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POSITIVE DEFINITENESS ON SPHERES AND HYPERBOLIC SPACES

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ABSTRACT. We consider two different concepts of positive definiteness, the metric version due to Schoenberg, and the often used algebraic version for a hypergroup. The two notions are the same for the unit sphere in euclidean space and the associated hypergroup of spherical random walks, but in general the metric concept is stronger. We determine explicit convolution structures on spheres and classical hyperbolic spaces geometrically and investigate large dimensional limits.

1. Introduction

Associated with the sphere $\mathbb{S}^n$ and hyperbolic space $\mathbb{H}^n_+$ are convolution structures, called commutative hypergroups, on the intervals $[0, \pi]$ and $[0, \infty]$ respectively. The characters of these hypergroups are Gegenbauer (ultraspherical) polynomials in the case of the sphere, and conical (associated Legendre) functions in the case of hyperbolic space. These characters are essentially the same as the spherical functions of these symmetric spaces when viewed as homogeneous spaces. There is also a related notion of positive definite function, and by a form of Bochner's theorem any such hypergroup positive definite function is a suitable non-negative linear combination (possibly in the integral sense) of characters (see for example [6]).

In [9] Schoenberg introduced a quite different notion of positive definiteness for a metric space M. He described these functions for the finite-dimensional spheres $\mathbb{S}^n$ and also the infinite-dimensional sphere $\mathbb{S}^\infty$, and showed how relations between them lead to representations of positive definite functions in terms of Gegenbauer polynomials and powers of $\cos x$ respectively. This notion was also investigated in [1] and [5].

In this paper we would like to reconcile the two notions of positive definiteness for the sphere and extend these ideas to hyperbolic spaces. The next section introduces the various notions of positive definiteness. Section 4 describes the hypergroup associated with the sphere $\mathbb{S}^n$ and its characters. Our approach is geometric and does not rely on any symmetric space theory. In particular, no detailed knowledge of the isometry groups is required.

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Sections 5 and 6 treat the classical hyperbolic spaces $\mathbb{H}_{+}^{n}$ in a similar fashion. The discussion of the characters, or spherical functions, involves the distinction between the principal series and supplementary series.

In Section 7 we give a direct proof to show that for either of these families of spaces, Schoenberg positive definiteness implies hypergroup positive definiteness. For the sphere, results of Schoenberg together with known descriptions of the spherical functions also give the converse.

In Section 9 the corresponding result for the non-compact homogeneous space $SL(2,\mathbb{C}) /\!\!/ SU(2)$ (the Naimark hypergroup) is considered, but it isn't known whether the notions are equivalent. One problem is that the Plancherel measure on the dual of this double coset hypergroup has support only on the principal series characters.

Schoenberg's work leads us to ask if there is a hypergroup structure on the infinite sphere $\mathbb{S}^{\infty}$ with characters generating its positive definite functions. We show that in fact the relevant structure is a semigroup, and that this semigroup is the natural infinite limit of the hypergroups associated with the finite spheres $\mathbb{S}^{n}$ as n approaches ∞. The corresponding limiting structure for the classical hyperbolic spaces $\mathbb{H}_{+}^{n}$ is also investigated.

Our investigations focus on the important cases of spheres and hyperbolic spaces, which are rank one symmetric spaces. The study of spherical functions on higher rank symmetric spaces, called Gelfand pairs, along with their resulting asymptotics has been studied by others, for example one can consider compact Lie groups themselves as Gelfand pairs under conjugation and their asymptotics as the dimension goes to infinity. The main works here are [7] for compact symmetric spaces of type A, and [8] for those of types B and C, and following a somewhat different approach, [10] in the unitary case. For a comprehensive overview see [4].

2. Positive Definite Functions

A complex-valued function f on a group G is *positive definite* if

$$\sum_{i,j=1}^{n} f\left(x_i x_j^{-1}\right) \xi_i \overline{\xi_j} \geq 0$$

for all choices of $x_i \in G$, $\xi_i \in \mathbb{C}$ and $n \in \mathbb{N}$. Let $P(G)$ denote the set of all continuous positive definite functions on G.

If $G=\mathbb{R}$, then Bochner's theorem establishes that any positive definite function is the Fourier-Stieltjes transform of a bounded non-decreasing function F, that is

$$f(x)=\int_{-\infty}^{\infty} e^{ix\xi} F(d\xi).$$

More generally, if G is a locally compact abelian group, then the Weil-Povzner-Raikov theorem states that there is a bounded positive measure μ on $\widehat{G}$ such that f is the Fourier-Stieltjes transform of μ.

We now consider the corresponding notion on a commutative hypergroup K; see [2] for details of hypergroups. These structures are defined through a convolution structure on their measure algebra $M(K)$; there is no actual group multiplication.

However, the convolution product

$$f(x * y) := \int f \, d(\epsilon_x * \epsilon_y)$$

is defined for any measurable function f for which the integral exists. Note that $\epsilon_x * \epsilon_y$ as the convolution of two point masses is a probability measure, but rarely has a single point support, so that $x * y$ has no meaning on its own. Nevertheless we can use this to develop a theory of positive definiteness. A complex-valued function f on a hypergroup K is *positive definite* if

$$\sum_{i,j=1}^{n} f\left(x_i * x_j^-\right) \xi_i \overline{\xi_j} \geq 0$$

for all choices of $x_i \in K$, $\xi_i \in \mathbb{C}$ and $n \in \mathbb{N}$. Let $P(K)$ denote the set of all continuous positive definite functions on K. Unlike the group case, functions in $P(K)$ are not necessarily bounded. Nevertheless, they satisfy the properties

(1) $f(e) \geq 0$,
(2) $f\left(x_i * x_i^-\right) \geq 0$ for all $x \in K$,
(3) $f(x^-) = \overline{f(x)}$ for all $x \in K$,
(4) $f(e) = \|f\|_\infty$ whenever f is bounded.

It is easily seen that $P(K)$ is closed under

(1) linear combinations with non-negative coefficients,
(2) pointwise convergence to a continuous limit.

Schoenberg introduced another notion of positive definiteness in the case of a metric space M with distance function $d(x,y)$. A continuous real-valued even function g defined on the interval $[-d(x,y), d(x,y)]$ is *positive definite* if

$$\sum_{i,j=1}^{n} g(d(x_i, x_j)) \xi_i \overline{\xi_j} \geq 0$$

for all $x_i \in M$, $\xi_i \in \mathbb{C}$ and $n \in \mathbb{N}$. Schoenberg showed that the set $\mathfrak{P}(M)$ of positive definite functions on M is closed under

(1) linear combinations with non-negative coefficients,
(2) pointwise convergence to a continuous limit,
(3) pointwise multiplication.

It should be emphasised that the third property is very strong, one that is not in general enjoyed by hypergroups.

If $M \subset N$ then $\mathfrak{P}(M) \supset \mathfrak{P}(N)$. This observation led Schoenberg to relations between Gegenbauer (ultraspherical functions) $P_n^{(\alpha)}(\cos t)$, generated for $\alpha > 0$ by the expansion

$$\left(1 - 2r\cos t + r^2\right)^{-\alpha} = \sum_{n=0}^{\infty} r^n P_n^{(\alpha)}(\cos t), \tag{2.1}$$

and for $\alpha = 0$ by

$$P_n^{(0)}(\cos t) = \cos nt = T_n(\cos t),$$

where T_n denotes the Tchebychev polynomial of the first kind of degree n. Let $\mathbb{S}^m$ denote the unit sphere in $(m+1)$-space and $\mathbb{S}^\infty$ the unit sphere in Hilbert space. Since we may assume that

$$\mathbb{S}^1 \subset \mathbb{S}^2 \subset \cdots \subset \mathbb{S}^m \subset \cdots \subset \mathbb{S}^\infty$$

it follows that

$$\mathfrak{P}\left(\mathbb{S}^1\right) \supset \mathfrak{P}\left(\mathbb{S}^2\right) \supset \cdots \supset \mathfrak{P}\left(\mathbb{S}^m\right) \supset \cdots \supset \mathfrak{P}\left(\mathbb{S}^\infty\right)$$

and in fact $\mathfrak{P}\left(\mathbb{S}^\infty\right)$ is exactly the intersection of all the sets $\mathfrak{P}\left(\mathbb{S}^m\right), m = 1, 2, \cdots$.

Schoenberg showed that each function $g \in \mathfrak{P}\left(\mathbb{S}^2\right)$ can be represented as $g(t) = \sum_{n=0}^\infty a_n P_n(\cos t)$, where $a_n \geq 0$ satisfies $\sum_{n=0}^\infty a_n < \infty$ and P_n is a normalised Legendre polynomial of degree n. More generally, each function $g \in \mathfrak{P}\left(\mathbb{S}^m\right)$ has a similar representation as

$$g(t) = \sum_{n=0}^\infty a_n W_n^{\frac{1}{2}(m-1)}(\cos t)$$

where the W_n^α are normalisations of the Gegenbauer polynomials via

$$W_n^\alpha(x) = P_n^{(\alpha)}(x) / P_n^{(\alpha)}(1).$$

The functions g in $\mathfrak{P}\left(\mathbb{S}^\infty\right)$ have a similar representation as

$$g(t) = \sum_{n=0}^\infty a_n (\cos t)^n,$$

where $a_n \geq 0$ and $\sum_{n=0}^\infty a_n < \infty$.

3. Positive Definite Functions and Homogeneous Spaces

Let G be a locally compact group and H a compact subgroup with normalised Haar measure ω_H. (For the sphere $\mathbb{S}^n$ and hyperbolic space $\mathbb{H}_+^n$, G, H will be chosen so that $G /\!/ H \cong [0, \pi], \mathbb{R}_+$ respectively.) Consider the following sequence of mappings and an associated correspondence between functions:

$$\begin{array}{ccccc} G & \overset{\pi_{\mathbb{H}}}{\to} & G/H & \overset{\pi'_{\mathbb{H}}}{\to} & G/\!/H \\ f^\flat(g) & = & f^\natural(gH) & = & f(HgH) \end{array} \tag{3.1}$$

We note that $f^\flat$ is constant on double $H-$cosets.

Theorem 3.1. *If $f^\flat$ is positive definite on the group G, then f is positive definite on the hypergroup $G/\!/H$.*

Proof. Assume $f^\flat$ to be positive definite on the group G and recall that it is $H-$ bi-invariant. Then we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} f^\flat\left(g_i g_j^{-1}\right) \geq 0$$

for all choices of $g_i \in G, c_i \in \mathbb{C}$ and $n \in \mathbb{N}$. Now for $k_1, k_2, ..., k_n \in H$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j} f^\flat\left(\left(k_i g_i k_i^{-1}\right)\left(k_j g_j k_j^{-1}\right)^{-1}\right) \geq 0.$$

Integrating this expression n times over H gives

$$\begin{aligned}
&0 \leq \\
&\int_H\int_H ...\int_H \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j} f^\flat\left(\left(k_i g_i k_i^{-1}\right)\left(k_j g_j k_j^{-1}\right)^{-1}\right)\omega_H\left(dk_1\right)\omega_H\left(dk_2\right)...\omega_H\left(dk_n\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\int_H\int_H f^\flat\left(g_i\left(k_i^{-1}k_j\right)g_j^{-1}\right)\omega_H\left(dk_i\right)\omega_H\left(dk_j\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\int_H f^\flat\left(g_i k g_j^{-1}\right)\omega_H\left(dk\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\int_H \varepsilon_{g_i} * f^\flat * \varepsilon_{g_j^{-1}}\left(k\right)\omega_H\left(dk\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j} f\left(Hg_iH * Hg_j^{-1}H\right),
\end{aligned}$$

which shows that f is hypergroup positive definite. Note that in the first equality we have used the property that $f^\flat$ is constant on $H-$ double cosets. $\square$

4. Convolution on the Sphere $\mathbb{S}^n$

We follow the ideas in [1] to develop the hypergroup structure of the unit $n-$sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let $G = SO(n+1)$ denote the group of rotations of $\mathbb{R}^{n+1}$ and view $H = SO(n)$ as a subgroup of $SO(n+1)$. We may assume that G left acts transitively on the $n-$sphere $\mathbb{S}^n$, with the stabiliser subgroup of the *north* pole $s_0 = (0, 0, ..., 0, 1)$ being H. Thus $\mathbb{S}^n \cong G/H$, the space $\{gH\}$ of left cosets of H in G. The orbits of the subgroup H on $\mathbb{S}^n$ are the meridian *hypercircles* C_φ parallel to $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, indexed by the angle φ from 0 to π. At the endpoints 0 and π these *hypercircles* reduce to the north and south poles s_0 and $-s_0$ respectively. The orbits of H on G/H form the double coset space $G/\!/H$, which can be identified with the set of double cosets $\{HgH\}$ of H in G. Referring to (3.1) we see that $f^\natural$ is a function on $\mathbb{S}^n$ constant on meridian circles centered on the $(0, 0, ..., 0, 1)$-axis, and $f^\flat$ is a function on G constant on $H-$double cosets.

The space $K = G/\!/H$ is a natural commutative hypergroup in one of the following equivalent ways. Given two functions f and g on K define their convolution by

$$(f * g)^\flat = f^\flat * g^\flat$$

involving the usual convolution of functions on the group G. Alternatively we may define

$$(f * g)(x) = \int_K f(x * y)\, g(y^-)\omega_K(dy),$$

where

$$f(x*y)=\int_K f\,d(\epsilon_x*\epsilon_y)$$

is defined in terms of the hypergroup product $\epsilon_x*\epsilon_y$ of the point measures at x and y. In other words the convolution of functions is determined by knowledge of the products of pairs of point measures, which are by the definition of a hypergroup necessarily probability measures on K. This gives a clear probabilistic interpretation to convolution in the case of a double coset hypergroup $K=G/\!\!/H$.

Let us now consider how to determine the hypergroup structure geometrically for the $n-$sphere

$$\mathbb{S}^n\cong SO(n+1)/SO(n).$$

We begin at the north pole $s_0=(0,0,.....,0,1)$, take a random step of distance x to a point P and then another random step from P of distance y to a point Q. In this case random means that if we consider the meridian which is the $(n-1)$ sphere

$$T=\{s\in\mathbb{S}^n : d(P,s)=y\},$$

then T is a homogeneous space for the group $SO(n)$ acting as rotations which fix the point P, and we are taking the probability distribution for Q to be uniform on T with respect to this group. To determine the probability density function $g^{(n)}_{x,y}(r)$ for the distance r from s_0 to Q we analyze the spherical triangle s_0PQ. Here distance means angular distance on the surface of the unit sphere. The portion of the sphere T for which the angle $\angle s_0PQ$ lies in $[\theta,\theta+d\theta]$ has measure $\frac{1}{c_n}\sin^{n-2}\theta\,d\theta$, where

$$c_n=\int_0^\pi \sin^{n-2}\theta\,d\theta=\frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}. \tag{4.1}$$

If one of x,y is 0 or π, then the resulting probability is concentrated at one point. Assume then that neither x nor y is 0 or π. In the spherical triangle $\overline{s_0PQ}$ let θ denote the angle $\angle s_0PQ$. Then the spherical cosine law asserts that

$$\cos r=\cos x\cos y+\sin x\sin y\cos\theta, \tag{4.2}$$

and since $\sin x$, $\sin y\geq 0$ we have

$$\cos x\cos y-\sin x\sin y\leq\cos r\leq\cos x\cos y+\sin x\sin y.$$

This is just

$$\cos(x+y)\leq\cos r\leq\cos(x-y). \tag{4.3}$$

Consider x and y to be fixed and take infinitesimals in (4.2) to obtain

$$\sin r\,dr=\sin x\sin y\sin\theta\,d\theta. \tag{4.4}$$

Then (4.2) gives

$$\sin^2\theta = 1 - \left(\frac{\cos r - \cos x\cos y}{\sin x\sin y}\right)^2$$

$$= \frac{1-\cos^2 y - \cos^2 x - \cos^2 r + 2\cos r\cos x\cos y}{(\sin x\sin y)^2}$$

$$= \frac{(\cos(x-y)-\cos r)(\cos r - \cos(x+y))}{(\sin x\sin y)^2},$$

and so

$$\sin\theta = \frac{[(\cos(x-y)-\cos r)(\cos r-\cos(x+y))]^{\frac{1}{2}}}{\sin x\sin y}. \tag{4.5}$$

Now

$$g_{x,y}^{(n)}(r)\ dr = \frac{1}{c_n}\sin^{n-2}\theta\, d\theta, \tag{4.6}$$

and appealing to (4.4) and (4.5) we have

$$g_{x,y}^{(n)}(r) = \frac{1}{c_n}\frac{\sin r\sin^{n-2}\theta}{\sin x\sin y\sin\theta}$$

$$= \frac{\sin r}{c_n\sin x\sin y}\left(\frac{[(\cos(x-y)-\cos r)(\cos r-\cos(x+y))]^{\frac{1}{2}}}{\sin x\sin y}\right)^{n-3}$$

$$= \frac{\sin r}{c_n}\frac{[(\cos(x-y)-\cos r)(\cos r-\cos(x+y))]^{\frac{n-3}{2}}}{[\sin x\sin y]^{n-2}},$$

valid for r satisfying (4.3). It follows that $K=[0,1]$ becomes a hypergroup with

$$f(x*y) = \int_0^1 f(r)\, g_{x,y}^{(n)}(r)\ dr,$$

or equivalently, using (4.6) and appealing to (4.2),

$$f(x*y) = \frac{1}{\pi}\int_0^\pi f\left(\cos^{-1}(\cos x\cos y + \sin x\sin y\cos\theta)\right)\ d\theta \tag{4.7}$$

We identify $K^{(n)} = SO(n+1)\,/\!\!/\,S(n)$ with the interval $[0,\pi]$ of values of the angle φ parametrising the meridians of $\mathbb{S}^n$. The characters of the associated hypergroups $K^{(n)}\cong[0,\pi]$ will be those functions χ satisfying

$$\chi(x)\,\chi(y) = \int_0^\pi \chi(r)\, g_{x,y}^{(n)}(r)\ dr,$$

or equivalently, using (4.7),

$$\chi(x)\,\chi(y) = \frac{1}{c_n}\int_0^\pi \chi\left(\cos^{-1}(\cos x\cos y+\sin x\sin y\cos\theta)\right)\sin^{n-2}\theta\, d\theta \tag{4.8}$$

for all $x, y \in K^{(n)}$. The solutions of (4.8) are given by

$$\psi_k^n(r) = \frac{k!\Gamma(n-1)}{\Gamma(n-1+k)} P_k^{\left(\frac{n-1}{2}\right)}(\cos r)$$

for $k = 0, 1, 2, \cdots$, where P_k^l is the Gegenbauer polynomial of order l, k defined as in (2.1). The c_n are chosen to ensure that $\psi_k^n(0) = 1$.

For $n = 2$ the Gegenbauer polynomial reduces to the Legendre polynomial $P_k^{\frac{1}{2}}(t) = P_k(t)$, and for $n = 3$ it reduces to the Tchebychev polynomial $P_k^1(t) = U_k(t)$ of the second kind. The latter has the explicit formula

$$U_k(\cos r) = \frac{\sin(k+1)r}{\sin r}$$

(see [11]).

It follows that the set of positive definite functions on the hypergroup $K^{(n)}$ agrees with the set of positive definite functions in the sense of Schoenberg ([9]) for the sphere $\mathbb{S}^n$ with the (natural) spherical distance $d^{(n)}$.

5. Convolution on the Hyperbolic Plane

The probabilistic development of the spherical hypergroup of the sphere $\mathbb{S}^n$ given above has a direct analogue for the classical hyperbolic spaces $\mathbb{H}_+^n$. Our treatment largely avoids details about the structure of the isometry group $SO(n,1)$ of $\mathbb{H}_+^n$ which is a significant advantage over more traditional approaches. In fact just as in the spherical case the algebraic structure contained in the associated 'hyperbolic hypergroups' is thus a direct consequence of the geometry of the hyperbolic spaces.

We begin in three-dimensional space with the Lorentzian inner product

$$\langle v, v' \rangle = \langle (x, y, z), (x', y', z') \rangle = -xx' - yy' + zz'$$

and define the classical hyperbolic plane $\mathbb{H}_+^2$ to be the sheet of the hyperboloid

$$\{v : \langle v, v \rangle = 1\}$$

through the point $O = (0, 0, 1)$. This is a restricted, classical form of hyperbolic geometry concentrating only on the interior of the light cone, which is not as general as the universal hyperbolic geometry introduced in ([12]), which rather looks at the entire three-dimensional space. The classical metric on $\mathbb{H}_+^2$ is given by the hyperbolic distance $d(P, Q)$ defined by

$$\cosh d(P, Q) = \langle P, Q \rangle .$$

As a homogeneous space

$$\mathbb{H}^2 \cong SO(2,1)/SO(2) \cong [0, \infty),$$

with $T = SO(2)$ regarded as those isometries fixing the point O. The orbits of T on $\mathbb{H}^2$ are circles centered at O indexed by their hyperbolic distance from O, which can take any value in $K = [0, \infty)$. We may describe the convolution of the circles either indirectly, by appealing to the group convolution of T bi-invariant measures in $G = SO(2,1)$, or directly by analyzing the geometry of random walks on $\mathbb{H}^2$. As in the case of the sphere, it is this latter approach that we prefer on account of its more immediate and elementary nature.

To convolve the circles of radius x and y around O, note first that if one or both of x, y is 0, then it acts as the identity, so in what follows we assume both x and y are non-zero. Choose a point P on the circle of radius x around O, and then choose randomly a point Q on the circle of radius y around P. Here again, randomly means with respect to the invariant probability measure on such a circle coming from the circle of isometries fixing P. The notion of angle between two directions in the hyperbolic plane is the same as the Euclidean one and is preserved by isometries. The hyperbolic triangle OPQ with θ representing the angle $\angle OPQ$ and the length r of the side OQ satisfy the hyperbolic cosine law

$$\cosh r = \cosh x \cosh y - \sinh x \sinh y \cos\theta. \tag{5.1}$$

It follows that r satisfies the inequalities

$$\cosh(x-y) \leq \cosh r \leq \cosh(x+y). \tag{5.2}$$

With x and y fixed, taking differentials of (5.1) gives

$$\sinh r\, dr = \sinh x \sinh y \sin\theta\, d\theta. \tag{5.3}$$

As in the case of the sphere, the measure of that portion of the circle around P corresponding to $[\theta, \theta + d\theta]$ is $\frac{1}{\pi} d\theta$ so that the probability density for the distance r is

$$m_{x,y}(r) = \frac{1}{\pi} \frac{\sinh r}{\sinh x \sinh y \sin\theta}.$$

To obtain an expression involving only x, y, r, we have from (5.1)

$$\begin{aligned}
\sin^2\theta &= 1 - \left(\frac{\cosh x \cosh y - \cosh r}{\sinh x \sinh y}\right)^2 \\
&= \frac{1 - \cosh^2 x - \cosh^2 y - \cosh^2 r + 2\cosh r \cosh x \cosh y}{(\sinh x \sinh y)^2} \\
&= \frac{(\cosh r - \cosh(x-y))(\cosh(x+y) - \cosh r)}{(\sinh x \sinh y)^2}.
\end{aligned}$$

Take a square root to get

$$\sin\theta = \frac{[(\cosh r - \cosh(x-y))(\cosh(x+y) - \cosh r)]^{\frac{1}{2}}}{\sinh x \sinh y}. \tag{5.4}$$

Thus

$$m_{x,y}(r) = \frac{\sinh r}{\pi\left[(\cosh r - \cosh(x-y))(\cosh(x+y) - \cosh r)\right]^{\frac{1}{2}}},$$

which is valid for all r satisfying (5.2). It follows that $K = [0, \infty)$ becomes a hypergroup with

$$f(x * y) = \int_0^\infty f(r)\, m_{x,y}(r)\, dr$$

or equivalently, using $m_{x,y}(r)\ dr = d\theta/\pi$ and appealing to (5.1),

$$f(x*y) = \frac{1}{\pi}\int_0^\pi f\left(\cosh^{-1}(\cosh x\cosh y - \sinh x\sinh y\cos\theta)\right)\ d\theta.$$

This corresponds to [2], p.236 with $a=1$. Haar measure for this hypergroup is

$$\omega(dx) = \left(\sinh^2 x\right)\lambda_{\mathbb{R}_+}(dx).$$

A character on this (non-compact) hypergroup is defined to be a continuous function χ satisfying

$$\begin{aligned}\chi(x)\chi(y) &= \chi(x*y)\\ &= \frac{1}{\pi}\int_0^\pi \chi\left(\cosh^{-1}(\cosh x\cosh y - \sinh x\sinh y\cos\theta)\right)\ d\theta\\ &= \frac{1}{\pi}\int_0^\pi \chi\left(\cosh^{-1}(\cosh x\cosh y + \sinh x\sinh y\cos\theta)\right)\ d\theta,\end{aligned}$$

where the second equality arises on replacing θ by $\pi-\theta$. It can be shown that the bounded characters are the so-called conical functions

$$\chi_{-\frac{1}{2}+i\kappa}(r) = p_{-\frac{1}{2}+i\kappa}(\cosh r) = \frac{1}{2\pi}\int_0^{2\pi}(\cosh r + \sinh r\cos\theta)^{-\frac{1}{2}+i\kappa}\ d\theta,$$

where either $\kappa\in\mathbb{R}_+$ (the principal series) or $\kappa\in i\left[0,\frac{1}{2}\right]$ (the supplementary series). The identity character is the element of the supplementary series, where $\kappa=\frac{1}{2}$. It is well known that the Plancherel measure for this hypergroup is supported on the principal series characters. Note the symmetry $p_{-\frac{1}{2}+i\kappa} = p_{-\frac{1}{2}-i\kappa}$.

Conical functions are examples of associated Legendre functions, where the lower parameter is $-\frac{1}{2}+i\kappa$ for some $\kappa\ge 0$. In this case the upper parameter is 0 and in MAPLE these are given by $\operatorname{LegendreP}\left(-\frac{1}{2}+i\kappa, 0, \cosh x\right)$.

6. Higher-dimensional Hyperbolic Hypergroups

The discussion of the previous section generalises to the case of hyperbolic (real) n-space $\mathbb{H}_+^n$. In $(n+1)$-dimensional space with inner product

$$\langle v,v'\rangle = \langle(x_1,\cdots,x_n,x_{n+1}),(x_1,\cdots,x_n,x_{n+1})\rangle = -x_1x_1'-\cdots-x_nx_n'+x_{n+1}x_{n+1}'$$

and hyperbolic distance $d(v,v')$ defined by

$$\cosh d(v,v') = \langle v,v'\rangle,$$

define $\mathbb{H}_+^n$ to be the sheet of the hyperboloid

$$\{v : \langle v,v\rangle = 1\}$$

passing through the point $O=(0,\cdots 0,1)$. As a homogeneous space

$$\mathbb{H}_+^n \cong SO(n,1)/SO(n)$$

with the stabiliser subgroup $SO(n)$ of the point O acting as (ordinary) rotations about the x_{n+1} axis. The non-trivial orbits of $SO(n)$ on $\mathbb{H}_+^n$ are n-spheres which we call zonal meridians. The meridian consisting of the set of points in $\mathbb{H}_+^n$ with hyperbolic distance from O some fixed $d \geq 0$ may be described as the intersection of $\mathbb{H}_+^n$ with the hyperplane given by $x_{n+1} = \cosh^{-1} d$ or $\left\{v : \langle v, e_{n+1} \rangle = \cosh^{-1} d\right\}$. Since any such meridian is determined by the positive number d, the orbit space of all zonal meridians is

$$K \cong SO\,(n,1) \,/\!\!/ SO\,(n) \cong [0, \infty)\,.$$

Since the group $SO\,(n,1)$ acts on $\mathbb{H}_+^n$ as isometries, any point P on $\mathbb{H}_+^n$ may be obtained as gP for some $g \in SO\,(n,1)$ (actually we may take g to be in the connected component $SO\,(n,1)_+$ of the identity). Then the set of points with hyperbolic distance d from a general point P will be called the meridian with centre P and radius d, and is obtained by intersecting $\mathbb{H}_+^n$ with the $SO\,(n,1)-$translate of the hyperplane $\left\{v : \langle v, e_{n+1} \rangle = \cosh^{-1} d\right\}$, that is, the hyperplane $\left\{v : \langle v, ge_{n+1} \rangle = \cosh^{-1} d\right\}$. Note that ge_{n+1} is also on $\mathbb{H}_+^n$. The general meridian is topologically an n-sphere and in particular carries a unique probability measure that is invariant under the subgroup of isometries of $SO\,(n,1)$ fixing its centre.

Geodesics through O are precisely the intersections of $\mathbb{H}_+^n$ with planes through the centre $(0, \cdots, 0, 0)$. It follows that general geodesics are obtained by intersecting $\mathbb{H}_+^n$ with general planes passing through the centre.

Fix $x, y \in \mathbb{R}_+$ both non-zero (otherwise the product is trivial) and suppose we choose a point P randomly on the meridian of centre O and radius x and then choose a point Q randomly on the meridian of centre P and radius y. To determine the probability density function $m_{x,y}^{(n)}(r)$ for the hyperbolic distance r from O to Q we proceed exactly as in the case of $\mathbb{H}^2$ and analyze the hyperbolic triangle $\overline{OPQ}$ as above. The only difference is that the portion of the meridian sphere T for which the angle $\angle OPQ$ lies in $[\theta, \theta + d\theta]$ has measure $\frac{1}{c_n} \sin^{n-2} \theta \, d\theta$, where c_n is as before.

For $n \geq 2$, appealing to (5.3) and (5.4) we obtain

$$\begin{aligned}
m_{x,y}^{(n)}(r) &= \frac{1}{c_n} \frac{\sinh r \sin^{n-2} \theta}{\sinh x \sinh y \sin \theta} \\
&= \frac{\sinh r}{c_n \sinh x \sinh y} \left[\frac{(\cosh r - \cosh (x-y))\,(\cosh (x+y) - \cosh r)}{(\sinh x \sinh y)^2} \right]^{\frac{n-3}{2}} \\
&= \frac{\sinh r}{c_n} \frac{\left[(\cosh r - \cosh (x-y))\,(\cosh (x+y) - \cosh r)\right]^{\frac{n-3}{2}}}{\left[\sinh x \sinh y\right]^{n-2}},
\end{aligned}$$

valid for all r satisfying (5.2). As before, $K = [0, \infty)$ becomes a hypergroup with

$$f\,(x * y) = \int_0^\infty f\,(r)\, m_{x,y}^{(n)}(r)\; dr,$$

or equivalently, using $m_{x,y}^{(n)}(r)\ dr = \frac{1}{c_n}\sin^{n-2}\theta\, d\theta$ and appealing to (5.1) with the obvious change of variable,

$$f(x*y) = \frac{1}{c_n}\int_0^\pi f\left(\cosh^{-1}(\cosh x\cosh y + \sinh x\sinh y\cos\theta)\right)\sin^{n-2}\theta\, d\theta. \tag{6.1}$$

This hypergroup is sometimes called a hyperbolic hypergroup (see [2], p.237 and [13]). The characters for this convolution are given by

$$\psi_\lambda^{(n-1)/2}(x) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_0^\pi (\cosh x + \cos t\sinh x)^{i\kappa - \frac{n-1}{2}}\sin^{n-2} t\, dt,$$

where $\kappa = \sqrt{\lambda - (\frac{n-1}{2})^2}$. Those for the hyperbolic plane are obtained when $n = 2$.

7. Schoenberg Positive Definite Implies Hypergroup Positive Definite

In this section M will denote either the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}_+^n$, each considered as a metric space with distance function d. Then as a homogeneous space $M \simeq G/H$, where G is the group of isometries of M and H is the compact subgroup fixing a distinguished point O. The double coset hypergroup $K = G/\!/H$ is then either the finite interval $[0,\pi]$ in the case of $\mathbb{S}^n$ or the infinite interval $[0,\infty]$ in the case of $\mathbb{H}_+^n$, where an H orbit on M (or a double coset) is identified with its distance from O. Let dh denote the normalised invariant (Haar) measure on H so that

$$\int_H dh = 1.$$

Lemma 7.1. *Fix points m, m' on M and let $x = d(O,m)$, $y = d(O,m')$ so that $x, y \in K$. Then for any function f on K*

$$\int_H f(d(m,hm'))\, dh = f(x*y).$$

Proof. For fixed m and m' consider the quantity $d(m,hm')$ as h varies over the subgroup H. To go from m to hm' we can take a step of size x to O and then a step of distance y to hm'. This means the average of the values of $f(d(m,hm'))$ over H is equal to $f(x*y)$ by the definition of hypergroup convolution. □

Theorem 7.2. *If a function f on K is positive definite in the metric sense of Schoenberg, then it is a positive definite function on the hypergroup $K = G/\!/H$.*

Proof. Suppose that $f \in \mathfrak{P}(K)$, which means that

$$\sum_{i,j=1}^n f(d(m_i,m_j))\,\xi_i\overline{\xi_j} \geq 0$$

for any positive integer n, any points m_i on M and any complex numbers ξ_i, where $i = 1, 2, \cdots, n$. Averaging each of $m_1, \cdots, m_n$ along its respective orbit by H gives

$$\int_H \cdots \int_H \sum_{i,j=1}^n f(d(h_i m_i, h_j m_j))\,\xi_i\overline{\xi_j} dh_1 \cdots dh_n \geq 0$$

or

$$\sum_{i,j=1}^{n}\left(\int_H\int_H f\left(d\left(h_i m_i, h_j m_j\right)\right) dh_i dh_j\right)\xi_i\overline{\xi_j}\geq 0.$$

If we consider just one term in this double sum, say the one involving m_i and m_j, then by $H-$invariance of the spherical distance d and Lemma 7.1

$$\begin{aligned}\int_H\int_H f\left(d\left(h_i m_i, h_j m_j\right)\right) dh_i dh_j &= \int_H f\left(d\left(m_i, h m_j\right)\right) dh\\ &= f\left(x_i * x_j\right),\end{aligned}$$

where $x_i = d\left(O, m_i\right)$ is interpreted as an element of the hypergroup K. This shows that

$$\sum_{i,j=1}^{n} f\left(x_i * x_j\right)\xi_i\overline{\xi_j}\geq 0$$

for all choices of $x_i\in K$, $\xi_i\in\mathbb{C}$ and $n\in\mathbb{N}$. □

Remark 7.3. The proof immediately extends to rank-one symmetric spaces.

8. Hyperbolic Hypergroups

The hyperbolic hypergroup $(\mathbb{R}_+, *)$ convolution is given by (with the notation in [2], 3.5.65, p.237)

$$\epsilon_x * \epsilon_y = \frac{\Gamma(b+1)}{\sqrt{\pi}\Gamma(b+\frac{1}{2})}\int_0^\pi \epsilon_{\operatorname{arccosh}(\cosh x\cosh y+\cos t\sinh x\sinh y)}\sin^{2b} t\, dt, \tag{8.1}$$

where $b=\rho/2$ and $\rho>0$ so that $A(x)=\sinh^{2\rho} t=\sinh^{4b} t$. In [13] the convolution (taking into account the different notation) is exactly the same but with $\rho>-1$. However in the body of this paper (p.217), (8.1) takes the form

$$\epsilon_x * \epsilon_y = \frac{\Gamma(\beta+\frac{1}{2})}{\sqrt{\pi}\Gamma(\beta)}\int_{-1}^1 \epsilon_{\operatorname{arccosh}(\cosh x\cosh y+u\sinh x\sinh y)}\left(1-u^2\right)^{\beta-1} du,$$

which is easily obtained from (8.1) with the substitution $u=\cos t$ and replacing b by $\beta-\frac{1}{2}$, in which case $b>-\frac{1}{2}$ (that is $\rho>-1$) is equivalent to $\beta>0$.

We now consider explicit representations for these characters. In [13] the function

$$\psi_\lambda^\beta(x) = \frac{\Gamma(\beta+\frac{1}{2})}{\sqrt{\pi}\Gamma(\beta)}\int_{-1}^1 (\cosh x+u\sinh x)^{\sqrt{\beta^2-\lambda}-\beta}\left(1-u^2\right)^{\beta-1} du$$

satisfies $\psi_\lambda^\beta(0) = \left\|\psi_\lambda^\beta\right\|_\infty = 1$ when $4\beta^2\operatorname{Re}\lambda\geq(\operatorname{Im}\lambda)^2$ (in which case ψ_λ^β is bounded), ψ_λ^β is real-valued when $\lambda\geq 0$ (in which case ψ_λ^β is a character) and ψ_λ^β is non-negative when $\lambda\in\left[0,\beta^2\right]$. With the change of variable $u=\cos t$ we obtain

$$\psi_\lambda^\beta(x) = \frac{\Gamma(\beta+\frac{1}{2})}{\sqrt{\pi}\Gamma(\beta)}\int_0^\pi (\cosh x+\cos t\sinh x)^{\sqrt{\beta^2-\lambda}-\beta}\sin^{2\beta-1} t\, dt. \tag{8.2}$$

The integrand in (8.2) is just that given for the toroidal function

$$P^{\mu}_{\nu-\frac{1}{2}}(\cosh x) = \frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\sinh^{\mu} x}{\Gamma\left(\nu-\mu+\frac{1}{2}\right)2^{\mu}\sqrt{\pi}\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^{\pi}\frac{\sin^{2\mu} t}{(\cosh x+\cos t\sinh x)^{\nu+\mu+\frac{1}{2}}}\,dt$$

in [3], Section 3.13, p.173.

Now the above parametrisation of the characters of $(\mathbb{R}_+,*)$ in [13] differs from that in [2]. In [13] we have $\mathbb{R}_+^{\wedge}\cong\mathbb{R}_+$, where the supplementary series corresponds to $\left[0,\beta^2\right]$ and the principal series $\left[\beta^2,\infty\right)$, whereas in [2] the dual space is given by

$$\mathbb{R}_+^{\wedge}\cong\mathbb{R}_+\cup i\,[0,\rho]\,,$$

where the supplementary series corresponds to $i\,[0,\rho]$ and the principal series $[0,\infty)$. The correspondence is easily described as

	Correspondence	Supplementary	Principal	Identity
Zeuner	$\psi^{\rho}_{\lambda^2+\rho^2}$	$\left[0,\rho^2\right]$	$\left[\rho^2,\infty\right)$	ψ^{ρ}_{0}
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Bloom/Heyer	ϕ^{ρ}_{λ}	$i\,[0,\rho]$	$[0,\infty)$	$\phi^{\rho}_{i\rho}$

The difference is that the characters in [2] are solutions to the Sturm-Liouville equation

$$L^{\rho}_{A}\phi^{\rho}_{\lambda}=\left(\lambda^2+\rho^2\right)\phi^{\rho}_{\lambda},\ \ \phi^{\rho}_{\lambda}(0)=1,\ \ \left(\phi^{\rho}_{\lambda}\right)'(0)=0 \tag{8.3}$$

whereas those in [13] solve the equation

$$L^{\beta}_{A}\psi^{\beta}_{\lambda}=\lambda\psi^{\beta}_{\lambda},\ \ \psi^{\beta}_{\lambda}(0)=1,\ \ \left(\psi^{\beta}_{\lambda}\right)'(0)=0. \tag{8.4}$$

In both cases the differential operator is given by

$$L^{\alpha}_{A}f=-f''-\frac{A'}{A}f'=-f''-2\alpha\frac{\cosh}{\sinh}f'.$$

9. Positive Definiteness on the Naimark Hypergroup

In the case $\rho=1$ (so that $b=\frac{1}{2}$ and $A(x)=\sinh^2 x$) our hyperbolic hypergroup is the well-known Naimark hypergroup, which has convolution

$$\epsilon_x*\epsilon_y=\frac{1}{2\sinh x\sinh y}\int_{|x-y|}^{x+y}\epsilon_t\sinh t\,dt. \tag{9.1}$$

Indeed, with the substitution

$$\cosh u=\cosh x\cosh y+\sinh x\sinh y\cos t$$

we have

$$\sinh u\frac{du}{dt}=-\sinh x\sinh y\sin t,$$

and $t=0$ gives

$$\cosh u=\cosh x\cosh y+\sinh x\sinh y=\cosh(x+y),$$

so that $u=x+y$, and $t=\pi$ gives

$$\cosh u=\cosh x\cosh y-\sinh x\sinh y=\cosh(x-y),$$

so that $u = |x - y|$. Thus the integral (6.1) becomes (with $n = 3$)

$$\varepsilon_x * \varepsilon_y = \frac{1}{2} \int_{x+y}^{|x-y|} \varepsilon_u \sinh u (-\sinh x \sinh y)^{-1} \, du$$
$$= \frac{1}{2 \sinh x \sinh y} \int_{|x-y|}^{x+y} \varepsilon_u \sinh u \, du.$$

The characters of the Naimark hypergroup are given by

$$\phi_\lambda^1(x) = \begin{cases} \dfrac{\sin \lambda x}{\lambda \sinh x}, & \lambda \neq 0, \\[2ex] \dfrac{x}{\sinh x}, & \lambda = 0. \end{cases}$$

Note that with $\lambda = \beta = 1$ we have

$$\psi_1^1(x) = \frac{x}{\sinh x} = \phi_0^1(x).$$

Now it should be observed that for $\lambda \in (0, \infty)$ the characters oscillate, whereas for $\lambda = i\xi$ where $0 < \xi \leq 1$ we have

$$\phi_{i\xi}^1(x) = \frac{\sin i\xi x}{i\xi \sinh x} = \frac{\sinh \xi x}{\xi \sinh x} \geq 0.$$

In both cases (except when $\xi = 1$) the characters belong to $C_0(\mathbb{R}_+)$. All characters are automatically positive definite.

Now Zeuner ([13], Proposition 6.1) has shown that the product of two characters on the Naimark hypergroup is also positive definite. Since continuous positive definite functions are themselves positive mixtures of characters it follows that the product of two continuous positive definite functions on the Naimark hypergroup is also positive definite.

This result indicates that the positive definite functions on the Naimark hypergroup satisfy the stronger multiplicative property (3) of Schoenberg positive definiteness, and indeed it is to be expected that the two notions of positive definiteness would coincide for this hyperbolic hypergroup, but this remains open.

10. Convolution Measures in Higher Dimensions

Our formulae for the explicit hypergroup structures on spheres and classical hyperbolic spaces are worth looking at separately for the interesting dependence on the dimension that they exhibit. For example, the formula that describes the measures appearing in the hypergroup structure on the n dimensional sphere

$$g_{x,y}^{(n)}(r) = \frac{\sin r}{c_n} \frac{\left[(\cos(x-y) - \cos r)(\cos r - \cos(x+y)) \right]^{\frac{n-3}{2}}}{[\sin x \sin y]^{n-2}},$$

where c_n is given by (4.1), has quite a different character for different values of n. This connects with the well-known geometrical understanding that low-dimensional spheres have a lot of surface area around the poles, while higher-dimensional spheres have a lot of surface area around the equator. We consider two examples, with both distributions centring on $\cos^{-1}(\cos 1 \cos 2) \simeq 1.797\,6$. The first is $g_{1,2}^{(5)}(r)$ and the second is $g_{1,2}^{(1500)}(r)$. Both are probability measures obtained by convolving circles of radii 1 and 2 on respectively a 5-dimensional sphere and 1500-dimensional sphere. The distribution in the latter case concentrates much more on the value $\cos^{-1}(\cos 1 \cos 2)$.

The corresponding convolutions in hyperbolic space exhibit a similar phenomenon. The measure appearing in the hypergroup structure on the n dimensional classical hyperbolic space is given by

$$m_{x,y}^{(n)}(r) = \frac{\sinh r}{c_n} \frac{[(\cosh r - \cosh(x-y))(\cosh(x+y) - \cosh r)]^{\frac{n-3}{2}}}{[\sinh x \sinh y]^{n-2}}.$$

The probability measures $m_{1,2}^{(5)}(r)$ and $m_{1,2}^{(1500)}(r)$ concentrate on

$$\cosh^{-1}(\cosh 1 \cosh 2) \simeq 2.4444.$$

They are obtained by convolving circles of radii 1 and 2 on respectively a 5-dimensional classical hyperbolic space and 1500-dimensional hyperbolic space.

It is natural to ask what happens in the limit with these measures as the dimension n goes to infinity.

11. Infinite Limits

We first consider the convolution of spheres in $\mathbb{R}^n$.

Proposition 11.1. *For $x, y > 0$*

$$\lim_{n\to\infty} \frac{2r}{c_n} \frac{\left[\left(r^2 - (x-y)^2\right)\left((x+y)^2 - r^2\right)\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}} \chi_{[|x-y|,x+y]}(r) = \varepsilon_{\sqrt{x^2+y^2}}, \quad (11.1)$$

where the limit is taken distributionally.

Proof. First observe that $|x-y| \le \sqrt{x^2+y^2} \le x+y$ for all $x, y \ge 0$. We consider

$$\int_{|x-y|}^{x+y} \frac{2r}{c_n} \frac{\left[\left(r^2 - (x-y)^2\right)\left((x+y)^2 - r^2\right)\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}} dr$$

$$= \frac{1}{c_n (2xy)^{n-2}} \int_{(x-y)^2}^{(x+y)^2} \left(u - (x-y)^2\right)^{\frac{n-3}{2}} \left((x+y)^2 - u\right)^{\frac{n-3}{2}} du$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}\,(2xy)^{n-2}} \frac{\Gamma\left(\frac{n-1}{2}\right)^2 (4xy)^{n-2}}{\Gamma(n-1)} = 1,$$

the latter step using the Legendre duplication formula, so that the left-hand side of (11.1) is a limit of probability density functions on $\mathbb{R}_+$. We show that as a pointwise limit this vanishes for $r \ne \sqrt{x^2+y^2}$ and the result will then follow.

For $r \ne \sqrt{x^2+y^2}$ we first consider

$$|x-y| < r < \sqrt{x^2+y^2}.$$

Then $r^2 = x^2 + y^2 - \gamma$ for some $\gamma \in (0, 2xy)$ and

$$\frac{2r}{c_n} \frac{\left[\left(r^2 - (x-y)^2\right)\left((x+y)^2 - r^2\right)\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}}$$

$$= \frac{2\sqrt{x^2+y^2-\gamma}}{c_n} \frac{\left[\left(x^2+y^2-\gamma-(r_1-r_2)^2\right)\left((x+y)^2-\left(x^2+y^2-\gamma\right)\right)\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}}$$

$$= \frac{2\sqrt{x^2+y^2-\gamma}}{c_n} \frac{\left[(2xy-\gamma)(2xy+\gamma)\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}}$$

$$= \frac{2\sqrt{x^2+y^2-\gamma}}{c_n} \frac{\left[4x^2y^2-\gamma^2\right]^{\frac{n-3}{2}}}{(2xy)^{n-2}} = \frac{\sqrt{x^2+y^2-\gamma}}{c_n xy}\left[1-\frac{\gamma^2}{4x^2y^2}\right]^{\frac{n-3}{2}}$$

$$= \frac{\sqrt{x^2+y^2-\gamma}}{xy\sqrt{\pi}}\left[1-\frac{\gamma^2}{4x^2y^2}\right]^{\frac{n-3}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \tag{11.2}$$

$$\sim \frac{\sqrt{x^2+y^2-\gamma}}{xy\sqrt{\pi}}\left[1-\frac{\gamma^2}{4x^2y^2}\right]^{\frac{n-3}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}} \tag{11.3}$$

$$\to 0 \text{ as } n \to \infty,$$

where for (11.2) we refer to (4.1), and for (11.3) see in [3], 1.18(5). A similar argument holds for $\sqrt{x^2+y^2} < r < x+y$. □

The corresponding result, with an analogous proof, for the convolution of hypercircles in $\mathbb{S}^n$ is the following.

Proposition 11.2. *The limit*

$$\lim_{n\to\infty} \frac{\sin r}{c_n} \frac{\left[(\cos(x-y)-\cos r)(\cos r - \cos(x+y))\right]^{\frac{n-3}{2}}}{(\sin x \sin y)^{n-2}} \chi_{[|x-y|, x+y]}(r)$$
$$= \varepsilon_{\cos^{-1}(\cos x \cos y)}$$

exists distributionally.

Similarly, for the convolution of hypercircles in $\mathbb{H}^n$ we can provide an analogous computation to show the following.

Proposition 11.3. *The limit*

$$\lim_{n\to\infty}\frac{\sinh r}{c_n}\frac{\left[\left(\cosh\left(x-y\right)-\cosh r\right)\left(\cosh r-\cosh\left(x+y\right)\right)\right]^{\frac{n-3}{2}}}{\left(\sinh x\sinh y\right)^{n-2}}\chi_{[|x-y|,x+y]}\left(r\right)$$
$$=\varepsilon_{\cosh^{-1}(\cosh x\cosh y)}$$

exists distributionally.

12. Schoenberg Sets of Positive Definite Functions

We can relate the above calculations to a semigroup structure in the limiting cases. Let $\mathcal{P}_n = P\left(\mathbb{S}^n\right)$ denote the set of functions on $[0,\pi]$ that are of the form

$$f\left(r\right)=\sum_{k=0}^{\infty}a_k\psi_k^n\left(r\right)\quad\text{where}\quad a_k\geq 0,\quad \sum_{k=0}^{\infty}a_k<\infty.$$

(In [1] it is observed that the definition of these sets is valid for arbitrary strictly positive n using the definition of the Gegenbauer polynomials given above.) While this definition does not extend to the case $n=\infty$, it is true that $\psi_k^n\left(r\right)\to\left(\cos r\right)^k$ as $n\to\infty$. This leads us to define $\mathcal{P}_\infty$ to be the set of functions on $[0,\pi]$ that are of the form

$$f\left(r\right)=\sum_{k=0}^{\infty}a_k\left(\cos r\right)^k\quad\text{where}\quad a_k\geq 0,\quad \sum_{k=0}^{\infty}a_k<\infty,$$

which is consistent with Schoenberg's result that $\mathcal{P}_\infty$ so defined consists exactly of the positive definite functions on $[0,\pi]$ for the metric space $\left(\mathbb{S}^\infty,d\right)$.

From the general considerations of Schoenberg it follows that $\mathcal{P}_n$ is a semigroup for all values $n=1,2,\cdots,\infty$ and that

$$\mathcal{P}_1\supset\mathcal{P}_2\supset\mathcal{P}_3\cdots\supset\mathcal{P}_\infty.$$

From what we have observed for any positive integer n, $\mathcal{P}_n$ consists of exactly all the positive definite functions on the hypergroup

$$K^{(n)}=SO\left(n+1\right)/\!\!/SO\left(n\right)\cong\left[0,\pi\right].$$

It is then reasonable to ask if there is a hypergroup $K^{(\infty)}$ associated with meridian 'circles' on the infinite-dimensional sphere $\mathbb{S}^\infty$ and whether $\mathcal{P}_\infty$ is then the set of positive definite functions on $K^{(\infty)}$.

The problem with the definition of the hypergroup $K^{(\infty)}$ is that although $\mathbb{S}^{(\infty)}$ is a homogeneous space for an infinite-dimensional Lie group there is no invariant probability measure on it that will readily allow a convenient notion of convolution. Nevertheless there is in a precise sense a limiting object of the hypergroups $K^{(n)}$ as $n\to\infty$, which we see is not a hypergroup but rather a semigroup, and its positive definite functions are then exactly $\mathcal{P}_\infty$. The semigroup multiplication is given by

$$x\cdot y:=\cos^{-1}\left(\cos x\cos y\right),$$

which is a consequence of Proposition 11.2.

Conjecture 12.1. It would be expected that a similar observation would be valid for $P\left(\mathbb{H}^n\right)$ for which a hyperbolic version of the Schoenberg results would be needed. At this stage the question is still open.

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