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The box-TDI system associated with 2-edge connected spanning subgraphs

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ABSTRACT

Let G be a graph and let A be its cutset-edge incidence matrix. We prove that the linear system $\frac{1}{2}A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box totally dual integral (box-TDI) *if and only if* G is a series-parallel graph; a by-product of this characterization is a structural description of a box-TDI system on matroids. Our results strengthen two previous theorems obtained respectively by Cornuéjols, Fonlupt, and Naddef and by Mahjoub which assert that both polyhedra $\{\mathbf{x} \mid \frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0}\}$ and $\{\mathbf{x} \mid \frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{1} \geq \mathbf{x} \geq \mathbf{0}\}$ are integral if G is series-parallel.

1. Introduction

Let $G = (V, E)$ be a graph. A *tour* of G is an alternating sequence $W = v_0e_0v_1e_1 \ldots v_{k-1}e_{k-1}v_k$ of its vertices and edges, where $v_0 = v_k$, such that $e_i = v_iv_{i+1}$ for all $i < k$ and that each vertex of G appears in W at least once. Given a length function defined on E, the *graphical traveling salesman problem* (GTSP) is to find a tour of G with minimum total length; this problem is a relaxation of the classical TSP where a minimum Hamiltonian cycle is sought. The reader is referred to Cornuéjols, Fonlupt, and Naddef [3] for a variety of motivations for investigating this graphical version. Due to its theoretical interest and practical value, the GTSP has been a subject of extensive research over the past two decades.

For any $S \subseteq V$, we use $\delta(S)$ to denote the set of all edges with one end in S and one end in $V - S$. We call $\delta(S)$ a *cutset* or simply a *cut* if $\emptyset \neq S \neq V$. To each tour W of G we associate an integral vector $\mathbf{x} = (x_e \mid e \in E)$, where x_e is the number of times that edge e occurs in the tour; this $\mathbf{x}$ is naturally called the *incidence vector* of W. Clearly, $\mathbf{x}$ satisfies the following:

$$\bullet\ x_e \text{ is a nonnegative integer for all } e \in E, \quad \textit{and}$$
$$\bullet\ \sum_{e\in\delta(S)} x_e \geq 2 \text{ for every cut } \delta(S) \text{ of } G. \tag{1.1}$$

However, not every vector satisfying (1.1) is an incidence vector of a tour of G. In fact, it is easy to see that a vector $\mathbf{x}$ satisfies (1.1) if and only if replacing each $e \in E$ with x_e parallel edges (e is actually deleted if $x_e = 0$) results in a 2-edge connected graph with vertex set V. (Such a graph $G_\mathbf{x}$ corresponds to a tour of G only when $G_\mathbf{x}$ is Eulerian.)

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Let $P_1(G)$ be the convex hull of incidence vectors of tours of G, and let $P_2(G)$ be the convex hull of all vectors that satisfy (1.1). Let A be the cutset-edge incidence matrix of G. Set $P(G) = \{\mathbf{x} \mid \frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{0}$ is the zero vector and $\mathbf{1}$ is the all-one vector. The polyhedron $P(G)$ plays an important role in various polyhedral approaches to the GTSP; it is also closely related to several network design problems (see, for instances, [9–13]). Clearly, $P_1(G) \subseteq P_2(G) \subseteq P(G)$, and the classical traveling salesman polytope (the convex hull of the incidence vectors of all Hamiltonian cycles) is a subset of $P_1(G)$. Moreover, observe that $P_2(G)$ is actually the convex hull of incidence vectors of 2-edge connected spanning subgraphs of G (with edge repetition allowed), and the integral vectors in $P(G)$ are precisely those satisfying (1.1).

Let us introduce two notions before presenting known results concerning $P_1(G)$, $P_2(G)$, and $P(G)$. A *minor* of a graph G is a graph obtained from a subgraph of G by contracting edges. A graph is called *series-parallel* if it is constructed from a forest by repeatedly adding loops and making series- and parallel-extensions (meaning replacing an edge by two edges in series or in parallel). It is well known that a graph is series-parallel if and only if it does not contain K_4 (the complete graph on four vertices) as a minor [5].

Graphs G with $P_1(G) = P(G)$ were characterized by Fonlupt and Naddef [8] in terms of excluded minors. In [3], Cornuéjols, Fonlupt, and Naddef suggested characterizing the class of graphs G for which $P(G)$ is integral, or equivalently, $P_2(G) = P(G)$. They showed that this class contains all *series-parallel graphs*.

Theorem A (*[3]*). *If G is a series-parallel graph, then $P(G)$ is an integral polyhedron.*

For a complete characterization of all graphs G with integral $P(G)$, see the recent work [4] of Ding and Zang. Notice that integral vectors in $P(G)$, or equivalently, vectors that satisfy (1.1), do not have to be 0–1 vectors, since they may have coordinates exceeding one. This could cause problems in certain applications. Clearly, a natural fix for this problem is to impose the *box condition* $\mathbf{1} \geq \mathbf{x} \geq \mathbf{0}$. This is what Mahjoub did in [11] and [12], where he investigated the convex hull of incidence vectors of 2-edge connected spanning subgraphs of G (with no edge repetition), and obtained the following.

Theorem B (*[11]*). *If G is a series-parallel graph, then $Q(G) = \{\mathbf{x} \mid \frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{1} \geq \mathbf{x} \geq \mathbf{0}\}$ is an integral polytope.*

The purpose of this paper is find a common strengthening of Theorems A and B. Let us begin with some definitions. A rational linear system $C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}$ is called *totally dual integral* (TDI), as introduced by Edmonds and Giles [6], if the maximization in the LP-duality equation

$$\min\{\mathbf{w}^{\mathrm{T}}\mathbf{x} \mid C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\} = \max\{\mathbf{y}^{\mathrm{T}}\mathbf{d} \mid \mathbf{y}^{\mathrm{T}}C \leq \mathbf{w}^{\mathrm{T}},\ \mathbf{y} \geq \mathbf{0}\} \tag{1.2}$$

has an integral optimal solution $\mathbf{y}$ for every integral vector $\mathbf{w}$ for which the maximum is finite. Furthermore, system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is called *box totally dual integral* (box-TDI) if the system $C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{x} \geq \mathbf{l}$ is TDI for all rational vectors $\mathbf{u}$ and $\mathbf{l}$, where coordinates of $\mathbf{u}$ are allowed to be $+\infty$.

We call a graph G *good* if the system $\frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0}$ is box-TDI. If G is disconnected then A has a zero-row and thus G is trivially good. The following theorem, our main result in this paper, gives a characterization of all connected good graphs.

Theorem 1.1. *A connected graph is good if and only if it is series-parallel.*

Edmonds and Giles [6] established that if $C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}$ is a TDI system and $\mathbf{d}$ is integral, then the polyhedron $\{\mathbf{x} \mid C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\}$ is integral. According to Theorem 1.1, if G is a series-parallel graph then the system $\frac{1}{2}A\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{x} \geq \mathbf{l}$ is TDI for all rational vectors $\mathbf{u}$ and $\mathbf{l}$, and hence it defines an integral polyhedron whenever $\mathbf{u}$ and $\mathbf{l}$ are integral by the above Edmonds–Giles theorem; in particular, both $P(G)$ and $Q(G)$ are integral, which demonstrates that Theorem 1.1 is a common strengthening of Theorems A and B. Since there are big differences between systems that define an integral polyhedron, that are TDI, and that are box-TDI (cf. [15,16] for a comprehensive and in-depth treatment of these topics), Theorem 1.1 strengthens both Theorems A and B in a significant way.

Theorem 1.1 also yields a structural description of a box-TDI system associated with the circuit covering problem on matroids. To help readers who are not familiar with matroid theory we provide a very brief introduction. We refer the reader to Oxley [14] for an in-depth treatment of this topic. Basically, a *matroid* M is a pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a set of subsets of E that satisfies certain axioms (see Section 1.1 of [14]). Members of E and $\mathcal{C}$ are called *elements* and *circuits* of M, respectively. Every graph $G = (V, E)$ defines a matroid $M = (E, \mathcal{C})$, where $\mathcal{C}$ consists of edge sets of all cycles of G. This matroid is denoted by $\mathcal{M}(G)$ and every matroid of this kind is called a *graphic* matroid. Another example of matroid is *uniform* matroid $U_{n,r} = (E, \mathcal{C})$, where $|E| = n$ and $\mathcal{C}$ consists of all subsets of E of size $r + 1$. For every matroid M there is another matroid, denoted by M^*, on the same set of elements, which is called the *dual* of M. The dual of a graphic matroid $\mathcal{M}(G)$ is the matroid $\mathcal{M}^*(G) = (E, \mathcal{D})$, where $\mathcal{D}$ consists of all nonempty minimal cuts of G. The dual of a graphic matroid is called a *cographic* matroid. If e is an element of a matroid M, then $M \setminus e$ and M/e are matroids on $E - \{e\}$, which are called the *deletion* and *contraction* of e, respectively. When $M = \mathcal{M}(G)$ is graphic, $M \setminus e = \mathcal{M}(G \setminus e)$ and $M/e = \mathcal{M}(G/e)$; when $M = \mathcal{M}^*(G)$ is cographic, $M \setminus e = \mathcal{M}^*(G/e)$ and $M/e = \mathcal{M}^*(G \setminus e)$. A *minor* of a matroid M is a matroid obtained from M by a sequence of deletions and contractions.

Let M be a matroid and let A be the circuit-element incidence matrix of M. We call M *good* if the linear system $\frac{1}{2}A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box-TDI. Based on Theorem 1.1, we can easily characterize all good matroids in terms of excluded minors.

Theorem 1.2. *A matroid is good if and only if none of its minors is isomorphic to $U_{2,4}$ or $\mathcal{M}(K_4)$.*

Now we outline the rest of the paper. In Section 2 we present some basic facts on TDI and box-TDI systems. In particular, we show that the property of being a box-TDI system is preserved under the operations of "deletion" and "contraction" of columns. When restricted to our special systems, these general results guarantee that minors of connected good graphs are still good. In the next two sections we show that the property of being a box-TDI system is also preserved under the operations of "series-" and "parallel-extensions" of columns. Again, these are results on general box-TDI systems. In the last section, we turn to graphs and matroids and prove the two main theorems listed above using results established in previous sections. Our proof will heavily rely on a result of Cook [2] (stated later in Theorem 2.2), which characterizes box-TDI systems. It is worth pointing out that almost all known box-TDI systems can be verified via totally unimodular matrices (cf. Section 5.20 of [16], where Cook's theorem is not even mentioned), but our system is an exception.

We close this section by clarifying our terminology. For any set X and any finite set K, let X^K denote the set of all X-valued vectors $\mathbf{x} = (x_k \mid k \in K)$ that are indexed by K. For any $\mathbf{x} \in X^K$ and $J \subseteq K$, the $|J|$-dimensional vector $\mathbf{x}|_J = (x_j \mid j \in J)$ stands for the restriction of $\mathbf{x}$ to J, which is in fact the projection of $\mathbf{x}$ on X^J. As customary, let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ denote the sets of real numbers, rational numbers, and integers, respectively. In addition, let $\mathbb{R}_+$, $\mathbb{Q}_+$, and $\mathbb{Z}_+$ denote the sets of nonnegative numbers in the corresponding sets. For any $\mathbf{x} = (x_k \mid k \in K) \in \mathbb{R}^K$, we use $\lfloor \mathbf{x} \rfloor$ and $\lceil \mathbf{x} \rceil$ as shorthands for the vectors $(\lfloor x_k \rfloor \mid k \in K)$ and $(\lceil x_k \rceil \mid k \in K)$, respectively.

2. TDI and box-TDI systems

In this section we present some basic results concerning TDI and box-TDI systems. The first is the classical result of Edmonds and Giles which says that total dual integrality implies primal integrality (cf. Corollary 22.1b of Schrijver [15]).

Theorem 2.1 (*[6]*). *Let $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ be a TDI system. If $\mathbf{d}$ is integral and the optimal value of (1.2) is finite, then the minimization problem also has an integral optimal solution.*

The next is a characterization of box-TDI systems due to Cook (cf. Corollary 22.9a of Schrijver [15]); our proof heavily relies on this important theorem.

Theorem 2.2 (*[2]*). *A rational system $C\mathbf{x} \geq \mathbf{d}, \mathbf{x} \geq \mathbf{0}$ defined on $\mathbb{R}^n$ is box-TDI if and only if it is TDI and for every $\mathbf{w} \in \mathbb{Q}^n$ there exists $\tilde{\mathbf{w}} \in \mathbb{Z}^n$ such that $\lfloor \mathbf{w} \rfloor \leq \tilde{\mathbf{w}} \leq \lceil \mathbf{w} \rceil$ and such that each optimal solution of $\min\{\mathbf{w}^{\mathrm{T}}\mathbf{x} \mid C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\}$ is also an optimal solution of $\min\{\tilde{\mathbf{w}}^{\mathrm{T}}\mathbf{x} \mid C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\}$.*

The following two lemmas follow immediately from the definition of box-TDI systems.

Lemma 2.1. *Let C' be obtained from C by adding a zero-column. If system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI, then so is the system $C'\mathbf{x}' \geq \mathbf{d}$, $\mathbf{x}' \geq \mathbf{0}$.*

Lemma 2.2. *Suppose $\mathbf{c}_1$ and $\mathbf{c}_2$ are rational vectors with $\mathbf{c}_1 \leq \mathbf{c}_2$, and d_1 and d_2 are rational numbers with $d_1 \geq d_2$. Then the system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{c}_1^{\mathrm{T}}\mathbf{x} \geq d_1$, $\mathbf{c}_2^{\mathrm{T}}\mathbf{x} \geq d_2$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI if and only if the system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{c}_1^{\mathrm{T}}\mathbf{x} \geq d_1$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI.*

Suppose the rows and columns of C are indexed by disjoint sets R and S, respectively. For any $s \in S$, let C/s be the matrix obtained from C by deleting the column indexed by s; let $C \setminus s$ be the matrix obtained from C by deleting column s and all rows r for which $C_{r,s} \neq 0$. The following simple fact was observed in [15], on page 323.

Lemma 2.3. *If $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI and $C' = C/s$, then $C'\mathbf{x}' \geq \mathbf{d}$, $\mathbf{x}' \geq \mathbf{0}$ is also box-TDI.*

Similarly, $C' = C \setminus s$ also defines a box-TDI system, as shown by the following lemma.

Lemma 2.4. *If $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI, $C' = C \setminus s$, and all entries of C are nonnegative, then $C'\mathbf{x}' \geq \mathbf{d}'$, $\mathbf{x}' \geq \mathbf{0}$ is also box-TDI, where $\mathbf{d}' = \mathbf{d}|_{R'}$ with $R' = \{r : C_{r,s} = 0\}$.*

Proof. Let $S' = S \setminus \{s\}$. We aim to show that the system $C'\mathbf{x}' \geq \mathbf{d}'$, $\mathbf{u}' \geq \mathbf{x}' \geq \mathbf{l}'$, $\mathbf{x}' \geq \mathbf{0}$ is TDI for all $\mathbf{u}' \in (\mathbb{Q} \cup \{+\infty\})^{S'}$ and $\mathbf{l}' \in \mathbb{Q}^{S'}$. For this purpose, let $\mathbf{w}'$ be an arbitrary vector in $\mathbb{Z}^{S'}$ such that the maximum (and hence the optimal value) of the following LP-duality equation

$$\min\left\{ (\mathbf{w}')^{\mathrm{T}}\mathbf{x}' \,\middle|\, \begin{bmatrix} C' \\ I \\ -I \end{bmatrix} \mathbf{x}' \geq \begin{bmatrix} \mathbf{d}' \\ \mathbf{l}' \\ -\mathbf{u}' \end{bmatrix} \mathbf{x}' \geq \mathbf{0} \right\} = \max\left\{ (\mathbf{y}')^{\mathrm{T}} \begin{bmatrix} \mathbf{d}' \\ \mathbf{l}' \\ -\mathbf{u}' \end{bmatrix} \,\middle|\, (\mathbf{y}')^{\mathrm{T}} \begin{bmatrix} C' \\ I \\ -I \end{bmatrix} \leq (\mathbf{w}')^{\mathrm{T}}, \mathbf{y}' \geq \mathbf{0} \right\} \tag{2.1}$$

is finite. To verify that the maximization problem in (2.1) has an integral optimal solution, we define $\mathbf{u} \in (\mathbb{Q} \cup \{+\infty\})^S$, $\mathbf{l} \in \mathbb{Q}^S$, and $\mathbf{w} \in \mathbb{Z}^S$ such that

(1) $\mathbf{u}|_{S'} = \mathbf{u}'$, $\mathbf{l}|_{S'} = \mathbf{l}'$, $\mathbf{w}|_{S'} = \mathbf{w}'$, $l_s = 0$, $u_s = +\infty$, and $w_s = 0$, and we consider the following primal-dual pair on C

$$\min\left\{\mathbf{w}^{\mathrm{T}}\mathbf{x}\,\middle|\,\begin{bmatrix} C \\ I \\ -I \end{bmatrix}\mathbf{x} \geq \begin{bmatrix} \mathbf{d} \\ \mathbf{l} \\ -\mathbf{u} \end{bmatrix}, \mathbf{x} \geq \mathbf{0}\right\} = \max\left\{\mathbf{y}^{\mathrm{T}}\begin{bmatrix} \mathbf{d} \\ \mathbf{l} \\ -\mathbf{u} \end{bmatrix}\,\middle|\,\mathbf{y}^{\mathrm{T}}\begin{bmatrix} C \\ I \\ -I \end{bmatrix} \leq \mathbf{w}^{\mathrm{T}}, \mathbf{y} \geq \mathbf{0}\right\}. \tag{2.2}$$

In the following, we refer to the four problems in (2.1) and (2.2) as (2.1)-Min, (2.1)-Max, (2.2)-Min, and (2.2)-Max, respectively. We will deduce the integrality of (2.1)-Max from the integrality of (2.2)-Max. To do so, we first claim that

(2) the two problems (2.1)-Min and (2.2)-Min have the same optimal value.

To establish the statement, for any feasible solution $\mathbf{x}'$ to (2.1)-Min, let $\mathbf{x} \in \mathbb{R}^S$ be the vector so that $\mathbf{x}|_{S'} = \mathbf{x}'$ and that $x_s = \max\{0, \max\{\frac{d_r}{C_{r,s}} \mid r \in R - R'\}\}$. From (1), the nonnegativity of C, and the definition of C' we see that $\mathbf{x}$ is a feasible solution to (2.2)-Min with $\mathbf{w}^{\mathrm{T}}\mathbf{x} = (\mathbf{w}')^{\mathrm{T}}\mathbf{x}'$. It follows that not only (2.2)-Min has at least one feasible solution, but also its optimal value, if it exists, is at most that of (2.1)-Min.

On the other hand, for any feasible solution $\mathbf{x}$ to (2.2)-Min, set $\mathbf{x}' = \mathbf{x}|_{S'}$. From (1) we deduce that $\mathbf{x}'$ is a feasible solution to (2.1)-Min with $(\mathbf{w}')^{\mathrm{T}}\mathbf{x}' = \mathbf{w}^{\mathrm{T}}\mathbf{x}$. Since (2.1)-Min has a finite minimum, it follows that (2.2)-Min has a finite minimum and this minimum is at least that of (2.1)-Min. Combining this conclusion with the conclusion obtained from the last paragraph, we get (2).

Since system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI and since the optimal value of (2.2) is finite by (2), (2.2)-Max has an integral optimal solution $\mathbf{y}^*$ by the definition of a TDI system. Let

- y_r be the coordinate of $\mathbf{y}^*$ corresponding to constraint $\sum_{j\in S} C_{r,j}x_j \geq d_r$, for each $r \in R$,
- y_t be the coordinate of $\mathbf{y}^*$ corresponding to constraint $x_t \geq l_t$, for each $t \in S$, and
- y_{-t} be the coordinate of $\mathbf{y}^*$ corresponding to constraint $-x_t \geq -u_t$, for each $t \in S$.

We proceed with the construction of an integral optimal solution $(\mathbf{y}')^*$ to (2.1)-Max:

Set $y'_r = y_r$ for every $r \in R'$; set $y'_t = y_t$ and $y'_{-t} = y_{-t}$ for every $t \in S'$.

From $\mathbf{w}' = \mathbf{w}|_{S'}$ and $\mathbf{y}^* \geq \mathbf{0}$, we see that $(\mathbf{y}')^*$ is feasible to (2.1)-Max. Moreover, since $u_s = +\infty$, constraint $-x_s \geq -u_s$ does not appear in (2.2) and so automatically $y_{-s} = 0$. It follows from this observation, the choice $w_s = 0$, and the nonnegativity of C that $y_s = 0$ and $y_r = 0$ for every $r \in R - R'$. Therefore, we deduce from (1) that $((\mathbf{d}')^{\mathrm{T}}, (\mathbf{l}')^{\mathrm{T}}, -(\mathbf{u}')^{\mathrm{T}})(\mathbf{y}')^* = (\mathbf{d}^{\mathrm{T}}, \mathbf{l}^{\mathrm{T}}, -\mathbf{u}^{\mathrm{T}})\mathbf{y}^*$. By (2), $(\mathbf{y}')^*$ is an integral optimal solution to (2.1)-Max, which proves the lemma. ■

3. Series extension

Let C be a matrix and let C' be obtained from C by adding a new column equal to an existing one. Then C' is called a *series extension* of C. This operation is so named because when C is the circuit-element incidence matrix of a matroid, the new matrix C' is exactly the circuit-element incidence matrix of a series extension of M. However, we point out that when we apply this operation to the cutset-edge matrix of a graph G, we are applying it to the cographic matroid $\mathcal{M}^*(G)$. Since a series extension in the dual of G is a parallel extension in G, it follows that C' turns out to be the cutset-edge matrix of a parallel extension of G. Similarly, in the next section, we will define the parallel extension of a matrix, which, when applied to cutset-edge matrix of G, will correspond to a series extension of G.

The next lemma follows easily from the definition of TDI systems.

Lemma 3.1. *If the system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is TDI and C' is a series extension of C, then the system $C'\mathbf{x}' \geq \mathbf{d}$, $\mathbf{x}' \geq \mathbf{0}$ is also TDI.*

Next, we prove an analogous result for box-TDI systems.

Lemma 3.2. *If the system $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ is box-TDI and C' is a series extension of C, then the system $C'\mathbf{x}' \geq \mathbf{d}$, $\mathbf{x}' \geq \mathbf{0}$ is also box-TDI.*

Remark. This lemma is an analog of a result of Edmonds and Giles [7] on repeating variables (cf. Theorem 22.10 of Schrijver [15]). We point out that our result does not follow from that of [7], and our proof, like that in [15], uses Theorem 2.2, a result of Cook [2].

Proof. Suppose S is the set of indices of columns of C and suppose $S' = S \cup \{t\}$, where t is the index of the new column of C', which equals a column of C with index, say, s. We will apply Theorem 2.2 to both systems $C\mathbf{x} \geq \mathbf{d}$, $\mathbf{x} \geq \mathbf{0}$ and $C'\mathbf{x}' \geq \mathbf{d}$, $\mathbf{x}' \geq \mathbf{0}$ with various objective functions. To unify our terminology, let $\boldsymbol{\alpha}' \in \mathbb{Q}^{S'}$ with $\alpha'_s \leq \alpha'_t$. Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}'|_S$. We consider the following two minimization problems:

$$\min\{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x} \mid C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\} \quad \text{and} \quad \min\{(\boldsymbol{\alpha}')^{\mathrm{T}}\mathbf{x}' \mid C'\mathbf{x}' \geq \mathbf{d},\ \mathbf{x}' \geq \mathbf{0}\},$$

which we denote by $\mathrm{Min}(C, \boldsymbol{\alpha})$ and $\mathrm{Min}(C', \boldsymbol{\alpha}')$, respectively. We first make a few observations.

(1) Suppose $\mathrm{Min}(C', \boldsymbol{\alpha}')$ has an optimal solution. Then
 (a) $\mathrm{Min}(C', \boldsymbol{\alpha}')$ has an optimal solution $\mathbf{x}'$ with $x'_t = 0$;

(b) if $\alpha'_s < \alpha'_t$, then every optimal solution $\mathbf{x}'$ to $\mathrm{Min}(C', \boldsymbol{\alpha}')$ satisfies $x'_t = 0$.

These observations follow from the fact that if $\mathbf{x}'$ is a feasible solution to $\mathrm{Min}(C', \boldsymbol{\alpha}')$ with $x'_t > 0$ then decreasing x'_t by a sufficiently small $\varepsilon > 0$ and increasing x'_s by the same ε would result in a new feasible solution $\bar{\mathbf{x}}'$ to $\mathrm{Min}(C', \boldsymbol{\alpha}')$ for which $(\boldsymbol{\alpha}')^{\mathrm{T}}\bar{\mathbf{x}}' = (\boldsymbol{\alpha}')^{\mathrm{T}}\mathbf{x}' + \varepsilon(\alpha'_s - \alpha'_t) \leq (\boldsymbol{\alpha}')^{\mathrm{T}}\mathbf{x}'$.

(2) $\mathbf{x}$ is an optimal solution to $\mathrm{Min}(C, \boldsymbol{\alpha})$ if and only if $\mathbf{x}'$ is an optimal solution to $\mathrm{Min}(C', \boldsymbol{\alpha}')$, where $\mathbf{x}'|_S = \mathbf{x}$ and $x'_t = 0$.

Clearly, $\mathbf{x}$ is an optimal solution to $\mathrm{Min}(C, \boldsymbol{\alpha})$ if and only if $\mathbf{x}'$ is an optimal solution to the problem $\min\{(\boldsymbol{\alpha}')^{\mathrm{T}}\mathbf{x}' \mid C'\mathbf{x}' \geq \mathbf{d},\ \mathbf{x}' \geq \mathbf{0},\ x'_t = 0\}$. Then we deduce (2) from (1a) immediately.

The following is another obvious observation.

(3) Suppose $\alpha'_s = \alpha'_t$. Suppose $\mathbf{x}', \hat{\mathbf{x}}' \in \mathbb{R}_+^{S'}$ with $\mathbf{x}'|_{S\setminus\{s\}} = \hat{\mathbf{x}}'|_{S\setminus\{s\}}$ and $x'_s + x'_t = \hat{x}'_s + \hat{x}'_t$. Then $\mathbf{x}'$ is an optimal solution to $\mathrm{Min}(C', \boldsymbol{\alpha}')$ if and only if $\hat{\mathbf{x}}'$ is an optimal solution to $\mathrm{Min}(C', \boldsymbol{\alpha}')$.

Now we start proving the lemma. By Theorem 2.2, $C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}$ is TDI. Thus, by Lemma 3.1, $C'\mathbf{x}' \geq \mathbf{d},\ \mathbf{x}' \geq \mathbf{0}$ is TDI. Next, let $\mathbf{w}' \in \mathbb{Q}^{S'}$. By symmetry, we may assume $w'_s \leq w'_t$. Let $\mathbf{w} = \mathbf{w}'|_S$. Since $C\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}$ is box-TDI, by Theorem 2.2, there exists $\tilde{\mathbf{w}}$ such that $\lfloor\mathbf{w}\rfloor \leq \tilde{\mathbf{w}} \leq \lceil\mathbf{w}\rceil$ and such that each optimal solution to $\mathrm{Min}(C, \mathbf{w})$ is also an optimal solution to $\mathrm{Min}(C, \tilde{\mathbf{w}})$. Let us define $\tilde{\mathbf{w}}'$ such that $\tilde{\mathbf{w}}'|_S = \tilde{\mathbf{w}}$, and $\tilde{w}'_t = \lceil w'_t\rceil$ if $w'_s < w'_t$ and $\tilde{w}'_t = \tilde{w}'_s$ if $w'_s = w'_t$. Clearly, $\tilde{w}'_s \leq \tilde{w}'_t$ and $\lfloor\mathbf{w}'\rfloor \leq \tilde{\mathbf{w}}' \leq \lceil\mathbf{w}'\rceil$. It remains to show that every optimal solution $\mathbf{x}'$ to $\mathrm{Min}(C', \mathbf{w}')$ is also an optimal solution to $\mathrm{Min}(C', \tilde{\mathbf{w}}')$.

Case 1. Suppose $x'_t = 0$. Then, by (2), $\mathbf{x}$ is an optimal solution to $\mathrm{Min}(C, \mathbf{w})$. It follows that $\mathbf{x}$ is an optimal solution to $\mathrm{Min}(C, \tilde{\mathbf{w}})$, which implies, by (2) again, that $\mathbf{x}'$ is an optimal solution to $\mathrm{Min}(C', \tilde{\mathbf{w}}')$.

Case 2. Suppose $x'_t > 0$. By (1b), $w'_s = w'_t$. Let $\hat{\mathbf{x}}' \in \mathbb{R}_+^{S'}$ with $\hat{\mathbf{x}}'|_{S\setminus\{s\}} = \mathbf{x}'|_{S\setminus\{s\}}$, $\hat{x}'_s = x'_s + x'_t$, and $\hat{x}'_t = 0$. By (3), $\hat{\mathbf{x}}'$ is an optimal solution to $\mathrm{Min}(C', \mathbf{w}')$. Then by Case 1, $\hat{\mathbf{x}}'$ is an optimal solution to $\mathrm{Min}(C', \tilde{\mathbf{w}}')$. Since $\tilde{w}'_t = \tilde{w}'_s$, we deduce from (3) again that $\mathbf{x}'$ is an optimal solution to $\mathrm{Min}(C', \tilde{\mathbf{w}}')$. ■

4. Parallel extension

$$\text{Let } C = \begin{bmatrix} C_1 & \frac{1}{2}\mathbf{1} \\ C_2 & \mathbf{0} \end{bmatrix}.$$

$$\text{Then } C' = \begin{bmatrix} \mathbf{0}^{\mathrm{T}} & \frac{1}{2} & \frac{1}{2} \\ C_1 & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ C_1 & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ C_2 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is called a *parallel extension* of C. In the following discussions, we denote the sets of indices of rows and columns of C by R and S, respectively. Moreover, let $s \in S$ be the index of the last column of C; let R be partitioned into R_1 and R_2, according to the partition of C. Let $S' = S \cup \{t\}$ be the set of indices of columns of C', where t is the index of the last column of C'; let $R' = \{r_0\} \cup R_1 \cup R'_1 \cup R_2$ be the set of indices of rows of C', where r_0 is the index of the first row of C' and $R'_1 = \{r' | r \in R_1\}$, which is a copy of R_1.

The goal of this section is to prove the following key lemma in this paper.

Lemma 4.1. *If $C\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box-TDI, then so is $C'\mathbf{x}' \geq \mathbf{1}, \mathbf{x}' \geq \mathbf{0}$.*

Since we will apply Theorem 2.2 in our proof to both systems $C\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ and $C'\mathbf{x}' \geq \mathbf{1}, \mathbf{x}' \geq \mathbf{0}$ with various objective functions, we introduce the following notations to unify our terminology. For any matrix D and vector $\boldsymbol{\alpha}$ of appropriate dimensions, we denote the two linear programming problems

$$\min\{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x} \mid D\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0}\} \quad \text{and} \quad \max\{\mathbf{y}^{\mathrm{T}}\mathbf{1} \mid \mathbf{y}^{\mathrm{T}}D \leq \boldsymbol{\alpha}^{\mathrm{T}},\ \mathbf{y} \geq \mathbf{0}\}$$

by $\mathrm{Min}(D, \boldsymbol{\alpha})$ and $\mathrm{Max}(D, \boldsymbol{\alpha})$, respectively. Let

$$\begin{aligned} &\text{(a) } \mathbf{w}' \in \mathbb{Q}^{S'} \text{ with } w'_s \geq w'_t; \\ &\text{(b) } \mathbf{y}' \in \mathbb{Q}_+^{R'} \text{ be an optimal solution of } \mathrm{Max}(C', \mathbf{w}'); \\ &\text{(c) } \mathbf{w} \in \mathbb{Q}^S \text{ such that } \mathbf{w}|_{S\setminus\{s\}} = \mathbf{w}'|_{S\setminus\{s\}} \text{ and } w_s = w'_s + w'_t - y'_{r_0}. \end{aligned} \tag{4.1}$$

We will use these terminology in the following lemmas, which are the technical parts in our proof of Lemma 4.1. To better understand the main idea of our proof, the reader is recommended to browse the last part of our proof (at the end of this section) where dots are connected and the whole picture emerges. For logical reasons, we have to place this part after the lemmas.

Lemma 4.2. *Let* $\tilde{w}_s \in \{\lfloor w_s \rfloor, \lceil w_s \rceil\}$. *Then there exist* $\tilde{w}'_s \in \{\lfloor w'_s \rfloor, \lceil w'_s \rceil\}$ *and* $\tilde{w}'_t \in \{\lfloor w'_t \rfloor, \lceil w'_t \rceil\}$ *such that* $\tilde{w}'_s + \tilde{w}'_t \geq \tilde{w}_s$ *and* $|\tilde{w}'_s - \tilde{w}'_t| \leq \tilde{w}_s$. *In addition,* $\tilde{w}'_s$ *and* $\tilde{w}'_t$ *can be chosen with the following extra property:*

(a) $\tilde{w}'_s + \tilde{w}'_t = \tilde{w}_s$ *if* $y'_{r_0} = 0$, *and*

(b) $\tilde{w}'_s - \tilde{w}'_t = \tilde{w}_s$ *if* $w'_t = \frac{1}{2}y'_{r_0}$.

Remark. If $w'_t = \frac{1}{2}y'_{r_0} = 0$ then $0 = \lfloor w'_t \rfloor \leq \tilde{w}'_t \leq \lceil w'_t \rceil = 0$ and so there is no conflict between (a) and (b).

Proof. The feasibility of $\mathbf{y}'$ implies $\mathbf{y}' \geq \mathbf{0}$, $\frac{1}{2}y'_{r_0} + \frac{1}{2}\sum_{r \in R_1} y'_r \leq w'_s$, and $\frac{1}{2}y'_{r_0} + \frac{1}{2}\sum_{r' \in R'_1} y'_{r'} \leq w'_t$. So

(1) $w'_s \geq w'_t \geq 0$ and $w_s \geq 0$.

If $\tilde{w}_s < \lceil w'_t \rceil - \lfloor w'_s \rfloor$ then $\tilde{w}_s + \lfloor w'_s \rfloor \leq \lceil w'_t \rceil - 1 \leq \lceil w'_s \rceil - 1 \leq \lfloor w'_s \rfloor$, and so $\tilde{w}_s = 0$ (by (1)) and $\lceil w'_t \rceil = \lceil w'_s \rceil$. Set $\tilde{w}'_s = \lceil w'_s \rceil$ and $\tilde{w}'_t = \lceil w'_t \rceil$. Then $|\tilde{w}'_s - \tilde{w}'_t| \leq \tilde{w}_s \leq \tilde{w}'_s + \tilde{w}'_t$ (by (1)), as required.

If $\tilde{w}_s > \lceil w'_t \rceil + \lfloor w'_s \rfloor$ then $\lceil w'_t \rceil + \lfloor w'_s \rfloor + 1 \leq \tilde{w}_s \leq \lceil w_s \rceil \leq \lceil w'_s \rceil + \lceil w'_t \rceil \leq \lceil w'_t \rceil + \lfloor w'_s \rfloor + 1$, which implies $\tilde{w}_s = \lceil w'_s \rceil + \lceil w'_t \rceil$. Set $\tilde{w}'_s = \lceil w'_s \rceil$ and $\tilde{w}'_t = \lceil w'_t \rceil$. Then $|\tilde{w}'_s - \tilde{w}'_t| \leq \tilde{w}_s \leq \tilde{w}'_s + \tilde{w}'_t$ (by (1)), as required.

It remains to consider the case when $\lceil w'_t \rceil - \lfloor w'_s \rfloor \leq \tilde{w}_s \leq \lceil w'_t \rceil + \lfloor w'_s \rfloor$. Set $\tilde{w}'_s = \lfloor w'_s \rfloor$ and $\tilde{w}'_t = \lceil w'_t \rceil$. Then $\tilde{w}'_t - \tilde{w}'_s \leq \tilde{w}_s \leq \tilde{w}'_t + \tilde{w}'_s$. To justify our choices of $\tilde{w}'_s$ and $\tilde{w}'_t$ we only need to show $\tilde{w}_s \geq \tilde{w}'_s - \tilde{w}'_t$. This is clear since $\tilde{w}_s + \tilde{w}'_t \geq \lfloor w'_s + w'_t - y'_{r_0} \rfloor + \lceil w'_t \rceil = \lfloor w'_s + w'_t - y'_{r_0} + \lceil w'_t \rceil \rfloor \geq \lfloor w'_s + 2(w'_t - \frac{1}{2}y'_{r_0}) \rfloor \geq \lfloor w'_s \rfloor = \tilde{w}'_s$.

Next, we adjust our choices of $\tilde{w}'_s$ and $\tilde{w}'_t$ in the two special cases. If $y'_{r_0} = 0$, then $\lfloor w'_s \rfloor + \lfloor w'_t \rfloor \leq \lfloor w'_s + w'_t \rfloor = \lfloor w_s \rfloor \leq \tilde{w}_s \leq \lceil w_s \rceil = \lceil w'_s + w'_t \rceil \leq \lceil w'_s \rceil + \lceil w'_t \rceil$. Hence $\tilde{w}_s = \tilde{w}'_s + \tilde{w}'_t$ for some $\tilde{w}'_s \in \{\lfloor w'_s \rfloor, \lceil w'_s \rceil\}$ and $\tilde{w}'_t \in \{\lfloor w'_t \rfloor, \lceil w'_t \rceil\}$. For this choice of $\tilde{w}'_s$ and $\tilde{w}'_t$, by (1), we still have $|\tilde{w}'_s - \tilde{w}'_t| \leq \tilde{w}_s \leq \tilde{w}'_s + \tilde{w}'_t$ and thus (a) is proved.

If $w'_t = \frac{1}{2}y'_{r_0}$ then $w_s = w'_s + w'_t - y'_{r_0} = w'_s - w'_t$. It follows that $\tilde{w}_s = \tilde{w}'_s - \tilde{w}'_t$ for some $\tilde{w}'_s \in \{\lfloor w'_s \rfloor, \lceil w'_s \rceil\}$ and $\tilde{w}'_t \in \{\lfloor w'_t \rfloor, \lceil w'_t \rceil\}$. By (1), $\tilde{w}_s \geq 0$ and hence $\tilde{w}'_s \geq \tilde{w}'_t$. Therefore, $|\tilde{w}'_s - \tilde{w}'_t| = \tilde{w}'_s - \tilde{w}'_t = \tilde{w}_s \leq \tilde{w}'_s + \tilde{w}'_t$, which proves case (b). ■

Lemma 4.3. $\frac{1}{2}y'_{r_0} + \frac{1}{2}\sum_{r' \in R'_1} y'_{r'} = w'_t$.

Proof. If $\frac{1}{2}y'_{r_0} + \frac{1}{2}\sum_{r' \in R'_1} y'_{r'} < w'_t$, then $\frac{1}{2}y'_{r_0} + \frac{1}{2}\sum_{r \in R_1} y'_r = w'_s$, for otherwise increasing y'_{r_0} would increase the optimal value of $\mathrm{Max}(C', \mathbf{w}')$. Hence we deduce from $w'_s \geq w'_t > \frac{1}{2}y'_{r_0}$ that $y'_r > 0$, for some $r \in R_1$. Let us define $\mathbf{z}' \in \mathbb{R}^{R'}$ such that $\mathbf{z}'|_{R' \setminus \{r_0, r, r'\}} = \mathbf{y}'|_{R' \setminus \{r_0, r, r'\}}$, $z'_r = y'_r - \varepsilon$, $z'_{r'} = y'_{r'} + \varepsilon$, and $z'_{r_0} = y'_{r_0} + \varepsilon$. It is straightforward to verify that if $\varepsilon > 0$ is small enough then $\mathbf{z}'$ is a feasible solution to $\mathrm{Max}(C', \mathbf{w}')$ with $(\mathbf{z}')^{\mathrm{T}}\mathbf{1} = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} + \varepsilon$, which contradicts the optimality of $\mathbf{y}'$ and so the lemma is proved. ■

Lemma 4.4. $\mathrm{Max}(C, \mathbf{w})$ *has a feasible solution* $\mathbf{y}$ *with* $\mathbf{y}^{\mathrm{T}}\mathbf{1} = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - y'_{r_0}$.

Proof. Let $\mathbf{y} \in \mathbb{Q}^R_+$ such that $\mathbf{y}|_{R_2} = \mathbf{y}'|_{R_2}$ and $y_r = y'_r + y'_{r'}$, for all $r \in R_1$. It is routine to verify that $\mathbf{y}^{\mathrm{T}} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \mathbf{y}' \begin{bmatrix} \mathbf{0}^{\mathrm{T}} \\ C_1 \\ C_1 \\ C_2 \end{bmatrix} \leq \mathbf{w}'|_{S \setminus \{s\}} = \mathbf{w}|_{S \setminus \{s\}}$ and $\mathbf{y}^{\mathrm{T}} \begin{bmatrix} \frac{1}{2}\mathbf{1} \\ \mathbf{0} \end{bmatrix} = \frac{1}{2}\sum_{r \in R_1}(y'_r + y'_{r'}) \leq w'_s + w'_t - y'_{r_0} = w_s$, which proves the feasibility of $\mathbf{y}$. In addition, $\mathbf{y}^{\mathrm{T}}\mathbf{1} = \sum_{r \in R_1} y_r + \sum_{r \in R_2} y_r = \sum_{r \in R_1}(y'_r + y'_{r'}) + \sum_{r \in R_2} y'_r = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - y'_{r_0}$, which proves the lemma. ■

Lemma 4.5. *Let* $\mathbf{x}'$ *be an optimal solution of* $\mathrm{Min}(C', \mathbf{w}')$. *Then*

(a) $x'_s = x'_t$ *if* $w'_t \neq \frac{1}{2}y'_{r_0}$; *and*

(b) $\frac{1}{2}y'_{r_0}(x'_s + x'_t) = y'_{r_0}$.

Proof. If $w'_t \neq \frac{1}{2}y'_{r_0}$, then Lemma 4.3 implies the existence of $r' \in R'_1$ with $y'_{r'} \neq 0$. Let $\mathbf{c}^{\mathrm{T}}$ be the row of C' that is indexed by r'. It follows from the complementary slackness that $\mathbf{c}^{\mathrm{T}}\mathbf{x}' = (\mathbf{c}|_{S \setminus \{s,t\}})^{\mathrm{T}}(\mathbf{x}'|_{S \setminus \{s,t\}}) + 0x'_s + \frac{1}{2}x'_t = 1$. On the other hand, since $\mathbf{x}'$ is an optimal solution of $\mathrm{Min}(C', \mathbf{w}')$, we deduce from $w'_s \geq w'_t$ that $x'_s \leq x'_t$, so $1 = (\mathbf{c}|_{S \setminus \{s,t\}})^{\mathrm{T}}(\mathbf{x}'|_{S \setminus \{s,t\}}) + 0x'_s + \frac{1}{2}x'_t \geq (\mathbf{c}|_{S \setminus \{s,t\}})^{\mathrm{T}}(\mathbf{x}'|_{S \setminus \{s,t\}}) + \frac{1}{2}x'_s + 0x'_t \geq 1$, which proves (a). Clearly, (b) is exactly the complementary slackness condition $y'_{r_0}(\frac{1}{2}x'_s + \frac{1}{2}x'_t - 1) = 0$. ■

Lemma 4.6. *If* $\mathbf{x}'$ *is an optimal solution to* $\mathrm{Min}(C', \mathbf{w}')$, *then* $\boldsymbol{\pi} = \mathbf{x}'|_S$ *is an optimal solution to* $\mathrm{Min}(C, \mathbf{w})$.

Proof. Clearly, $\boldsymbol{\pi}$ is a feasible solution to $\mathrm{Min}(C, \mathbf{w})$, as $\mathbf{x}'$ is a feasible solution to $\mathrm{Min}(C', \mathbf{w}')$. Moreover,

$$\begin{aligned} \mathbf{w}^{\mathrm{T}}\boldsymbol{\pi} &= (\mathbf{w}|_{S \setminus \{s\}})^{\mathrm{T}}(\mathbf{x}'|_{S \setminus \{s\}}) + w_s x'_s \\ &= (\mathbf{w}')^{\mathrm{T}}\mathbf{x}' - w'_s x'_s - w'_t x'_t + (w'_s + w'_t - y'_{r_0})x'_s \\ &= (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - w'_t(x'_t - x'_s) - y'_{r_0}x'_s. \end{aligned} \tag{4.2}$$

If $\frac{1}{2}y'_{r_0} = w'_t$, then (4.2) implies $\mathbf{w}^{\mathrm{T}}\boldsymbol{\pi} = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - \frac{1}{2}y'_{r_0}(x'_s + x'_t)$. By Lemma 4.5(b), $\mathbf{w}^{\mathrm{T}}\boldsymbol{\pi} = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - y'_{r_0}$. If $\frac{1}{2}y'_{r_0} \neq w'_t$, Lemma 4.5 implies $x'_s = x'_t$ and $y'_{r_0}x'_s = y'_{r_0}$. Thus (4.2)) implies $\mathbf{w}^{\mathrm{T}}\boldsymbol{\pi} = (\mathbf{y}')^{\mathrm{T}}\mathbf{1} - y'_{r_0}$ again. In both cases, we deduce from Lemma 4.4 that $\mathbf{w}^{\mathrm{T}}\boldsymbol{\pi} = \mathbf{y}^{\mathrm{T}}\mathbf{1}$, for a feasible solution $\mathbf{y}$ of Max$(C, \mathbf{w})$, so $\boldsymbol{\pi}$ is an optimal solution to Min$(C, \mathbf{w})$. ■

Lemma 4.7. *Suppose $\boldsymbol{\alpha} \in \mathbb{Z}^S$ and $\boldsymbol{\alpha}' \in \mathbb{Z}^{S'}$ such that $\boldsymbol{\alpha}|_{S\setminus\{s\}} = \boldsymbol{\alpha}'|_{S\setminus\{s\}}$ and $|\alpha'_s - \alpha'_t| \leq \alpha_s \leq \alpha'_s + \alpha'_t$. If $\boldsymbol{\eta}$ is an integral feasible solution to* Max$(C, \boldsymbol{\alpha})$*, then there exists an integral feasible solution $\boldsymbol{\eta}'$ to* Max$(C', \boldsymbol{\alpha}')$ *with $(\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1} = \boldsymbol{\eta}^{\mathrm{T}}\mathbf{1} + \alpha'_s + \alpha'_t - \alpha_s$.*

Proof. The feasibility of $\boldsymbol{\eta}$ implies $\frac{1}{2}\sum_{r\in R_1}\eta_r \leq \alpha_s = \frac{1}{2}(\alpha'_s - \alpha'_t + \alpha_s) + \frac{1}{2}(\alpha'_t - \alpha'_s + \alpha_s)$. Since $|\alpha'_s - \alpha'_t| \leq \alpha_s$, we can find $r_1 \in R_1$ and a partition (Q_1, Q_2) of $R_1 \setminus \{r_1\}$ such that $\sum_{r\in Q_1}\eta_r \leq \alpha'_s - \alpha'_t + \alpha_s$ and $\sum_{r\in Q_2}\eta_r \leq \alpha'_t - \alpha'_s + \alpha_s$. So there exist $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_+$ such that $\eta_{r_1} = \varepsilon_1 + \varepsilon_2$, $\varepsilon_1 + \sum_{r\in Q_1}\eta_r \leq \alpha'_s - \alpha'_t + \alpha_s$, and $\varepsilon_2 + \sum_{r\in Q_2}\eta_r \leq \alpha'_t - \alpha'_s + \alpha_s$. Let us define $\boldsymbol{\eta}' \in \mathbb{Z}_+^{R'}$ such that

- $\boldsymbol{\eta}'|_{R_2} = \boldsymbol{\eta}|_{R_2}$,
- $\eta'_{r_0} = \alpha'_s + \alpha'_t - \alpha_s$,
- $\eta'_{r_1} = \varepsilon_1$ and $\eta'_{r'_1} = \varepsilon_2$,
- $\eta'_r = \eta_r$ and $\eta'_{r'} = 0$, for all $r \in Q_1$,
- $\eta'_r = 0$ and $\eta'_{r'} = \eta_r$, for all $r \in Q_2$.

Then it is straightforward to verify that $\boldsymbol{\eta}'$ satisfies the requirements of the lemma. ■

Proof of Lemma 4.1. Suppose $\mathbf{w}' \in \mathbb{Q}^{S'}$ such that Max$(C', \mathbf{w}')$ has a finite maximum. We prove that there exists $\tilde{\mathbf{w}}' \in \mathbb{Z}^{S'}$ such that

(i) $\lfloor \mathbf{w}' \rfloor \leq \tilde{\mathbf{w}}' \leq \lceil \mathbf{w}' \rceil$,
(ii) every optimal solution to Min$(C', \mathbf{w}')$ is also an optimal solution to Min$(C', \tilde{\mathbf{w}}')$, and
(iii) Max$(C', \tilde{\mathbf{w}}')$ has an integral optimal solution.

We point out that (i)–(iii) imply the lemma. If $\mathbf{w}'$ is integral, then (i) implies $\tilde{\mathbf{w}}' = \mathbf{w}'$ and thus (iii) implies that $C'\mathbf{x}' \geq \mathbf{1}$, $\mathbf{x}' \geq \mathbf{0}$ is TDI. In addition, by Theorem 2.2, (i) and (ii) imply that $C'\mathbf{x}' \geq \mathbf{1}, \mathbf{x}' \geq \mathbf{0}$ is box-TDI. Therefore, we only need to show the existence of $\tilde{\mathbf{w}}'$ that satisfies (i)–(iii).

Let $\mathbf{y}'$ be an optimal solution to Max$(C', \mathbf{w}')$. By symmetry, we may assume that $w'_s \geq w'_t$. Hence assumptions (a) and (b) in (4.1) hold. Let $\mathbf{w}$ be defined as in (4.1)(c). Since $C\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box-TDI, by Theorem 2.2, there exists $\tilde{\mathbf{w}}$ such that $\lfloor \mathbf{w} \rfloor \leq \tilde{\mathbf{w}} \leq \lceil \mathbf{w} \rceil$ and all optimal solutions to Min$(C, \mathbf{w})$ are also optimal solutions to Min$(C, \tilde{\mathbf{w}})$. Let $\tilde{\mathbf{w}}' \in \mathbb{Z}^{S'}$ such that $\tilde{w}'_s$ and $\tilde{w}'_t$ are determined as in Lemma 4.2 and $\tilde{\mathbf{w}}'|_{S\setminus\{s\}} = \tilde{\mathbf{w}}|_{S\setminus\{s\}}$. Consequently, (i) is satisfied.

To prove (ii) and (iii), let $\mathbf{x}'$ be an optimal solution to Min$(C', \mathbf{w}')$. By Lemma 4.6, $\boldsymbol{\pi} = \mathbf{x}'|_S$ is an optimal solution to Min$(C, \mathbf{w})$. Then the choice of $\tilde{\mathbf{w}}$ implies that $\boldsymbol{\pi}$ is an optimal solution to Min$(C, \tilde{\mathbf{w}})$. Since $C\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box-TDI, Max$(C, \tilde{\mathbf{w}})$ has an integral optimal solution $\boldsymbol{\eta}$. From the choice of $\tilde{\mathbf{w}}'$ we deduce that $\boldsymbol{\alpha} = \tilde{\mathbf{w}}$ and $\boldsymbol{\alpha}' = \tilde{\mathbf{w}}'$ satisfy the assumptions in Lemma 4.7. Thus there exists an integral feasible solution $\boldsymbol{\eta}'$ to Max$(C', \tilde{\mathbf{w}}')$ such that $(\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1} = \boldsymbol{\eta}^{\mathrm{T}}\mathbf{1} + \tilde{w}'_s + \tilde{w}'_t - \tilde{w}_s$.

Now we prove that $\mathbf{x}'$ and $\boldsymbol{\eta}'$ are optimal solutions to Min$(C', \tilde{\mathbf{w}}')$ and Max$(C', \tilde{\mathbf{w}}')$, respectively, which would prove (ii) and (iii) and thus the lemma. Since $\mathbf{x}'$ and $\boldsymbol{\eta}'$ are feasible solutions, we only need to verify $(\tilde{\mathbf{w}}')^{\mathrm{T}}\mathbf{x}' = (\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1}$. Observe that

$$\begin{aligned}(\tilde{\mathbf{w}}')^{\mathrm{T}}\mathbf{x}' - (\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1} &= (\tilde{\mathbf{w}}'|_{S\setminus\{s\}})^{\mathrm{T}}(\mathbf{x}'|_{S\setminus\{s\}}) + \tilde{w}'_s x'_s + \tilde{w}'_t x'_t - \boldsymbol{\eta}^{\mathrm{T}}\mathbf{1} - \tilde{w}'_s - \tilde{w}'_t + \tilde{w}_s \\ &= \tilde{\mathbf{w}}^{\mathrm{T}}\boldsymbol{\pi} - \tilde{w}_s x'_s + \tilde{w}'_s x'_s + \tilde{w}'_t x'_t - \boldsymbol{\eta}^{\mathrm{T}}\mathbf{1} - \tilde{w}'_s - \tilde{w}'_t + \tilde{w}_s \\ &= -\tilde{w}_s x'_s + \tilde{w}'_s x'_s + \tilde{w}'_t x'_t - \tilde{w}'_s - \tilde{w}'_t + \tilde{w}_s. \end{aligned} \tag{4.3}$$

Case 1. Suppose $w'_t \neq \frac{1}{2}y'_{r_0}$. Then Lemma 4.5(a) implies $x'_s = x'_t$. Moreover, if $y'_{r_0} > 0$, then Lemma 4.5(b) implies $x'_s = x'_t = 1$; if $y'_{r_0} = 0$, then Lemma 4.2(a) implies $\tilde{w}_s = \tilde{w}'_s + \tilde{w}'_t$. In both cases, (4.3)) implies $(\tilde{\mathbf{w}}')^{\mathrm{T}}\mathbf{x}' = (\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1}$.

Case 2. Suppose $w'_t = \frac{1}{2}y'_{r_0}$. Then Lemma 4.2(b) implies $\tilde{w}_s = \tilde{w}'_s - \tilde{w}'_t$. Now it follows from (4.3) and Lemma 4.5(b) that $(\tilde{\mathbf{w}}')^{\mathrm{T}}\mathbf{x}' - (\boldsymbol{\eta}')^{\mathrm{T}}\mathbf{1} = \tilde{w}'_t(x'_s + x'_t - 2) = \frac{1}{2}y'_{r_0}(x'_s + x'_t - 2) = 0$. ■

The reader is referred to Cook [1] for various other operations that preserve total dual integrality.

5. Graphs and matroids

In this section we prove the two main results stated in Section 1. We denote the cutset-edge incidence matrix of a graph G by A_G, instead of A, to show its dependency on G. Recall that $G \setminus e$ and G/e stand for the results of deleting and contracting, respectively, an edge e in G.

Proof of Theorem 1.1. *Necessity*. It was observed in [11] that the polyhedron $\{\mathbf{x} \mid \frac{1}{2}A_{K_4}\mathbf{x} \geq \mathbf{1}, \mathbf{1} \geq \mathbf{x} \geq \mathbf{0}\}$ is not integral, which, by Theorem 2.1, implies that K_4 is not good. On the other hand, it is straightforward to verify that $A_{H\backslash e} = A_H/e$ and $A_{H/e} = A_H\backslash e$, for all graphs H and all $e \in E(H)$. Therefore, Lemmas 2.3 and 2.4 imply that K_4 cannot be obtained from G by deleting and contracting edges. Since G is connected, it follows that G must be series-parallel.

Sufficiency. We first consider the case when $G = (V, E)$ is a tree. Let $\mathcal{C}$ be the set of cuts of G of size one and let B_G be the $\mathcal{C} - E$ incidence matrix. Clearly, B_G is the identity matrix, as G is a tree, and thus the system $\frac{1}{2}B_G\mathbf{x} \geq \mathbf{1}, \ \mathbf{x} \geq \mathbf{0}$ is box-TDI. Notice that every row of A_G dominates some row of B_G, so we deduce from Lemma 2.2 that G is good.

Since a general series-parallel connected graph is constructed from a tree by repeatedly applying three simple operations, we only need to show that these operations keep a good graph good. In fact, it is routine to verify that, if H' is obtained from H by adding a loop then $A_{H'}$ is obtained from A_H by adding a zero-column; if H' is a series- or parallel-extension H then $A_{H'}$ is a parallel- or series-extension of A_H, respectively. Therefore, the theorem follows from Lemmas 2.1, 3.2 and 4.1. ■

Let A_M be the circuit-element incidence matrix of a matroid M. Recall that M is good if $\frac{1}{2}A_M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ is box-TDI. Clearly, a connected graph G is good if and only if its cographic matroid $\mathcal{M}^*(G)$ is good.

Proof of Theorem 1.2. *Necessity*. Since K_4 is not a good graph, $\mathcal{M}^*(K_4)$ is not good and hence $\mathcal{M}(K_4)$ is not good either, as $\mathcal{M}^*(K_4)$ and $\mathcal{M}(K_4)$ are isomorphic. Notice that $\mathbf{x}^* = \frac{2}{3}\mathbf{1}$ is a nonintegral vertex of the polyhedron $\{\mathbf{x} \mid \frac{1}{2}A_{U_{2,4}}\mathbf{x} \geq \mathbf{1}, \ \mathbf{x} \geq \mathbf{0}\}$, since $\mathbf{x}^*$ belongs to the polyhedron and $\frac{1}{2}A_{U_{2,4}}\mathbf{x}^* = \mathbf{1}$ with $A_{U_{2,4}}$ being full rank. Hence $\frac{1}{2}A_{U_{2,4}}\mathbf{x} \geq \mathbf{1}, \ \mathbf{x} \geq \mathbf{0}$ is not TDI and so $U_{2,4}$ is not good. On the other hand, it is straightforward to verify that $A_{M\backslash e} = A_M \setminus e$ and $A_{M/e} = A_M/e$, for all matroids M and all elements e of M. Therefore, Lemmas 2.3 and 2.4 imply that $U_{2,4}$ and $\mathcal{M}(K_4)$ are not minors of any good matroid.

Sufficiency. Suppose M has no minors isomorphic to $U_{2,4}$ or $\mathcal{M}(K_4)$. Then its dual M^* has no minors isomorphic to $U_{2,4}$ or $\mathcal{M}(K_4)$. It is well known (cf. Corollary 11.2.15 of [14]) that $M^* = \mathcal{M}(G)$ for some series-parallel connected graph G. Since circuits of M are precisely minimal cuts of G, the result follows from Lemma 2.2 and Theorem 1.1 immediately. ■

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