and the initial condition $Q(0+, x, \varphi) = \varphi(x)$ for all $x \in \mathbb{R}^d$.

An intermediate stage of this result is the following integral equation for Q

$$Q(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, y) \varphi(y) dy + \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t - \tau, x, y) Q(\tau, y, \varphi) v(y) dy,$$

where $t > 0$, $x \in \mathbb{R}^d$.

3. Solving Problem A

3.1. Let the hyperplane S and the bounded continuous function $(r(x))_{x \in S}$ be such as above. One can easily verify that the function

$$f_t(x) = \int_0^t d\tau \int_S g_0(t - \tau, x, y) r(y) d\sigma_y$$

is a W-function for the process $(x(t))_{t \geq 0}$ (see [4, Chapter 6, §3]) satisfying the inequality

$$f_t(x) \leq N \|r\| \frac{\alpha}{\alpha - 1} t^{1-1/\alpha}$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$ (see (2.4)), where $\|r\| = \sup_{x \in S} r(x)$. Therefore, according to Theorem 6.6 from [4], there exists a W-functional $(\eta_t(r))_{t \geq 0}$ of the process $(x(t))_{t \geq 0}$ such that $\mathbb{E}_x \eta_t(r) = f_t(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$.

For $r_0(x) \equiv 1$ we put $\eta_t = \eta_t(r_0)$, $t \geq 0$. The functional $(\eta_t)_{t \geq 0}$ is called the local time on S for the process $(x(t))_{t \geq 0}$. It is evident that $\eta_t(r) = \int_0^t r(x(s)) d\eta_s$, $t \geq 0$.

3.2. We now approximate the functional $(\eta_t(r))_{t \geq 0}$ by somewhat simpler ones. For $h > 0$, we define a function $(v_h(x))_{x \in \mathbb{R}^d}$ by setting $v_h(x) = \int_S g_0(h, x, y) r(y) d\sigma_y$, $x \in \mathbb{R}^d$, and a functional $(\eta_t^{(h)}(r))_{t \geq 0}$ by the equality $\eta_t^{(h)}(r) = \int_0^t v_h(x(s)) ds$, $t \geq 0$.

The function v_h for fixed $h > 0$ is continuous and bounded, so the W-functional $(\eta_t^{(h)}(r))_{t \geq 0}$ is well-defined. Its characteristic is given by

$$\begin{aligned} f_t^{(h)}(x) &= \mathbb{E}_x \eta_t^{(h)}(r) = \int_0^t d\tau \int_{\mathbb{R}^d} g_0(\tau, x, y) v_h(y) dy \\ &= \int_h^{t+h} d\tau \int_S g_0(\tau, x, y) r(y) d\sigma_y. \end{aligned}$$

Hence,

$$\mathbb{E}_x \eta_t^{(h)}(r) - \mathbb{E}_x \eta_t(r) = \int_t^{t+h} d\tau \int_S g_0(\tau, x, y) r(y) d\sigma_y - \int_0^h d\tau \int_S g_0(\tau, x, y) r(y) d\sigma_y.$$

Taking into account (2.4), we arrive at the inequality

$$\begin{aligned} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} |\mathbb{E}_x \eta_t^{(h)}(r) - \mathbb{E}_x \eta_t(r)| &\leq N \|r\| \frac{\alpha}{\alpha - 1} \left[h^{1-1/\alpha} \right. \\ &\quad \left. + \sup_{0 \leq t \leq T} \left((t+h)^{1-1/\alpha} - t^{1-1/\alpha} \right) \right] \end{aligned}$$

valid for all $T > 0$ and $h > 0$. Denote by $q_T(h)$ the expression on the right-hand side of this inequality. Obviously, $q_T(h) \rightarrow 0$, as $h \rightarrow 0+$, for any fixed $T > 0$. According to Lemma 6.5 from [4], the following inequality

$$\mathbb{E}_x (\eta_t^{(h)}(r) - \eta_t(r))^2 \leq 2(f_t^{(h)}(x) + f_t(x)) q_T(h)$$

holds true for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. Since for those (t, x) we have

$$f_t^{(h)}(x) \leq N \|r\| \frac{\alpha}{\alpha - 1} (T+h)^{1-1/\alpha}; \quad f_t(x) \leq N \|r\| \frac{\alpha}{\alpha - 1} T^{1-1/\alpha},$$

we can assert that the inequality

$$\mathbb{E}_x (\eta_t^{(h)}(r) - \eta_t(r))^2 \leq 4N \|r\| \frac{\alpha}{\alpha - 1} (T+h_0)^{1-1/\alpha} q_T(h) \quad (3.1)$$

is fulfilled for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $h \in (0, h_0]$ ($T > 0$ and $h_0 > 0$ are arbitrary fixed numbers).

3.3. For $t > 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, we put

$$u^{(h)}(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta_t^{(h)}(r)} \right), \quad u(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta_t(r)} \right).$$

Proposition 3.1. *There exists a sequence $(h_n)_{n \geq 1}$ such that $h_n \rightarrow 0$, as $n \rightarrow \infty$, and*

$$\lim_{n \rightarrow \infty} u^{(h_n)}(t, x, \varphi) = u(t, x, \varphi)$$

uniformly with respect to $x \in \mathbb{R}^d$ and locally uniformly with respect to $t \in [0, \infty)$.

Proof. Since $|e^{-a} - e^{-b}| \leq |a - b|$ for all $a \geq 0$ and $b \geq 0$, we can write down the chain of inequalities (for an arbitrary $T > 0$)

$$\begin{aligned} |u^{(h)}(t, x, \varphi) - u(t, x, \varphi)| &\leq \|\varphi\| \mathbb{E}_x |\eta_t^{(h)}(r) - \eta_t(r)| \\ &\leq \|\varphi\| \left[\mathbb{E}_x (\eta_t^{(h)}(r) - \eta_t(r))^2 \right]^{1/2} \leq K_T(h_0) (q_T(h))^{1/2} \|\varphi\| \end{aligned}$$

valid for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $h \in (0, h_0]$, where $K_T(h_0)$ is a constant finite for $T < \infty$. To complete the proof one should make use of the diagonal method. \square

3.4. The function $u^{(h)}$ (for a fixed $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$) is a unique bounded solution to the integral equation (see Section 2.5)

$$u^{(h)}(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, y) \varphi(y) dy - \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t - \tau, x, y) u^{(h)}(\tau, y, \varphi) v_h(y) dy. \quad (3.2)$$

It is an easy exercise to verify that the relation

$$\lim_{h \rightarrow 0+} \int_{\mathbb{R}^d} \psi(y) v_h(y) dy = \int_S \psi(y) r(y) d\sigma_y \quad (3.3)$$

is fulfilled for any continuous function $(\psi(y))_{y \in \mathbb{R}^d}$ such that $\int_{\mathbb{R}^d} |\psi(y)| dy < \infty$.

Proposition 3.2. *For a given $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, the function $(u(t, x, \varphi))_{t \geq 0, x \in \mathbb{R}^d}$ is a unique bounded solution of the equation*

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, y) \varphi(y) dy - \int_0^t d\tau \int_S g_0(t - \tau, x, y) u(\tau, y, \varphi) r(y) d\sigma_y. \quad (3.4)$$

Proof. In order to pass to the limit, as $h_n \rightarrow 0$, in equation (3.2) (written for $h = h_n$), one should observe that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t - \tau, x, y) u(\tau, y, \varphi) v_{h_n}(y) dy \\ &= \int_0^t d\tau \int_S g_0(t - \tau, x, y) u(\tau, y, \varphi) r(y) d\sigma_y \end{aligned}$$

according to (3.3). Besides,

$$\int_0^t d\tau \int_{\mathbb{R}^d} g_0(t - \tau, x, y) v_h(y) dy = f_t^{(h)}(x) \leq N \|r\| \frac{\alpha}{\alpha - 1} (T + h)^{1-1/\alpha},$$

as was established in Section 3.2. Taking into account Proposition 3.1, we arrive at equation (3.4) for the function u .

A solution to the equation (3.4) can be constructed by the method of successive approximations. If we put

$$u_0(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, y) \varphi(y) dy, \quad t > 0, x \in \mathbb{R}^d, \varphi \in \mathbb{C}_b(\mathbb{R}^d),$$

and for $k \geq 1$

$$u_k(t, x, \varphi) = \int_0^t d\tau \int_S g_0(t - \tau, x, y) u_{k-1}(\tau, y, \varphi) r(y) d\sigma_y,$$

then by induction on k , we can easily obtain the following estimate

$$|u_k(t, x, \varphi)| \leq \frac{\|\varphi\| \|r\|^k}{(c^{1/\alpha} \alpha \sin \frac{\pi}{\alpha})^k} \frac{t^{k(1-1/\alpha)}}{\Gamma(k(1-1/\alpha) + 1)} \quad (3.5)$$

held true for all $t > 0$, $x \in \mathbb{R}^d$, $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ and $k = 0, 1, 2, \dots$. As a consequence of (3.5), we have that the series

$$\sum_{k=0}^{\infty} (-1)^k u_k(t, x, \varphi) \quad (3.6)$$

is a continuous solution of (3.4) satisfying the condition

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} |u(t, x, \varphi)| < \infty$$

for any $T > 0$. Another consequence of (3.5) is that such a solution is unique. Therefore, the function u can be represented by the series (3.6). The proposition is proved. \square

3.5. We now can formulate the main result of Section 3

Theorem 3.3. *For a fixed $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ the function*

$$u(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta(r)} \right), \quad t \geq 0, x \in \mathbb{R}^d,$$

solves the Problem A.

Proof. The first item on the right hand side of (3.4) satisfies the equation (i) in the whole region $t > 0$ and $x \in \mathbb{R}^d$. It also satisfies the initial condition (ii). The second item on the right-hand side of (3.4) is a single-layer potential. According to 2.4.A, it satisfies (i) and its initial value vanishes. The relations 2.4.B imply now the equalities

$$\mathbf{B}_\nu u(t, \cdot, \varphi)(x \pm) = \frac{2}{\alpha t} \int_{\mathbb{R}^d} (y, \nu) \varphi(y) g_0(t, x, y) dy \pm r(x) u(t, x, \varphi)$$

valid for $t > 0$ and $x \in S$, and the condition (iii) follows from these relations immediately. The theorem has been proved. \square

3.6. If $d = 1$, then $S = \{0\}$ and $r = r(0)$ is a non-negative number. The equation for the function u in this case can be written as follows

$$u(t, x, \varphi) = \int_{\mathbb{R}^1} g_0(t, x, y) \varphi(y) dy - r \int_0^t g_0(t - \tau, x, 0) u(\tau, 0, \varphi) d\tau. \quad (3.7)$$

Denote by \tilde{u} and \tilde{g}_0 the Laplace transformations of the functions u and g_0 , respectively ($\lambda > 0$)

$$\tilde{u}(\lambda, x, \varphi) = \int_0^\infty u(t, x, \varphi) e^{-\lambda t} dt, \quad \tilde{g}_0(\lambda, x, y) = \int_0^\infty g_0(t, x, y) e^{-\lambda t} dt.$$

Then (3.7) implies the equality

$$\tilde{u}(\lambda, x, \varphi) = \int_{\mathbb{R}^1} \left[\tilde{g}_0(\lambda, x, y) - \frac{r \tilde{g}_0(\lambda, x, 0) \tilde{g}_0(\lambda, 0, y)}{1 + r \tilde{g}_0(\lambda, 0, 0)} \right] \varphi(y) dy,$$

where $\tilde{g}_0(\lambda, 0, 0) = [c^{1/\alpha} \alpha \sin \frac{\pi}{\alpha}]^{-1} \lambda^{1/\alpha-1}$. It means that the resolvent kernel $\tilde{g}^*(\lambda, x, y)$ of the process $(x^*(t))_{t \geq 0}$ (see Section 1) is given by

$$\tilde{g}^*(\lambda, x, y) = \tilde{g}_0(\lambda, x, y) - \frac{r \tilde{g}_0(\lambda, x, 0) \tilde{g}_0(\lambda, 0, y)}{1 + r \tilde{g}_0(\lambda, 0, 0)}$$

for $\lambda > 0$, $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$. One can obtain from this equality, in particular, the Laplace transform for the distribution function of ζ (the life time of the process $(x^*(t))_{t \geq 0}$)

$$\mathbb{E}_x^* e^{-\lambda \zeta} = \frac{r \tilde{g}_0(\lambda, x, 0)}{1 + r \tilde{g}_0(\lambda, 0, 0)}, \quad x \in \mathbb{R}^1, \lambda > 0.$$

4. Solving Problem B

4.1. We are now given by a continuous bounded function $(p(x))_{x \in S}$ with non-negative values. Consider the Markov process $(\hat{x}(t), \hat{\mathcal{M}}_t, \mathbb{P}_x)$ defined in Section 1. The resolvent operator for this process can be calculated in the following way (see [6, Chapter II, §6])

$$\begin{aligned} \mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(\hat{x}(t)) dt &= \mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(x(\zeta_t)) dt \\ &= \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) dt + \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) d\eta_t(p), \end{aligned} \quad (4.1)$$

where $x \in \mathbb{R}^d$, $\lambda > 0$, $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ (we have taken into account that the equality $\zeta_t = t'$ implies $t = t' + \eta_{t'}(p)$).

4.2. If we put

$$Q_\lambda(t, x, \varphi) = \mathbb{E}_x(\varphi(x(t)) e^{-\lambda \eta_t(p)}), \quad t > 0, \lambda > 0, x \in \mathbb{R}^d, \varphi \in \mathbb{C}_b(\mathbb{R}^d),$$

then in accordance with Section 3, we have the following equation for Q_λ

$$Q_\lambda(t, x, \varphi) = \int_{\mathbb{R}^d} g_0(t, x, y) \varphi(y) dy - \lambda \int_0^t d\tau \int_S g_0(t - \tau, x, y) Q_\lambda(\tau, y, \varphi) p(y) d\sigma_y.$$

Multiplying both sides of this equation by $e^{-\lambda t}$ and integrating with respect to t over $(0, \infty)$, we get the equation

$$U_1(\lambda, x, \varphi) = \int_{\mathbb{R}^d} \tilde{g}_0(\lambda, x, y) \varphi(y) dy - \lambda \int_S \tilde{g}_0(\lambda, x, y) U_1(\lambda, y, \varphi) p(y) d\sigma_y, \quad (4.2)$$

where $\tilde{g}_0(\lambda, x, y) = \int_0^\infty g_0(t, x, y) e^{-\lambda t} dt$ and

$$U_1(\lambda, x, \varphi) = \int_0^\infty Q_\lambda(t, x, y) e^{-\lambda t} dt = \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) dt.$$

4.3. To calculate the second item on the right hand side of (4.1), we observe that

$$\mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) d\eta_t(p) = \lim_{h \rightarrow 0^+} \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) v_h(x(t)) dt,$$

where this time $v_h(x) = \int_S g_0(h, x, y) p(y) d\sigma_y$, $h > 0$, $x \in \mathbb{R}^d$. According to Section 4.2, we have

$$\mathbb{E}_x \int_0^\infty e^{-\lambda(t+\eta_t(p))} \varphi(x(t)) v_h(x(t)) dt = U_1(\lambda, x, \varphi \cdot v_h).$$

It is a very simple conclusion that for $\lambda > 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, the relation $\lim_{h \rightarrow 0^+} U_1(\lambda, x, \varphi \cdot v_h) = U_2(\lambda, x, \varphi)$ fulfilled, where U_2 is the solution to the equation

$$U_2(\lambda, x, \varphi) = \int_S \tilde{g}_0(\lambda, x, y) \varphi(y) p(y) d\sigma_y - \lambda \int_S \tilde{g}_0(\lambda, x, y) U_2(\lambda, y, \varphi) p(y) d\sigma_y, \quad (4.3)$$

4.4. As a consequence of 2.4.B, we have the following relations

$$\mathbf{B}_\nu \left(\int_S \tilde{g}_0(\lambda, \cdot, y) \tilde{\psi}(\lambda, y) d\sigma_y \right) (x \pm) = \mp \tilde{\psi}(\lambda, x)$$

valid for $\lambda > 0$, $x \in S$ and any continuous function $(\psi(t, x))_{t \geq 0, x \in S}$ such as in Section 2.4. These relations imply the following ones ($x \in S$, $\lambda > 0$)

$$\begin{aligned} \mathbf{B}_\nu U_1(\lambda, \cdot, \varphi)(x \pm) &= \int_{\mathbb{R}^d} \mathbf{B}_\nu \tilde{g}_0(\lambda, \cdot, y) \varphi(y) dy \pm \lambda p(x) U_1(\lambda, x, \varphi), \\ \mathbf{B}_\nu U_2(\lambda, \cdot, \varphi)(x \pm) &= \mp p(x) \varphi(x) \pm \lambda p(x) U_2(\lambda, x, \varphi). \end{aligned}$$

4.5. We put $U(\lambda, x, \varphi) = U_1(\lambda, x, \varphi) + U_2(\lambda, x, \varphi)$. Then

$$\mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(\hat{x}(t)) dt = U(\lambda, x, \varphi).$$

It follows from the equations (4.2), (4.3) that the function U satisfies the equation

$$\mathbf{A}U = \lambda U - \varphi(x)$$

in the region $x \notin S$. Besides, it satisfies the boundary condition ($\lambda > 0$, $x \in S$)

$$\frac{1}{2} \mathbf{B}_\nu U(\lambda, \cdot, \varphi)(x+) - \frac{1}{2} \mathbf{B}_\nu U(\lambda, \cdot, \varphi)(x-) = p(x) (\lambda U(\lambda, x, \varphi) - \varphi(x)).$$

We have thus proved the following assertion

Theorem 4.1. *The function*

$$\hat{U}(t, x, \varphi) = \mathbb{E}_x \varphi(\hat{x}(t)), \quad t > 0, \quad x \in \mathbb{R}^d$$

solves the Problem B.

4.6. If $d = 1$, then

$$\begin{aligned} \mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(\hat{x}(t)) dt &= \frac{p \tilde{g}_0(\lambda, x, 0)}{1 + \lambda p \tilde{g}_0(\lambda, 0, 0)} \varphi(0) \\ &+ \int_{\mathbb{R}^1} \left[\tilde{g}_0(\lambda, x, y) - \frac{\lambda p \tilde{g}_0(\lambda, x, 0) \tilde{g}_0(\lambda, 0, y)}{1 + \lambda p \tilde{g}_0(\lambda, 0, 0)} \right] \varphi(y) dy \end{aligned}$$

for all $\lambda > 0$, $x \in \mathbb{R}^1$ and $\varphi \in \mathbb{C}_b(\mathbb{R}^1)$, where $p = p(0)$ is a non-negative number.

In the case of $p \rightarrow \infty$ the point $x = 0$ becomes an absorbing one. In this case

$$\begin{aligned} \mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(\hat{x}_\infty(t)) dt &= \frac{\tilde{g}_0(\lambda, x, 0)}{\lambda \tilde{g}_0(\lambda, 0, 0)} \varphi(0) \\ &+ \int_{\mathbb{R}^1} \left[\tilde{g}_0(\lambda, x, y) - \frac{\tilde{g}_0(\lambda, x, 0) \tilde{g}_0(\lambda, 0, y)}{\tilde{g}_0(\lambda, 0, 0)} \right] \varphi(y) dy. \end{aligned}$$

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