3-28-2009

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Recommended Citation
Ding, G. (2009). On canonical antichains. *Discrete Mathematics, 309* (5), 1123-1134. https://doi.org/10.1016/j.disc.2007.12.018

Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 1123–1134
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On canonical antichains

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Received 20 December 2005; accepted 3 December 2007
Available online 18 January 2008

Abstract

An antichain A of a well-founded quasi-order Q is *canonical* if for every ideal F of Q, F has an infinite antichain if and only if $F \cap A$ is infinite. In this paper we characterize the obstructions to having a canonical antichain. As an application we show that, under the induced subgraph relation, the class of finite graphs does not have a canonical antichain. In contrast, this class does have a canonical antichain with respect to the subgraph relation.

Keywords: Well-quasi-ordering; Infinite antichain; Subgraph; Induced subgraph

1. Introduction

We begin with some conventional terminology. Let $\leq$ be a binary relation on a set E and let Q denote the pair $(E, \leq)$. Then Q is a *quasi-order* if $\leq$ is reflexive and transitive. A *minimal* element of a subset F of E is an element $x_0 \in F$ for which $x \leq x_0$ implies $x_0 \leq x$ for all x in F. We call Q *well founded* if every nonempty subset of E has a minimal element. If F is a subset of E such that $x \leq y \in F$ implies $x \in F$ for all choices of x and y in E, then F is called an *ideal* of Q. Two members x and y of E are *comparable* if $x \leq y$ or $y \leq x$. For a subset F of E, an *antichain* of F is a subset of F for which no two distinct members are comparable. An *antichain* of Q is simply an antichain of E.

This research was motivated by the studies on the existence and nonexistence of infinite antichains of various combinatorial objects, in particular, of graphs. These studies received a lot of attentions in recent years [4]. One of the main reasons for the attentions is the following. Suppose that $Q = (E, \leq)$ is a well-founded quasi-order for which there is no infinite antichain. Then it is not difficult to see that every ideal F of Q can be characterized by a Kuratowski-type theorem. Namely, there exist finitely many elements $e_1, \ldots, e_k$ of E such that an element e of E is in F if and only if $e_i \not\leq e$ for all e_i. If E is a class of combinatorial objects, then the algorithmic implication of this observation is that the membership recognition problem for every such F can be solved in polynomial time, provided that "$x_0 \leq x$" can be tested in polynomial time for every fixed element x_0 of E.

Traditionally, researchers were more interested in constructing larger quasi-orders from smaller ones and preserving the property of having no infinite antichains [3]. However, from the algorithmic point of view, it is also very important to characterize for a quasi-order all those ideals that do not have infinite antichains. As an attempt to move towards this direction, we introduce the concept of *canonical antichain*.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.018

Throughout this paper, we use $\mathbb{N}$ to denote the set of positive integers and use $\mathbb{N}^*$ to denote the set of nonnegative integers. If $\mathcal{X}$ is a set of sets, we will write $\cup\{X : X \in \mathcal{X}\}$ for $\cup_{X\in\mathcal{X}} X$.

1.1. Definition and implications

Let $Q = (E, \leq)$ be a quasi-order. For any subset F of E, let $\mathrm{Excl}(F) = E \setminus \{x : x \geq f \text{ for some } f \in F\}$. Then an antichain A of Q is *canonical* if for every finite subset A_0 of A, all antichains of $\mathrm{Excl}(A \setminus A_0)$ are finite. We point out in the next proposition that a canonical antichain can be used to characterize the ideals of Q that do not have infinite antichains.

Proposition 1.1. *Suppose that a quasi-order Q has a canonical antichain A. Then the following are equivalent for every ideal F of Q:*

(1) *all antichains of F are finite;*
(2) $F \subseteq \mathrm{Excl}(A \setminus A_0)$ *for some finite subset A_0 of A;*
(3) *$F \cap A$ is finite.*

Proof. Implication (1) $\Rightarrow$ (3) is trivial. For implication (3) $\Rightarrow$ (2) we only need to take $A_0 = F \cap A$. Finally, implication (2) $\Rightarrow$ (1) follows from the definition of a canonical antichain. □

Notice that (2) and (3) in Proposition 1.1 provide two different types of characterizations of ideals of Q that do not have infinite antichains, one is in terms of "exclusion" and the other is in terms of "inclusion". In fact, canonical antichains can be defined via these characterizations. In particular, the definition we used in the abstract agrees with the definition we just gave above, as stated in the following proposition.

Proposition 1.2. *The following are equivalent for every antichain A of a quasi-order Q:*

(1) *A is a canonical antichain;*
(2) *for any ideal F of Q, F has an infinite antichain if and only if $F \cap A$ is infinite.*

Proof. Implication (1) $\Rightarrow$ (2) follows from Proposition 1.1. To prove the reverse implication (2) $\Rightarrow$ (1), we only need to take $F = \mathrm{Excl}(A \setminus A_0)$, for every finite subset A_0 of A. □

We have seen how a canonical antichain can be used to characterize ideals that do not have infinite antichains. The next proposition explores the negation of this concept. It basically says that if an antichain A is not canonical, then there exists an antichain which is either "bigger" or more "elementary" than A.

Proposition 1.3. *Suppose that an antichain A of a quasi-order Q is not canonical. Then there exists an infinite antichain A' such that at least one of the following holds:*

(1) *$A \setminus A'$ is finite but $A' \setminus A$ is infinite;*
(2) *there exists an injection φ from A' to A such that $x < \varphi(x)$, for all $x \in A'$.*

Proof. Since A is not canonical, there exists a finite subset A_0 of A such that $\mathrm{Excl}(A \setminus A_0)$ has an infinite antichain B. Notice that two distinct members of $V = B \cup (A \setminus A_0)$ are comparable only if one is in $A \setminus A_0$, say a, and one is in B, say b, and $b < a$ holds. Let G be the infinite graph with vertex set V and edges xy, for all distinct comparable elements x, y of V. If G has an infinite matching $\{a_i b_i : i \in \mathbb{N}\}$, then $A' = \{b_i : i \in \mathbb{N}\}$ and $\varphi(b_i) = a_i$ ($i \in \mathbb{N}$) satisfy (2). If G has no infinite matching then G has a finite maximal matching $\{a_i b_i : i = 1, 2, \ldots, k\}$. In this case, it is straightforward to verify that antichain $A' = (B \setminus \{b_i : i = 1, 2, \ldots, k\}) \cup (A \setminus (A_0 \cup \{a_i : i = 1, 2, \ldots, k\}))$ satisfies (1). □

As a consequence of Proposition 1.3, if a quasi-order Q has no canonical antichain, then for any of its antichain A, there is always another antichain that is either "bigger" or more "elementary" than A. In a sense, this describes the wildness of antichains in Q.

The following is one more justification for introducing the concept "canonical antichain". Notice that if both A_1 and A_2 are antichains of a quasi-order Q and such that $(A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ is finite, then one of them being canonical implies that the other is canonical as well. Therefore, "Q has a finite canonical antichain" which is equivalent to "the

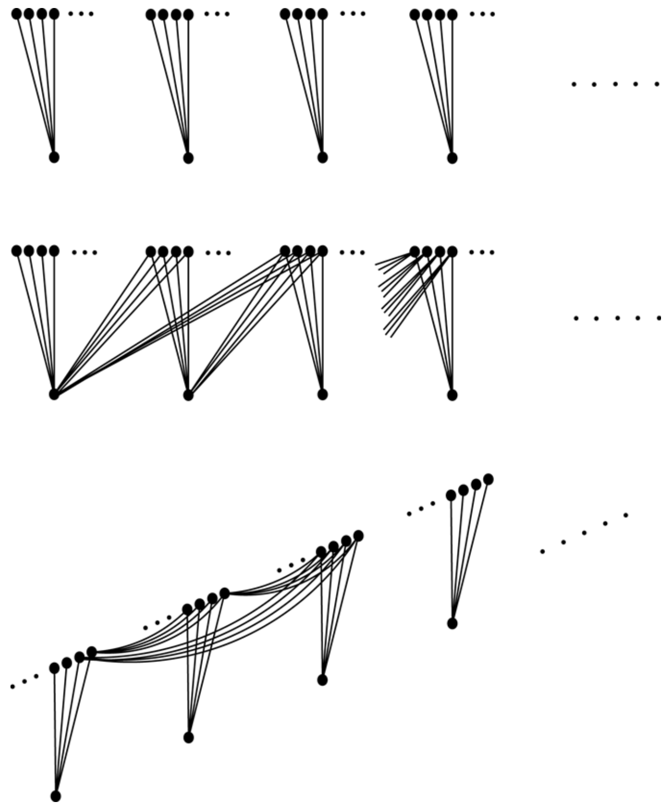

Fig. 1. Binary relations $<_k$, for $k = 1, 2, 3$.

empty set is a canonical antichain of Q", and that is equivalent to "no antichain of Q is infinite". From this point of view, the study of canonical antichains is a natural extension of the study of the existence of infinite antichains. In fact, this approach has been applied to graph theory. It was known for a long time that there are infinite antichains with respect to the subgraph relation. In [1], the author identified a canonical antichain and thus characterized all ideals that do not have infinite antichains.

1.2. Main results

The purpose of this paper is to characterize those quasi-orders that have a canonical antichain. Let S be the set of ordered pairs (x, y) of integers for which $x \geq 1$ and $y \geq 0$. For each k in $\{1, 2, 3\}$, we define a binary relation $\leq_k$ as follows (see Fig. 1). For any (x, y) and (x', y') in S, first let

- $(x, y) <_1 (x', y')$ if $y = 0$ and $x = y'$;
- $(x, y) <_2 (x', y')$ if $y = 0$ and $x \leq y'$; and
- $(x, y) <_3 (x', y')$ if $x + y = y'$.

Then we define $\leq_k$ as the transitive closure of $<_k$. That is, $(x, y) \leq_k (x', y')$ if either $(x, y) = (x', y')$ or $(x, y) <_k (x_1, y_1) <_k \cdots <_k (x_n, y_n) <_k (x', y')$, for some finite (could be empty) sequence $(x_1, y_1), \ldots, (x_n, y_n)$. Let $Q_k = (S, \leq_k)$. It is not difficult to see that Q_1, Q_2, and Q_3 are quasi-orders that do not have canonical antichains. Our main theorem in this paper will say that if a quasi-order does not have a canonical antichain, then it must "contain" some Q_k. In other words, Q_1, Q_2, and Q_3 are the only obstacles to the existence of a canonical antichain. To make this more precise, we need a few definitions.

A *partition* of a set E is a set $\mathcal{P}$ of nonempty subsets of E for which $E = \cup\{X : X \in \mathcal{P}\}$ and $X \cap Y = \emptyset$ for all distinct members X and Y of $\mathcal{P}$. A quasi-order $(E, \leq)$ together with a partition $\mathcal{P}$ of E shall be called a *partitioned* quasi-order and shall be denoted by $(\mathcal{P}, \leq)$. Let $S_y = \{(x, y) : x \in \mathbb{N}\}$ $(y \in \mathbb{N}^*)$ and let $\mathcal{S} = \{S_y : y \in \mathbb{N}^*\}$. Then

$(\mathcal{S}, \leq_k)$ is a partitioned quasi-order for every k in $\{1, 2, 3\}$. Let $(E, \leq)$ and $(E', \leq')$ be two quasi-orders, and let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of E and E', respectively. Then $(\mathcal{P}, \leq)$ and $(\mathcal{P}', \leq')$ are *isomorphic* if there is an isomorphism σ between $(E, \leq)$ and $(E', \leq)$ for which $\mathcal{P}' = \{\sigma(X) : X \in \mathcal{P}\}$.

For any antichain A of $Q = (V, \leq)$, let $\text{Incl}(A) = \{x \in E : x < a$ for some $a \in A\}$. We will say that A is *fundamental* if $\text{Incl}(A)$ has no infinite antichains. Now we can state the main result of this paper.

Theorem 1.1. *A well-founded quasi-order $Q = (E, \leq)$ does not have a canonical antichain if and only if there exists a set $\mathcal{A}$ of pairwise disjoint fundamental antichains of Q such that the partitioned quasi-order $(\mathcal{A}, \leq)$ is isomorphic to $(\mathcal{S}, \leq_k)$ for some k in $\{1, 2, 3\}$.*

The proof of this theorem will be given in the next two sections. We will use this theorem to prove in the last section that the class of finite graphs does not have a canonical antichain with respect to the induced subgraph relation. Now we close this section by presenting an open problem. Let $Q = (E, \leq)$ be a quasi-order and let X be a subset of E for which the following condition is satisfied:
(*) an ideal F of Q has an infinite antichain if and only if $F \cap X$ has an infinite antichain.
Such an X always exists no matter what Q is, since we may take X to be E. In fact, we can choose X in many case so that $(X, \leq)$ is very simple. For instance, if Q has a canonical antichain, then we can choose X to be an antichain. Therefore, it is natural to raise the problem of finding an X with property (*) such that $(X, \leq)$ is as simple as possible. As we shall see in the next section that the existence of an infinite antichain implies the existence of an infinite fundamental antichain. Thus we can choose X to be the union of all infinite fundamental antichains. But I do not know how to simplify $(X, \leq)$ any further.

2. Constructing a canonical antichain

A quasi-order $(E, \leq)$ is a partial order if $\leq$ is antisymmetric. It is clear that Q_1, Q_2, and Q_3 are actually partial orders. Thus Theorem 1.1 holds if it holds for partial orders. For this reason, we shall only consider partial orders from now on.

Let $Q = (E, \leq)$ be a partial order. An element e of E is *critical* if there exists an infinite fundamental antichain A of Q such that $e < a$ for all a in A. The following is the main result of this section.

Theorem 2.1. *A well-founded partial order has a canonical antichain if and only if every fundamental antichain contains only finitely many critical elements.*

In order to prove Theorem 2.1 we need a few lemmas.

Lemma 2.1. *Let X be a fundamental antichain of a partial order $Q = (E, \leq)$. If Q has a canonical antichain A, then $X \setminus A$ must be finite.*

Proof. Let $A_0 = A \cap \text{Incl}(X)$. Since X is fundamental and A_0 is an antichain of $\text{Incl}(X)$, we conclude that A_0 is finite. Thus, as A is canonical, all antichains of $\text{Excl}(A \setminus A_0)$ are finite. In particular, the antichain $X \cap \text{Excl}(A \setminus A_0)$ is finite. Therefore, to prove the lemma, it is enough for us to show that $X \setminus A \subseteq X \cap \text{Excl}(A \setminus A_0)$, or equivalently, $X \setminus A \subseteq \text{Excl}(A \setminus A_0)$.

Let x be an element of $X \setminus A$. Since $\text{Incl}(\{x\})$ is contained in $\text{Incl}(X)$, we conclude that $A \cap \text{Incl}(\{x\})$ is contained in $A \cap \text{Incl}(X)$, which is A_0. Thus $A \setminus A_0 \subseteq A \setminus \text{Incl}(\{x\})$, and so $(A \setminus A_0) \cap \text{Incl}(\{x\}) = \emptyset$. It follows that there is no element a of $A \setminus A_0$ for which $a < x$. Consequently, since $x \notin A$, there is no element a of $A \setminus A_0$ for which $a \leq x$. Therefore, x belongs to $\text{Excl}(A \setminus A_0)$, as required. □

Let $Q = (E, \leq)$ be a partial order. If A is an antichain of Q, then we call A a *maximal* antichain if no proper superset of A is an antichain. Equivalently, A is maximal if every element of E is comparable with some element of A. Let X be a subset of E. An element x_0 of X is *maximal* if $x_0 \leq x$ implies $x = x_0$ for all x in X. An *upper bound* of X is an element x_0 of E for which $x \leq x_0$ for all x in X. Similarly, a *lower bound* of X is an element x_0 of E for which $x_0 \leq x$ for all x in X. We shall call X a *chain* if every pair of elements of X are comparable.

Zorn's Lemma. *If every chain of a partial order $(E, \leq)$ has an upper (lower) bound, then E has a maximal (minimal) element.*

The following is a simple corollary of Zorn's Lemma.

Lemma 2.2. *Every antichain can be extended into a maximal antichain.*

Proof. Let A be an antichain of a partial order $Q = (E, \leq)$ and let E' be the set of elements of E that are not comparable with any element of A. Let $\mathcal{A}$ be the set of all antichains of E'. Then $\mathcal{A} \neq \emptyset$ since $\emptyset \in \mathcal{A}$. It is clear that $\mathcal{Q} = (\mathcal{A}, \subseteq)$ is a partial order. Moreover, for every chain $\mathcal{C}$ of $\mathcal{Q}$, the union of members of $\mathcal{C}$ is an upper bound of $\mathcal{C}$. Therefore, we conclude from Zorn's Lemma that $(E', \leq)$ has a maximal antichain A'. Now it is easy to verify that $A \cup A'$ is a maximal antichain of E. $\square$

The next is a technical lemma.

Lemma 2.3. *Let X be a maximal antichain of a partial order $Q = (E, \leq)$, and let Y be a maximal antichain of $X \cup \mathrm{Incl}(X)$. Then Y is a maximal antichain of Q.*

Proof. From the assumption on Y it is clear that every x in X is comparable with some y_x in Y. We first show that $y_x \leq x$ for every x in X. Since Y is a subset of $X \cup \mathrm{Incl}(X)$, there must exist an element x^* of X for which $y_x \leq x^*$. Suppose that $y_x \leq x$ does not hold. From the choice of y_x we deduce that $x < y_x$. But this implies $x < x^*$, contradicting the fact that X is an antichain.

Now let e be an element of E. We need to show that e is comparable with some element of Y. If e is in $X \cup \mathrm{Incl}(X)$, then the claim is clear because of our assumption on Y. If e is not in $X \cup \mathrm{Incl}(X)$, since X is a maximal antichain of Q, it follows that $x \leq e$ for some x in X. Then we have $y_x \leq e$, which finishes the proof of the lemma. $\square$

The next lemma guarantees the existence of an infinite fundamental antichain.

Lemma 2.4. *If a well-founded partial order $Q = (E, \leq)$ has an infinite antichain, then Q has an infinite maximal antichain A such that every infinite antichain of $A \cup \mathrm{Incl}(A)$ is a subset of A.*

Proof. Let $\mathcal{A}$ be the set of all infinite maximal antichains of Q. Then we deduce from Lemma 2.2 that $\mathcal{A}$ is not empty. For any two members A_1 and A_2 of $\mathcal{A}$, let $A_1 \preceq A_2$ if for every element a_1 of A_1, there exists an element a_2 of A_2 such that $a_1 \leq a_2$. Since all members of $\mathcal{A}$ are antichains, $A_1 \preceq A_2 \preceq A_1$ must imply that $A_1 = A_2$. It follows that $\mathcal{Q} = (\mathcal{A}, \preceq)$ is a partial order.

Next we prove that every chain $\mathcal{C}$ of $\mathcal{Q}$ has a lower bound. Since $\mathcal{A}$ is not empty, we may certainly assume that $\mathcal{C}$ is not empty. Let E' be the union of members of $\mathcal{C}$ and let A' be the set of minimal elements of E'. Clearly, A' is an antichain of Q. By Lemma 2.2, we may extend A' into a maximal antichain A of Q. Now we show that A is a lower bound of $\mathcal{C}$.

First, we prove that A is in $\mathcal{A}$. Owing to the way A is constructed, it is enough for us to show that A' is infinite. Suppose that A' is finite. Then we can choose X in $\mathcal{C}$ such that $|X \cap A'|$ is maximized. In addition, since X is infinite, we can choose an element x from $X \setminus A'$. Recall that Q is well founded. Hence we conclude from the definition of A' that there is an element a of A' such that $a \leq x$. As x is not in A', we must have $a \neq x$, and so $a < x$. Therefore, a is not in X since X is an antichain. Let Y be a member of $\mathcal{C}$ that contains a. Then $X \preceq Y$ does not hold. For otherwise, there is an element y in Y such that $x \leq y$. Since Y is an antichain, we deduce from $a \leq x \leq y$ that $a = x$, contradicting the choice of a and x. Thus $X \npreceq Y$ and hence $Y \preceq X$ as $\mathcal{C}$ is a chain. But Y is a maximal antichain of Q, it follows that every element a' of A' is comparable with some element $y_{a'}$ of Y. By the definition of A' we must have $a' \leq y_{a'}$. Furthermore, since $Y \preceq X$, there must be an element $x_{a'}$ of X for which $y_{a'} \leq x_{a'}$. As X is an antichain, for every $a' \in A' \cap X$, we deduce from $a' \leq y_{a'} \leq x_{a'}$ that $a' = y_{a'}$. Thus $X \cap A' \subseteq Y$, contradicting the choice of X since a belongs to $(A' \cap Y) \setminus X$. This contradiction finishes the proof of $A \in \mathcal{A}$.

Second, we prove that $A \preceq X$ for every X in $\mathcal{C}$. Since X is a maximal antichain of Q, every element a of A must be comparable with some element x_a of X. We claim that $a \leq x_a$ for all a in A. Suppose that there exists an element a of A for which $a \nleq x_a$. Then we must have $x_a < a$. Recall that Q is well founded. Thus we conclude from the definition of A' that there is an element a' of A' for which $a' \leq x_a$. Consequently, $a' < a$, contradicting the fact that A is an antichain. This contradiction proves that $A \preceq X$ for all X in $\mathcal{C}$, and hence A is a lower bound of $\mathcal{C}$, as we claimed.

From Zorn's Lemma we conclude that $\mathcal{Q}$ has a minimal element A. Now we show that A meets the requirements of the lemma. Suppose that there exists an infinite antichain X of $A \cup \mathrm{Incl}(A)$ for which $X \setminus A \neq \emptyset$. By Lemma 2.2, X can be extended into a maximal antichain Y of $A \cup \mathrm{Incl}(A)$. Then we deduce from Lemma 2.3 that Y actually belongs to $\mathcal{A}$. However, since Y is a subset of $A \cup \text{Incl}(A)$, for every y in Y there exists a member a of A such that $y \le a$. In addition, $Y \setminus A \neq \emptyset$ since $Y \setminus A \supseteq X \setminus A$. Thus $Y \preceq A$ and $Y \neq A$, contradicting the choice of A. This contradiction finishes the proof of the lemma. □

Now we prove the main theorem of this section.

Proof of Theorem 2.1. Let $Q = (E, \le)$ be a partial order. We first consider the "only if" part. Let X be a fundamental antichain of Q such that X contains infinitely many critical elements. Suppose that Q has a canonical antichain A. Then by Lemma 2.1, $X \setminus A$ is finite. As a consequence, some critical element x of X belongs to A. Let Y be an infinite fundamental antichain such that $x < y$ for all y in Y. By Lemma 2.1 again, $Y \setminus A$ is finite. It follows that some y in Y belongs to A. Thus we have $\{x, y\} \subseteq A$ and $x < y$, which contradict the fact that A is an antichain. This contradiction proves that Q has no canonical antichain.

Next we consider the "if" part. Without loss of generality, we may assume that Q has infinite antichains. Let $\mathcal{A}$ be the set of all infinite fundamental antichains of Q. It follows from Lemma 2.4 that $\mathcal{A}$ is not empty. For any two members A_1 and A_2 of $\mathcal{A}$, let $A_1 \prec A_2$ if

(i) $A_1 \setminus A_2$ is finite but $A_2 \setminus A_1$ is infinite;
(ii) $A_1 \setminus A_2 \subseteq \text{Incl}(A_2 \setminus A_1)$; and
(iii) for every x in $\text{Incl}(A_2 \setminus A_1)$, there exist infinitely many y in $A_2 \setminus A_1$ for which $x < y$.

Then let $A_1 \preceq A_2$ if $A_1 = A_2$ or $A_1 \prec A_2$. We claim that $\mathcal{Q} = (\mathcal{A}, \preceq)$ is a partial order.

Obviously, $\preceq$ is reflexive. It is also obvious from (i) that $\preceq$ is antisymmetric. To show that $\preceq$ is transitive, we only consider the case when $A_1 \prec A_2 \prec A_3$ because all other cases are clear. We shall prove that $A_1 \prec A_3$ by verifying (i), (ii), and (iii). For every x in $\text{Incl}(A_2 \setminus A_1)$, let $A_2(x)$ be the set of elements y of $A_2 \setminus A_1$ for which $x < y$. We define $A_3(x)$ similarly for every x in $\text{Incl}(A_3 \setminus A_2)$. Since $A_1 \prec A_2 \prec A_3$, each $A_2(x)$ and each $A_3(x)$ must be infinite. Observe that $X \setminus Y \subseteq (X \setminus Z) \cup (Z \setminus Y)$ holds for all sets X, Y and Z. In particular, we have

(a) $A_1 \setminus A_3 \subseteq (A_1 \setminus A_2) \cup (A_2 \setminus A_3)$,
(b) $A_3 \setminus A_2 \subseteq (A_3 \setminus A_1) \cup (A_1 \setminus A_2)$,
(c) $A_2 \setminus A_1 \subseteq (A_2 \setminus A_3) \cup (A_3 \setminus A_1)$, and
(d) $A_3 \setminus A_1 \subseteq (A_3 \setminus A_2) \cup (A_2 \setminus A_1)$.

From (a) we deduce that $A_1 \setminus A_3$ is finite, and from (b) we deduce that $A_3 \setminus A_1$ is infinite, which establishes (i) for A_1 and A_3. By (a) we have $A_1 \setminus A_3 \subseteq \text{Incl}(A_2 \setminus A_1) \cup \text{Incl}(A_3 \setminus A_2)$, so every x in $A_1 \setminus A_3$ belongs to either $\text{Incl}(A_2 \setminus A_1)$ or $\text{Incl}(A_3 \setminus A_2)$. In the first case we deduce from (c) that $A_2(x) \cap (A_3 \setminus A_1) \neq \emptyset$ and in the second case we deduce from (b) that $A_3(x) \cap (A_3 \setminus A_1) \neq \emptyset$. Thus we have $x \in \text{Incl}(A_3 \setminus A_1)$ in both cases and so $A_1 \setminus A_3 \subseteq \text{Incl}(A_3 \setminus A_1)$. This proves (ii) for A_1 and A_3. Next we conclude from (d) that $\text{Incl}(A_3 \setminus A_1) \subseteq \text{Incl}(A_3 \setminus A_2) \cup \text{Incl}(A_2 \setminus A_1)$. It follows that every x in $\text{Incl}(A_3 \setminus A_1)$ is a member of either $\text{Incl}(A_3 \setminus A_2)$ or $\text{Incl}(A_2 \setminus A_1)$. But again, in the first case we deduce from (b) that $A_3(x) \cap (A_3 \setminus A_1)$ is infinite, and in the second case we deduce from (c) that $A_2(x) \cap (A_3 \setminus A_1)$ is infinite. Hence (iii) holds for A_1 and A_3. This completes the proof of $A_1 \prec A_3$, which means that $\preceq$ is transitive, and so $\mathcal{Q}$ is a partial order as we claimed.

Since a subset of a fundamental antichain is also fundamental, we conclude from (iii) and the choice of $\mathcal{A}$ that
(1) *if* $A_1 \prec A_2$ *then all elements of* $\text{Incl}(A_2 \setminus A_1)$ *are critical.*

Next we prove that every chain $\mathcal{C}$ of $\mathcal{Q}$ has an upper bound. Since $\mathcal{A}$ is not empty, we may assume that $\mathcal{C}$ is not empty either. Without loss of generality, we may also assume that for every member A of $\mathcal{C}$ there exists another member X of $\mathcal{C}$ such that $A \prec X$. For any $A \in \mathcal{C}$, let $\mathcal{C}_A = \{X : X \in \mathcal{C} \text{ and } A \preceq X\}$, $F_A = \cup\{A \setminus X : X \in \mathcal{C}_A\}$, and $A^* = \cup\{A \setminus F_A : A \in \mathcal{C}\}$. We claim that A^* is an upper bound of $\mathcal{C}$.

We first make a few observations. From (ii) and (1) it is clear that each F_A is a set of critical elements. Since $F_A \subseteq A$, we conclude that
(2) *each* F_A *is finite.*

Moreover,

$$A \setminus F_A = A \cap \overline{F_A} = A \cap \left(\bigcap_{X \in \mathcal{C}_A} \overline{A \setminus X} \right) = \bigcap_{X \in \mathcal{C}_A} (A \cap \overline{A \setminus X}) = \bigcap_{X \in \mathcal{C}_A} (A \cap X) = \bigcap_{X \in \mathcal{C}_A} X.$$

Immediately, we have

(3) *if* A_1 *and* A_2 *are members of* $\mathcal{C}$ *for which* $A_1 \preceq A_2$, *then* $A_1 \setminus F_{A_1} \subseteq A_2 \setminus F_{A_2}$.

Now we show that A^* is in $\mathcal{A}$. Since $A \setminus F_A$ is an antichain for every A in $\mathcal{C}$, we conclude from (3) that A^* is an antichain. By the assumption $\mathcal{C} \neq \emptyset$ we certainly can fix a member A of $\mathcal{C}$. Then we deduce from $A^* \supseteq A \setminus F_A$ and (2) that A^* is infinite. Clearly, $X \setminus F_X \subseteq X \subseteq (X \setminus A) \cup A$ for every X in $\mathcal{C}$. Let $Z = \cup\{\mathrm{Incl}(X \setminus A) : X \in \mathcal{C}_A\}$. Then by (3) we have

$$\mathrm{Incl}(A^*) = \mathrm{Incl}\left(\bigcup_{X \in \mathcal{C}_A} X \setminus F_X\right) \subseteq \bigcup_{X \in \mathcal{C}_A} \mathrm{Incl}((X \setminus A) \cup A) = Z \cup \mathrm{Incl}(A).$$

Notice that A is fundamental and by (1), all elements of Z are critical. Thus all antichains of $\mathrm{Incl}(A^*)$ are finite, which finish the proof of $A^* \in \mathcal{A}$.

Let $A \in \mathcal{C}$. We now prove that $A \prec A^*$ by verifying (i), (ii), and (iii). From our assumption on $\mathcal{C}$ we know that there exists a member B of $\mathcal{C}$ for which $A \prec B$. Clearly, $B \setminus A$ is infinite by (i), and F_B is finite by (2). Thus, as $A^* \setminus A \supseteq (B \setminus F_B) \setminus A = (B \setminus A) \setminus F_B$, we conclude that $A^* \setminus A$ is infinite. On the other hand, as $A \setminus A^* \subseteq A \setminus (A \setminus F_A) \subseteq F_A$, it follows from (2) that $A \setminus A^*$ is finite, so (i) is verified for A and A^*. Now for every x in $A \setminus A^*$, we prove that $x \in \mathrm{Incl}(A^*)$. Since $A \setminus A^* \subseteq F_A$, there must be a member C in $\mathcal{C}_A$ such that $x \in A \setminus C$. From (ii) we deduce that $x \in \mathrm{Incl}(C \setminus A)$. Then we deduce from (iii) that there are infinitely many elements y of $C \setminus A$ for which $x < y$. Since F_C is finite, there must be an element y of $C \setminus F_C$ with $x < y$. This means that $x \in \mathrm{Incl}(C \setminus F_C)$, and thus $x \in \mathrm{Incl}(A^*)$ as we wanted. Since A is an antichain and $x \in A$, it follows that $x \in \mathrm{Incl}(A^* \setminus A)$, which proves (ii) for A and A^*. Finally, we prove that (iii) holds for A and A^*. From (3) it follows that

$$\mathrm{Incl}(A^* \setminus A) = \mathrm{Incl}\left(\bigcup_{X \in \mathcal{C}_A} (X \setminus F_X) \setminus A\right) \subseteq \bigcup_{X \in \mathcal{C}_A} \mathrm{Incl}(X \setminus A).$$

Therefore, if x is in $\mathrm{Incl}(A^* \setminus A)$, then x is in $\mathrm{Incl}(X \setminus A)$ for some X in $\mathcal{C}_A$. It follows that there are infinitely many elements y of $X \setminus A$ for which $x < y$. Thus, as F_X is finite, $x < y$ for infinitely many y in $(X \setminus F_X) \setminus A$, which implies that $x < y$ for infinitely many y in $A^* \setminus A$. The proof of $A \prec A^*$ is complete.

By Zorn's lemma, $\mathcal{Q}$ has a maximal element A. We shall prove that A is a canonical antichain of Q and this will finish the proof of Theorem 2.1. For a contradiction, suppose that A has a finite subset A_0 such that $\mathrm{Excl}(A \setminus A_0)$ has an infinite antichain. Then we may choose A_0 with this property so that $|A_0|$ is as small as possible. Let $E' = \mathrm{Excl}(A \setminus A_0) \setminus \mathrm{Incl}(A_0)$. Since $A_0 \subseteq A$ and A is fundamental, E' must also have an infinite antichain. By Lemma 2.4 there exists an infinite maximal antichain B of $(E', \leq)$ such that every infinite antichain of $B \cup \mathrm{Incl}_{E'}(B)$ is a subset of B, where $\mathrm{Incl}_{E'}(B) = E' \cap \mathrm{Incl}(B)$. Let $B_0 = B \cap \mathrm{Incl}(A)$ and $X = (A \setminus A_0) \cup (B \setminus B_0)$. For any $a \in A \setminus A_0$ and $b \in B \setminus B_0$, we deduce from $B \subseteq E' \subseteq \mathrm{Excl}(A \setminus A_0)$ that $a \not\leq b$, and we deduce from $(B \setminus B_0) \cap \mathrm{Incl}(A) = \emptyset$ that $b \not< a$. Thus X is an antichain. Since A is infinite and A_0 is finite, X must be infinite. In addition, from $B \subseteq E' \subseteq \mathrm{Excl}(A \setminus A_0)$ we deduce that $\mathrm{Incl}(B) \subseteq \mathrm{Excl}(A \setminus A_0) = E' \cup \mathrm{Incl}(A_0)$, and so $\mathrm{Incl}(B) \subseteq \mathrm{Incl}_{E'}(B) \cup \mathrm{Incl}(A_0)$. It follows that $\mathrm{Incl}(X) \subseteq \mathrm{Incl}_{E'}(B) \cup \mathrm{Incl}(A)$. But this implies that all antichains of $\mathrm{Incl}(X)$ are finite, and so we conclude that $X \in \mathcal{A}$.

Now we deduce the desired contradiction by showing that $A \prec X$. Since $A \setminus X \subseteq A_0$, the set $A \setminus X$ must be finite. On the other hand, since A is fundamental, B_0 must be finite and so we deduce from $X \setminus A \supseteq B \setminus (A_0 \cup B_0)$ that $X \setminus A$ is infinite. This proves (i) for A and X. By the minimality of $|A_0|$ we must have $B \cap A_0 = \emptyset$. Thus $A \setminus X = A_0$ and $X \setminus A = B \setminus B_0$. As $A_0 \subseteq E'$ and B is a maximal antichain of E', every element a of A_0 must be comparable with some element b of B. From $B \cap A_0 = \emptyset$ we deduce that $a \neq b$, and from $B \cap \mathrm{Incl}(A_0) \subseteq E' \cap \mathrm{Incl}(A_0) = \emptyset$ we deduce that $b \not< a$. This means that $a < b$. But $b \notin B_0$, for otherwise $a < b \leq a'$ for some $a' \in A$, contradicting the fact that A is an antichain. This contradiction proves that $A_0 \subseteq \mathrm{Incl}(B \setminus B_0)$, which is $A \setminus X \subseteq \mathrm{Incl}(X \setminus A)$, so (ii) is established for A and X. Finally, we show that (iii) holds for A and X. From $A_0 \subseteq E'$ and $A_0 \subseteq \mathrm{Incl}(B \setminus B_0)$ it is clear that $A_0 \subseteq \mathrm{Incl}_{E'}(B)$. Then recall $\mathrm{Incl}(B) \subseteq \mathrm{Incl}_{E'}(B) \cup \mathrm{Incl}(A_0)$. It follows that every x in $\mathrm{Incl}(X \setminus A) = \mathrm{Incl}(B \setminus B_0)$ is in $\mathrm{Incl}_{E'}(B)$ or $\mathrm{Incl}(A_0)$. Let $y = x$ in first case and let $y \in A_0 \subseteq \mathrm{Incl}_{E'}(B)$ with $y \geq x$ in the second case. So we have $y \in \mathrm{Incl}_{E'}(B)$ and $y \geq x$ in both cases. Let B_y be the set of elements b of B for which $y < b$. If B_y is finite, then $(B \setminus B_y) \cup \{y\}$ is an infinite antichain of $B \cup \mathrm{Incl}_{E'}(B)$ that is not contained in B, a contradiction. Consequently, B_y, and thus $B_y \setminus B_0$, is infinite, and $x \leq y < b$ for all b in $B_y \setminus B_0 \subseteq B \setminus B_0 = X \setminus A$. Therefore, $A \prec X$ is proved.

However, A was chosen as a maximal element of $\mathcal{Q}$. Thus X, and hence A_0 does not exist. Therefore, A is indeed a canonical antichain and the proof of Theorem 2.1 is complete. $\square$

3. Identifying the obstacles

Let $Q = (E, \leq)$ be a fixed partial order. In this section, we complete the proof of Theorem 1.1 by showing that if Q has an infinite fundamental antichain for which all elements are critical, then Q contains some Q_k in the way as describes in the theorem. We first prove a few lemmas.

Lemma 3.1. *Let A and B be antichains of Q with $A \subseteq \mathrm{Excl}(B)$. If B is infinite and fundamental, then B has an infinite subset B' such that for every $a \in A$, either a is incomparable with any $b \in B'$ or $a < b$ for all $b \in B'$.*

Proof. Let $A_0 = A \cap \mathrm{Incl}(B)$. Since A is an antichain and B is fundamental, A_0 must be finite. We prove the lemma by induction on $|A_0|$. The result is clear if $|A_0| = 0$ since we may take B' to be B. Suppose that $A_0 \neq \emptyset$. Let $a \in A_0$ and let B_a be the set of elements in B that are comparable with a. Clearly, $a < b$ for all b in B_a. If B_a is finite, replace B by $B \setminus B_a$, then the result follows from our induction hypothesis. If B_a is infinite, then replace A and B by $A \setminus \{a\}$ and B_a, respectively. Again, the result follows from our induction hypothesis. $\square$

Lemma 3.2. *Let A and B be disjoint subsets of E. Then either there exists a finite subset C of $A \cup B$ such that no element in $A \setminus C$ is comparable with any element in $B \setminus C$, or there exist $\{a_i : i \in \mathbb{N}\} \subseteq A$ and $\{b_i : i \in \mathbb{N}\} \subseteq B$ such that a_i and b_i are comparable for all $i \in \mathbb{N}$.*

Proof. Let G be the (possibly infinite) graph with vertex set $A \cup B$ and edges ab, for all $a \in A$ and $b \in B$ such that a and b are comparable. If G has an infinite matching $\{a_i b_i : i \in \mathbb{N}\}$ then the second alternative in the lemma holds. Else G has a finite maximal matching $\{a_i b_i : i = 1, 2, \ldots, k\}$. Then the first alternative in the lemma holds since we may take $C = \{a_i, b_i : i = 1, 2, \ldots, k\}$. $\square$

To prove the next lemma, we need a classical result of Ramsey. For any set X, let X^2 denote the set of two-element subsets of X.

Ramsey's Theorem. *If $\{P_1, \ldots, P_p\}$ is a partition of $\mathbb{N}^2$, where $p \in \mathbb{N}$, then $\mathbb{N}$ has an infinite subset X such that $X^2 \subseteq P_i$, for some i.*

Lemma 3.3. *Let $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$ be disjoint subsets of E such that each a_i is comparable with b_i. Then there exists an infinite subset M of $\mathbb{N}$ that satisfies at least one of the following:*

(i) *for all $i, j \in M$, a_i is comparable with b_j if and only if $i = j$;*
(ii) *for all $i, j \in M$, a_i is comparable with b_j if and only if $i \geq j$;*
(iii) *for all $i, j \in M$, a_i is comparable with b_j if and only if $i \leq j$;*
(iv) *for all $i, j \in M$, a_i is comparable with b_j.*

Proof. For any two members i and j of $\mathbb{N}$ with $i < j$, it is clear that the following are all the possibilities concerning the comparison relations between a_i and b_j, and between a_j and b_i.

(1) a_i and b_j are not comparable, and a_j and b_i are not comparable either;
(2) a_i and b_j are not comparable, but a_j and b_i are comparable;
(3) a_i and b_j are comparable, but a_j and b_i are not comparable;
(4) a_i and b_j are comparable, and a_j and b_i are comparable as well.

For $k = 1, 2, 3, 4$, let P_k be the set of those pairs $\{i, j\}$ that satisfy (k). Then it is clear that $\{P_1, P_2, P_3, P_4\}$ is a partition of $\mathbb{N}^2$. By Ramsey's Theorem, $\mathbb{N}$ has an infinite subset M such that $M^2 \subseteq P_k$ for some k. Now it is straightforward to verify that for $k = 1, 2, 3$, and 4, M satisfies (i), (ii), (iii), and (iv), respectively. $\square$

Let X be an infinite fundamental antichain and let $\mathcal{A}$ be a countable set of infinite fundamental antichains. Then $\mathcal{A}$ is called X-*nice* if members of $\mathcal{A}$ can be listed as $A_1, A_2, \ldots$ and there is an infinite sequence $x_1, x_2, \ldots$ of distinct elements of X such that for every $a \in \cup\{A_i : i \in \mathbb{N}\}$, a is comparable with x_i if and only if $x_i < a$ and $a \in A_j$ for some $j \geq i$. The set $\{x_i : i \in \mathbb{N}\}$ shall be denoted by $X_{\mathcal{A}}$, and $x_1, x_2, \ldots$ and $A_1, A_2, \ldots$ shall be referred as *nice permutations* of $X_{\mathcal{A}}$ and $\mathcal{A}$, respectively. The following is a proposition on nice sets. It is an easy corollary of the above definition and so we omit its proof.

Lemma 3.4. *Suppose that $\mathcal{A}$ is X-nice. If $\mathcal{A}_1$ is an infinite subset of $\mathcal{A}$, then $\mathcal{A}_1$ is also X-nice. If each A in $\mathcal{A}$ has an infinite subset B_A and $\mathcal{B}_{\mathcal{A}} = \{B_A : A \in \mathcal{A}\}$, then $\mathcal{B}_{\mathcal{A}}$ is X-nice as well.*

The next is the last lemma we need in proving our main theorem.

Lemma 3.5. *Let $A, B_1, B_2, \ldots$ be an infinite sequence of infinite fundamental antichains. Suppose that for every $a \in A$ and every B_i, either a is incomparable with any $b \in B_i$ or $a < b$ for all $b \in B_i$. Then at least one of the following holds:*

(a) *there exist a set $\mathcal{A}$ of pairwise disjoint fundamental antichains of Q such that the partitioned partial order $(\mathcal{A}, \leq)$ is isomorphic to $(\mathcal{S}, \leq_1)$;*
(b) *there exists a finite subset A_0 of A and a finite subset N_0 of $\mathbb{N}$ such that no element in $A \setminus A_0$ is comparable with any element in $\cup\{B_i : i \notin N_0\}$;*
(c) *there exists an infinite subset M of $\mathbb{N}$ such that $\{B_i : i \in M\}$ is A-nice.*

Proof. For each i in $\mathbb{N}$, take an element b_i from B_i as its representative. Since each B_i is infinite, it is not difficult to see that these representatives can be chosen so that they are distinct. Let $B^* = \{b_i : i \in \mathbb{N}\}$. It is clear from our assumption that $A \cap B_i = \emptyset$ for all i, and thus $A \cap B^* = \emptyset$. Suppose that there is a finite subset C of $A \cup B^*$ such that no element in $A \setminus C$ is comparable with any element in $B^* \setminus C$. Then (b) holds if we take $A_0 = A \cap C$ and $N_0 = \{i \in \mathbb{N} : b_i \in C\}$. By Lemma 3.2, we may assume that there exist $\{a_j : j \in \mathbb{N}\} \subseteq A$ and $\{b_{i_j} : j \in \mathbb{N}\} \subseteq B^*$ such that $a_j < b_{i_j}$ for all $j \in \mathbb{N}$. Let us apply Lemma 3.3 to $\{a_j : j \in \mathbb{N}\}$ and $\{b_{i_j} : j \in \mathbb{N}\}$ and let M be the infinite set produced by the lemma, which satisfies one of (i)–(iv). Let us assume that $m_1 < m_2 < \cdots$ are the members of M. Since A is an antichain and each B_i is fundamental, it is clear that neither (ii) nor (iv) occurs. If M satisfies (i), then (a) holds since we can take $\mathcal{A} = \{A_t : t \in \mathbb{N}^*\}$ with $A_0 = \{a_{m_k} : k \in \mathbb{N}\}$ and $A_t = B_{i_{m_t}}$ for all $t \in \mathbb{N}$. If M satisfies (iii), it is straightforward to verify that (c) holds with our choice of M. □

Finally, we are ready to prove the main theorem of this paper.

Proof of Theorem 1.1. We first consider the "if" part. Suppose that Q has disjoint fundamental antichains $A_0, A_1, \ldots$ such that the partitioned quasi-order $(\{A_i : i \geq 0\}, \leq)$ is isomorphic to some $(\mathcal{S}, \leq_k)$. Without loss of generality, let us assume that each A_i is mapped into S_i. Then it is not difficult to see that A_0 consists of critical elements. Thus by Theorem 2.1 Q has no canonical antichains.

Next we consider the "only if" part. In the following proof, we assume that there does not exist a set $\mathcal{A}$ of disjoint fundamental antichains for which the partitioned partial order $(\mathcal{A}, \leq)$ is isomorphic to $(\mathcal{S}, \leq_1)$. From Theorem 2.1 we deduce that Q has an infinite fundamental antichain A for which all elements are critical. Let x be an element of A. Then there exists an infinite fundamental antichain B_x such that $x < b$ for all b in B_x. Clearly, since A is an antichain, there are no $b \in B_x$ and $a \in A$ with $b \leq a$. Equivalently, $A \subseteq \text{Excl}(B_x)$, for all x in A. By Lemma 3.1 we may assume that for any a and x in A, either $a < b$ for all $b \in B_x$ or a is incomparable with any $b \in B_x$. It follows from Lemma 3.5 and the choice of the family $\{B_x : x \in A\}$ that (c) must hold. In other words, there are distinct elements $a_1, a_2, \ldots$ of A such that $\{B_{a_i} : i \in \mathbb{N}\}$ is A-nice. Let us denote each B_{a_i} by B_i. Without loss of generality, let us assume that

(1) $A = \{a_i : i \in \mathbb{N}\}$, *the set* $\{B_i : i \in \mathbb{N}\}$ *is* A*-nice, and* $a_1, a_2, \ldots$ *and* $B_1, B_2, \ldots$ *are nice permutations of* A *and* $\{B_i : i \in \mathbb{N}\}$, *respectively.*

Let i and j be indices for which $i < j$. Since no $x \in B_i$ is comparable with a_j yet $a_j < y$ for all $y \in B_j$, it follows that there are no $x \in B_i$ and $y \in B_j$ with $y \leq x$. Thus by applying Lemma 3.1 repeatedly and by Lemma 3.4 we may assume that

(2) *if* $i < j$ *and* $x \in B_i$, *then either* $x < y$ *for all* $y \in B_j$ *or* x *is incomparable with any* $y \in B_j$.

Next we construct a sequence $A_0, A_1, A_2, \ldots$ of infinite fundamental antichains such that for every index i, either $\{A_j : j > i\}$ is A_i-nice or no element of A_i is comparable with any element in $\cup\{A_j : j > i\}$. We shall define this sequence inductively. In the ith step, $A_0, \ldots, A_{i-1}$ should have been defined. Also, there should be an infinite set J_{i-1} of indices such that the following induction hypothesis is satisfied.

(IH) *For* $i' = 0, 1, \ldots, i-1$, *let* $\mathcal{A}_{i',i-1} = \{A_j : i' < j \leq i-1\} \cup \{B_j : j \in J_{i-1}\}$. *Then either* $\mathcal{A}_{i',i-1}$ *is* $A_{i'}$*-nice or no element of* $A_{i'}$ *is comparable with any element in* $\cup\{X : X \in \mathcal{A}_{i',i-1}\}$.

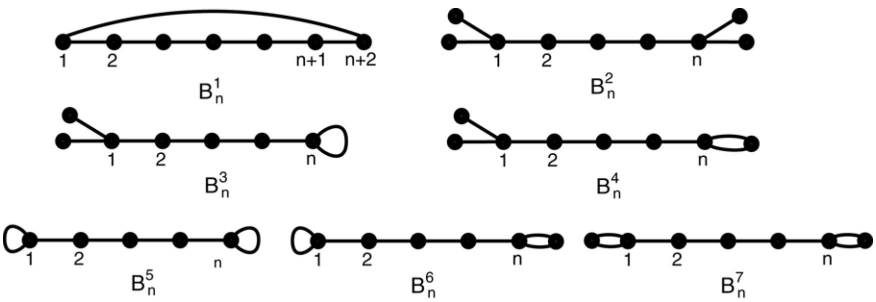

Fig. 2. A canonical antichain for the subgraph relation.

Let $A_0 = A$ and $J_0 = \mathbb{N}$. Then it is clear from (1) that (IH) is satisfied for $i = 1$. Now we consider the general step. Let us assume that $j_0 < j_1 < j_2 < \cdots$ are all members of J_{i-1}. By (2) we may apply Lemma 3.5 to $B_{j_0}, B_{j_1}, B_{j_2}, \ldots$ and we conclude that (b) or (c) holds. If (b) holds, then there exists a finite subset C of B_{j_0} and a finite subset K of $J_{i-1} \setminus \{j_0\}$ such that no element of $B_{j_0} \setminus C$ is comparable with any element of $\cup\{B_j : j \in (J_{i-1} \setminus \{j_0\}) \setminus K\}$. Let $A_i = B_{j_0} \setminus C$ and $J_i = (J_{i-1} \setminus \{j_0\}) \setminus K$. Then, as $\mathcal{A}_{i',i} \subseteq \mathcal{A}_{i',i-1}$, it is clear from Lemma 3.4 that (IH) is satisfied for $i + 1$. If (c) holds instead, then $\{B_j : j \in K\}$ is B_{j_0}-nice for some infinite subset K of $J_{i-1} \setminus \{j_0\}$. In this case, let $A_i = B_{j_0}$ and $J_i = K$. Again, as $\mathcal{A}_{i',i} \subseteq \mathcal{A}_{i',i-1}$, it is clear from Lemma 3.4 that (IH) is satisfied for $i + 1$. Therefore, the required sequence $A_0, A_1, \ldots$ can be generated. It is straightforward to see that this sequence has the required properties.

For each index i, let $\mathcal{A}_i = \{A_j : j > i\}$. Clearly, there are two kinds of terms A_i: either $\mathcal{A}_i$ is A_i-nice or no element of A_i is comparable with any element of $\cup\{A_j : j > i\}$. We shall refer them as type-1 and type-2, respectively. By our construction, A_0 is type-1. If there are infinitely many type-1 terms, then by Lemma 3.4 we may assume that all terms are type-1, and in addition, $(A_i)_{\mathcal{A}_i} = A_i$ for all i. Now it is not difficult to see that the partitioned partial order $(\{A_i : i \geq 0\}, \leq)$ is isomorphic to $(\mathcal{S}, \leq_3)$. On the other hand, if there are infinitely many type-2 terms, then, without loss of generality, we may assume that all terms, except the first, are type-2 terms. Also without loss of generality, we may assume that $(A_0)_{\mathcal{A}_0} = A_0$. In this case, it is not difficult to see that the partitioned partial order $(\{A_i : i \geq 0\}, \leq)$ is isomorphic to $(\mathcal{S}, \leq_3)$. Thus the proof of Theorem 1.1 is complete. □

4. Graphs

Let $\mathcal{G}$ be the class of all finite graphs and let $\preceq$ be a graph containment relation. If $\preceq$ is the minor relation, it has been proved by Robertson and Seymour in a long series of papers [5] that $(\mathcal{G}, \preceq)$ has no infinite antichains. In other words, the empty set is one of its canonical antichains. Next, let $\mathcal{B}_0 = \{B^i_n : 1 \leq i \leq 2 \text{ and } n \geq 1\}$ and $\mathcal{B} = \{B^i_n : 1 \leq i \leq 7 \text{ and } n \geq 1\}$, where each B^i_n is the graph illustrated in Fig. 2. If $\preceq$ be the subgraph relation, it has been proved by the author in [1] that $\mathcal{B}_0$ is a canonical antichain of the class of all finite simple graphs. We remark that, by adopting the techniques that are used in [1], one can show that $\mathcal{B}$ is a canonical antichain of $(\mathcal{G}, \preceq)$.

Suppose that $\preceq$ is the induced subgraph relation. Then the situation is totally different. Our goal in this section is to show the following result.

Theorem 4.1. $(\mathcal{G}, \preceq)$ *has no canonical antichains.*

We prove this theorem using results established in earlier sections. In particular, we need to construct antichains and show that they are fundamental. This will be the main step in our proof, which requires some preparations.

For any set E, let E^* denote the set of all finite sequences of members of E. Suppose that $\leq$ is a binary relation on E. Then we define a binary relation $\leq^*$ on E^* as follows. For any two members $e = [e_1, \ldots, e_s]$ and $f = [f_1, \ldots, f_t]$ of E^*, let $e \leq^* f$ if there exist indices $i_1, i_2, \ldots, i_s$ such that $1 \leq i_1 < i_2 < \cdots < i_s \leq t$ and $e_1 \leq f_{i_1}, e_2 \leq f_{i_2}, \ldots, e_s \leq f_{i_s}$. It is easy to see that $Q^* = (E^*, \leq^*)$ is a quasi-order if $Q = (E, \leq)$ is. As usual, we call Q a *well-quasi-order* (a *wqo*) if it is a well-founded quasi-order with no infinite antichains. The following is a classical result of Higman [2].

Higman's Theorem. Q^* *is a wqo if* Q *is.*

The following lemma is an immediate corollary of this theorem.

Lemma 4.1. *Let $\mathcal{H}$ be an ideal of $(\mathcal{G}, \preceq)$ and let $\mathcal{H}_0$ be the class of connected graphs in $\mathcal{H}$. Then $(\mathcal{H}, \preceq)$ is a wqo if and only if $(\mathcal{H}_0, \preceq)$ is.*

Proof. The "only if" part is trivial. As for the "if" part, let us notice that members of $\mathcal{H}$ can be considered as finite sequences of members of $\mathcal{H}_0$. Then the result follows from Higman's Theorem immediately. □

Notice that if a well-founded quasi-order has a canonical antichain then all its ideals have canonical antichains. Therefore, to prove the nonexistence of a canonical antichain we only need to prove it for a special ideal. To prove Theorem 4.1 we choose an ideal $\mathcal{F}$ that consists of only simple graphs and, for every $F \in \mathcal{F}$ there exists a vertex v such that all connected components of $F \setminus v$ are paths.

Let k and n be positive integers. Let $F_{k,n}$ be the graph with vertex-set $\{0, 1, \ldots, n(k+1)+3\}$ and edge-set $\{(i, i+1) : i = 0, 1, \ldots, n(k+1)+2\} \cup \{(0, i(k+1)+2) : i = 0, 1, \ldots, n\} \cup \{(0, n(k+1)+3)\}$. Vertex 0 will be called the *center* of $F_{k,n}$. For all $k \in \mathbb{N}$, let $\mathcal{F}^k = \{F_{k,n} : n \in \mathbb{N}\}$.

Lemma 4.2. *$\mathcal{F}^k$ is a fundamental antichain, for all $k \in \mathbb{N}$.*

Proof. It is obvious that each $\mathcal{F}^k$ is an antichain. So we only need to show that $\mathrm{Incl}(\mathcal{F}^k)$ has no infinite antichains. Let $\mathcal{F}_0^k$ be the class of connected graphs in $\mathrm{Incl}(\mathcal{F}^k)$. By Lemma 4.1, we only need to show that $(\mathcal{F}_0^k, \preceq)$ is a wqo. For any graph $F \in \mathcal{F}^k$, a subgraph of F is *centered* if it contains the center of F. Let $\mathcal{F}_1^k$ be the class of centered graphs in $\mathcal{F}_0^k$ and let $\mathcal{F}_2^k = \mathcal{F}_0^k \setminus \mathcal{F}_1^k$. Clearly, $(\mathcal{F}_2^k, \preceq)$ is a wqo since $\mathcal{F}_2^k$ consists of paths. Therefore, to prove the lemma, we only need to show that $(\mathcal{F}_1^k, \preceq)$ is a wqo.

Let $Z = \{\mathrm{END}, 0, 1, 2, \ldots\}$. For any two members z and z' of Z, let $z \le z'$ if $z = z' = END$, or if both z and z' are integers such that $z \le z'$ in the ordinary sense. Clearly, $(Z, \le)$ is a wqo. It follows from Higman's Theorem that $((Z^*)^*, \le')$ is a wqo, where $\le'$ is $(\le^*)^*$. Next, we encode each graph $X \in \mathcal{F}_1^k$ as a member $p(X)$ of $(Z^*)^*$ such that $p(F) \le' p(G)$ implies $F \preceq G$ for all graphs F and G of $\mathcal{F}_1^k$. The existence of such a coding shows that $(\mathcal{F}_1^k, \preceq)$ is a wqo, which will prove the lemma.

Let X be a graph in $\mathcal{F}_1^k$. Let $n \in \mathbb{N}$ such that X is an induced subgraph of $F_{k,n}$. Let $X_1, \ldots, X_m$ be the connected components of $X \setminus 0$. We define $p(X)$ to be the sequence $[p(X_1), \ldots, p(X_m)]$, where each $p(X_i)$ is defined to be $[\alpha_i, \beta_i, \gamma_i]$ as follows. Let a_i and b_i be integers such that $V(X_i) = \{j : a_i \le j \le b_i\}$. Let N_i be the set of vertices of $V(X_i)$ that are adjacent to 0. It is clear from the choice of X that N_i is not empty. Let $c_i = \min\{j : j \in N_i\}$ and $d_i = \max\{j : j \in N_i\}$. Then we define

(1) $\beta_i = |N_i|$;
(2) $\alpha_i = END$ if $a_i = 1$, and $\alpha_i = c_i - a_i$ if $a_i \ne 1$;
(3) $\gamma_i = END$ if $b_i = n(k+1)+3$, and $\gamma_i = b_i - d_i$ if $b_i \ne n(k+1)+3$.

Let F and G be graphs in $\mathcal{F}_1^k$ with $p(F) \le' p(G)$. Let $p(F) = [p(F_1), \ldots, p(F_s)]$ and $p(G) = [p(G_1), \ldots, p(G_t)]$. Then there exist indices $i_1, \ldots, i_s$ such that $1 \le i_1 < \cdots < i_s \le t$ and $p(F_1) \le^* p(G_{i_1}), \ldots, p(F_s) \le^* p(G_{i_s})$. Now it is straightforward to verify that $F_j^+ \preceq G_{i_j}^+$ for all j, where F_j^+ (or $G_{i_j}^+$) is the subgraph of F (or G) induced by $V(F_j) \cup \{0\}$ (or $V(G_{i_j}) \cup \{0\}$). It follows that $F \preceq G$ and thus the lemma is proved. □

Proof of Theorem 4.1. Let $\mathcal{F}^0$ be the class of all cycles of length at least four. It is easy to see (using Lemma 4.1) that $\mathcal{F}^0$ is a fundamental antichain of $(\mathcal{G}, \preceq)$. Notice that the partitioned quasi-order $(\{\mathcal{F}^k : k \in \mathbb{N}^*\}, \preceq)$ is isomorphic to $(\mathcal{S}, \le_1)$. Hence the result follows from Theorem 1.1 and Lemma 4.2. □

Remark. From the proof of Theorem 4.1 it is not difficult to see that many ideals $\mathcal{F}$ of $(\mathcal{G}, \preceq)$ do not have canonical antichains either. For instance, if $\mathcal{F}_2$ is the class of bipartite graphs in $\mathcal{F}$, then no antichain of $\mathcal{F}_2$ is canonical. Clearly, this observation can be generalized as follows. For any $t \in \mathbb{N}$, if $\mathcal{F}_t$ is the class of graphs $F \in \mathcal{F}$ such that the length of all cycles of F are multiples of t, then $\mathcal{F}_t$ has no canonical antichains.

Acknowledgements

The author's research was partially supported by NSA grant H98230-05-1-0081, NSF grants DMS-0556091, and NSF grant ITR-0326387.

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