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Matroids arisen from matrogenic graphs

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Abstract

Let G be a finite simple graph and let $\mathscr{I}(G)$ be the set of subsets X of $V(G)$ such that the subgraph of G induced by X is threshold. If $\mathscr{I}(G)$ is the independence system of a matroid, then G is called matrogenic [3]. In this paper, we characterize matroids arising from matrogenic graphs.

1. Introduction

In this paper, we shall only consider finite simple graphs. A graph G is called *threshold* if there is a mapping f, from the vertex set of G to the set of real numbers, such that two distinct vertices u and v of G are adjacent if and only if $f(u)+f(v)>0$. Threshold graphs were introduced by Chvátal and Hammer [1] and were characterized in many ways [1, 5, 6, 9]. The following is one of these characterizations.

Theorem 1.1 (Chvátal and Hammer [1]). *A graph is threshold if and only if none of its induced subgraphs is isomorphic to $2K_2$, C_4 or P_4. (see Fig. 1)*

Let G be a graph and let $\mathscr{I}(G)$ be the set of subsets X of $V(G)$ such that the subgraph of G induced by X is threshold. Then G is called *matrogenic* [3] if $\mathscr{I}(G)$ is the independence system of a matroid. If G is matrogenic, we shall denote the associated matroid by $M(G)$. In this paper, we characterize those matroids M for which there exists a matrogenic graph G with $M=M(G)$.

In Section 2, we summarize the results in [3] which characterize matrogenic graphs. Then, in Section 3, we prove the main theorem of this paper that characterizes matroids

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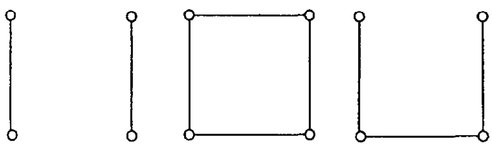

Fig. 1.

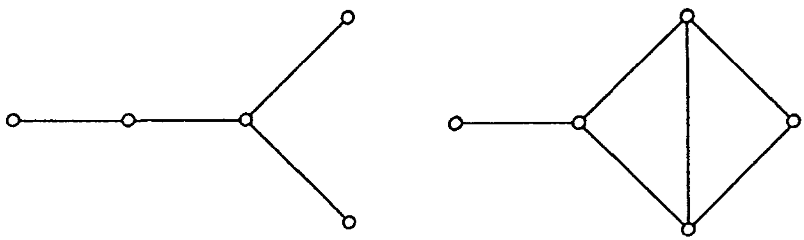

Fig. 2.

of the form $M(G)$ for some matrogenic graph G. In the last section, we discuss the more general $\mathscr{G}$-matrogenic graphs.

2. Matrogenic graphs

As shown by the following theorem, matrogenic graphs can be characterized by excluding certain induced subgraphs.

Theorem 2.1 (Földes and Hammer [3]). *A graph is matrogenic if and only if none of its induced subgraphs is isomorphic to a graph $H = (V, E)$, where $V = \{u, v, x, y, z\}$, such that ux, uy and vz are in E but vx, vy and uz are not in E.*

Let G be a graph and let X be a subset of $V(G)$. Then we denote by $G(X)$ the subgraph of G induced by X. As usual, the complement of G is denoted by $\overline{G}$. If $\overline{G}(X)$ or $G(X)$ is a complete graph, then X is called *stable* or *complete*, respectively. We call G a *split graph* if $V(G)$ can be partitioned into a stable set and a complete set. The following is an obvious corollary of Theorem 2.1.

Corollary 2.2 (Földes and Hammer [3]). *A split graph is matrogenic if and only if none of its induced subgraphs is isomorphic to one of the two graphs illustrated in Fig. 2 above.*

To present a structural characterization of matrogenic graphs, we need some definitions. Let G_1 be a graph and let G_2 be a split graph. Suppose that $V(G_1) \cap V(G_2) = \emptyset$

and that $V(G_2)$ is partitioned into a stable set S and a complete set T. Then the *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E$, where $E = \{xy: x \in V(G_1),\ y \in T\}$. Clearly, the resulting graph G depends not only on the two graphs G_1 and G_2 themselves, but also on the partition (S, T) of $V(G_2)$ (since the partition may not be unique). Thus, we shall specify this partition whenever it is necessary. Another remark needs to be made is that $G_1 + G_2$ may differ from $G_2 + G_1$ (in fact, $G_2 + G_1$ is not necessarily well defined). Finally, let us call a graph G with at least four vertices, a *crown* if G is isomorphic to C_5, nK_2 (the perfect matching with n edges) or $\overline{nK_2}$.

Theorem 2.3 (Földes and Hammer [3]). *A graph G is matrogenic if and only if there exists a matrogenic split graph G_2, where $V(G_2)$ might be empty, such that either $G = G_2$ or $G = G_1 + G_2$ for some crown G_1.*

By Theorem 2.3, to understand the structure of a matrogenic graph, it is enough to understand the structure of a matrogenic split graph. Now we need some more definitions. Let k be a nonnegative integer. A family $\{N_i: i \in I\}$ of sets indexed by a set I is called a *k-system* if

(i) the cardinality of each N_i is k,

(ii) the cardinality of $(N_i - N_j) \cup (N_j - N_i)$ is 2 if $N_i \neq N_j$,

(iii) if $N_i = N_j$ for some distinct indices i and j in I, then $N_i = N_j$ for all the indices i and j in I.

Let G be a graph. For each vertex x of G, we denote by $N(x)$ the set of vertices of G that are adjacent to x. If X is a subset of $V(G)$, then we denote by X_k the set of vertices x in X such that the degree of x in G is k.

Theorem 2.4 (Földes and Hammer [3]). *Let G be a split graph and let $V(G)$ be partitioned into the stable set S and the complete set T. Then G is matrogenic if and only if*

(i) *for each $k \geqslant 0$, the family $\{N(x): x \in S_k\}$ is a k-system, and*

(ii) *for all $k' > k \geqslant 0$, if $x \in S_k$ and $y \in S_{k'}$, then $N(x) \subset N(y)$.*

The following is a corollary of Theorem 2.4.

Corollary 2.5 (Földes and Hammer [3]). *Let G be a matrogenic split graph. Then either G or $\bar{G}$ has a vertex of degree at most* 1.

3. Matroids arisen from matrogenic graphs

Let G be a graph and let $\mathscr{C}(G)$ be the set of subsets X of $V(G)$ such that the subgraph of G induced by X is isomorphic to $2K_2$, C_4 or P_4. Then, by Theorem 1.1, we have the following lemma.

Lemma 3.1. *G is matrogenic if and only $\mathscr{C}(G)$ is the set of circuits of a matroid.*

Consequently, we have another lemma.

Lemma 3.2. *For any matrogenic graph G,*
(i) *every induced subgraph of G is also matrogenic;*
(ii) *if G' is obtained from G by adding an isolated vertex, then G' is matrogenic and $M(G')$ can be obtained from $M(G)$ by adding a coloop (that is, by adding an element that does not belong to any circuit of the matroid $M(G')$).*

Since $\overline{2K_2} = C_4$ and $\overline{P_4} = P_4$, it follows that $\mathscr{C}(\overline{G}) = \mathscr{C}(G)$ for all graphs G. Thus, we deduce the following from Lemma 3.1.

Lemma 3.3. *If G is matrogenic, then $\overline{G}$ is also matrogenic and $M(\overline{G}) = M(G)$.*

Remark. Lemma 3.2(i) and the first half of Lemma 3.3 are also obtained in [3] as corollaries of Theorem 2.1.

By the definition of $\mathscr{C}(G)$, it is straightforward to verify the next lemma.

Lemma 3.4. *If $G_1 + G_2$ is well defined, then $\mathscr{C}(G_1 + G_2) = \mathscr{C}(G_1) \cup \mathscr{C}(G_2)$.*

As a consequence, we have the following lemma.

Lemma 3.5. *Suppose that $G_1 + G_2$ is well defined. Then $G_1 + G_2$ is matrogenic if and only if both G_1 and G_2 are. Moreover, in the case when both G_1 and G_2 are matrogenic, $M(G_1 + G_2)$ is the direct sum of $M(G_1)$ and $M(G_2)$.*

Now, for every positive integer n, let H_n be the split graph with $V(H_n) = S \cup T$, where $S = \{x_1, \ldots, x_n\}$ is stable and $T = \{y_1, \ldots, y_n\}$ is complete, such that $S \cap T = \emptyset$ and $x_1y_1, \ldots, x_ny_n$ are the only edges of H_n between S and T. The following is our key lemma.

Lemma 3.6. *Let G be a matrogenic split graph such that neither G nor $\overline{G}$ has an isolated vertex. Then there exists an integer n exceeding one such that either G or $\overline{G}$ is the join of a matrogenic split graph G' and H_n.*

Proof. It is clear from Corollary 2.5 and the assumptions of Lemma 3.6 that G or $\overline{G}$ (say G) has a vertex of degree 1. Let x_0 be this vertex and let y_0 be the vertex adjacent to x_0. Suppose that $V(G)$ is partitioned into a stable set S and a complete set T. Then x_0 belongs to S since neither G nor $\overline{G}$ has an isolated vertex. It follows that y_0 belongs to T. Since y_0 is not isolated in $\overline{G}$, there must be a vertex x_1 in S adjacent to y_0 in $\overline{G}$. But x_1 is not isolated in G, thus, there exists a vertex y_1

in $T - \{y_0\}$ adjacent to x_1 in G. Therefore, we conclude that there is an induced subgraph H of G such that x_0 is in $V(H)$ and H is isomorphic to some H_n (for instance, $G(\{x_0, x_1, y_0, y_1\})$ is such a graph). Let us choose such an H with $V(H)$ maximal. We shall prove that G is the join of G', which is defined as $G - V(H)$, and H.

Let $S_1 = S \cap V(G')$, $T_1 = T \cap V(G')$, $S_2 = S - S_1$ and $T_2 = T - T_1$. We first prove that every x in S_2 is not adjacent to any vertex y in $V(G')$. For suppose there exists x in S_2 that is adjacent to a vertex y in $V(G')$. Then y belongs to T_1 (since S is stable) and x differs from x_0 (since x_0 is adjacent only to y_0 and y_0 is in T_2, not in T_1). Let z be the vertex in T_2 that is adjacent to x. Then it is easy to see that $G(\{x_0, y_0, x, y, z\})$ is isomorphic to the second graph illustrated in Fig. 2. But by Corollary 2.2, this induced subgraph of G is not matrogenic. Thus, we conclude from Lemma 3.2(i) that G is not matrogenic either, a contradiction.

We next prove that every y in T_2 is adjacent to every x in $V(G')$. For suppose there exists y in T_2 that is not adjacent to some vertex x in $V(G')$. It is clear that x belongs to S_1 (since T is complete). From the maximality of H and the assumption that x is not isolated we deduce that x is adjacent to a vertex y_1 in T_2. Let z and z_1 be the vertices in S_2 that are adjacent to y and y_1, respectively. It is easy to see that $G(\{x, y, y_1, z, z_1\})$ is isomorphic to the first graph illustrated in Fig. 2. By Corollary 2.2, this induced subgraph of G is not matrogenic. On the other hand, this graph should be matrogenic because of Lemma 3.2(i) and the assumption that G is matrogenic. Again, a contradiction. Therefore, as required, we conclude that G is the join of G' and H. □

Let G be a graph. The join of G and H_n will be called an *n-augmentation* of G. Then the following theorem is clear from Lemmas 3.2, 3.3, 3.5 and 3.6.

Theorem 3.7. *G is a matrogenic split graph if and only if G can be constructed from the empty graph by a sequence of the following three operations*: (i) *adding an isolated vertex*; (ii) *taking the complement*; (iii) *making an n-augmentation.*

In stating the main result of this paper, we shall need the following definitions. Let $U_{3,5}$ denote the rank-three uniform matroid on five elements. Then it is clear that $M(C_5)$ is $U_{3,5}$. Let n be a positive integer and let $K_{2,n}$ be the complete bipartite graph with two vertices in one color class and n vertices in the other color class. Then it is also clear that $M(nK_2)$, as well as $M(\overline{nK_2})$, is the cycle matroid of $K_{2,n}$. We shall denote this matroid by M_n in this paper. Finally, it is easy to see that $\mathscr{C}(H_n) = \mathscr{C}(nK_2)$. Therefore, H_n is matrogenic and $M(H_n) = M_n$.

Now, our main theorem follows from Lemma 3.5, Theorems 2.3 and 3.7.

Theorem 3.8 (Main). *Let M be a matroid. Then there exists a matrogenic graph G with $M = M(G)$ if and only if M is the direct sum of coloops, copies of M_n and at most one copy of $U_{3,5}$.*

4. $\mathscr{G}$-matrogenic graphs

Let $\mathscr{G}$ be a finite set of graphs such that no member is an induced subgraph of another. For any graph G, let us denote by $\mathscr{C}_{\mathscr{G}}(G)$ the set of subsets X of $V(G)$ such that $G(X)$ is isomorphic to a member of $\mathscr{G}$. Then we may define G to be *$\mathscr{G}$-matrogenic* if $\mathscr{C}_{\mathscr{G}}(G)$ is the set of circuits of a matroid. We shall denote the associated matroid by $M_{\mathscr{G}}(G)$ when G is *$\mathscr{G}$-matrogenic*. From the circuit exchange property of matroids we deduce directly the following theorem.

Theorem 4.1. *There exists a finite set $\mathscr{G}^*$ of graphs such that a graph G is $\mathscr{G}$-matrogenic if and only if G has no induced subgraph isomorphic to a member of $\mathscr{G}^*$.*

For instance, if $\mathscr{G} = \{2K_2, C_4, P_4\}$, then $\mathscr{G}^*$ consists of the ten graphs described in Theorem 2.1. If we take $\mathscr{G} = \{P_4\}$, then it is not difficult to show that $\mathscr{G}^*$ consists of the six graphs illustrated in Fig. 3. Results analogous to Theorems 3.7 and 3.8 can be established similarly. We do not present them here because they are routine. In fact, we should mention the following, which was pointed out by J.L. Fouquet.

Our $\{P_4\}$-matrogenic graphs are precisely the *extended P_4-sparse* graphs introduced by Giakoumakis [4] as an extension of *P_4-sparse* graphs of Hoáng [7]. A P_4-sparse graph does not contain C_5 or any of the six graphs in Fig. 3 as an induced subgraph. It can be shown that every C_5 in a $\{P_4\}$-matrogenic graph (i.e. an extended P_4-sparse graph) is an *homogeneous* part (all vertices of this C_5 have the same neighborhood outside). For a $\{P_4\}$-matrogenic graph, all maximal P_4-free induced subgraphs have the same number of vertices yet these subgraphs may not be isomorphic (consider C_5). In contrast, it is straightforward to verify that these subgraphs are exactly *canonical cographs* defined in [2] and thus we conclude from a theorem in [8] that a graph G is P_4-sparse if and only if for every induced subgraph H of G, all maximal P_4-free induced subgraphs of H are isomorphic.

Finally, let us call a matroid M *graph-generated* if there is a graph G and a finite set $\mathscr{G}$ of graphs with $M = M_{\mathscr{G}}(G)$. Then it is natural to ask if a given matroid is graph-generated. We do not know the answer to this question if the matroid is simple.

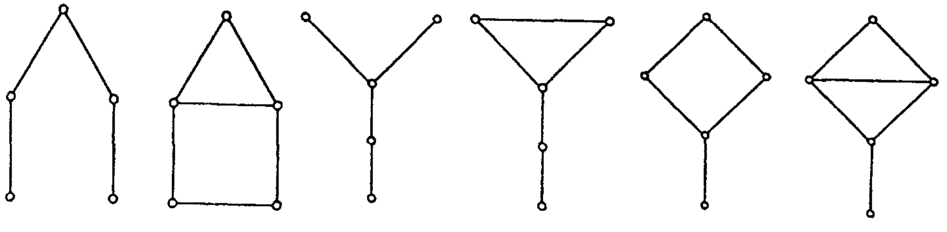

Fig. 3.

However, for matroids which are not simple, it is not difficult to verify the following theorem.

Theorem 4.2. *Let M be a matroid and let k be the size of the smallest circuit of M.*

(i) *Suppose $k = 1$. Then M is graph-generated if and only if the rank of M is zero.*

(ii) *Suppose $k = 2$. Then M is graph-generated if and only if every connected component of M has rank one.*

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