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Nash Twist and Gaussian Noise Measure for Isometric C1 Maps

Amites Dasgupta
Indian Statistical Institute, amites@isical.ac.in

Mahuya Datta
Indian Statistical Institute, mahuya@isical.ac.in

NASH TWIST AND GAUSSIAN NOISE MEASURE FOR ISOMETRIC C^1 MAPS

AMITES DASGUPTA AND MAHUYA DATTA

ABSTRACT. Starting with a short map $f_0 : I \to \mathbb{R}^3$ on the unit interval I, we construct random isometric map $f_n : I \to \mathbb{R}^3$ (with respect to some fixed Riemannian metrics) for each positive integer n, such that the difference $(f_n - f_0)$ goes to zero in the C^0 norm. The construction of f_n uses the Nash twist. We show that the distribution of $n^{3/2}(f_n - f_0)$ converges (weakly) to a Gaussian noise measure.

1. Introduction

The problem of associating a measure to the solution space of a differential equation has been mentioned by Gromov in an interview with M. Berger [1]. Our point of interest lies in the space of isometric immersions of a Riemannian manifold (M, g) into a Euclidean space $\mathbb{R}^q$ with the canonical metric h. In 1954, Nash proved that if a manifold M with a Riemannian metric g can be embdedded in a Euclidean space $\mathbb{R}^q$, $q > n+1$, then one can construct a large class of isometric C^1 embeddings ([9]). If the initial embedding $f_0 : M \to \mathbb{R}^q$ is strictly short, that is if $g - f^*h$ is a Riemannian metric on M then the isometric embeddings can be made to lie in an arbitrary C^0 neighbourhood of the initial embedding. In the following year, Kuiper showed that the bound can be improved to $q \geq n + 1$ ([6]). The Nash process is an iterative process; each stage of the iteration consists of several small steps each of which involves a choice of a rapidly oscillating function defining a perturbation, called a Nash twist. Starting with the short map f_0, one constructs a sequence $\{f_n\}$ of short immersions, where f_n is obtained from f_{n-1} possibly through infinite steps each involving a Nash twist. Successive Nash twists performed on f_n results in a correction to the induced metric f_n^*h. These corrections do not yield an isometric immersion at any stage but each f_n still remains strictly short; however, f_{n+1} is better than f_n in the sense that the induced metric f_{n+1}^*h is closer to g than that in the previous stage. The Nash twist is a controlled perturbation - the C^1 distance between any two consecutive maps f_n and f_{n+1} remains bounded by the distance between g and the induced metric f_n^*h; furthermore f_n can be made to lie in an arbitrary C^0 neighbourhood of f_0. As a result the sequence converges to a g-isometric C^1 immersion.

Nash-Kuiper theory was later generalised by Gromov into the theory of convex integration ([3]). Camillo De Lellis and László Székelyhidi, Jr., have briefly mentioned about the probabilistic approach to convex integration ([7]) by pointing to the fact that Convex integration can be seen as a control problem: at each step of the iteration, one has to choose an admissible perturbation, consisting essentially of a (plane-)wave direction and a frequency.

In the present article we shall consider the domain space of the maps to be 1-dimensional. In dimension 1, the solution to the C^1-isometric embedding problem does not require an infinite Nash process. Isometric maps $f : \mathbb{I} \to \mathbb{R}^3$ with respect to the standard Riemannian metrics can be obtained simply by integrating a curve in the 2-sphere and this reduces the Nash process to a single stage. However, Nash twists play an important role in controlling the C^0-distance between the initial and the perturbed map. In order to keep the solutions sufficiently close to the original embedding, Nash introduced a periodic function of high frequency (or rapidly oscillating function) under the integration process. We may remark here that in higher dimension, each step in the Nash process can be reduced to a parametric version of the 1-dimensional Nash process described above. The C^0-closeness will then translate into $C^\perp$-closeness (refer to [3, page 170]). However, the problem in dimension greater than 1 is considerably more difficult and we plan to take it up in future.

It is indeed the case that as the frequency in the Nash twist goes to infinity the distance between the initial short map and the resulting isometric immersion goes to zero in the C^0-norm. This motivates us to study the distribution of $f - f_0$ with respect to an appropriate measure on the space of isometric immersions $f : \mathbb{I} \to \mathbb{R}^3$.

We naturally incorporate a randomness in the Nash twist which translates into a Gaussian noise measure for the difference function $f - f_0$. For each positive integer n, we construct random functions f_n such that the difference $(f_n(.) - f_0(.))$ goes to zero (in C^0 norm). We scale it up and examine the distribution. We show that the distribution of $n^{3/2}(f_n - f_0)$ converges (weakly) to a Gaussian noise measure. Thus the random solutions $f_n(.)$ can be thought of as distributed like $f_0 + n^{-3/2}$(Gaussian noise) for large n. In Theorem 2.1 we state this rigorously, identifying the weak limit of $n^{3/2} \int_0^t (f_n - f_0)(s)\, ds$ as a Gaussian process. Section 3 is devoted to the proof which requires essentially weak convergence of random walks. In the last section, we compare the above process with a class of extensively studied Gaussian processes.

2. Notation and Main Result

Let M be a smooth manifold with a Riemannian metric g. The isometric immersions $f : M \to \mathbb{R}^q$ are solutions to the following system of partial differential equations:

$$\left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle = g_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where $u_1, u_2, \ldots u_n$ is a local coordinate system on M, g_{ij}, $i, j = 1, 2, \ldots, n$, are the matrix coefficients of g and the $\langle \ , \ \rangle$ denotes the inner product on $\mathbb{R}^q$.

To motivate the concept of randomness and measure we consider the following simple case where the domain space M is the unit interval $[0,1]$ and hence an arbitrary metric on M is of the form $g\,dt^2$, where $g : [0,1] \to \mathbb{R}_+$ is a smooth positive function on $[0,1]$. The shortness condition on a smooth regular curve $f_0 : [0,1] \to \mathbb{R}^3$ then translates into the pointwise inequality $0 < ||\partial_u f_0|| < \sqrt{g}$ on $[0,1]$. Given such an f_0 we want to find a function $f_n : [0,1] \to \mathbb{R}^3$ such that $||\partial_u f_n|| = \sqrt{g}$ (which means that f_n is isometric) and the C^0-distance between f_n and f_0 decreases with n.

The following considerations illustrate the *Nash twist* in dimension 1. Suppose that (X, Y, Z) is the Frenet-Serret frame along f_0 (assuming that such a frame exists at all points $u \in [0,1]$), where X is the unit tangent along the curve. Then (Y, Z) span a plane field J along f_0 perpendicular to X. Consider the curve $Y(u)\cos 2\pi s + Z(u)\sin 2\pi s$, $0 \le s \le 1$, on $J(u)$ for each fixed $u \in [0,1]$. Then with $r^2 = g - ||\partial_u f_0||^2$ the function

$$\partial_u f_0 + r(u)(Y(u)\cos 2\pi s + Z(u)\sin 2\pi s)$$

has the required euclidean norm $\sqrt{g}$ and over $s \in [0,1]$ integrates to $\partial_u f_0$ (the convex integration condition). Now, a Nash twist of f_0 is given by

$$f_n(t) = f_0(0) + \int_0^t \left\{ \partial_u f_0(u) + r(u)(Y(u)\cos 2\pi n u + Z(u)\sin 2\pi n u) \right\} du,$$

where n connects with the frequency of the periodic functions, namely $\cos 2\pi n u$ and $\sin 2\pi n u$, mentioned in the previous section. Clearly, f_n is a solution of the isometry equation since $(Y(u), Z(u))$ is an orthonormal basis of $J(u)$. We want to show that the function (or the difference curve) $\int_0^t r(u)(Y(u)\cos 2\pi n u + Z(u)\sin 2\pi n u)du$ is uniformly small over $[0,1]$. We do this for the two integrals separately with some notational abuse.

Applying integration by parts we get

$$\int_0^1 r(u)\cos 2\pi n u\, du \quad = \quad \frac{1}{2\pi n} r(u)\sin 2\pi n t - \frac{1}{2\pi n}\int_0^t r'(u)\sin 2\pi n u\, du$$

which in absolute value is bounded by $\frac{1}{2n\pi}$ as r is a smooth function on the interval $[0,1]$. The same estimates also apply to $\int_0^t r(u)\sin 2\pi n u\, du$ and thus we conclude uniform closeness of f_n and f_0.

This difference curve can also be considered for a random path by changing the function $H_n(u) = nu$ over random choices. The H_n in the above example can be obtained by integrating the constant function $h_n = n$ over $[0,1]$. Instead we take $h_n = \pm n$ on each subinterval $(k/n, (k+1)/n]$ independently with equal probability. These choices are actually explicitly mentioned by Gromov except for the probability part. Then integration of the resulting function will give a random function $H_n(\omega, u)$ which is the graph of a simple random walk. To see this calculation we consider a sequence of independent and identically distributed random variables X_k which take the values ± 1 with equal probability on a probability space $(\Omega, \mathcal{F}, P)$, where Ω can be taken as the infinite product space $\{-1, +1\}^{\mathbb{N}}$ and consider the function $h_n(\omega, x) = n X_k, (k-1)/n \le x < k/n$. Therefore, each

subinterval of length $1/n$ contributes ± 1 and hence

$$H_n(\omega, u) = \int_0^u h_n(\omega, x)dx = S_k \pm n(u - (k/n)), \quad k/n \le u < (k+1)/n,$$

where $S_k = X_1 + \cdots + X_k$. The random sum $S_n(\omega, t)$ can be interpreted as the random walk obtained by linearly joining S_k, $1 \le k \le n$.

The components of the random difference curve

$$f_n(t) - f_0(t) = \int_0^t r(u)[Y(u)\cos 2\pi H_n(\omega, u) + Z(u)\sin 2\pi H_n(\omega, u)]du,$$

go to zero in C^0-norm in the same way as in the nonrandom case discussed above. From the probabilistic point of view it becomes natural to study the limiting distribution of $\{f_n(t) - f_0(t), 0 \le t \le 1\}$, possibly after rescaling. Here we consider the integrated process $\{\int_0^t (f_n(s) - f_0(s))ds, 0 \le t \le 1\}$, and show that after appropriate scaling it converges weakly to a Gaussian process. The integrated process may be viewed as the signed area of the graph of the original process.

In our case the graph of $\int_0^t e^{2\pi i H_n(\omega, u)}du$ (omitting $r(u), Y(u), Z(u)$ for simplicity now) looks similar (in a probabilistic sense of considering all possible paths) over equal intervals and is independent over disjoint intervals. However, the limit of its normalization is not a function but a random distribution (in the sense of generalized functions), it is called the Brownian white noise measure. To keep track of the random difference curve one tracks its rescaled integral, which converges weakly to Brownian motion. A rigorous formulation (bringing in $r(u)$, $Y(u)$, $Z(u)$) is the following

Theorem 2.1. *The sequence of processes $2\pi n^{3/2} \int_0^t (f_n - f_0)\, ds, 0 \le t \le 1$, converges weakly to*

$$\int_0^t r(u)Z(u)dW(u) - \int_0^t \int_0^s \partial_u(r(u)Z(u))dW(u)\, ds, 0 \le t \le 1,$$

as $n \to \infty$, where $W(\cdot)$ denotes the Wiener measure on $C[0,1]$, the space of real valued continuous functions on the interval $[0,1]$.

In this sense, the limit of the scaled random difference curve is locally the Gaussian noise measure $r(t)Z(t)dW(t) - (\int_0^t \partial_u(r(u)Z(u))dW(u))\, dt$ (see [8]). The rate of convergence may also be interesting from the probabilistic point of view.

3. Proof of the Main Result

We first keep track of the integral (omitting $r(u), Y(u), Z(u)$)

$$\int_0^s e^{2\pi i H_n(u)}du = \int_0^s e^{2\pi i nu}du = (1/2\pi n)[\sin(2\pi ns) - i\{\cos(2\pi ns) - 1\}], 0 \le s \le 1.$$
$$(3.1)$$

The main observation about this function on the right is that over each $(k/n, (k+1)/n]$ interval the imaginary part is the graph of $1 - \cos(2\pi nx), 0 < x \le 1/n$, and the real part is the graph of $\sin(2\pi nx), 0 < x \le 1/n$. The periodic behavior along with the factor $(1/2\pi n)$ indicates the C^0-closeness as s varies in $[0,1]$.

Plugging in $H_n(\omega, u)$ described in the previous section the corresponding integral $\int_0^s e^{2\pi i H_n(\omega,u)} du$ differs from (3.1) in randomly inverting the imaginary part of the graph over each $(k/n, (k+1)/n]$ interval. To see this consider $s \in (k/n, (k+1)/n]$. Then

$$\int_{\frac{k}{n}}^{s} e^{2\pi i(S_k \pm n(u-k/n))} du$$

$$= (1/2\pi n)\Big[\sin(2\pi n(s-k/n)) \pm (-i)\{\cos(2\pi n(s-k/n)) - 1\}\Big].$$

Noting that the integral over each $(k/n, (k+1)/n]$ is zero and using the periodicity of sin and cos functions we have established that the graph of $\int_0^s e^{2\pi i H_n(\omega,u)} du$ is obtained by randomly inverting the imaginary part of the graph of (3.1) over each $(k/n, (k+1)/n]$ interval. We now prove that

$$2\pi n^{3/2} \int_0^t \Big\{ \int_0^s \sin 2\pi H_n(\omega, u) du \Big\} ds, \; 0 \le t \le 1,$$

converges weakly to Brownian motion as $n \to \infty$ and

$$2\pi n^{3/2} \int_0^t \Big\{ \int_0^s \cos 2\pi H_n(\omega, u) du \Big\} ds, \; 0 \le t \le 1,$$

converges to the zero process.

To see the exact form of this (random) function we note that, as proved, the function $\int_0^s e^{2\pi i H_n(\omega,u)} du$ has a graph which is the graph of (3.1) with the imaginary part randomly inverted over each $[k/n, (k+1)/n)$ interval. The integral of the (periodic) function in (3.1) over each $[k/n, (k+1)/n)$ interval is $(i/2\pi n^2)$ (the real part integrates to contribute zero). Thus the integral of the function $\int_0^s e^{2\pi i H_n(\omega,u)} du$ is $\pm i(1/2\pi n^2)$ over the same interval, the $\pm$ sign coming from the random inverting. Written explicitly, for $k/n \le t < (k+1)/n$,

$$\int_0^t \{ \int_0^s e^{2\pi i H_n(\omega,u)} du \} ds = \sum_{j=0}^{k-1} \int_{j/n}^{(j+1)/n} \Big\{ \int_0^s e^{2\pi i H_n(\omega,u)} du \Big\} ds$$

$$+ \int_{k/n}^t \Big\{ \int_0^s e^{2\pi i H_n(\omega,u)} du \Big\} ds$$

$$= \frac{i}{2\pi n^2} \sum_{j=1}^k X_j + O\Big(\frac{1}{n^2}\Big),$$

where X_j are independent ± 1 random variables. Note that the integral from k/n to t adds a continuous function of order $O(\frac{1}{n^2})$ to the random walk obtained from the X_j's. Multiplying by $2\pi n^{3/2}$ we get the weak convergence to Brownian motion. In this sense the random difference $\int_0^s e^{2\pi i H_n(\omega,u)} du, 0 \le s \le 1$, when normalized converges to the generalized derivative of Brownian motion, called Brownian white noise.

For the general case, we consider (with some abuse of notation)

$$\int_0^s r(u) e^{2\pi i H_n(\omega,u)} du, 0 \le s \le 1,$$

where r is a sufficiently differentiable function. In this case, depending on r, after two integrations we get back a different random walk minus the area under another random walk, and consider weak convergence again.

We shall deal with the real and the imaginary part of the integral separately. As observed before the randomness has no role to play in the real part. A straight-forward calculation shows that

$$\int_0^s r(u)\cos(2\pi H_n(\omega,u))du = \int_0^s r(u)\cos(2\pi nu)du$$

$$= \frac{1}{2n\pi}r(s)\sin 2n\pi s - \frac{1}{2n\pi}\int_0^s r'(u)\sin 2n\pi u\, du$$

$$= \frac{1}{2n\pi}r(s)\sin 2n\pi s + \frac{1}{4n^2\pi^2}r'(s)\cos 2n\pi s$$

$$- \frac{1}{4n^2\pi^2}r'(0) - \frac{1}{4n^2\pi^2}\int_0^s r''(u)\cos 2n\pi u\, du$$

Therefore,

$$\int_0^t\left\{\int_0^s r(u)\cos(2\pi H_n(\omega,u))du\right\}ds = \frac{1}{2\pi n}\int_0^t r(s)\sin(2\pi ns)\,ds + O\left(\frac{1}{n^2}\right).$$

Since the first term on the right hand side is $O(1/n^2)$ it follows that

$$\lim_{n\to\infty} n^{3/2}\int_0^t\left\{\int_0^s r(u)\cos 2\pi H_n(\omega,u)du\right\}ds = 0.$$

Thus the real part of the integral when scaled by $n^{3/2}$ converges to zero uniformly.

To deal with the imaginary part of the integral, for $l/n \le s < (l+1)/n$, we split it as follows

$$\sum_{k=0}^{[ns]-1}\int_{k/n}^{(k+1)/n} r(u)\sin(2\pi H_n(\omega,u))du + \int_{l/n}^s r(u)\sin(2\pi H_n(\omega,u))du, \qquad (3.2)$$

and then writing $r(u) = r(k/n)+(r(u)-r(k/n))$ on the subinterval $(k/n,(k+1)/n]$ we get the following for (3.2):

$$\sum_{k=0}^{[ns]-1}\int_{k/n}^{(k+1)/n} (r(u)-r(k/n))\sin(2\pi H_n(\omega,u))du$$

$$+ \quad r(l/n)\int_{l/n}^s \sin(2\pi H_n(\omega,u))du$$

$$+ \quad \int_{l/n}^s (r(u)-r(l/n))\sin(2\pi H_n(\omega,u))du. \qquad (3.3)$$

We denote the function (represented by the sum) on the first row by $\psi_1(s)$ and the two terms on the second and third rows by $\psi_2(s)$ and $\psi_3(s)$ respectively. Since $n^{3/2}\int_0^t \psi_2(s)ds$ gives a random walk with steps $\pm\frac{1}{\sqrt{n}}r(k/n)$ over the interval $[k/n,(k+1)/n)$, $n^{3/2}\int_0^t \psi_2(s)ds$ converges weakly to $\frac{1}{2\pi}\int_0^t r(s)dW(s)$.

Next we consider the part ψ_1. The summands of ψ_1 are not necessarily zero as r is non-constant. Substituting $u = k/n + z/n$ in the k-th summand and disregarding the $\pm$ signs, we get

$$\frac{1}{n} \int_0^1 \left[r\left(\frac{k}{n} + \frac{z}{n}\right) - r\left(\frac{k}{n}\right) \right] \sin(2\pi z) dz.$$

To clearly understand this summand we divide the integral into two parts where the sine function has the same sign, so that one can apply the mean value theorem for integrals. Thus, we can get $z_1 \in (0, 1/2)$ so that

$$\frac{1}{n} \int_0^1 [r\left(\frac{k}{n} + \frac{z}{n}\right) - r\left(\frac{k}{n}\right)] \sin(2\pi z) dz$$

$$= \frac{1}{n} \int_0^{1/2} \left[r\left(\frac{k}{n} + \frac{z}{n}\right) - r\left(\frac{k}{n}\right) \right] \sin(2\pi z) dz$$

$$+ \frac{1}{n} \int_{1/2}^1 \left[r\left(\frac{k}{n} + \frac{z}{n}\right) - r\left(\frac{k}{n}\right) \right] \sin(2\pi z) dz$$

$$= \frac{1}{n} \int_0^{1/2} \left[r\left(\frac{k}{n} + \frac{z}{n}\right) - r\left(\frac{k}{n} + \frac{z}{n} + \frac{1}{2n}\right) \right] \sin(2\pi z) \, dz$$

$$= \frac{1}{n\pi} \left[r\left(\frac{k}{n} + \frac{z_1}{n}\right) - r\left(\frac{k}{n} + \frac{z_1}{n} + \frac{1}{2n}\right) \right]$$

$$= -\frac{1}{2n^2\pi} r'\left(\frac{k}{n} + \frac{z_1}{n} + \frac{\theta}{2n}\right),$$

where $0 < \theta < 1$. If we add these integrals after multiplying each of them by ± 1 from random inversions, and scale the sum by $n^{3/2}$ then it corresponds to a random walk converging weakly to

$$-\frac{1}{2\pi} \int_0^s r'(u) dW(u).$$

The continuous random curve $\psi_1 + \psi_3$ matches this random walk at the points k/n and is otherwise at a distance at most

$$O\left(\frac{1}{n} |r(k/n + 1/2n) - r(k/n)|\right) = O(1/n^2)$$

from it. Thus after scaling by $n^{3/2}$ the random curve $\psi_1 + \psi_3$ converges weakly to the same limit and the integral of $\psi_1 + \psi_3$ converges weakly to

$$-\frac{1}{2\pi} \int_0^t \left\{ \int_0^s r'(u) dW(u) \right\} ds$$

by the continuous mapping theorem. This completes the proof of the main result. For completeness we indicate how the random walk

$$\sum_{k=1}^{[nt_1]} r(k/n) Y_k \tag{3.4}$$

(here Y_k are i.i.d. $\pm\frac{1}{\sqrt{n}}$ random variables) and the area under the random walk

$$-\sum_{i=1}^{[ns]} r'(i/n)Y_i \tag{3.5}$$

up to time t_2 converges jointly in distribution, after which tightness on product space can be used to conclude weak convergence on $C[0,1] \times C[0,1]$. To obtain the area under (3.5), each random walk height is multiplied by $1/n$ and then added. The interchange of summation gives the area as

$$-\sum_{i=0}^{k-1} (\frac{k-i}{n}) r'(i/n)Y_i, \tag{3.6}$$

where $k = [nt_2]$. From (3.4) and (3.6) the limiting joint finite dimensional distribution follows. The limit of the expression in (3.6) is seen to be $-\frac{1}{2\pi}\int_0^t(t-u)r'(u)\,dW(u)$ which equals $-\frac{1}{2\pi}\int_0^t\{\int_0^s r'(u)dW(u)\}ds$. Now the sum of the two processes (3.4) and (3.6) converges weakly. $\qquad\square$

4. Concluding Remarks

It is seen from the proof of Theorem 2.1 that the random Nash twist on the initial short curve f_0 to obtain an increase of $r^2(u)\,du^2$ to the induced metric f_0^*h, leads to a process whose structure is similar to the following process (refer to [4, Theorem 6.3])

$$X(t) = W(t) - \int_0^t \int_0^s \ell(s,u)dW(u)\,ds,$$

where $\ell(s,u)$ is a Volterra kernel with appropriate conditions. In our case, componentwise we need a Gaussian process $\int_0^t r(u)\,dW(u)$ and a function $r'(u)/r(u)$ to replace $W(t)$ and $\ell(s,u)$ in the above formula.

In higher dimension the difference metric $g - f_0^*h$ can be written as a finite sum of monomials $r^2 d\varphi^2$, where φ is a rank 1 function. Applying Nash twist along $(\operatorname{Im} Df_0)^\perp$, one is able to add $r^2 d\varphi^2$ only approximately. To look into the problem of C^0 distance one may need several independent Brownian motions in different directions and we refer to Theorem 3.2.5 of Kallianpur and Xiong ([5]) for such a construction. However a precise formulation combining various directions is not clear and we hope to explore these aspects in future.

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Amites Dasgupta: Statistics and Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Calcutta 700108, India.
E-mail address: amites@isical.ac.in

Mahuya Datta: Statistics and Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Calcutta 700108, India.
E-mail address: mahuya@isical.ac.in