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# PROBABILITY DISTRIBUTIONS AND ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE ONE-PARAMETER FIBONACCI GROUP

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Abstract. Starting from $F$, the matrix generating Fibonacci numbers, we find the one-parameter Lie group generated by $F^2$. The matrix elements of the group provide "special functions" identities that include special relationships for Fibonacci and Lucas numbers. The generator of the group provides a self-adjoint operator that we study in the context of quantum probability, recovering an interesting family of binomial distributions involving the golden ratio and a related class of Krawtchouk polynomials.

## 1. Introduction

Solutions to the Fibonacci recurrence

$$x_n = x_{n-1} + x_{n-2}$$

may be found as entries in powers of the companion matrix

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

In particular, the Fibonacci and Lucas sequences arise via appropriate choices of initial values. Many properties of those sequences are related to corresponding properties of the sequence of powers of the matrix $F$.

This point of view of "special functions", following Vilenkin, Klimyk, et al. [8, 9], is that they appear as matrix elements [entries] of Lie groups. To this end, we embed $F^2$ into a one-parameter Lie group, *Fibonacci group*, and consider properties of this group, a subgroup of $SL(2, \mathbb{R})$.

We start with basic properties of the matrix $F$ and continue in the succeeding section with properties of the sequence of powers of the golden ratio, $\varphi$. We present all 2-by-2 symmetric matrix solutions to the basic equation $F^2 = I + F$ to fill out background. Section 5 presents the Lie group generated by $F^2$, starting with the matrix $|F|$ determined as the symmetric matrix with the same spectral resolution as $F$, but with eigenvalues replaced by their absolute values. Section 6 takes a look at the algebra of matrices under consideration by their special form. The

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next section takes a brief look at the corresponding unitary group. The self-adjoint generator, call it $A$, of the Fibonacci group provides a way to bring in quantum probability. This is accomplished by looking at the Lie algebra and corresponding matrix factorization in $SL(2,\mathbb{R})$. After recalling Bochner's Theorem, we determine the distribution of $A$ as a quantum random variable and find a connection with a class of Krawtchouk polynomials involving the golden ratio.

## 2. Basic Properties

Let

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

be the *Fibonacci matrix*. It is easily verified that

$$F^2 = I + F$$

which imply, for $n \geq 1$,

$$F^{n+1} = F^{n-1} + F^n$$

as well as

$$F^{-1} = F - I \ .$$

The following basic features are readily proven:

**Proposition 2.1.** Properties of $F$
*1. For $n \geq 0$,*

$$F^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

*where $F_n$ denotes the $n^{\text{th}}$* Fibonacci number*, with initial conditions $F_{-1} = 1$, $F_0 = 0$.*
*2. The trace* $\operatorname{tr} F^n = F_{n+1} + F_{n-1} = L_n$*, the $n^{\text{th}}$* Lucas number*, and* $\det F^n = (-1)^n$.
*3. $F$ has eigenvalues $\varphi$, the golden ratio equal to $(1+\sqrt{5})/2$, and $-\varphi^{-1}$.*

For an account of Fibonacci and Lucas related matrices we refer to [4, 5].

## 3. Powers of the Golden Ratio

A fundamental role is played by powers of the golden ratio $\varphi$ [7].

**Proposition 3.1.** *1. We have the defining property $\varphi^2 = 1 + \varphi$. Some further relations:*

$$2\varphi - 1 = \sqrt{5} \ , \ \varphi\sqrt{5} = 2 + \varphi = 1 + \varphi^2 \ , \ \varphi^{-1} = -1 + \varphi$$

*2. For $n \geq 0$, we have*

$$\varphi^n = F_{n-1} + F_n\varphi \ , \ (-\varphi^{-1})^n = F_{n+1} - F_n\varphi$$

*Proof.* These follow directly by induction on $n$. □

Writing these in matrix form yields the important spectral result for $F^n$:

**Proposition 3.2.** *We have the spectral decomposition*

$$F^n = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & (-\varphi^{-1})^n \end{pmatrix} \begin{pmatrix} \varphi^{-1} & 1 \\ -1 & \varphi^{-1} \end{pmatrix}$$

*Proof.* By Propositions 2.1 and 3.1 ,

$$F^n \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \varphi^n & (-\varphi^{-1})^{n-1} \\ \varphi^{n+1} & (-\varphi^{-1})^n \end{pmatrix} = \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & (-\varphi^{-1})^n \end{pmatrix}$$

and the result is obtained after multiplying both sides of the above from the right by

$$\begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{-1} & 1 \\ -1 & \varphi^{-1} \end{pmatrix}$$

□

## 4. The Matrix Equation $F^2 - F - I = 0$

**Proposition 4.1.** *The real* $(2 \times 2)$ *symmetric matrices* $F$ *that satisfy the matrix equation* $F^2 - F - I = 0$ *are, up to an orthogonal similarity transformation, of the form*

$$F = \left( \begin{array}{cc} \varphi & 0 \\ 0 & -\varphi^{-1} \end{array} \right) \ or \ \left( \begin{array}{cc} -\varphi^{-1} & 0 \\ 0 & \varphi \end{array} \right) \ or \ \left( \begin{array}{cc} \varphi & 0 \\ 0 & \varphi \end{array} \right) \ or \ \left( \begin{array}{cc} -\varphi^{-1} & 0 \\ 0 & -\varphi^{-1} \end{array} \right)$$

*Proof.* Since $F$ is symmetric, it has real eigenvalues $\lambda_1, \lambda_2$ and there exists an orthogonal $(2 \times 2)$ matrix $P$ such that

$$F = PDP^T \ , \ D = \text{diag}(\lambda_1, \lambda_2)$$

Then

$$F^2 = F + I \Leftrightarrow PD^2P^T = PDP^T + PIP^T$$
$$\Leftrightarrow P\left(D^2 - D - I\right)P^T = \mathbf{0} \Leftrightarrow D^2 - D - I = \mathbf{0}$$

which has the only solutions

$$D = \text{diag}(\varphi, -\varphi^{-1}) \text{ or } \text{diag}(-\varphi^{-1}, \varphi) \text{ or } \text{diag}(\varphi, \varphi) \text{ or } \text{diag}(-\varphi^{-1}, -\varphi^{-1})$$

Moreover, if $F$ is a symmetric matrix solution of the equation $F^2 - F - I = \mathbf{0}$, then any matrix $G := P^T FP$, where $P$ is an orthogonal $(2 \times 2)$ matrix, is also a symmetric solution of the equation since

$$G^2 - G - I = (P^T FP)^2 - P^T FP - I = P^T (F^2 - F - I)F = \mathbf{0}$$

□

*Remark* 4.2. Viewed as linear transformations on the plane, the first two matrices $F$ in Proposition 4.1 are area preserving and orientation reversing since their determinant is equal to $-1$. The other two are orientation preserving, since they have positive determinants, with areas magnified by $\varphi^2$ and $\frac{1}{\varphi^2}$ respectively.

**Example 4.3.** Examples of orthogonal similarity matrices $P$ and the resulting matrix $F$ are:

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

in which case

$$\begin{aligned} F &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \varphi\cos^2\theta - \varphi^{-1}\sin^2\theta & (\varphi+\varphi^{-1})\sin\theta\cos\theta \\ (\varphi+\varphi^{-1})\sin\theta\cos\theta & \varphi\sin^2\theta - \varphi^{-1}\cos^2\theta \end{pmatrix} \end{aligned}$$

and

$$P = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

in which case

$$\begin{aligned} F &= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \varphi\cos^2\theta - \varphi^{-1}\sin^2\theta & (\varphi+\varphi^{-1})\sin\theta\cos\theta \\ (\varphi+\varphi^{-1})\sin\theta\cos\theta & \varphi\sin^2\theta - \varphi^{-1}\cos^2\theta \end{pmatrix} \end{aligned}$$

*Remark* 4.4. We can also find the real $(2 \times 2)$ symmetric matrices $F$ that satisfy the matrix equation $F^2 - F - I = 0$ by reducing to a system of polynomials. If

$$F = \begin{pmatrix} x & y \\ y & w \end{pmatrix}$$

then the matrix equation $F^2 - F - I = 0$ is equivalent to the system

$$-1 - x + x^2 + y^2 = 0 \tag{4.1}$$

$$-y + wy + xy = 0 \tag{4.2}$$

$$-1 - w + w^2 + y^2 = 0 \tag{4.3}$$

For $y = 0$ we find that $x$ and $w$ are either $\varphi$ or $-\varphi^{-1}$ and we get the solution triples

$$(x, y, w) \in \{(\varphi, 0, \varphi), (\varphi, 0, -\varphi^{-1}), (-\varphi^{-1}, 0, -\varphi^{-1}), (-\varphi^{-1}, 0, \varphi)\}$$

For an arbitrary $y \neq 0$, equation (4.1) implies $x = \frac{1 \pm \sqrt{5-y^2}}{2}$ and then (4.2) implies $w = 1 - x = \frac{1 \mp \sqrt{5-y^2}}{2}$ where $y \in [-\sqrt{5}, \sqrt{5}]$.

## 5. Fibonacci Group

We will construct a one-parameter matrix group starting from $F^2$ so that we have a Lie subgroup of $SL(2, \mathbb{R})$. Observe that

$$2 + \varphi = \varphi\sqrt{5}\ ,\ 2 - \varphi^{-1} = \varphi^{-1}\sqrt{5}$$

In other words, the spectrum of the matrix $(2I + F)/\sqrt{5}$ is $\{\varphi, \varphi^{-1}\}$. It satisfies

$$(2I + F)/\sqrt{5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \quad , \quad ((2I + F)/\sqrt{5})^2 = F^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Thus, $(2I + F)/\sqrt{5}$ is the positive definite symmetric square root of $F^2$.

**Proposition 5.1.** *The matrix*

$$|F| := (2I + F)/\sqrt{5}$$

*admits the spectral decomposition* $|F| = PDP^{-1}$ *where*

$$D = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix} \ , \ P = \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix} \ , \ P^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{-1} & 1 \\ -1 & \varphi^{-1} \end{pmatrix}$$

*Proof.* The proof follows, through standard diagonalization, from the fact that $(1, \varphi)^T$ and $(-\varphi, 1)^T$ are eigenvectors of $|F|$ corresponding to its eigenvalues $\varphi$ and $\varphi^{-1}$ respectively and

$$\begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{-1} & 1 \\ -1 & \varphi^{-1} \end{pmatrix}$$

□

For the remainder of this paper, we use the notation

$$\ell = \log \varphi \ .$$

**Theorem 5.2.** *The one-parameter Fibonacci group* $\mathcal{F} = \{|F|^t \, / \, t \in \mathbb{R}\}$ *is a Lie subgroup of* SL(2,ℝ) *of the form*

$$|F|^t = e^{tA}$$

*where the generator* $A$ *is given by*

$$A = \frac{\ell}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{\ell}{\sqrt{5}} (-I + 2F)$$

*The group element* $|F|^t$ *has the explicit form*

$$|F|^t = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{t-1} + \varphi^{1-t} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{1+t} + \varphi^{-(1+t)} \end{pmatrix}$$

*Proof.* In the notation of Proposition 5.1

$$|F|^t = e^{t \log |F|} = e^{t \log (PDP^{-1})} = e^{tA}$$

where

$$A = \log(PDP^{-1}) = P(\log D)P^{-1} = P \begin{pmatrix} \log \varphi & 0 \\ 0 & \log\left(\varphi^{-1}\right) \end{pmatrix} P^{-1}$$

$$= \ell \, P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1} = \frac{\ell}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

We may also compute $A$ as follows:

$$A = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |F|^t = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{t-1} + \varphi^{1-t} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{1+t} + \varphi^{-(1+t)} \end{pmatrix}$$

$$= \frac{\ell}{\sqrt{5}} \begin{pmatrix} \varphi^{-1} - \varphi & 2 \\ 2 & \varphi - \varphi^{-1} \end{pmatrix} = \frac{\ell}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

since $\varphi^2 - 1 = \varphi$. Indeed, for each $n = 0, 1, 2, ...,$

$$A^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi(-\ell)^n + \varphi^{-1}(\ell)^n & (\ell)^n - (-\ell)^n \\ (\ell)^n - (-\ell)^n & \varphi(\ell)^n + \varphi^{-1}(-\ell)^n \end{pmatrix}$$

and summing the exponential series we obtain

$$|F|^t = e^{tA} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{t-1} + \varphi^{1-t} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{1+t} + \varphi^{-(1+t)} \end{pmatrix}$$

We notice that

$$\det(|F|^t) = e^{t \operatorname{tr} A} = e^0 = 1$$

so $|F|^t \in SL(2,\mathbb{R})$ for each $t$. We also have that $I = |F|^0 \in \mathcal{F}$, $|F|^t|F|^s = e^{(t+s)A} \in \mathcal{F}$ and $|F|^{-t} = e^{-tA} \in \mathcal{F}$. Thus $\mathcal{F}$ is a subgroup of $SL(2,\mathbb{R})$ and therefore of $GL(2)$ as well. Moreover, if $(|F|_n^t)_{n\in\mathbb{N}}$, $|F|_n^t = e^{t_n A}$, is a sequence in $\mathcal{F}$ converging to a matrix $\Phi \in GL(2,\mathbb{R})$ of the form $\Phi = e^{t_0 A} \in \mathcal{F}$ where $t_0 = lim_{n\to\infty} t_n$, i.e., $\mathcal{F}$ is a closed matrix subgroup of $GL(2,\mathbb{R})$. By von Neumann's theorem [6] $\mathcal{F}$ is a Lie group. □

*Remark* 5.3. If $\tan\theta = \varphi$ then $\tan 2\theta = -2$. Thus $(1/\ell)A$ is a reflection through a line with slope $\varphi$ passing through the origin.

**5.1. Integer values of $t$.** We recover the powers of $F$ for even integer values of $t$. For odd integer values, it turns out that we find Lucas numbers. In fact

**Proposition 5.4.** *1. For even integer $t = 2m$, we have*

$$|F|^{2m} = \begin{pmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{pmatrix} .$$

*2. For odd integer $t = 2m+1$, we have*

$$|F|^{2m+1} = \frac{1}{\sqrt{5}} \begin{pmatrix} L_{2m} & L_{2m+1} \\ L_{2m+1} & L_{2m+2} \end{pmatrix} .$$

*Proof.* By Theorem 5.2, for $t = 2m$

$$|F|^{2m} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{2m-1} + \varphi^{1-2m} & \varphi^{2m} - \varphi^{-2m} \\ \varphi^{2m} - \varphi^{-2m} & \varphi^{1+2m} + \varphi^{-(1+2m)} \end{pmatrix} = \begin{pmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{pmatrix} .$$

Multiplying $|F|^{2m}$ by $|F|$ we get

$$|F|^{2m+1} = \begin{pmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2F_{2m-1} + F_{2m} & 2F_{2m} + F_{2m+1} \\ 2F_{2m} + F_{2m+1} & F_{2m} + 3F_{2m+1} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} L_{2m} & L_{2m+1} \\ L_{2m+1} & L_{2m+2} \end{pmatrix}$$

because $L_{2m+2} = F_{2m+3} + F_{2m+1} = F_{2m+2} + 2F_{2m+1} = F_{2m} + 3F_{2m+1}$ where we have made repeated use of the defining recursion of the Fibonacci numbers: $F_k = F_{k-1} + F_{k-2}$ and of the Lucas numbers: $L_k = F_{k+1} + F_{k-1}$. □

*Remark* 5.5. In particular, note that the original $|F|$ is indeed

$$|F| = \frac{1}{\sqrt{5}} \begin{pmatrix} L_0 & L_1 \\ L_1 & L_2 \end{pmatrix} .$$

We can recover $F$ as well.

**Proposition 5.6.** *The group $|F|^t$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|F|^t = A|F|^t$$

*and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=1}|F|^t = \ell F \ .$$

*Proof.* The first line is by construction. The second follows by direct calculation. □

For a survey of Fibonacci related groups we refer to [2].

## 6. Algebra Generated by $F$

Notice that all of the matrices involved in this study are of the form

$$\begin{pmatrix} x & y \\ y & x+y \end{pmatrix} = xI + yF$$

They form an algebra over the reals with multiplication

$$(aI + bF)(cI + dF) = (ac + bd)I + (ad + bc + bd)F$$

What is of special interest here is that the continuous parameter family $|F|^t$ are all of this form as well.

**Proposition 6.1.** *Letting*

$$x_t := \frac{1}{\sqrt{5}}(\varphi^{1-t} + \varphi^{t-1}) \qquad \text{and} \qquad y_t := \frac{1}{\sqrt{5}}(\varphi^t - \varphi^{-t}) \ .$$

*we have*

$$|F|^t = x_t I + y_t F$$

*Proof.*

$$x_t I + y_t F = x_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y_t \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x_t & y_t \\ y_t & x_t + y_t \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{1-t} + \varphi^{t-1} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{1-t} + \varphi^{t-1} + \varphi^t - \varphi^{-t} \end{pmatrix}$$

In view of Theorem 5.2, we have to verify the identity

$$\varphi^{1-t} + \varphi^{t-1} + \varphi^t - \varphi^{-t} = \varphi^{t+1} + \varphi^{-(t+1)}$$

Multiplying both sides by $\varphi^t$ we obtain

$$(\varphi^2 - \varphi - 1)(\varphi^{2t} - 1) = 0$$

which is true since $\varphi^2 - \varphi - 1 = 0$. Thus

$$x_t I + y_t F = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{1-t} + \varphi^{t-1} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{t+1} + \varphi^{-(t+1)} \end{pmatrix} = |F|^t$$

□

**Proposition 6.2.** *In the notation of Proposition 6.1, $x_t$ and $y_t$ satisfy the initial condition $x_0 = 1$ , $y_0 = 0$. Moreover, for $t, s \in \mathbb{R}$:*
*1. Group law: $x_{s+t} = x_s x_t + y_s y_t$ , $y_{s+t} = x_s y_t + x_t y_s + y_s y_t$*
*2. Doubling: $x_{2t} = x_t^2 + y_t^2$ , $y_{2t} = 2x_t y_t + y_t^2$*
*3. Inversion: $x_{-t} = x_t + y_t$ , $y_{-t} = -y_t$*

*Proof.* As in the proof of Proposition 6.1

$$|F|^t = \begin{pmatrix} x_t & y_t \\ y_t & x_t + y_t \end{pmatrix}$$

Since $|F|^t = e^{tA}$ we have $|F|^t |F|^s = e^{(t+s)A} = |F|^{t+s}$ which after writing it in matrix form, carrying out the multiplication and equating corresponding entries of the resulting matrices, yields the group law identities. For $s = t$ we obtain the doubling identities and for $t = 0$ the defining formulas for $x_t$ and $y_t$ give $x_0 = (\varphi + \varphi^{-1})/\sqrt{5} = (\varphi^2 + 1)/\sqrt{5} = 1$ and $y_0 = 0$. Finally,

$$x_t + y_t = (\varphi^{1-t} + \varphi^{t-1} + \varphi^t - \varphi^{-t})/\sqrt{5} = (\varphi^{t+1} + \varphi^{-(t+1)})/\sqrt{5} = x_{-t}$$

□

## 6.1. Hyperbolic form.

**Proposition 6.3.** *The Fibonacci group $\mathcal{F}$ has the hyperbolic form*

$$|F|^t = \frac{2}{\sqrt{5}} \begin{pmatrix} \cosh(\ell t - \ell) & \sinh \ell t \\ \sinh \ell t & \cosh(\ell t + \ell) \end{pmatrix}$$

*Proof.* The proof follows from Theorem 5.2 after writing the entries of

$$|F|^t = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{t-1} + \varphi^{1-t} & \varphi^t - \varphi^{-t} \\ \varphi^t - \varphi^{-t} & \varphi^{1+t} + \varphi^{-(1+t)} \end{pmatrix}$$

in terms of the hyperbolic functions. For example,

$$\varphi^{t-1} + \varphi^{1-t} = e^{(t-1)\ell} + e^{(1-t)\ell} = 2\cosh((t-1)\ell)$$

□

*Remark* 6.4. $\mathcal{F}$ is a self-adjoint group of operators over the reals. The determinant is one for all $t$. In view of

$$|F|^t = \begin{pmatrix} x_t & y_t \\ y_t & x_t + y_t \end{pmatrix}$$

and Proposition 6.3, letting $x_t = \frac{2}{\sqrt{5}} \cosh(\ell t - \ell)$ and $y_t = \frac{2}{\sqrt{5}} \sinh \ell t$ the group law, doubling and inverse identities of Proposition 6.2 become:

$$\cosh(\ell(s+t) - \ell) = \frac{2}{\sqrt{5}} \cosh(\ell s - \ell)\cosh(\ell t - \ell) + \frac{2}{\sqrt{5}} \sinh \ell s \sinh \ell t$$

$$\sinh \ell(s+t) = \frac{2}{\sqrt{5}} \cosh(\ell s - \ell)\sinh \ell t + \frac{2}{\sqrt{5}} \cosh(\ell t - \ell)\sinh \ell s + \frac{2}{\sqrt{5}} \sinh \ell s \sinh \ell t$$

$$\cosh(2\ell t - \ell) = \frac{2}{\sqrt{5}} \cosh^2(\ell t - \ell) + \frac{2}{\sqrt{5}} \sinh^2 \ell t$$

$$\sinh 2\ell t = \frac{4}{\sqrt{5}} \cosh(\ell t - \ell)\sinh \ell t + \frac{2}{\sqrt{5}} \sinh^2 \ell t$$

$$\cosh(-t\ell - \ell) = \cosh(\ell t - \ell) + \sinh \ell t$$
$$\sinh \ell(-t) = -\sinh \ell t$$

## 7. Unitary Fibonacci Group

Extending to an algebra over the complex numbers, we have the unitary Fibonacci group $\mathcal{U} = \{|F|^{it} \,/\, t \in \mathbb{R}\}$ where

$$|F|^{it} = e^{itA} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{i(t+i)} + \varphi^{-i(t+i)} & \varphi^{it} - \varphi^{-it} \\ \varphi^{it} - \varphi^{-it} & \varphi^{i(t-i)} + \varphi^{-i(t-i)} \end{pmatrix}$$

**Proposition 7.1.** *The elements of the unitary Fibonacci group $\mathcal{U}$ have the trigonometric form*

$$|F|^{it} = \frac{2}{\sqrt{5}} \begin{pmatrix} \cos(\ell(t+i)) & i\sin(\ell t) \\ i\sin(\ell t) & \cos(\ell(t-i)) \end{pmatrix}$$

*Proof.* The proof follows from Proposition 6.3 after replacing $t$ by $it$ and using the identities $\cosh t = \cos it$ and $\sinh t = -i \sin it$ or, equivalently, $\cosh it = \cos t$ and $\sinh it = i \sin t$. □

*Remark* 7.2. The unitarity property $|F|^{it} \left(|F|^{-it}\right)^T = I$ implies the "Pythagorean Identity"

$$|\cos(\ell(t+i))|^2 + \sin^2 \ell t = \frac{5}{4}\ .$$

*Remark* 7.3. The group law, doubling and inverse identities become:

$$\cos(\ell(s+t) + i\ell) = \frac{2}{\sqrt{5}} \cos(\ell s + i\ell)\cos(\ell t + i\ell) - \frac{2}{\sqrt{5}} \sin \ell s \sin \ell t$$
$$\sin \ell(s+t) = \frac{2}{\sqrt{5}} \cos(\ell s + i\ell)\sin \ell t + \frac{2}{\sqrt{5}} \cos(\ell t + i\ell)\sin \ell s + i\frac{2}{\sqrt{5}} \sin \ell s \sin \ell t$$
$$\cos(2\ell t + i\ell) = \frac{2}{\sqrt{5}} \cos^2(\ell t + i\ell) - \frac{2}{\sqrt{5}} \sin^2 \ell t$$
$$\sin 2\ell t = \frac{4}{\sqrt{5}} \cos(\ell t + i\ell)\sin \ell t + i\frac{2}{\sqrt{5}} \sin^2 \ell t$$
$$\cos(\ell t - i\ell) = \cos(\ell t + i\ell) + i \sin \ell t$$
$$\sin \ell t = -\sin(-\ell t)$$

## 8. Lie Algebra and Group Elements

We use as standard basis for the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$

$$R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

raising, neutral, and lowering operators respectively [3] with commutation relations

$$[R, \rho] = 2R\,,\ [\rho, L] = 2L\,,\ [L, R] = \rho$$

The corresponding $SL(2, \mathbb{R})$ group elements $g$ take the form

$$g = e^{VR} h^{\rho} e^{VL} = \begin{pmatrix} 1 & 0 \\ V & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix}$$

for parameters $V$ and $h$, corresponding to the LDU (Gauss) decomposition.

**Proposition 8.1.** *For the $SL(2,\mathbb{R})$ group element $g = (g_{ij})_{1\le i,j\le 2}$ we recover the parameters $h$ and $V$ by*

$$h = g_{11} \qquad \text{and} \qquad V = g_{12}/g_{11}$$

*Proof.* The proof follows from the fact that

$$g = e^{VR} h^{\rho} e^{VL} = \begin{pmatrix} 1 & 0 \\ V & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h & hV \\ Vh & hV^2 + h^{-1} \end{pmatrix}$$

□

**Theorem 8.2.** *The Fibonacci group element $|F|^t$ has the $SL(2,\mathbb{R})$ form $|F|^t = e^{VR} h^{\rho} e^{VL}$ with*

$$h = h_t = \frac{\varphi^{t-1} + \varphi^{1-t}}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cosh(t\ell - \ell)$$

*and*

$$V = V_t = \frac{\varphi^{t} - \varphi^{-t}}{\varphi^{t-1} + \varphi^{1-t}} = \frac{\sinh t\ell}{\cosh(t\ell - \ell)}$$

*where $\ell = \ell$. We note the "Fibonacci condition" as the relation*

$$h^{-2} = 1 + V - V^2 = (1 + \varphi V)(1 - \varphi^{-1} V) \ .$$

*Proof.* The Fibonacci condition is $g_{11} + g_{12} = g_{22}$. The formulas for $h$ and $V$ follow from the form of $|F|^t$ in Propositions 8.1, 6.3 and in Theorem 5.2, after equating the corresponding entries. □

*Remark* 8.3. Note that at $t = 2$, $h = V = 1$, hence the relation

$$F^2 = e^R e^L$$

easily checked directly from the matrices.

## 9. Bochner's Theorem and Quantum Random Variables

A continuous function $f : \mathbb{R} \mapsto \mathbb{C}$ is *positive definite* if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s) \phi(t) \bar{\phi}(s) \, dt \, ds \ge 0$$

for every continuous function $\phi : \mathbb{R} \mapsto \mathbb{C}$ with compact support. Bochner's theorem (see [10] p. 346) states that such a function can be represented as

$$f(t) = \int_{\mathbb{R}} e^{it\lambda} \, dv(\lambda)$$

where $v$ is a non-decreasing right-continuous bounded function. If $f(0) = 1$ then such a function $v$ defines a probability measure on $\mathbb{R}$ and Bochner's theorem says that $f$ is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable that follows the probability distribution defined by $v$. Moreover, the condition of positive definiteness of $f$ is necessary and sufficient for such a representation.

An example of such a positive definite function is provided by $f(t) = \langle \Omega, e^{itA} \Omega \rangle$ where $\Omega$ is the normalized vacuum vector of a Fock-Hilbert space $\mathcal{F}$ and $A$ is an *observable* (self-adjoint operator on $\mathcal{F}$) also called a *quantum random variable* in which case

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\,dt\,ds = \| \int_{\mathbb{R}} e^{-itA}\phi(t)\,dt\ \Omega\ \|^2 \geq 0$$

In the next section we compute the *vacuum characteristic function* $\langle\Omega, e^{itA}\Omega\rangle$, where $\Omega$ is the vacuum vector of the Fock space associated with the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra [3] and $A$ is the self-adjoint generator of the Fibonacci group.

## 10. Quantum Probability Distribution and Associated Orthogonal Polynomials

**10.1. Distribution of $A$ as a quantum random variable.** To develop the quantum probability connections, we form a GNS/Fock space construction. Let $\Omega$ be a norm one "vacuum vector", i.e., vacuum state, satisfying

$$L\Omega = 0 \qquad \text{and} \qquad \rho\Omega = N\Omega$$

with $N$ a positive integer. The self-adjoint operator $A$ becomes multiplication by the real variable $x$. The vacuum expectation value of the group $e^{zA}$ gives the moment generating function of $A$ interpreted as a random variable. (We use $z$ rather than $t$ here for the group parameter to distinguish it from the time-parameter of the corresponding stochastic process.)

We get

$$\langle\Omega, e^{zA}\Omega\rangle = \langle\Omega, e^{VR}h^{\rho}\Omega\rangle = \langle e^{VL}\Omega, h^N\Omega\rangle = h^N$$

the $N^{\text{th}}$ convolution power of the distribution corresponding to $h$. Now,

$$\begin{aligned} h(z) &= \frac{2}{\sqrt{5}}\cosh(z\ell-\ell) = \frac{e^{\ell z-\ell}}{\sqrt{5}} + \frac{e^{-\ell z+\ell}}{\sqrt{5}} \\ &= \frac{\varphi^{-1}}{\sqrt{5}}e^{\ell z} + \frac{\varphi}{\sqrt{5}}e^{-\ell z} \\ &= pe^{\ell z} + qe^{-\ell z} \end{aligned}$$

which is the moment generating function for a Bernoulli distribution taking values $\pm\ell$ with probabilities

$$p = \frac{\varphi^{-1}}{\sqrt{5}} \qquad \text{and} \qquad q = \frac{\varphi}{\sqrt{5}}$$

respectively. Thus

**Proposition 10.1.** *For integer $N > 0$, the operator $A$ has a binomial distribution with parameters $p$ and $N$.*

We abstract to where the only properties of the operators $R$, $\rho$, and $L$ we use are the commutation relations defining the Lie algebra. The basis states for the representation are

$$\psi_n = R^n\Omega$$

with the actions of $L$ and $\rho$ induced via $\mathfrak{sl}(2,\mathbb{R})$ commutation relations. By induction we find

$$R\psi_n = \psi_{n+1}, \qquad \rho\psi_n = (N-2n)\psi_n, \qquad L\psi_n = n(N+1-n)\psi_{n-1}\ .$$

Now write $A$ in terms of $R, \rho, L$:

$$A = \frac{\ell}{\sqrt{5}}(2R - \rho + 2L)$$

The action of $A$ as multiplication by $x$ gives:

**Proposition 10.2.** *We have the "three-term" recurrence formula*

$$\frac{x\sqrt{5}}{\ell}\psi_n = 2\psi_{n+1} + (2n - N)\psi_n + 2n(N + 1 - n)\psi_{n-1}$$

which hints at orthogonal polynomials.

**10.2. Associated family of orthogonal polynomials.** Our goal is to find the generating function for the basis $\{\psi_n\}_{n\geq 0}$. Start with

$$e^{zA}\Omega = e^{zx}\Omega = e^{VR}h^{\rho}e^{VL}\Omega = h^N e^{VR}\Omega$$

If we solve $v = V(z)$ for $z$ we will have the generating function

$$e^{vR}\Omega = \sum_{n=0}^{\infty} \frac{v^n}{n!}\,\psi_n \ .$$

On the other hand, denoting the function inverse to $V(z)$ by $z = U(v)$,

$$e^{vR}\Omega = h^{-N}e^{zx}\Omega = (1 + v - v^2)^{N/2}e^{xU(v)}\Omega$$

using the relation found above via the Fibonacci condition. Once we have functions of $x$ and $v$ we may drop the $\Omega$'s. Thus

**Theorem 10.3.** *The generating function for the basis $\{\psi_n\}$ has the form*

$$(1 + \varphi v)^{(\frac{x}{2\ell}+\frac{N}{2})}(1 - \varphi^{-1}v)^{(-\frac{x}{2\ell}+\frac{N}{2})}$$

*Proof.* For the factors involving $N$ in the exponent, we use the factorization

$$1 + v - v^2 = (1 + \varphi v)(1 - \varphi^{-1}v) \ .$$

For the factors involving $x$ in the exponent, we must solve for the inverse function $U$. Write

$$V = \frac{\sinh z\ell}{\cosh(z\ell - \ell)} = \frac{e^{z\ell} - e^{-z\ell}}{e^{z\ell-\ell} - e^{\ell-z\ell}} = \frac{e^{2z\ell} - 1}{e^{2z\ell-\ell} + e^{\ell}} = v$$

Solving for $z$ gives

$$\begin{aligned} U(v) &= \frac{1}{2\ell}\,\log\frac{1 + ve^{\ell}}{1 - ve^{-\ell}} \\ &= \frac{1}{2\ell}\,\log\frac{1 + \varphi v}{1 - \varphi^{-1}v} \ . \end{aligned}$$

Exponentiating to $e^{xU(v)}$ and multiplying in the remaining factors yields the result. □

Now consider the random walk moving right or left length $\ell$ for $N$ steps arriving at position $x = k\ell$. We let

$$j = \text{number of steps moving left} = (N-k)/2$$

Note that the number of steps to the right is $(N+k)/2$. Rewriting in terms of $j$, the exponents take the form $N-j$ and $j$ respectively. Thus we have the form, denoting the generating function by $G(v)$

$$G(v) = (1+\varphi v)^{N-j}(1-\varphi^{-1}v)^j \ .$$

Denote the corresponding random variable by $J$. Then $J$ is binomial with distribution

$$\text{Prob}\{J=j\} = \frac{1}{5^{N/2}}\binom{N}{j}(\varphi^{-1})^{N-j}\varphi^j = \frac{1}{5^{N/2}}\binom{N}{j}\varphi^{2j-N} \ .$$

Orthogonality means that the expected value of $G(v)G(w)$ depends only on $vw$. In fact, we calculate

$$\langle G(v)G(w)\rangle = \frac{1}{5^N}((\varphi+\varphi^{-1})(1+vw))^N = (1+vw)^N \ .$$

We recognize the generating function for *Krawtchouk polynomials* which are families of polynomials orthogonal with respect to binomial distributions [1, 3].

## 11. Conclusion

We have shown how to consider Fibonacci and Lucas numbers, related sequences, and the golden ratio as special functions related to a particular Lie subgroup of $SL(2,\mathbb{R})$. One possibility to consider is that the algebra generated by $F$ is an interesting candidate for a parallel study of a "quantum mechanics" alternative to that based on the complex numbers. It would provide an approach different from generalizations based on Clifford algebras, for example.

The quantum probability angle leads to the question of developing a corresponding field theory. To start, the study of tensor powers of $F$ and representations of the associated algebra on symmetric tensor powers (boson Fock space) would provide interesting connections with Boolean algebra that provide many possibilities for further study.

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