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A GENERAL ITÔ FORMULA FOR ADAPTED AND INSTANTLY INDEPENDENT STOCHASTIC PROCESSES

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Abstract. We review a new stochastic integral for adapted and instantly independent stochastic processes and show that it is well-defined. Then we prove a unified Itô formula for the new stochastic integral. This general formula is used to produce several interesting special cases of the Itô formula. Then we apply these formulas to study exponential processes and stochastic differential equations involving the new stochastic integral.

1. Introduction

Let $B(t)$, $t \geq 0$, be a Brownian motion starting at 0 and $\{\mathcal{F}_t;\, 0 \leq a \leq t \leq b\}$ a filtration satisfying the conditions:

(a) $B(t)$ is $\{\mathcal{F}_t\}$-adapted, i.e., $B(t)$ is $\mathcal{F}_t$-measurable for each $t \in [a, b]$;
(b) $B(t) - B(s)$ and $\mathcal{F}_s$ are independent for any $s \leq t$ in $[a, b]$.

An *Itô process* is a stochastic process X_t of the form:

$$X_t = X_a + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \quad a \leq t \leq b, \tag{1.1}$$

where X_a is $\mathcal{F}_a$-measurable, $f(t)$ is an $\{\mathcal{F}_t\}$-adapted stochastic process with almost all sample paths being in $L^2([a, b])$, and $g(t)$ is an $\{\mathcal{F}_t\}$-adapted stochastic process with almost all sample paths being in $L^1([a, b])$. The first integral in Equation (1.1) is an Itô integral. If in addition $f(t)$ is a continuous stochastic process, then we have the equality

$$\int_a^b f(t)\, dB(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n f(t_{i-1})\big(B(t_i) - B(t_{i-1})\big), \tag{1.2}$$

in probability, see e.g., Theorem 5.3.3 in the book [14]. Notice that the evaluation points are the left endpoints of subintervals.

The stochastic process X_t in Equation (1.1) is often expressed symbolically in the following form of *stochastic differential*:

$$dX_t = f(t)\, dB(t) + g(t)\, dt, \quad a \leq t \leq b,$$

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with the understanding that the initial condition X_a is $\mathcal{F}_a$-measurable.

Suppose θ is a C^2-function on $\mathbb{R}$. The *Itô formula* states that

$$\theta(X_t) = \theta(X_a) + \int_a^t \theta'(X_s) f(s)\, dB(s) + \int_a^t \Big(\theta'(X_s) g(s) + \frac{1}{2}\,\theta''(X_s) f(s)^2\Big)\, ds. \tag{1.3}$$

Hence $\theta(X_t)$ is also an Itô process. In terms of stochastic differentials, Equation (1.3) can be written as

$$d\theta(X_t) = \theta'(X_t)\, dX_t + \frac{1}{2}\,\theta''(X_t)\,(dX_t)^2,$$

where $(dX_t)^2$ is computed by using the following multiplication rule:

$$(dB(t))^2 = dt, \;\; dB(t)dt = 0, \;\; (dt)^2 = 0.$$

More generally, suppose we have Itô processes

$$dX_t^{(i)} = f_i(t)\, dB(t) + g_i(t)\, dt, \quad 1 \le i \le n.$$

Let $\theta(t, x_1, \ldots, x_n)$ be a continuous function with continuous partial derivatives $\theta_t, \theta_{x_i}, \theta_{x_i x_j}$, $1 \le i, j \le n$. Then we have the Itô formula in terms of stochastic differentials

$$\begin{aligned} & d\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}) \\ &= \theta_t(t, X_t^{(1)}, \ldots, X_t^{(n)})\, dt + \sum_{i=1}^n \theta_{x_i}(t, X_t^{(1)}, \ldots, X_t^{(n)})\, dX_t^{(i)} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \theta_{x_i x_j}(t, X_t^{(1)}, \ldots, X_t^{(n)})\, (dX_t^{(i)})(dX_t^{(j)}). \end{aligned} \tag{1.4}$$

We want to emphasize that there are two crucial conditions in the Itô theory of stochastic integration:

(i) The integrand $f(t)$ in Equation (1.1) is $\{\mathcal{F}_t\}$-adapted;
(ii) the evaluation points for $f(t)$ in Equation (1.2) are the left endpoints of subintervals.

Under these two conditions we have (1) the martingale property of the stochastic process $\int_a^t f(s)\, dB(s)$, $a \le t \le b$, when $E\int_a^b |f(t)|^2\, dt < \infty$, and (2) the Markov property of the solution of a stochastic differential equation.

There are many extensions of the Itô integral $\int_a^b f(t)\, dB(t)$ for integrands $f(t)$ which may not be $\{\mathcal{F}_t\}$-adapted, see for instance the papers [1, 2, 3, 5, 6, 7, 8, 10, 13, 18, 19, 21, 22, 23, 24, 25]. In particular, we mention an extension by using white noise analysis. Suppose $f(t)$ is $\{\mathcal{F}_t\}$-adapted and $E\int_a^b |f(t)|^2\, dt < \infty$. A theorem due to Kubo and Takenaka [12] states that

$$\int_a^b \partial_t^* f(t)\, dt = \int_a^b f(t)\, dB(t), \tag{1.5}$$

where ∂_t^* is the adjoint of the white noise differentiation operator ∂_t, the left-hand side is a white noise integral, and the right-hand side is the Itô integral of $f(t)$.

For detail on the white noise integral and the proof of Equation (1.5), see Chapter 13 and Theorem 13.12 in the book [13].

Notice that the white noise integral in the left-hand side of Equation (1.5) can be defined as a Pettis integral without having to assume that $f(t)$ is $\{\mathcal{F}_t\}$-adapted. However, in general $\int_a^b \partial_t^* f(t)\,dt$ is a generalized white noise function. When it is realized as a square integrable random variable, then the integral is an extension of the Itô integral, namely, we can define the stochastic integral

$$\int_a^b f(t)\,dB(t) := \int_a^b \partial_t^* f(t)\,dt. \tag{1.6}$$

For example, when $f(t) = B(1)$, we have

$$\int_0^1 B(1)\,dB(t) := \int_0^1 \partial_t^* B(1)\,dt = B(1)^2 - 1.$$

For the derivation of the last equality, see Example 13.14 in the book [13].

The stochastic integral defined in Equation (1.6) turns out to be the same as the stochastic integral introduced by Hitsuda [8] in 1972 and by Skorokhod [25] in 1975 with different methods.

In [8] Hitsuda states an extension of the Itô formula which can expressed in white noise formulation as follows:

$$\begin{aligned}\theta\big(B(t), B(b)\big) &= \theta\big(B(a), B(b)\big) + \int_a^t \partial_s^*\Big(\frac{\partial\theta}{\partial x}\big(B(s), B(b)\big)\Big)\,ds \\ &\quad + \int_a^t \Big(\frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(B(s), B(b)\big) + \frac{\partial^2\theta}{\partial x\partial y}\big(B(s), B(b)\big)\Big)\,ds, \quad a \le t \le b. \end{aligned}\tag{1.7}$$

For the proof, see Theorem 13.19 in the book [13].

Now, in [1, 2] Ayed and Kuo introduced a new stochastic integral more in the spirit of the Itô theory of stochastic integration. A stochastic process $\varphi(t)$ is called *instantly independent* of $\{\mathcal{F}_t\}$ if $\varphi(t)$ and $\mathcal{F}_t$ are independent for each $t \in [a, b]$. Suppose $f(t)$ is an $\{\mathcal{F}_t\}$-adapted continuous stochastic process and $\varphi(t)$ is continuous stochastic process being instantly independent of $\{\mathcal{F}_t\}$. Then we define a new stochastic integral as follows:

$$\int_a^b f(t)\varphi(t)\,dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big), \tag{1.8}$$

provided that the limit in probability exists. Notice that the evaluation points for $\varphi(t)$ are the right endpoints of subintervals.

In Section 2 we will provide more information on this new stochastic integral. In [1] an Itô formula for this anticipating integral was proved:

$$\begin{aligned}\theta\big(B(t), B(b)\big) &= \theta\big(B(a), B(b)\big) + \int_a^t \theta_x\big(B(s), B(b)\big)\,dB(s) \\ &\quad + \int_a^t \Big(\frac{1}{2}\theta_{xx}\big(B(s), B(b)\big) + \theta_{xy}\big(B(s), B(b)\big)\Big)\,ds, \quad a \le t \le b, \end{aligned}\tag{1.9}$$

which can be written in terms of stochastic differentials as

$$d\theta\big(B(t), B(b)\big) = \theta_x\big(B(t), B(b)\big)\, dB(t) + \Big(\frac{1}{2}\theta_{xx}\big(B(t), B(b)\big) + \theta_{xy}\big(B(t), B(b)\big)\Big)\, dt.$$

By comparing Equations (1.7) and (1.9), we expect that our new stochastic integral and the Hitsuda–Skorokhod integral coincide in the common domain.

The Itô formula in Equation (1.9) has been generalized to several different cases in [16, 17, 20]. However, the derivations of the Itô formula in various forms in these papers are somewhat tedious and not so transparent. The main purpose of the present paper is to give a unified formulation of the Itô formula in Section 3. The new derivation is rather simple and transparent. We will show that the various forms of the Itô formula in the previous papers can be easily derived from the unified formulation as special cases. Moreover, in Section 4 we will give several applications of the Itô formula.

2. A General Stochastic Integral

First we need to prove a lemma in order to define our new stochastic integral for general integrands beyond those in Equation (1.8).

Lemma 2.1. *Let $f_i(t), 1 \le i \le m,\ g_j(t), 1 \le j \le n$, be $\{\mathcal{F}_t\}$-adapted continuous stochastic processes and let $\varphi_i(t), 1 \le i \le m,\ \xi_j(t), 1 \le j \le n$, be continuous stochastic processes being instantly independent of $\{\mathcal{F}_t\}$. Suppose the stochastic integrals $\int_a^b f_i(t)\varphi_i(t)\, dB(t)$ and $\int_a^b g_j(t)\xi_j(t)\, dB(t)$ exist for $1 \le i \le m$, $1 \le j \le n$. Assume that*

$$\sum_{i=1}^m f_i(t)\varphi_i(t) = \sum_{j=1}^n g_j(t)\xi_j(t), \quad a \le t \le b.$$

Then the following equality holds:

$$\sum_{i=1}^m \int_a^b f_i(t)\varphi_i(t)\, dB(t) = \sum_{j=1}^n \int_a^b g_j(t)\xi_j(t)\, dB(t). \tag{2.1}$$

Remark 2.2. When $f(t)\varphi(t) = g(t)\xi(t)$ (for Case 1 in the proof below), we do not need to assume the existence of the stochastic integrals $\int_a^b f(t)\varphi(t)\, dB(t)$ and $\int_a^b g(t)\xi(t)\, dB(t)$. Equation (2.1) for this case is understood to mean that if one side of the equality exists, then the other side also exists and the equality holds.

Proof. We divide the proof into several cases:

Case 1. $m = n = 1$.

We suppress the indices so that $f(t)\varphi(t) = g(t)\xi(t)$, $a \le t \le b$. For the sake of technical simplicity, we assume that $f(t), g(t), \varphi(t)$, and $\xi(t)$ do not take the value 0. Then we have

$$\frac{f(t)}{g(t)} = \frac{\xi(t)}{\varphi(t)}.$$

Note that the left-hand side is $\{\mathcal{F}_t\}$-adapted and the right-hand side is instantly independent of $\{\mathcal{F}_t\}$. But it is easy to see that if an $\{\mathcal{F}_t\}$-adapted stochastic process is instantly independent of $\{\mathcal{F}_t\}$, then it must be a deterministic function. Hence there is a continuous function $a(t)$ on $[a,b]$ such that

$$\frac{f(t)}{g(t)} = \frac{\xi(t)}{\varphi(t)} = a(t),$$

and so

$$f(t) = a(t)g(t), \quad \varphi(t) = \frac{1}{a(t)}\xi(t).$$

Therefore

$$\begin{aligned}\int_a^b f(t)\varphi(t)\,dB(t) &\approx \sum_{i=1}^n f(t_{i-1})\varphi(t_i)\big(B(t_i)-B(t_{i-1})\big)\\ &\approx \sum_{i=1}^n \big[a(t_{i-1})g(t_{i-1})\big]\Big[\frac{1}{a(t_i)}\xi(t_i)\Big]\big(B(t_i)-B(t_{i-1})\big)\\ &\approx \sum_{i=1}^n g(t_{i-1})\xi(t_i)\big(B(t_i)-B(t_{i-1})\big)\\ &\approx \int_a^b g(t)\xi(t)\,dB(t),\end{aligned}$$

which is Equation (2.1) for this case.

Case 2. $m=1, n=2$.

In this case, we have the following assumption

$$f(t)\varphi(t) = g_1(t)\xi_1(t) + g_2(t)\xi_2(t). \tag{2.2}$$

Let $a(t) = E\varphi(t), b_1(t) = E\xi_1(t), b_2(t) = E\xi_2(t)$. For technical simplicity, we assume that the functions $a(t), b_1(t), b_2(t)$ do not vanish. Then by taking conditional expectation of Equation (2.2) with respect to $\{\mathcal{F}_t\}$, we get

$$f(t)a(t) = g_1(t)b_1(t) + g_2(t)b_2(t).$$

Let $\widetilde{a}(t) = a(t)/b_2(t), \widetilde{b}_1(t) = b_1(t)/b_2(t)$. Then we have

$$g_2(t) = f(t)\widetilde{a}(t) - g_1(t)\widetilde{b}_1(t). \tag{2.3}$$

Put $g_2(t)$ in Equation (2.3) into Equation (2.2) to get

$$f(t)\big[\varphi(t) - \widetilde{a}(t)\xi_2(t)\big] = g_1(t)\big[\xi_1(t) - \widetilde{b}_1(t)\xi_2(t)\big]. \tag{2.4}$$

By Remark 2.2, we can apply Case 1. Hence

$$\int_a^b f(t)\big[\varphi(t) - \widetilde{a}(t)\xi_2(t)\big]\,dB(t) = \int_a^b g_1(t)\big[\xi_1(t) - \widetilde{b}_1(t)\xi_2(t)\big]\,dB(t), \tag{2.5}$$

But we can use the definition in Equation (1.8) to see that

$$\begin{aligned}&\int_a^b f(t)\big[\varphi(t) - \widetilde{a}(t)\xi_2(t)\big]\,dB(t)\\ &\quad = \int_a^b f(t)\varphi(t)\,dB(t) - \int_a^b f(t)\widetilde{a}(t)\xi_2(t)\,dB(t)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\int_a^b g_1(t)\big[\xi_1(t) - \widetilde{b}_1(t)\xi_2(t)\big]\, dB(t) \\ = \int_a^b g_1(t)\xi_1(t)\, dB(t) - \int_a^b g_1(t)\widetilde{b}_1(t)\xi_2(t)\, dB(t)\end{aligned} \tag{2.7}$$

It follows from Equations (2.3), (2.5), (2.6), and (2.7) that

$$\begin{aligned}&\int_a^b f(t)\varphi(t)\, dB(t) \\ &= \int_a^b g_1(t)\xi_1(t)\, dB(t) + \int_a^b \big[f(t)\widetilde{a}(t) - g_1(t)\widetilde{b}_1(t)\big]\xi_2(t)\, dB(t) \\ &= \int_a^b g_1(t)\xi_1(t)\, dB(t) + \int_a^b g_2(t)\xi_2(t)\, dB(t),\end{aligned}$$

which is Equation (2.1) for this case.

Case 3. $m = 1, n \geq 3$.

In this case, we have

$$f(t)\varphi(t) = \sum_{i=1}^{n} g_i(t)\xi_i(t).$$

Then similar to Equation (2.3) we have

$$g_n(t) = f(t)\widetilde{a}(t) - \sum_{i=1}^{n-1} g_i(t)\widetilde{b}_i(t),$$

where $\widetilde{a}(t) = E\varphi(t)/E\xi_n(t)$ and $\widetilde{b}_i(t) = E\xi_i(t)/E\xi_n(t), 1 \leq i \leq n-1$. Then, similar to Equation (2.4), we have

$$f(t)\big[\varphi(t) - \widetilde{a}(t)\xi_n(t)\big] = \sum_{i=1}^{n-1} g_i(t)\big[\xi_i(t) - \widetilde{b}_i(t)\xi_n(t)\big].$$

Thus the $\{1 : n\}$-situation has been reduced to the $\{1 : (n-1)\}$-situation. Hence we can use the same arguments as those in Case 2 and the induction to prove Equation (2.1) for this case.

Case 4. $m \geq 2, n \geq 1$.

In this case we have

$$\sum_{i=1}^{m} f_i(t)\varphi_i(t) = \sum_{j=1}^{n} g_j(t)\xi_j(t)$$

which can be rewritten as

$$f_1(t)\varphi_1(t) = \sum_{j=1}^{n} g_j(t)\xi_j(t) - \sum_{i=2}^{m} f_i(t)\varphi_i(t).$$

Thus the $\{m : n\}$-situation becomes the $\{1 : (n+m-1)\}$-situation. Then we simply apply Case 3 to derive Equation (2.1) for this case. □

Now, suppose $\Phi(t)$ is a stochastic process of the following form

$$\Phi(t) = \sum_{i=1}^{m} f_i(t)\varphi_i(t), \quad a \le t \le b, \tag{2.8}$$

where $f_i(t)$'s are $\{\mathcal{F}_t\}$-adapted continuous stochastic processes and $\varphi_i(t)$'s are continuous stochastic processes being instantly independent of $\{\mathcal{F}_t\}$. Define the stochastic integral of $\Phi(t)$ by

$$\int_a^b \Phi(t)\, dB(t) = \sum_{i=1}^{m} \int_a^b f_i(t)\varphi_i(t)\, dB(t), \tag{2.9}$$

where each one of the integral in the right-hand side is defined by Equation (1.8). By Lemma 2.1 the stochastic integral $\int_a^b \Phi(t)\, dB(t)$ is well-defined.

Definition 2.3. Suppose $\Phi(t), a \le t \le b$, is a stochastic process and there exists a sequence $\{\Phi_n(t)\}_{n=1}^{\infty}$ of stochastic processes of the form in Equation (2.8) satisfying the conditions:

(a) $\int_a^b |\Phi(t) - \Phi_n(t)|^2\, dt \longrightarrow 0$ almost surely.

(b) $\int_a^b \Phi_n(t)\, dB(t)$ converges in probability.

Then the *stochastic integral* of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\, dB(t) = \lim_{n\to\infty} \int_a^b \Phi_n(t)\, dB(t), \quad \text{in probability.} \tag{2.10}$$

We give two examples to demonstrate the crucial ideas of this new stochastic integral, namely, taking the left endpoints of subintervals as the evaluation points for the Itô part (meaning adapted) and the right endpoints of subintervals as the evaluation points for the counter part (meaning instantly independent).

Example 2.4. It is shown in [1] that

$$\int_0^t B(1)\, dB(s) = B(1)B(t) - t, \quad 0 \le t \le 1. \tag{2.11}$$

Here we give a simple and transparent proof of this equality. The integrand $B(1)$ has the following decomposition

$$B(1) = B(t) + \big(B(1) - B(t)\big),$$

where the first term $B(t)$ is the Itô part and the second term $B(1) - B(t)$ is the counterpart of $B(1)$. By using Lemma 2.1 and Equation (1.8), we see that

$$\begin{aligned}
\int_0^t B(1)\, dB(s) &= \lim_{n\to\infty} \sum_{i=1}^{n} \big[B(s_{i-1}) + \big(B(1) - B(s_i)\big)\big]\big(B(s_i) - B(s_{i-1})\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \big[B(1) - \big(B(s_i) - B(s_{i-1})\big)\big]\big(B(s_i) - B(s_{i-1})\big) \\
&= \lim_{n\to\infty} \Big[B(1) \sum_{i=1}^{n} \big(B(s_i) - B(s_{i-1})\big) - \sum_{i=1}^{n} \big(B(s_i) - B(s_{i-1})\big)^2\Big] \\
&= B(1)B(t) - t,
\end{aligned}$$

where in the last equality we have used the quadratic variation of the Brownian motion $B(t)$.

Example 2.5. Let us compute the stochastic integral $\int_0^1 \big(\int_0^1 B(s)\,ds\big)\,dB(t)$. First we derive a decomposition of the random variable $\int_0^1 B(s)\,ds$ in terms of the Itô part and the conterpart. By the Itô formula, we have

$$d(tB(t)) = B(t)\,dt + t\,dB(t),$$

which yields that

$$\int_0^1 B(s)\,ds = B(1) - \int_0^1 s\,dB(s). \tag{2.12}$$

Note that for any $0 \le t \le 1$,

$$B(1) = B(t) + \big(B(1) - B(t)\big),$$
$$\int_0^1 s\,dB(s) = \int_0^t s\,dB(s) + \int_t^1 s\,dB(s).$$

Therefore, we obtain the decomposition

$$\int_0^1 B(s)\,ds = \Big(B(t) - \int_0^t s\,dB(s)\Big) + \Big(B(1) - B(t) - \int_t^1 s\,dB(s)\Big), \tag{2.13}$$

where the first term is the Itô part and the second term is the counterpart. For simplicity, let $\Delta B_i = B(t_i) - B(t_{i-1})$. Then by using Equation (2.13) we have

$$\begin{aligned}
&\int_0^1 \Big(\int_0^1 B(s)\,ds\Big)\,dB(t) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \Big[\Big(B(t_{i-1}) - \int_0^{t_{i-1}} s\,dB(s)\Big) + \Big(B(1) - B(t_i) - \int_{t_i}^1 s\,dB(s)\Big)\Big]\Delta B_i \\
&= \lim_{n\to\infty} \sum_{i=1}^n \Big[B(1) - \big(B(t_i) - B(t_{i-1})\big) - \Big(\int_0^1 s\,dB(s) - \int_{t_{i-1}}^{t_i} s\,dB(s)\Big)\Big]\Delta B_i \\
&= B(1)^2 - 1 - B(1)\int_0^1 s\,dB(s) + \lim_{n\to\infty} \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} s\,dB(s)\Big)\Delta B_i \\
&= B(1)\int_0^1 B(s)\,ds - 1 + \lim_{n\to\infty} \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} s\,dB(s)\Big)\Delta B_i,
\end{aligned} \tag{2.14}$$

where in the last equality we have used the fact that $\int_0^1 s\,dB(s) = B(1) - \int_0^1 B(s)\,ds$ from Equation (2.12). Then we use the fact that $(\Delta B_i)^2 \approx \Delta t_i$ to show that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} s\,dB(s)\Big)\Delta B_i &= \lim_{n\to\infty} \sum_{i=1}^n t_{i-1}(\Delta B_i)^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^n t_{i-1}\Delta t_i = \int_0^1 t\,dt = \frac{1}{2}.
\end{aligned} \tag{2.15}$$

Upon putting Equation (2.15) into Equation (2.14), we obtain the value of the stochastic integral

$$\int_0^1 \Big(\int_0^1 B(s)\, ds \Big)\, dB(t) = B(1) \int_0^1 B(s)\, ds - \frac{1}{2}. \tag{2.16}$$

Next, we consider the multiple Wiener–Itô integral introduced by Itô [9] in 1951. It is related to an iterated stochastic integral by the following theorem. For a simple proof, see the book [14].

Theorem 2.6. *For $f \in L^2([a,b]^n)$, we have the equality*

$$\begin{aligned} &\int_{[a,b]^n} f(t_1, t_2, \ldots, t_n)\, dB(t_1) dB(t_1) \cdots dB(t_n) \\ &= n! \int_a^b \cdots \int_a^{t_{n-2}} \Big[\int_a^{t_{n-1}} \widehat{f}(t_1, \ldots, t_{n-1}, t_n)\, dB(t_n) \Big]\, dB(t_{n-1}) \cdots dB(t_1). \end{aligned}$$

where $\widehat{f}$ is the symmetrization of f.

Observe that the iterated integral over the restriced region in the right-hand side is to make sure that in each step the integrand is adapted so that an Itô integral is defined. However, for our new stochasic integral, there is no need to restrict the region since in each step the integral is defined as our new stochastic integral. The next theorem is proved in [2].

Theorem 2.7. *Let $f \in L^2([a,b]^n)$. Then*

$$\begin{aligned} &\int_{[a,b]^n} f(t_1, t_2, \ldots, t_n)\, dB(t_1) dB(t_1) \cdots dB(t_n) \\ &= \int_a^b \cdots \int_a^b \Big[\int_a^b f(t_1, \ldots, t_{n-1}, t_n)\, dB(t_n) \Big]\, dB(t_{n-1}) \cdots dB(t_1). \end{aligned}$$

From this theorem we see that we can compute a multiple Wiener–Itô integral as an iterated stochastic integral just like ordinary calculus. In fact, let us consider another example and see what we can conclude further.

Example 2.8. What is the value of the iterated integral $\int_0^1 \big(\int_0^1 B(s)\, dB(t) \big)\, ds$? For the first stochastic integral we have

$$\int_0^1 B(s)\, dB(t) = \int_0^s B(s)\, dB(t) + \int_s^1 B(s)\, dB(t). \tag{2.17}$$

Replace 1 by b in Equation (2.11), which can be easily seen to be valid, interchange s and t, and then put $b = s$ to get

$$\int_0^s B(s)\, dB(t) = B(s)^2 - s. \quad 0 \le s \le 1. \tag{2.18}$$

On the other hand, we have

$$\int_s^1 B(s)\, dB(t) = B(s) \int_s^1 dB(t) = B(s)\big(B(1) - B(s)\big). \tag{2.19}$$

Put Equations (2.18) and (2.19) into Equation (2.17) to obtain

$$\int_0^1 B(s)\, dB(t) = B(1)B(s) - s,$$

which, upon integrationg in s, yields immediately the equality

$$\int_0^1 \Big(\int_0^1 B(s)\, dB(t) \Big)\, ds = B(1) \int_0^1 B(s)\, ds - \frac{1}{2}. \tag{2.20}$$

Remark 2.9. Observe that from Equations (2.16) and (2.20) we can deduce the following equality:

$$\int_0^1 \Big(\int_0^1 B(s)\, ds \Big)\, dB(t) = \int_0^1 \Big(\int_0^1 B(s)\, dB(t) \Big)\, ds,$$

i.e., we can change the order of integration! Thus we can define double stochastic integrals with respect to $dB(t)$ and ds in terms of iterated integrals. In fact, we will use the Itô formula in Section 3 to give more examples for multiple stochastic integrals with respect to $dB(t_1), dB(t_2), \ldots, dB(t_n)$, and ds.

3. A General Itô Formula

The formula in Equation (1.9) is not a general form of the Itô formula for the new stochastic integral. In fact, the formulation is somewhat misleading since it seems to indicate that there are other forms of the Itô formula. Note that the anticipating part $B(b)$ in the function $\theta\big(B(t), B(b)\big)$ has the decomposition

$$B(b) = B(t) + \big(B(b) - B(t)\big), \tag{3.1}$$

with the Itô part $B(t)$ and the counterpart $B(b) - B(t)$ which we need to take care of separately in order to obtain a general Itô formula.

A filtration $\{\mathcal{F}_t; a \le t \le b\}$ is fixed. Adaptedness and instantly independence will be understood to be with respect to this filtration. Being motivated by the decomposition of an anticipating stochastic process into a function of an Itô part and a counterpart such as $B(b)$ in Equation (3.1), we consider the following two stochastic processes

$$X_t = X_a + \int_a^t g(s)\, dB(s) + \int_a^t h(s)\, ds \tag{3.2}$$

$$Y^{(t)} = Y^{(a)} + \int_t^b \xi(s)\, dB(s) + \int_t^b \eta(s)\, ds, \tag{3.3}$$

where $g(t)$ and $h(t)$ are adapted so that X_t is an Itô process, and $\xi(t)$ and $\eta(t)$ are instantly independent such that $Y^{(t)}$ is also instantly independent. Thus $Y^{(t)}$ is a stochastic process in the counterpart. For clarity, we use sub-t and sup-(t) to denote integrals $\int_a^t$ and $\int_t^b$, respectively.

Let $\theta(x, y)$ be a real-valued C^2-function on $\mathbb{R}^2$ so that we have the estimate

$$\begin{aligned}\theta(x, y) \approx\ & \theta(x_0, y_0) + \theta_x(x_0, y_0)(x - x_0) + \theta_y(x_0, y_0)(y - y_0) \\ & + \frac{1}{2}\theta_{xx}(x_0, y_0)(x - x_0)^2 + \frac{1}{2}\theta_{yy}(x_0, y_0)(y - y_0)^2 \\ & + \theta_{xy}(x_0, y_0)(x - x_0)(y - y_0). \qquad (3.4)\end{aligned}$$

We will informally derive the stochastic differential $d\theta(X_t, Y^{(t)})$. The evaluation points for X_t and $Y^{(t)}$ in a new stochastic integrand will be the left endpoints and the right endpoints, respectively, of subintervals in a partition.

For any partition $\{a = t_0, t_1, \ldots, t_n = t\}$ of the interval $[a, t]$, we have

$$\theta(X_t, Y^{(t)}) = \theta(X_a, Y^{(a)}) + \sum_{i=1}^{n} \Big[\theta(X_{t_i}, Y^{(t_i)}) - \theta(X_{t_{i-1}}, Y^{(t_{i-1})})\Big] \qquad (3.5)$$

Then for each $i = 1, 2, \ldots, n$, we use Equation (3.4) to get

$$\begin{aligned}& \theta(X_{t_i}, Y^{(t_i)}) - \theta(X_{t_{i-1}}, Y^{(t_{i-1})}) \\ & \approx \theta_x(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}}) + \theta_y(X_{t_{i-1}}, Y^{(t_{i-1})})(Y^{(t_i)} - Y^{(t_{i-1})}) \\ & \quad + \frac{1}{2}\theta_{xx}(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}})^2 \\ & \quad + \frac{1}{2}\theta_{yy}(X_{t_{i-1}}, Y^{(t_{i-1})})(Y^{(t_i)} - Y^{(t_{i-1})})^2 \\ & \quad + \theta_{xy}(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}})(Y^{(t_i)} - Y^{(t_{i-1})}) \\ & = \mathrm{I}_i + \mathrm{II}_i + \mathrm{III}_i + \mathrm{IV}_i + \mathrm{V}_i, \qquad (3.6)\end{aligned}$$

where $\mathrm{I}_i, \mathrm{II}_i, \mathrm{III}_i, \mathrm{IV}_i$, and V_i are defined term by term in the corresponding order, respectively.

Now, we take care of summations of these terms over i. For the first term in Equation (3.6), we have

$$\begin{aligned}& \sum_{i=1}^{n} \mathrm{I}_i = \sum_{i=1}^{n} \theta_x(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}}) \\ & \approx \sum_{i=1}^{n} \Big\{\theta_x(X_{t_{i-1}}, Y^{(t_i)}) + \theta_{xy}(X_{t_{i-1}}, Y^{(t_i)})(Y^{(t_{i-1})} - Y^{(t_i)})\Big\}(X_{t_i} - X_{t_{i-1}}) \\ & \to \int_a^t \theta_x(X_s, Y^{(s)})\, dX_s - \int_a^t \theta_{xy}(X_s, Y^{(s)})\, (dX_s)(dY^{(s)}). \qquad (3.7)\end{aligned}$$

For the second term in Equation (3.6), we have

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{II}_i &= \sum_{i=1}^{n}\theta_y(X_{t_{i-1}}, Y^{(t_{i-1})})(Y^{(t_i)} - Y^{(t_{i-1})})\\ &\approx \sum_{i=1}^{n}\Big\{\theta_y(X_{t_{i-1}}, Y^{(t_i)}) + \theta_{yy}(X_{t_{i-1}}, Y^{(t_i)})(Y^{(t_{i-1})} - Y^{(t_i)})\Big\}(Y^{(t_i)} - Y^{(t_{i-1})})\\ &\to \int_a^t \theta_y(X_s, Y^{(s)})\, dY^{(s)} - \int_a^t \theta_{yy}(X_s, Y^{(s)})\,(dY^{(s)})^2. \qquad (3.8)\end{aligned}$$

For the third term in Equation (3.6), we have

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{III}_i &= \frac{1}{2}\sum_{i=1}^{n}\theta_{xx}(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}})^2\\ &\to \frac{1}{2}\int_a^t \theta_{xx}(X_s, Y^{(s)})\,(dX_s)^2. \qquad (3.9)\end{aligned}$$

Note that we do not have to change $\theta_{xx}(X_{t_{i-1}}, Y^{(t_{i-1})})$ to $\theta_{xx}(X_{t_{i-1}}, Y^{(t_i)})$ since the integrator $(dX_s)^2 = g(s)^2\, ds$ from Equation (3.2). Similarly, for the fourth term in Equation (3.6), we have

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{IV}_i &= \frac{1}{2}\sum_{i=1}^{n}\theta_{yy}(X_{t_{i-1}}, Y^{(t_{i-1})})(Y^{(t_i)} - Y^{(t_{i-1})})^2\\ &\to \frac{1}{2}\int_a^t \theta_{yy}(X_s, Y^{(s)})\,\big(dY^{(s)}\big)^2. \qquad (3.10)\end{aligned}$$

For the last term in Equation (3.6), we have

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{V}_i &= \sum_{i=1}^{n}\theta_{xy}(X_{t_{i-1}}, Y^{(t_{i-1})})(X_{t_i} - X_{t_{i-1}})(Y^{(t_i)} - Y^{(t_{i-1})})\\ &\to \int_a^t \theta_{xy}(X_s, Y^{(s)})\,(dX_s)(dY^{(s)}). \qquad (3.11)\end{aligned}$$

Observe the following two facts from Equations (3.7) to (3.11):

(1) The terms with θ_{xy} in Equations (3.7) and (3.11) cancel out! This is due to the nature of our new stochastic integral.
(2) The coefficients of the integrals of θ_{yy} in Equations (3.8) and (3.10) add up to $-\frac{1}{2}$.

Finally, we sum up Equations (3.7) to (3.11) to get the stochastic differential of $\theta(X_t, Y^{(t)})$:

$$\begin{aligned}d\theta(X_t, Y^{(t)}) &= \theta_x(X_t, Y^{(t)})\, dX_t + \frac{1}{2}\theta_{xx}(X_t, Y^{(t)})\,(dX_t)^2\\ &\quad + \theta_y(X_t, Y^{(t)})\, dY^{(t)} - \frac{1}{2}\theta_{yy}(X_t, Y^{(t)})\,(dY^{(t)})^2.\end{aligned}$$

Thus we have derived a general Itô formula in the next theorem.

Theorem 3.1. *Let $X_t, a \le t \le b$, be an Itô process given by Equation (3.2) and $Y^{(t)}, a \le t \le b$, an instantly independent process given by Equation (3.3). Suppose $\theta(x, y)$ is a real-valued C^2-function on $\mathbb{R}^2$. Then the following equality holds for $a \le t \le b$:*

$$\theta(X_t, Y^{(t)}) = \theta(X_a, Y^{(a)}) + \int_a^t \theta_x(X_s, Y^{(s)})\, dX_s + \frac{1}{2}\int_a^t \theta_{xx}(X_s, Y^{(s)})\,(dX_s)^2 + \int_a^t \theta_y(X_s, Y^{(s)})\, dY^{(s)} - \frac{1}{2}\int_a^t \theta_{yy}(X_s, Y^{(s)})\,(dY^{(s)})^2,$$

which can be expressed symbolically in terms of stochastic differentials as

$$d\theta(X_t, Y^{(t)}) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}\,(dX_t)^2 + \theta_y\, dY^{(t)} - \frac{1}{2}\theta_{yy}\,(dY^{(t)})^2.$$

From the derivation of Theorem 3.1, we can easily obtain the Itô formula for multiple stochastic processes in the next theorem.

Theorem 3.2. *Let $X_t^{(i)}, 1 \le i \le n$, and $Y_j^{(t)}, 1 \le j \le m$, be stochastic processes of the form given by Equation (3.2) and (3.3), respectively. Suppose $\theta(t, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a real-valued function being C^1 in t and C^2 in x_i's and y_j's. Then the stochastic differential of $\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)})$ is given by*

$$\begin{aligned} &d\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)}) \\ &= \theta_t\, dt + \sum_{i=1}^n \theta_{x_i}\, dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^n \theta_{x_i x_j}\,(dX_t^{(i)})(dX_t^{(j)}) \\ &\quad + \sum_{k=1}^m \theta_{y_k}\, dY_k^{(t)} - \frac{1}{2}\sum_{k,\ell=1}^m \theta_{y_k y_\ell}\,(dY_k^{(t)})(dY_\ell^{(t)}). \end{aligned} \tag{3.12}$$

Remark 3.3. Let X_t and Y_t be Itô processes of the form in Equation (3.2) and $X^{(t)}$ and $Y^{(t)}$ instantly independent processes of the form in Equation (3.3). Then we have the following *product rules*:

$$\begin{aligned} d(X_t Y_t) &= Y_t\, dX_t + X_t\, dY_t + (dX_t)(dY_t), \\ d(X_t Y^{(t)}) &= Y^{(t)}\, dX_t + X_t\, dY^{(t)}, \\ d(X^{(t)} Y^{(t)}) &= Y^{(t)}\, dX^{(t)} + X^{(t)}\, dY^{(t)} - (dX^{(t)})(dY^{(t)}). \end{aligned}$$

Example 3.4. We use Theorem 3.2 to derive $d\theta(B(t), B(b))$, $a \le t \le b$. Note that we have the decomposition of $B(b)$ as

$$B(b) = B(t) + \big(B(b) - B(t)\big). \tag{3.13}$$

Let $X_t = B(t)$, $Y^{(t)} = B(b) - B(t)$, and $f(x, y) = \theta(x, x + y)$. Then we have

$$f_x = \theta_1 + \theta_2, \quad f_{xx} = \theta_{11} + 2\theta_{12} + \theta_{22}, \quad f_y = \theta_2, \quad f_{yy} = \theta_{22},$$

where we have used sub-numbers to denote the partial derivatives of θ to avoid confusing. Apply Theorem 3.2 to $X_t, Y^{(t)}$, and the function $f(x,y)$ to get

$$\begin{aligned} df(X_t, Y^{(t)}) &= f_x\, dX_t + \frac{1}{2} f_{xx}\,(dX_t)^2 + f_y\, dY^{(t)} - \frac{1}{2} f_{yy}\,(dY^{(t)})^2 \\ &= (\theta_1 + \theta_2)\, dB(t) + \frac{1}{2}(\theta_{11} + 2\theta_{12} + \theta_{22})\, dt - \theta_2\, dB(t) - \frac{1}{2}\theta_{22}\, dt \\ &= \theta_1\, dB(t) + \Big(\frac{1}{2}\theta_{11} + \theta_{12}\Big)\, dt. \end{aligned}$$

But $f(X_t, Y^{(t)}) = \theta(B(t), B(b))$. Hence we have the stochastic differential of $\theta(B(t), B(b))$ given by

$$d\theta(B(t), B(b)) = \theta_x\, dB(t) + \Big(\frac{1}{2}\theta_{xx} + \theta_{xy}\Big)\, dt,$$

where we have changed θ_1 to θ_x, etc. This equality is the Itô formula in Equation (1.9) in terms of stochastic differentials. Hence the Itô formula in [1] can be derived from Theorem 3.2.

Example 3.5. Consider the stochastic process $\theta\big(B(t), \int_0^1 B(s)\, ds\big), 0 \le t \le 1$. We can apply Theorem 3.2 to find its stochastic differential. Let $X_t^{(1)} = B(t)$. By Equation (2.13) we have the following decomposition:

$$\int_0^1 B(s)\, ds = X_t^{(2)} + Y^{(t)}, \tag{3.14}$$

where $X_t^{(2)}$ and $Y^{(t)}$ are defined by

$$X_t^{(2)} = B(t) - \int_0^t s\, dB(s), \quad Y^{(t)} = B(1) - B(t) - \int_t^1 s\, dB(s).$$

Then we have the stochastic differentials

$$dX_t^{(1)} = dB(t), \quad dX_t^{(2)} = (1-t)\, dB(t), \quad dY^{(t)} = (t-1)\, dB(t).$$

Define a function $f(x_1, x_2, y) = \theta(x_1, x_2 + y)$. Then we have

$$f_{x_1} = \theta_1, \quad f_{x_1x_1} = \theta_{11}, \quad f_{x_2} = \theta_2, \quad f_{x_2x_2} = \theta_{22},$$
$$f_{x_1x_2} = \theta_{12}, \quad f_y = \theta_2, \quad f_{yy} = \theta_{22}.$$

Apply Theorem 3.2 to $X_t^{(1)}, X_t^{(2)}, Y^{(t)}$, and the function $f(x_1, x_2, y)$ to get

$$\begin{aligned} df(X_t^{(1)}, X_t^{(2)}, Y^{(t)}) &= f_{x_1}\, dX_t^{(1)} + \frac{1}{2} f_{x_1x_1}\,(dX_t^{(1)})^2 + f_{x_2}\, dX_t^{(2)} \\ &\quad + \frac{1}{2} f_{x_2x_2}\,(dX_t^{(2)})^2 + f_{x_1x_2}\,(dX_t^{(1)})(dX_t^{(2)}) \\ &\quad + f_y\, dY^{(t)} - \frac{1}{2} f_{yy}\,(dY^{(t)})^2. \end{aligned} \tag{3.15}$$

Then use the above stochastic differentials and partical derivatives to simplify Equation (3.15) to

$$df(X_t^{(1)}, X_t^{(2)}, Y^{(t)}) = \theta_1\, dB(t) + \Big(\frac{1}{2}\theta_{11} + (1-t)\theta_{12}\Big)\, dt.$$

But $f(X_t^{(1)}, X_t^{(2)}, Y^{(t)}) = \theta\big(B(t), \int_0^1 B(s)\,ds\big)$. Hence we have proved the equality:

$$d\theta\Big(B(t), \int_0^1 B(s)\,ds\Big) = \theta_x\,dB(t) + \Big(\frac{1}{2}\theta_{xx} + (1-t)\theta_{xy}\Big)\,dt. \tag{3.16}$$

Here we have changed notation for partial derivatives because now there is no confusion. In particular, let us take the function $\theta(x,y) = xy$. Then Equation (3.16) becomes

$$d\Big(B(t)\int_0^1 B(s)\,ds\Big) = \Big(\int_0^1 B(s)\,ds\Big)\,dB(t) + (1-t)\,dt.$$

Upon integrating from 0 to 1, we obtain

$$B(1)\int_0^1 B(s)\,ds = \int_0^1 \Big(\int_0^1 B(s)\,ds\Big)\,dB(t) + \int_0^1 (1-t)\,dt$$

so that we have the equality

$$\int_0^1 \Big(\int_0^1 B(s)\,ds\Big)\,dB(t) = B(1)\int_0^1 B(s)\,ds - \frac{1}{2}.$$

Thus we have another way to derive the equality in Equation (2.16).

Examples 3.4 and 3.5 lead to the following more general situation. Suppose a random variable ζ can be decomposed as

$$\zeta = \zeta_t + \zeta^{(t)}, \quad \forall\ a \le t \le b, \tag{3.17}$$

where ζ_t is an Itô process of the form given by Equation (3.2) and $\zeta^{(t)}$ an instantly independent process of the form given by Equation (3.3). For examples, $B(b)$ can be decomposed as in Equation (3.13) and $\int_0^1 B(s)\,ds$ as in Equation (3.14).

Suppose $\Phi(x)$ and $\theta(x,y)$ are C^2-functions on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let X_t be an Itô process. We want to derive the stochastic differential of $\theta\big(X_t, \Phi(\zeta)\big)$. Use the decomposition of ζ in Equation (3.17) to rewrite

$$\theta\big(X_t, \Phi(\zeta)\big) = \theta\big(X_t, \Phi(\zeta_t + \zeta^{(t)})\big).$$

In order to apply Theorem 3.2, we let $X_t^{(1)} = X_t, X_t^{(2)} = \zeta_t, Y^{(t)} = \zeta^{(t)}$ and define a function

$$f(x_1, x_2, y) = \theta\big(x_1, \Phi(x_2 + y)\big).$$

Then we have the following partial derivatives:

$$f_{x_1} = \theta_1, \quad f_{x_1x_1} = \theta_{11}, \quad f_{x_2} = \theta_2\Phi', \quad f_{x_2x_2} = \theta_{22}(\Phi')^2 + \theta_2\Phi'',$$
$$f_{x_1x_2} = \theta_{12}\Phi', \quad f_y = \theta_2\Phi', \quad f_{yy} = \theta_{22}(\Phi')^2 + \theta_2\Phi''.$$

Therefore, by Theorem 3.2 we have

$$\begin{aligned} df(X_t, \zeta_t, \zeta^{(t)}) = {} & f_{x_1}\,dX_t + \frac{1}{2}f_{x_1x_1}\,(dX_t)^2 \\ & + f_{x_2}\,d\zeta_t + \frac{1}{2}f_{x_2x_2}\,(d\zeta_t)^2 + f_{x_1x_2}\,(dX_t)(d\zeta_t) \\ & + f_y\,d\zeta^{(t)} - \frac{1}{2}f_{yy}\,(d\zeta^{(t)})^2. \end{aligned} \tag{3.18}$$

Note that the random variable ζ does not depend on t. Hence we have $d\zeta^{(t)} = -d\zeta_t$. Use this fact and the above relationships on the partial derivatives to simplify Equation (3.18) to

$$df(X_t, \zeta_t, \zeta^{(t)}) = \theta_1\, dX_t + \frac{1}{2}\theta_{11}\,(dX_t)^2 + \theta_{12}\Phi'(\zeta)\,(dX_t)(d\zeta_t).$$

Recall that $f(X_t, \zeta_t, \zeta^{(t)}) = \theta\big(X_t, \Phi(\zeta)\big)$. Hence we have derived the stochastic differential of $\theta\big(X_t, \Phi(\zeta)\big)$.

Similarly, we can also derive the stochastic differential of $\theta\big(Y^{(t)}, \Phi(\zeta)\big)$. We sum up the above discussion as the next theorem.

Theorem 3.6. *Suppose ζ is a random variable such that $\zeta = \zeta_t + \zeta^{(t)}$ for all $a \le t \le b$ with ζ_t being an Itô process and $\zeta^{(t)}$ an instantly independent process. Assume that $\Phi(x)$ is a C^1-function and $\theta(x, y)$ a C^2-function. Then we have*

$$d\theta\big(X_t, \Phi(\zeta)\big) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}\,(dX_t)^2 + \theta_{xy}\Phi'(\zeta)\,(dX_t)(d\zeta_t),$$
$$d\theta\big(Y^{(t)}, \Phi(\zeta)\big) = \theta_x\, dY^{(t)} - \frac{1}{2}\theta_{xx}\,(dY^{(t)})^2 - \theta_{xy}\Phi'(\zeta)\,(dY^{(t)})(d\zeta^{(t)}).$$

In particular, when $\Phi(x) = x$, we have the formulas:

$$d\theta(X_t, \zeta) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}\,(dX_t)^2 + \theta_{xy}\,(dX_t)(d\zeta_t),$$
$$d\theta(Y^{(t)}, \zeta) = \theta_x\, dY^{(t)} - \frac{1}{2}\theta_{xx}\,(dY^{(t)})^2 - \theta_{xy}\,(dY^{(t)})(d\zeta^{(t)}).$$

We can also consider the case when a random variable η is a product $\eta = \eta_t\, \eta^{(t)}$ for all $t \in [a, b]$ with η_t being an Itô process and $\eta^{(t)}$ an instantly independent process. For example

$$e^{B(1)} = e^{B(t)}\, e^{B(1)-B(t)}, \quad 0 \le t \le 1.$$

By applying Theorem 3.2 we can easily obtain the next theorem.

Theorem 3.7. *Let η be a random variable such that $\eta = \eta_t\, \eta^{(t)}$ for all $a \le t \le b$ with η_t being an Itô process and $\eta^{(t)}$ an instantly independent process. Suppose $\Psi(x)$ is a C^1-function and $\theta(x, y)$ a C^2-function. Then we have*

$$d\theta\big(X_t, \Psi(\eta)\big) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}\,(dX_t)^2 + \theta_{xy}\Psi'(\eta)\eta^{(t)}\,(dX_t)(d\eta_t),$$
$$d\theta\big(Y^{(t)}, \Psi(\eta)\big) = \theta_x\, dY^{(t)} - \frac{1}{2}\theta_{xx}\,(dY^{(t)})^2 - \theta_{xy}\Psi'(\eta)\eta_t\,(dY^{(t)})(d\zeta^{(t)}).$$

For the special case $\Phi(x) = x$, we have

$$d\theta(X_t, \eta) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}\,(dX_t)^2 + \theta_{xy}\eta^{(t)}\,(dX_t)(d\zeta_t),$$
$$d\theta(Y^{(t)}, \eta) = \theta_x\, dY^{(t)} - \frac{1}{2}\theta_{xx}\,(dY^{(t)})^2 - \theta_{xy}\eta_t\,(dY^{(t)})(d\eta^{(t)}).$$

4. Some Applications

In this section we use the Itô formula to study exponential processes. First we solve the following stochastic differential equation

$$\begin{cases} d\mathcal{E}(t) = B(1)\mathcal{E}(t)\, dB(t), \quad 0 \le t \le 1, \\ \mathcal{E}(0) = 1. \end{cases} \tag{4.1}$$

Being motivated by the solution of a linear stochastic differential equation in the Itô theory (see, e.g., Section 11.1 in the book [13]), we try a stochastic process of the form:

$$\mathcal{E}(t) = \exp\left[B(1)f(t)\int_0^t \frac{1}{f(s)}\, dB(s) - \frac{1}{2}B(1)^2 f(t)^2 \int_0^t \frac{1}{f(s)^2}\, ds - g(t)\right], \tag{4.2}$$

where $f(t)$ and $g(t)$ are deterministic functions to be determined so that $\mathcal{E}(t)$ is a solution of Equation (4.1). Note that $g(0) = 0$ in order for $\mathcal{E}(t)$ to satisfy the initial condition $\mathcal{E}(0) = 1$.

Apply Theorem 3.6 (with an obvious addition of the t-variable) to the stochastic process $X_t = \int_0^t \frac{1}{f(s)}\, dB(s), \Phi(x) = x, \eta = B(1)$ and the following function

$$\theta(t,x,y) = \exp\left[yf(t)x - \frac{1}{2}y^2 f(t)^2 \int_0^t \frac{1}{f(s)^2}\, ds - g(t)\right].$$

Then we have $\mathcal{E}(t) = \theta(t, X_t, B(1))$ with the stochastic differential given by

$$\begin{aligned} d\mathcal{E}(t) &= B(1)\mathcal{E}(t)\, dB(t) \\ &+ \mathcal{E}(t)\left\{1 - g'(t) + \big(f(t) + f'(t)\big)\left[B(1)X_t - f(t)^2 B(1)^2 \int_0^t \frac{1}{f(s)^2}\, ds\right]\right\} dt. \end{aligned}$$

Hence in order for $\mathcal{E}(t)$ to be a solution of Equation (4.1) we must have

$$g'(t) = 1, \quad f(t) + f'(t) = 0,$$

which yields $g(t) = t$ since $g(0) = 0$ and $f(t) = ce^{-t}$ for some constant c. Put these two functions into Equation (4.2) to produce

$$\mathcal{E}(t) = \exp\left[B(1)\int_0^t e^{-(t-s)}\, dB(s) - \frac{1}{4}B(1)^2\big(1 - e^{-2t}\big) - t\right].$$

Thus $\mathcal{E}(t)$ is a solution of Equation (4.1) with $\mathcal{E}(0) = 1$. On the other hand, the uniqueness of a solution is obvious. We state this fact as a theorem.

Theorem 4.1. *The stochastic process*

$$\mathcal{E}(t) = \exp\left[B(1)\int_0^t e^{-(t-s)}\, dB(s) - \frac{1}{4}B(1)^2\big(1 - e^{-2t}\big) - t\right]$$

is the solution of the stochastic differential equation

$$\begin{cases} d\mathcal{E}(t) = B(1)\mathcal{E}(t)\, dB(t), \quad 0 \le t \le 1. \\ \mathcal{E}(0) = 1. \end{cases}$$

Remark 4.2. This stochastic differential equation was first studied by Buckdahn [3] by a different method. The white noise version is a special case of Theorem 13.34 in the book [13].

Now, consider general exponential processes arising from the Itô part and the counter part. Let X_t be an Itô process

$$X_t = \int_a^t g(s)\, dB(s) - \frac{1}{2}\int_a^t g(s)^2\, ds, \quad a \le t \le b.$$

On the other hand, suppose $h(t)$ is an instantly independent process such that $E\int_a^b |h(t)|^2\, dt < \infty$. Assume that the stochastic process

$$Y^{(t)} = -\int_t^b h(s)\, dB(s) - \frac{1}{2}\int_t^b h(s)^2\, ds, \quad a \le t \le b,$$

is also instantly independent. Define an *exponential process* $\mathcal{E}(t)$ associated with $g(t)$ and $h(t)$ by

$$\mathcal{E}(t) = e^{X_t}\, e^{Y^{(t)}}, \quad a \le t \le b. \tag{4.3}$$

Apply Theorem 3.2 to the above stochastic processes X_t and $Y^{(t)}$ and the function $\theta(x, y) = e^x\, e^y$. Then we get

$$d\theta(X_t, y^{(t)}) = \big(g(t) + h(t)\big)\theta(X_t, y^{(t)})\, dB(t).$$

But $\theta(X_t, y^{(t)}) = \mathcal{E}(t)$. Hence we have

$$d\mathcal{E}(t) = \big(g(t) + h(t)\big)\mathcal{E}(t)\, dB(t).$$

Thus we have proved the next theorem.

Theorem 4.3. *Let $g(t)$ be an adapted stochastic process with $E\int_a^b |g(t)|^2\, dt < \infty$ and let $h(t)$ be an instantly independent process such that $E\int_a^b |h(t)|^2\, dt < \infty$ and $\int_t^b h(s)\, dB(s)$ is instantly independent. Then the exponential process*

$$\mathcal{E}(t) = \exp\left[\int_a^t g(s)\, dB(s) - \frac{1}{2}\int_a^t g(s)^2\, ds - \int_t^b h(s)\, dB(s) - \frac{1}{2}\int_t^b h(s)^2\, ds\right]$$

satisfies the stochastic differential equation

$$d\mathcal{E}(t) = \big(g(t) + h(t)\big)\mathcal{E}(t)\, dB(t).$$

Remark 4.4. Note that the values of $\mathcal{E}(t)$ at a and b are given by

$$\mathcal{E}(a) = \exp\left[-\int_a^b h(s)\, dB(s) - \frac{1}{2}\int_a^b h(s)^2\, ds\right]$$

$$\mathcal{E}(b) = \exp\left[\int_a^b g(s)\, dB(s) - \frac{1}{2}\int_a^b g(s)^2\, ds\right].$$

We have two special cases:

(1) When $h = 0$, the exponential process

$$\mathcal{E}(t) = \exp\left[\int_a^t g(s)\, dB(s) - \frac{1}{2}\int_a^t g(s)^2\, ds\right]$$

is the one in the Itô theory solving the equation:

$$\begin{cases} d\mathcal{E}(t) = g(t)\mathcal{E}(t)\, dB(t), & a \le t \le b. \\ \mathcal{E}(a) = 1. \end{cases}$$

(2) When $g = 0$, we have the exponential process

$$\mathcal{E}(t) = \exp\left[-\int_t^b h(s)\, dB(s) - \frac{1}{2}\int_t^b h(s)^2\, ds \right],$$

which solves the backward stochastic differential equation:

$$\begin{cases} d\mathcal{E}(t) = h(t)\mathcal{E}(t)\, dB(t), \quad a \le t \le b. \\ \mathcal{E}(b) = 1. \end{cases}$$

Example 4.5. Consider the stochastic differential equation

$$d\mathcal{E}(t) = B(1)\mathcal{E}(t)\, dB(t), \quad 0 \le t \le 1. \tag{4.4}$$

By Theorem 4.1 the stochastic process

$$\mathcal{E}(t) = \exp\left[B(1)\int_0^t e^{-(t-s)}\, dB(s) - \frac{1}{4}B(1)^2\big(1 - e^{-2t}\big) - t \right] \tag{4.5}$$

is a solution of Equation (4.4). On the other hand, we can write $B(1)$ as

$$B(1) = B(t) + \big(B(1) - B(t)\big)$$

and apply Theorem 4.3 with $g(t) = B(t)$ and $h(t) = B(1) - B(t)$ to see that the stochastic process

$$\begin{aligned} \mathcal{E}(t) = \exp\bigg[&\int_0^t B(s)\, dB(s) - \frac{1}{2}\int_0^t B(s)^2\, ds \\ &- \int_t^1 \big(B(1) - B(s)\big)\, dB(s) - \frac{1}{2}\int_t^1 \big(B(1) - B(s)\big)^2\, ds \bigg] \end{aligned} \tag{4.6}$$

is also a solution of Equation (4.4). Note that the stochastic processes in Equations (4.5) and (4.6) are very different and have different initial conditions.

Example 4.6. Consider a stochastic differential equation with an anticipating initial condition. In [3], Buckdahn proved that the solution of the equation

$$\begin{cases} dX_t = X_t\, dB(t), \quad 0 \le t \le 1, \\ X_0 = \operatorname{sgn}\big(B(1)\big), \end{cases}$$

is given by

$$X_t = \operatorname{sgn}\big(B(1) - t\big)e^{B(t) - \frac{1}{2}t}. \tag{4.7}$$

In [13] we study the white noise formulation of this equation, i.e., $dX_t = \partial_t^* X_t\, dt$ and use the S-transform to derive the solution X_t in Equation (4.7), see Example 13.30 in the book [13].

Example 4.7. Consider the same stochastic differential equation in Example 4.6, but with a different initial condition, namely,

$$\begin{cases} dX_t = X_t\, dB(t), \quad 0 \le t \le 1, \\ X_0 = B(1), \end{cases}$$

In [15] we use the iteration method to derive the solution

$$X_t = \big(B(1) - t\big)e^{B(t) - \frac{1}{2}t}. \tag{4.8}$$

Examples 4.6 and 4.7 lead to more general linear stochastic differential equation of the form:

$$\begin{cases} dX_t = \alpha(t)X_t\,dB(t) + \beta(t)X_t\,dt, \quad a \le t \le b, \\ X_a = \rho\big(B(b) - B(a)\big), \end{cases} \tag{4.9}$$

where $\alpha(t)$ is a deterministic function in $L^2([a,b])$, $\beta(t)$ is an adapted stochastic process such that $E\int_a^b |\beta(t)|^2\,dt < \infty$, and ρ is a continuous function on $\mathbb{R}$.

Note that within the Itô theory the solution of Equation (4.9) with the initial condition $X_a = x$ is given by

$$X_t = x\exp\bigg[\int_a^t \alpha(s)\,dB(s) + \int_a^t \Big(\beta(s) - \frac{1}{2}\alpha(s)^2\Big)\,ds\bigg].$$

On the other hand, observe the forms of the solutions in Equations (4.7) and (4.8) for the initial conditions $X_0 = \mathrm{sgn}\big(B(1)\big)$ and $X_0 = B(1)$, respectively. Thus we expect the solution of Equation (4.9) with the initial condition $\rho\big(B(b) - B(a)\big)$ to be of the form:

$$X_t = \rho\big(B(b) - B(a) - k(t)\big)e^{Z_t}, \tag{4.10}$$

where $k(t)$ is a deterministic function to be derived and Z_t is the Itô process

$$Z_t = \int_a^t \alpha(s)\,dB(s) + \int_a^t \Big(\beta(s) - \frac{1}{2}\alpha(s)^2\Big)\,ds.$$

For simplicity, we assume that the function ρ is a C^1-function. Apply Theorem 3.6 (with an obvious addition of the t-variable) to the function

$$\theta(t,z,y) = \theta\big(y - k(t)\big)e^z$$

with z for Z_t, y for $B(b) - B(a)$. Then we get the stochastic differential of X_t:

$$\begin{aligned} dX_t &= \alpha(t)X_t\,dB(t) + \beta(t)X_t\,dt \\ &\qquad - \rho'\big(B(b) - B(a) - k(t)\big)\big(k'(t) - \alpha(t)\big)e^{Z_t}\,dt. \end{aligned}$$

This shows that in order for X_t to be a solution of Equation (4.9), we must have $k'(t) = \alpha(t)$. Hence $k(t) = \int_a^t \alpha(s)\,ds + C$. But

$$X_a = \rho\big(B(b) - B(a) - k(a)\big) = \rho\big(B(b) - B(a)\big).$$

Thus $k(a) = C = 0$ and we have $k(t) = \int_a^t \alpha(s)\,ds$. Put this function $k(t)$ into Equation (4.10) to get a solution of Equation (4.9).

Theorem 4.8. *Let $\alpha(t)$ be a deterministic function in $L^2([a,b])$, $\beta(t)$ an adapted stochastic process such that $E\int_a^b |\beta(t)|^2\,dt < \infty$, and ρ a continuous function on $\mathbb{R}$. Then the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t)X_t\,dB(t) + \beta(t)X_t\,dt, \quad a \le t \le b, \\ X_a = \rho\big(B(b) - B(a)\big), \end{cases} \tag{4.11}$$

has a unique solution given by

$$X_t = \rho\Big(B(b) - B(a) - \textstyle\int_a^t \alpha(s)\,ds\Big)\exp\bigg[\displaystyle\int_a^t \alpha(s)\,dB(s) + \int_a^t \Big(\beta(s) - \frac{1}{2}\alpha(s)^2\Big)\,ds\bigg]. \tag{4.12}$$

Proof. We have already shown the existence of a solution in the above discussion. The uniqueness of a solution is obvious, e.g., by the iteration method. □

Remark 4.9. The solution X_t in Equation (4.12) is derived in [11] by somewhat complicated calculations. Observe that the solution X_t depends on the drift term $\beta(t)$ only in the exponential process, which is an Itô part. We also point out that when $\alpha(t)$ is a non-deterministic stochastic process, Equation (4.11) seems to be very hard to solve.

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