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A Study of Mathematical Equivalence: The Importance of the Equal Sign

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A STUDY OF MATHEMATICAL EQUIVALENCE: THE IMPORTANCE OF THE EQUAL SIGN

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Abstract

The purpose of this study was to investigate students' understanding and knowledge of the equal sign, so that instructional resources could be identified to improve student's conceptual understanding about mathematical equivalence. A test, consisting of a combination of items taken from previous studies, as well as items developed by the researchers, was designed to gauge students' understanding of the equality symbol. The test was administered to 54 seventh-graders in Spring 2015. The results of the test indicated a significant number of students in our district have a limited understanding of mathematical equivalence. This papers ends with some suggested activities recommended to prepare students for formal courses in algebra by offering teachers opportunities to understand and develop their student's grasp of equality and the equal sign.

Chapter 1: Introduction

The equal sign is present in all levels of mathematics, yet as (Knuth et al. 2008) states, "research suggests that many students at all grade levels have not developed adequate understandings of the meaning of the equal sign." In algebra, it used together with variables in order to describe or express facts about numbers. For example, "$x^2 + x = 6$" is a statement about x, which can be translated into the statement "$x = 3$ or $x = 2$." Without proper understanding of the equal sign, it is virtually impossible to make sense of how the symbolic language of mathematics works and what it can express.

Numerous research studies support the idea that students' understanding of the equal sign can be placed into two basic categories:

- relational, the notion that the equal sign means that the two expressions on either side refer to the same quantity; or
- operational, the notion that the equal sign means "find the total" or the "and the answer is…" (McNeil et al., 2006).

Unfortunately, research suggests that many students fail to recognize the dual function of the equal sign. Instead many students recognize its function exclusively as an operator believing that it indicates where to write an answer (Knuth et al. 2008). They do not see it as a signal that the expressions on either side have the same value.

This has implications for their future work in algebra and the number system. "Limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding the equal sign as a relation" (Carpenter, Franke, and Levi 2003). In an attempt to connect students' prior

knowledge to new knowledge and to solidify students' understanding of equality, I designed my thesis to investigate students' understanding and knowledge of the equal sign. Based on this, I provide materials to improve instructional practices and foster a formal understanding of the equal sign in a middle school setting.

There are several contributing factors that perpetuate misconceptions of the equal sign, starting with a failure to define what the equal sign is and what is means in a middle school mathematics classroom and beyond. Knuth et al. (2006) states, "the way that mathematics has historically been taught in the United States – in elementary school (and in particular, arithmetic), students learn to reason about operations as procedures to follow, and they do not learn to reason about operations as expressions of quantitative relationships until middle school (and, in particular, pre-algebra). Yet, a relational view of the equal sign is essential to understanding that transformations performed in the process of solving an equation preserve the equivalence relation (i.e., the transformed equations are equivalent) – an idea that many students find difficult, and that is not an explicit focus of typical instruction" (Knuth et al., 2005).

Algebra has long been the focus of research and reform efforts in mathematics education. This attention can be attributed to growing concerns about student's lack of preparation for algebra and the importance algebra plays in readiness for college and employment. Current research highlights the need for students to be able to reason abstractly about the equal sign and understand that it works differently from other mathematical symbols that signal an operation to be performed (such as the addition, subtraction, multiplication, and division symbols, +, -, x, ÷). "If difficulties with the equal sign are due to knowledge built from early experience with arithmetic, then student's ability to acquire the relational concept of the equal sign may depend on the learning context" (Knuth et al., 2006). Activities centered on Common Core Standards in

Mathematics are designed to develop a deeper conceptual understanding by focusing on fewer topics, emphasizing coherence and rigor by developing an authentic command of mathematical concepts. Such activities may help to correct misunderstandings of the equal sign, integrating arithmetic and algebraic reasoning synergistically.

This paper will attempt to answer the following questions: What common misconceptions do students in my own school system share about the equal sign? What learning problems in algebra can be attributed to student's failure to understand what the equal sign expresses? What kind of understanding is necessary for a student to develop a sophisticated grasp of the equal sign and equality? And lastly, how can instructional materials be designed to reach this level of understanding?

1.1 History of the Equal Sign

The symbol (=) appears first in Robert Recorde's 1557 *The Whetstone of Witte*. In publications before Recorde's use of "a paire of parralles," to show equality, it was typically expressed by using one of the following written words: "*aequales, aequantur, esgale, faciunt, ghelijck*, or *gleich*, and sometimes by the abbreviated form *aeq"* (Cajori 1928). When we trace the historical evolution of the equal sign ("=") and its significance, we find that the adaptation and application of its use grew significantly after it appeared in works by English mathematicians Thomas Harriot, William Oughtred, and Richard Norwood.

Today, the equal sign is used in several different ways:

- As a "give the answer sign" in a lot of arithmetic instruction,
- As a sign saying that the two expressions on either side denote the same number (or function),

- As a symbol for making definitions or for stating formulas. For example:
  - One might write $A = 325{,}341$ to create an abbreviation, to avoid having to write the number over and over.
  - A possible definition of the function f is: $f(x) = x^2 + 1$.
  - To state a formula, as in "volume $V = lwh$, where l = length, w = width , h = height."
  - Finally, in computer programming, $x = 2$ causes the value 2 to be stored in the address x.

The subsequent chapters are arranged as follows. Chapter 2 reviews evidence known about how children tend to view the equal sign and how these views affect future performance and learning in mathematics. Chapter 3 describes the methods used to conduct the study, the demographics of the school and district, the participants involved, the assessment instrument used, and the data obtained. Chapter 4 discusses research findings, conclusions, and directions for future research. Chapter 5 contains a collection of activities identified to improve student's conceptual understanding about mathematical equivalence and the equal sign.

Chapter 2: Literature Review

This chapter begins with a review of what is known about how children tend to view the equal sign in and out of context. After this, we describe the research on how these views affect future performance and learning in mathematics.

Articles appearing in the review of literature were collected from the following online databases: Google Scholar, JSTOR, and Academic Search Complete using the following search terms: "equal sign," "equals sign," "equality," "misconceptions of the equal sign," and related phrases. Based on my search, the earliest discussions of the operation/relation misconception appear to be (Behr et al. 1976), and (Kieran 1981). I collected the articles that cited one of them. I include only peer-reviewed articles that have empirical findings (observational studies or experiments), focusing on primary and secondary education**.**

2.1 Different Interpretations of the Equal Sign

The algebraic representation of the equal sign can easily be interpreted two ways: operationally, as a "do something" symbol, or structurally, as a fixed relationship between two magnitudes. This directly corresponds to the extensively noted and discussed duality of interpretation of the meaning of equality. The "=" symbol can be understood as a "command" to carry out operations appearing on the left or, alternatively, as a sign of sameness, a symbol of identity.

Behr et al. (1976) is the first published study that investigates the misconception about equality and the equal sign. This study investigated how students in grades 1 to 6 interpret equality sentences. Through the use of unstructured individual interviews, the authors discovered that children do not tend to view the equal sign as relationship of sameness but as a

signal to carry out computations from left to right. When presented with problems of the form a + b = □, students "perceive it as a stimulus calling for answer to be placed in a box" (Behr et al. (1976). They found no evidence to suggest children's knowledge about equality progresses as they matriculate through the grades. The study finds that there is a strong propensity among the children "to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, -, etc.) precede it" (Behr et al., 1976). Children insisted that equations follow a particular from (e. g. 2 + 5 = 7), "rather than to reflect, make judgments, and infer meaning" (Behr et al., 1976). As a result, students struggle with making sense of equations that do not adhere to that particular form and tend to reject equations such as 15 = 7 + 8, 5 + 4 = 7 + 2, and 2 = 2. The authors say that this restrictive and rigid concept of equality exhibited with respect to written number sentences may inherently affect learning of other mathematical concepts.

Kieran (1981) uses a cross-sectional analysis to examine the handling of the equal sign among preschool through college students. She reports that the idea of the equal sign as a operator symbol, a "do something, starts even before formal education begins and continues throughout high school.[1]

Kieran (1981) begins by describing the intuitive behaviors of preschoolers determining the equality of two sets. Based on the observed behavior of children, Kieran suggests that before most students enter school, they have already developed an intuitive understanding of equal sign as an operator signal. When students enter grade school, they bring with them these preconceived notions, misconceptions, and intuitive inklings about equality that influence the way they perceive and encounter basic arithmetic operations, later undermining their success in,

[1] The examples of these misapplications illustrated below are common to the 7^{th} and 8^{th} grade students encountered during my teaching experience.

and understanding of algebra. These notions also influence their interpretation of symbolic expressions and equations when first introduced in grade school.

Most primary school children seem to have difficulty interpreting the equal sign as a formal relation, exhibiting a view of the equal sign as an operator symbol, or as the arrival of an answer. Kieran (1981) speculates that although children seem to learn to read and write elementary arithmetic symbolism with ease, they still do not necessarily understand it the same way as adults do. Consequently, we as adults have to be very careful when attempting to interpret children's symbolism, since unlike children, adults easily interpret equality sentences as equivalence relations (Kieran, 1981). Kieran seems to be suggesting that one of the problems children have in learning the relational meaning of the equals sign springs from the natural, inborn patterns of thinking, more than from the nature of the instruction they experience.

Kieran reports on a teaching experiment conducted by (Denmark et al., 1976). The authors tried to teach the concept of equality to first-graders before they encountered the + and = signs in school by using activities with a balance scale and the corresponding written equations. According the Kieran, "students were able to acquire some flexibility in accepting the use of the equal sign in a variety of sentence structures (e.g., $3 = 3$, $3 + 2 = 4 + 1$); however, the equal sign was still viewed as an operator, not a relational symbol. Furthermore, the data do not support the conjecture that if students (first graders) are provided with appropriate instructional experiences in which they encounter the use of the equal sign in a variety of sentence forms, they will acquire a conceptualization of equality as a relation between two names for a number" (Kieran, 1981).

Kieran also reports findings of another teaching experiment conducted by (Collis, 1974) that provides evidence that confirm students' inability to respond appropriately to the "name for a number" idea. According to Kieran, students cannot make sense of equations such as $4 + 5 = 3$

+ 6. “The child needs literally to be able “to see” a unique result before the operations on numbers mean anything to him, that is 4 + 5 = 3 + 6 must be written 4 + 5 = 9 or 3 + 6 = 9. After about age 13, Collis points out, the learner is willing to infer beyond physical models and to use specific cases for forming adequate generalizations” (Kieran, 1981).

Kieran also illustrates misuse of the equal sign during the transition period, “transition between requiring the answer after the equal sign and accepting the equal sign as a symbol for equivalence” (Kieran, 1981). For example, “The statement, 1063 + 217 = 1280 – 425, reflects a well-known pattern, that of writing down the operations in the order in which they are being thought and that keeping a running total” (Kieran, 1981).

Kieran (1981) provides multiple examples to highlight students’ misuse of the equal sign across all levels of learning. She provides evidence from previously documented research findings to suggest that students’ initial understandings of equality are based on intuitive notions of the equal sign as a “do something” symbol or as a symbol that indicates where to “put the answer” even before they begin formal school. She is also careful to note that these intuitive notions can be gradually transformed into a relational understanding of equality.

Baroody & Ginsburg (1983) study fifteen participants from a school of middle to upper class students in grades 1 through 3 investigating the impacts of a math curriculum developed by Wynroth (1975). The curriculum is individualized, consisting of sequences of games to focus on or two concepts at a time.

In contrast to Kieran, Baroody & Ginsburg (1983) suggest that children’s difficulties with equivalence are partly due to early experiences in mathematics, formally interpreting addition as a unidirectional process. These researchers also suggest “children expect written (horizontal) equations to have a particular form: an arithmetic problem consisting of two (or perhaps more)

terms on the left, the result on the right, and in between, a connecting ("equals") symbol (e.g., 4 + 5 = 9)" (Baroody & Ginsburg, 1983).

These authors go on to suggest that three fundamental properties of equivalence are required to establish a formal relational view of mathematical equality:

- "First a child would have to accept 13 = 7 + 6 as well as 7 + 6 = 13 (the property of symmetry).
- Second, a child would have to accept identity statements such as 8 = 8 (the reflexive property).
- Finally, if a child appreciated a familiar form such as 7 + 6 = 13 and could accept 13 = XIII, they should be able to deduce that 7 + 6 = XIII (the transitive property)" (Baroody & Ginsburg, 1983).

Baroody & Ginsburg are careful to use "equals" to mean "the same number." They wanted to avoid children learning of the 'equals' as 'the answer is'" (Baroody & Ginsburg, 1983). Their work provides evidence that concentrated conceptual instruction can improve children's understanding of mathematical equivalence. "Most children accepted equations such as 7 + 6 = 4 + 9, 2 + 4 = 3 × 2, and 7 + 6 = 14 – 1 as sensible without actually seeing a written result (answer)" (Baroody & Ginsburg, 1983).

Baroody and Ginsburg concluded that it might be beneficial for children to see arithmetic problems in nontraditional formats (e.g. 5 = 5, 2 + 3 = 4 +1, 7 = 4 + 3, etc.) before the traditional (e.g. 6 + 7 = 13), thereby solidifying a relational understanding of mathematical equivalence. Baroody and Ginsburg suggest that students' view of the equal sign could be developed into a more formal, rational, understanding, with appropriate instruction and exposure to atypical arithmetic problems from the start.

The work of Baroody and Ginsburg suggests that the ways that children interpret symbolic representations are strongly dependent on the context from previous learning experiences. Presently, it seems that the operational view of the equal sign is reinforced with each year of progression in primary school. A study by Knuth et al. (2008) revealed that the format of equations in most early education textbooks are presented with "operations on the left side of the equal sign" and a place for the answer on the right. This study suggest that as result of exposure to problems of this format students may be begin to favor less sophisticated interpretations of the equal sign. This impacts students' learning in algebra, interfering with learning the relational of equality. Knuth's study also showed that students at each grade level who tend to view of the equal sign as relational not only out-performed their counterparts who did not, but these students also employed more sophisticated strategies to solve items when presented questions typical of a beginning algebra course.

One implication from Knuth et al. (2008) is that "helping students acquire a view of the equal sign as a symbol that represents an equivalence relation between two quantities many, in turn, help prepare them for success in algebra (and beyond)." The authors attempted to implement this insight through professional development events for middles school teachers, where teachers were encouraged to "look for opportunities within their existing classroom practices to engage students in conversations about the equal sign as well as to create such opportunities intentionally" (Knuth et al. (2008). They advise teachers to look for naturally occurring opportunities to help foster a formal, relational, understanding of equivalence. "An example of 'naturally' occurring opportunity, most mathematics teachers have likely witnessed the equality 'strings' that students often produce (e.g., 3 + 5 = 8 + 2 = 10 + 5 = 15); these equality strings provide an excellent opportunity to discuss with students the meaning of the

equal sign and its proper use. To create opportunities intentionally, teachers might provide students with arithmetic (or algebraic) equations to solve in which numbers and operations appear on both sides of the equal sign" (Knuth et al. 2008). They suggest that the use of these strategies will foster more sophisticated and appropriate interpretations of the equality, increasing success in algebra courses.

2.2 The Importance of the Algebra

K–8 curricula have traditionally delayed the introduction of algebra until after arithmetic. One rationale for this is to give students a chance to acquire arithmetic skills thought to be needed for success in algebra. Nonetheless, most students still have difficulties with algebraic reasoning. Others rationalize the decision to delay algebra in terms of "presumed constrictions in students' cognitive competence" (Carraher, 2006).

Algebra, to some extent, is the study of patterns and structures that are present in arithmetic. Curricula that treat them as two distinct topics will leave students unable to draw connections between arithmetic practice and algebraic reasoning. Carraher (2006) says, "Transitional or 'pre-algebra' approaches attempt to ameliorate the strains imposed by a rigid separation of arithmetic and algebra. However, 'bridging or transitional proposals' are predicated on an impoverished view of elementary mathematics—impoverished in their postponement of mathematical generalization until the onset of algebra instruction."

Carraher et al. (2006) longitudinal study of students from grades 2 to 4, provides evidence that students as young as third grade, "can make use of algebraic ideas and representations that are typically omitted from early mathematics curriculum and thought to be cognitively beyond their reach" (Carraher et al. 2006). Carraher et al. (2006) documents this evidence by charting the growth and development of the students' conceptual understanding of

algebraic relations and notations, proving students capability to make generalizations about numbers. For example, Figure 1 shows one of the problems presented to students. Carraher et al. (2006) reported the following converstation between a student, who represented the initial amount in each bank with *N* (see Figure 2), and observer after allowing students to work alone or in pairs, to try to illustrate in writing what was decribed in the problem:

Mary and John each have a piggy bank.
On *Sunday,* they both had the same amount in their piggy banks.
On *Monday*, their Grandmother comes to visit them and gives $3 to each of them.
On *Tuesday*, they go together to the bookstore. Mary spends $3 on Harry Potter's new book. John spends $5 on a 2001 calendar with dog pictures on it.
On *Wednesday*, John washes his neighbor's car and makes $4. Mary also made $4 babysitting. They run to put their money in their piggy banks.
On *Thursday,* Mary opens her piggy bank and finds that she has $9.

Figure 1. The Piggy Bank problem

Observer: So, what does it say over here?
Student: *N*.
Observer: Why did you write that down?
Student: Because you don't know. You don't know how much amount they have.
Observer: So, does *N* . . . What does that mean to you?
Student: *N* means any number.
Observer: Do they have *N*, or do they have *N* together?
Student: [Does not respond.]
Observer: How much does Mary have?
Student: *N*.

(Carraher et al., 2006)

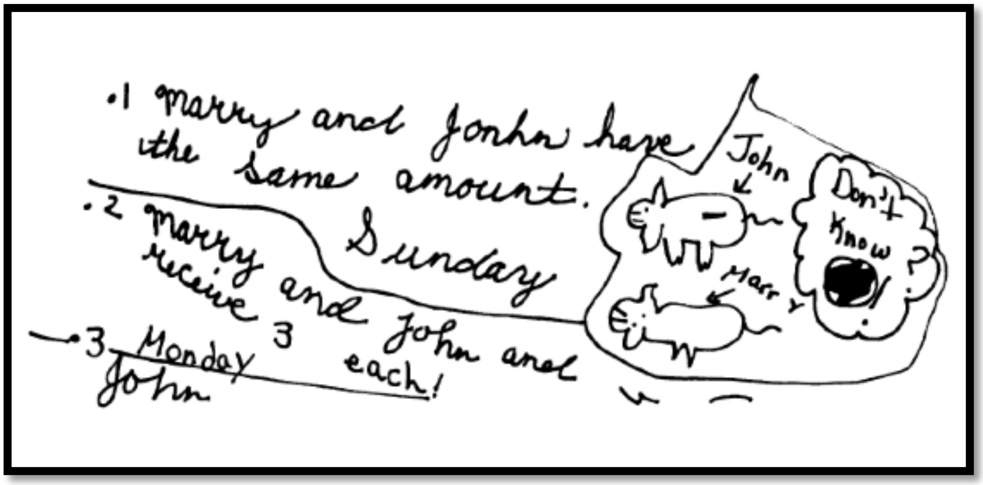

Figure 2. Student's intial representation of the Piggy Bank Problem

Observer: And how about John?

Student: *N*.

Observer: Is that the same N or do they have different *N*s?

Student: They're the same, because it said on Sunday that they had the same amount of money.

Observer: And so, if we say that John has *N*, is it that they have, like, $10 each?

Student: No.

Observer: Why not?

Student: Because we don't know how much they have.

(Carraher et al., 2006)

These researchers view "the introduction of algebra in elementary school as a move from particular numbers and measures toward relations among set of numbers and measures, especially, especially functional notation" (Carraher, 2006). Carraher suggests that this change in focus would help to facilitate a subtle transformation of existing curriculum to integrate algebra, by ascribing algebraic meaning to existing mathematics activities. "Even in early grades, algebraic notation can play a supportive role in learning mathematics. Symbolic notation, number lines, function tables and graphs are powerful tools for children to understand

and express functional relationships across a wide variety of problems" (Carraher 2006). He suggests that we treat arithmetic operations as function from the start.

Carraher (2006) also noted that, evidence of student's "difficulties in algebra are rooted in missed opportunities and notions originated in their early mathematics instruction, that must later be 'undone,' such as the view that the equals sign means 'yields'." Students who hold on to operational view on the equal sign are less successful on algebra type questions than students who understand the as a relationship between two quantities. Providing children opportunities that focus on the relationships between two quantities, instead of the typical arithmetic schemas, will help serve them by furthering a more sophisticated understanding of the equal sign and making the transition from arithmetic to algebra easier with less cognitive strain.

Chapter 3: The Study

This chapter reports on a study designed to investigate students' understanding of the equal sign in my district. We will describe the methods used to conduct the study, the demographics of the school and district, the participants involved, the assessment instrument used and the data obtained.

The high school at which the study was performed was located in a small Louisiana town. It had a population of about 830 students, serving students from grades 7–12. The district had a total of ten schools with approximately 4,000 students from Pre-K to 12th grade. About 60 percent of the high school population was African American and about 40 percent was Caucasian. The 7th and 8th grade Math and ELA classes were each 96 minutes long. All other classes were 48 minutes long. The school was in its first year of full implementation of Common Core standards, after one year of a transitional curriculum, an integration of grade level expectations from the Louisiana Comprehensive Curriculum and Common Core Standards.

3.1 Participants

A test designed to gauge understanding of the equality symbol was administered to 54 seventh-graders in Spring 2015. The participants were from three classes, all having the same teacher. The classes were all of roughly the same size, with a total of 62 students. There was no control group. The students in each of the groups tested had been exposed to the same content and class structure. The math skills and ability of the students in each group were about the same. The entire curriculum had been covered before the test was administered. All three classes had been exposed to various types of equality statements, as suggested by Knuth et al. (2008), and had had opportunities to discuss and reason about the equal sign in a variety of situations.

3.2 Assessment

The test consisted of a combination of items taken from previous studies, as well as items developed by the researchers. Students were asked to respond to four assessment items during the last 48 minutes of class.

The first item, shown in Fig. 3, was taken from Knuth et al. (2008). It consisted of three prompts. The first prompt required students to identify the sign by name. The rationale for the first prompt was to prevent students from providing the name of the symbol in the second prompt. The second prompt required students to define the equal sign. The third prompt asked students to provide additional meanings they may associate with the sign.

Problem 1. The following question is about this statement:

$$3 + 4 = 7$$
$$\uparrow$$

a) The arrow above points to a symbol. What is the name of the symbol? ______________________

b) What does the symbol mean?

c) Can the symbol mean anything else? Please explain.

Figure 3. Assessment Item 1: interpreting the equal sign.

The second item, shown in Fig. 4, was taken from Knuth et al. (2008). It required students to make a judgment about the solution to a set of equivalent equations. Students had to first determine if the solution to one equation was also a solution for the other. Students had to be able to recognize that the transformations performed on the second equation preserved the quantitative relationship of the first.

Problem 2. Is the number that goes in the $\square$ the same in the following two equations? (yes / no).

$$2 \times \square + 15 = 31 \qquad 2 \times \square + 15 - 9 = 31 - 9$$

Explain your reasoning.

Figure 4. Assessment Item 2: using the concept of mathematical equivalence.

Researchers designed the third assessment item, shown in figure 5. It required students to create a statement of equality. The rationale for this question was to provide students the opportunity to demonstrate their understanding of equality, illustrating the concept of this idea in a problem of their own design.

Problem 3. Make up a problem for a test that has = in it.

Figure 5. Assessment Item 3: creating a statement of equality.

Researchers also designed the last item, shown in figure 6. It required students to interpret the meaning of an equality statement with expressions involving various operations on either side. Students were also asked to determine if the statement correct and they had an opportunity to explain why.

Problem 4. What does the following mean? Is it correct? Why?

$$7 \times 2 - 3 = 5 \times 2 + 1$$

Figure 6. Assessment Item 4: interpretation of an equality statement.

3.3 Results

Assessment Item 1. Student responses to prompts (b) and (c) for the first assessment item were classified into three categories: operational, relational, or other (Knuth et al., 2008). A

response was classified as operational if the students defined the equal sign as meaning to "find the total" or "put the answer." The following student responses were classified as operational:

- "The equal sign means that whatever numbers are added, subtracted, divided, or multiplied, it's equal to the outcome."
- "It means the total of something."

Student responses that defined the equal sign as expressing a relation between two sides or quantities were classified as relational. The following examples were taken from student responses whose answers were classified as relational:

- "It means the same."
- "It means that the expression on the left is equal to the one on the right."
- "It means that both expressions are the same."
- "It can mean equivalent like $^2/_4 = ½$."
- "It means that the expression before is equivalent to the number following."
- "The equal sign compares whatever is on either side of it in a way that shows they have the same value."

Student responses were classified as other if students defined the equal sign as meaning "equal to" or provided a literal translation of the equivalence statement, for example, 3 plus 4 is 7. Student responses to prompt (b) and (c) were classified separately then placed into a category based on the definition that was more clearly developed.

Table 1 shows students responses, by class, categorized by the definition that appeared to dominate in the responses to parts (b) and (c). The majority of students in classes 1 and 2 provided a definition of the equal sign that could be classified as relational, 68 percent and 56 percent respectively. Only 41 percent of students in the third group provided a definition that

could be classified as relational. Figure 7 shows the number of students who provided each type of equal sign definition as their dominant definition. Of the fifty-four students, 57 percent (31 out of 54) provided a formal definition of the equal sign as a relation. About 15 percent (8 out of 54) of the fifty-four students provided a definition of the equal sign that was classified as operational and 28 percent (15 out of 54) of the students provided a very limited interpretation of the equal sign that was subsequently classified as other.

Table 1. Number of students in each class who provided each type of equal sign definition (n = 54).

Dominant Definition	Class 1	Class 2	Class 3	Row Total
Operational	4	2	2	8
Relational	13	11	7	31
Other	2	5	8	15

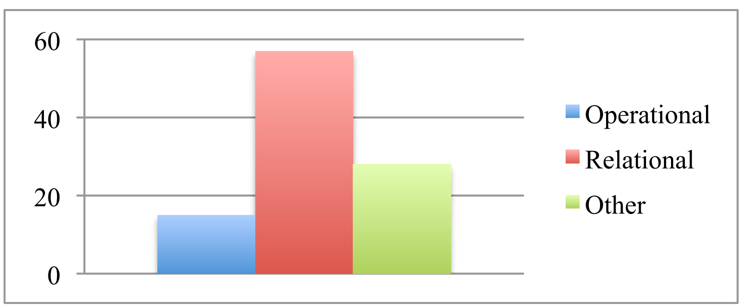

Figure 7. Percent of 7th grade students who provided each type of equal sign definition as their dominant definition (n = 54).

Assessment Item 2. Figure 8 shows the proportion of students who were able to judge correctly the set of equivalent equations in the second assessment item as having the same solution, as dependent on their interpretation of the equal sign as operational, relational or other on assessment item 1. The majority of students, who had their responses classified as "relational" or "other" on the first assessment item, were more likely to respond that the set of

equations shared the same solution, 8. Only 20 percent of the 8 students who had their response classified as operational on the first assessment item were able to respond correctly.

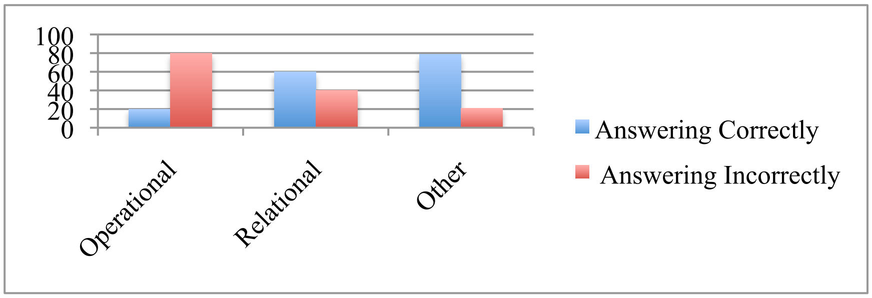

Figure 8. Percent of students by equal sign interpretation (Item 1) answering Assessment Item 2 correctly.

Table 2 shows the various strategies students employed to solve the set of equivalent equations in the second assessment item. The majority of student responses for the second assessment item were categorized into one of the following four categories: recognize equivalence, solve and compare, answer after equal sign or other. The following examples of student responses were representative of each of the four categories:

- "Yes, because 9 is subtracted from both sides in the second equation so you end up with the same equation which is equivalent to itself." (recognize equivalence)
- "Yes, because if you solve both equations for *x* you get 8, so 8 goes into both boxes." (solve and compare)
- "Yes, because $2 \times 8 + 15$ is always 31." (answer after the equal sign)
- "No, because they have different answers [student provided no evidence to support their findings}" (other)

A strategy that could not be determined from their response was subsequently categorized as "other."

Table 2. Students by equal sign interpretation who used various strategies on Assessment Item 2.

Strategy Used on Item 2	Response to Item 1			Row Total
	Operational	Relational	Other	
Recognize equivalence	0	8	0	8
Solve and compare	4	8	2	14
Answer after the equal sign	2	6	8	16
Other	2	8	2	12
No response/ Didn't know	0	1	3	4
Column Total	8	31	15	54

Students who had who had their responses classified as "relational" on the first assessment item, made use of more strategies than any other group to determine the answer to the set of equations. Only students who had their interpretation of the equal sign classified as relational on the first assessment item, employed the recognize equivalence strategy. About 26 percent (8 out of 31) of students in this group were able to recognize that operations performed on the second equation preserved its equivalence to the first equation; subsequently students were able to determine the value of the missing quantity as being the same. Half of the responses classified as operational on the first assessment item, made use of the "solve and compare" strategy. More than half students who had their interpretation of the equal sign classified as other on the first assessment item, made use of the answer after the equal sign strategy.

Assessment Item 3. All Students provided a response for this item, showing some appropriate use of the equal sign. Thirty seven percent (20 of the 54) of the responses included a

variable. Students who had their interpretation of the equal sign classified as relational on the on the first assessment item, accounted for more than half (13 out of 20) of responses that included a variable. Typical responses using a variable were 2x + 6 = 26, 6x = 3x, 2(3n + 4) = (6n + 8), and 2(3n + 5). Typical responses without a variable were 3 × 3 = 9, 1 + 1 = 3 – 1, and 2(2) – 6 = 4 – 6.

Table 3. Presence of variable in statement of equality.

Nature of Statements		Variable Present in Response to Item 3		Row Total
		Yes	No	
Response to Item 1	Operational	2	6	8
	Relational	13	18	31
	Other	5	10	15
Column Total		20	34	54

Table 3b. Nature of statements in item 1 compared to types of statements created in item 3.

Nature of Statements		Item 3		Row Total
		Operational	Relational	
Item 1	Operational	6	2	8
	Relational	20	11	31
	Other	11	4	15
Column Total		37	17	54

Student responses were classified as relational under either of the following conditions:

- It included more than on operation on both sides of the equal sign with or without a variable (e.g., $1 + 1 = 3 - 1$, $2(2) - 6 = 4 - 6$, $5x + 3x = 8x$, $2(3n + 4) = (6n + 8)$), or

- it included one or more operations on either side of the equal sign with a variable, and students demonstrated that the same transformation performed on both sides of the equal preserved the quantitative relationship of both expressions (e.g., $2x + 6 = 26$, $2x = 5 + -15$).

Responses were classified as operational under the following conditions:

- It included a numerical expression on either side and did not include a variable (e.g., $7 + 5 + 3 = 15$, $-5 + 14 = 9$, $5 \times 2 + 10 = 20$), or
- it included a numerical expression on either side with a variable, but students did not attempt to solve (e.g., $4b + 6 = 18$, $x + 3 = 7$).

<u>Assessment Item 4.</u> All Students were able to interpret correctly the equality statement and provide appropriate reasoning to explain why it was, in fact, true. The following student responses were typical for fourth assessment item:

- "The following (equality statement) means that the given expressions are simplified to be equivalent to one another. The problem is true because both expressions have a value of 11."
- "Yes, because both answers will give you eleven. The product of seven times two minus three is equal to five times two plus one."
- "It is true. It means that 11 is equal to 11."
- "It means that $7 \times 2 - 3$ is equal to $5 \times 2 + 1$. Yes it is correct [student shows computations simplifying each expression to prove the statement is correct].

Chapter 4: Discussion

The purpose of this study was to investigate students' understanding and knowledge of the equal sign, so that instructional resources could be identified to improve student's conceptual understanding about mathematical equivalence in my district.

4.1 Discussion

In previous research, assessment items 1 and 2 were established as good indicators of students' understanding of the equal sign (Knuth et al., 2008). From the test, we find that less than 60% of the students tested will make a clear relational interpretation. The widespread conceptual gap noted in the literature is present among my students, even after lessons designed to remedy it.

What can we say about the new items? It is interesting to note that some findings from the data appear to indicate a relationship between items on the test, however statistical tests show that the results could have simply arisen by chance. Perhaps the different questions addressed different aspects of the student's grasp of the meaning of the equal sign. A conclusion we can draw from this is that it may be difficult to design questions that accurately reflect the student's understanding, or that understanding may not be so easily classified. Baroody and Ginsburg suggest that the ways children interpret symbolic representations are strongly dependent on context. Maybe whether students perceive the equal sign as relational depends on context. With this in mind, we might consider altering Assessment item 3 by requiring students to translate a word problem into a mathematical statement, and Assessment item 4 could be altered by requiring students to translate a mathematical statement into a word problem. These changes

would provide a context, and this might lead to questions that more meaningfully reflect student understanding of the equal sign.

What do the results of the test suggest? Finding from Assessment items 1 and 2 are consistent with Knuth's findings: "Students' understanding of the equal sign was associated with their performance on the equivalence equations problem (assessment item 2), both in terms of their judgments for the solution to the problem and the strategies they used to arrive at those judgments" (Knuth et. al 2008). Students who exhibited a relational view of the equal sign were more likely than students who did not to solve the set of equivalent equations on assessment item 2 correctly and made use of more strategies than any other group in determining the answer (but the statistical significance was not great). For example, only students who exhibited a relational view of the equal used the "recognize equivalence strategy" recognizing that operations performed on the second equation preserved its equivalence to the first equation. This is worth attention and more study, considering the critical importance of this skill to solve algebraic equations. Students who exhibited a relational view of the equal sign appeared to be more comfortable creating and solving statements of equalities involving variables on Assessment item 3.

4.2 Conclusion

The literature shows that children's understanding of mathematical equivalence is one the most important concepts linked to the development of algebraic reasoning and subsequently, increased student success in algebra. In 2008, The National Mathematics Advisory Panel recognized that "preparation of students for entry into, and success in, algebra" to be of critical importance to our country, as success in algebra is considered to be the gatekeeper to college and career opportunities. Researchers suggest that by providing learning opportunities that focus on

the relationships between two quantities, exposing children to nontraditional arithmetic problem formats, using words that describe relations such as "is the same number" or "is the same amount as" in place of the equal sign in equivalence statements, ascribing algebraic meaning to arithmetic problems, and organizing problems into practice sets based on equivalent values, we can improve students' conceptual understanding of mathematical equivalence (Baroody & Ginsburg, 1983, Carraher, 2006, Knuth et al., 2008).

Results from the test that we administered suggest a broad range of understandings that students exhibit when asked to interpret the meaning of the equal sign. In future studies, the focus could be designing a more elaborate test to identify the specific understandings or misunderstandings a student may exhibit under varying conditions.

It evident from the results of this study that a significant number of students in our district have limited understanding of mathematical equivalence, thus highlighting the need for continued efforts, attention (and research) concerning the perception of the equal sign in elementary as well as middle-school grades. Instruction provided in the elementary grades may head off misconceptions or even prevent misunderstandings from developing. The next step would be to create a curriculum or set of lesson teachers could use to help students achieve mastery level understanding of the equal sign as a relation, as an implication from Knuth et al. (2008) draws a direct correlation between understanding of the equal sign as a relation and success in algebra.

What are the long-term consequences of having a poor understanding of mathematical equivalence? We know that children's misconceptions about mathematical equivalence are deep rooted, persisting among students throughout primary, secondary and even post-secondary schooling. The general assumption is that the better understanding of mathematical equivalence

in early grades leads to greater success in mathematics as students advance through school, into algebra and beyond.

Chapter 5: Instructional Resources

This chapter contains a collection of activities identified to improve student's conceptual understanding about mathematical equivalence and the equal sign.

Given the focus of this thesis, it should be noted that the authors of Common Core suggest that the development of equivalence and algebraic reasoning should begin as early as kindergarten by first developing understanding of mathematical properties and establishing relationships between whole numbers and quantities regardless of arrangement. This connection continues through first grade, later extending knowledge of relations between quantities to rational numbers, then to expressions and equations. After students study basic arithmetic operations (e.g., addition, subtraction, multiplication and division) in elementary school, relationships between operations (e.g., writing subtraction as addition of an opposite and division as multiplication by a reciprocal) are established in middle school, particularly sixth and seventh grade. "Pervasive classroom use of these mathematical practices in each grade affords students opportunities to develop understanding of operations and algebraic thinking" (Progressions for the Common Core State Standards in Mathematics, 2011).

5.1: Suggested Activities

Taken together, the current mathematical standards, the findings of previous research and the findings of the study reported here suggest more attention to certain activities and lessons, which we illustrate below. The suggested activities are intended to prepare students for formal courses in algebra by offering teachers opportunities to understand and develop their student's grasp of equality and the equal sign. By incorporating these activities in classroom instruction and practices, teachers will expose their students to appropriate interpretations of the equal sign.

The activities are designed for use in all or any classroom or grade level to establish or reinforce the concept of mathematical equivalency and proper use of the equal sign. These activities can also be tailored by adjusting the complexity of problem sets to fit the needs of individual students or whole group.

True/False Arithmetic Statements. Activity of this type focus on the function of the equal sign as a symbol expressing a relationship of equality without requiring students to carry out calculations but instead to make judgments about the validity of statements by determining if the expression on the left is equal to the quantity expressed on the right. Problems presented to students may resemble the following sequence of true/false statements:

$4 + 5 = 9$ $\quad$ $6 + 4 = 9$ $\quad$ $9 = 4 + 5$ $\quad$ $5 + 4 = 6 + 3$

This type of activity allows teachers' opportunities to identify specific misconceptions students may have about the equal sign by examining their responses. Most students will regard the first sentence, $4 + 5 = 9$, as true. Students who attend to the structure of the second problem, $6 + 4 = 9$, may regard the sentence as true. Students may reject the third sentence, $9 = 4 + 5$, and fourth sentence, $5 + 4 = 6 + 3$, because they do not adhere to the typical "compute and write the answer" format. The teacher can use this as an opportunity to engage students into a conversation about the meaning of the equal sign and its proper use.

When students have the appropriate arithmetic skills, more complicated sentences might be used to continue modeling appropriate understanding in interesting contexts:

$1 + 2 + 3 + 4 + 5 = 6 + 9$ $\quad$ $(123)(11) = 143$ $\quad$ $(1 + 2 + 3)^2 = 1 + 8 + 27$

Finding the Missing Number. Activities of this type require students to solve different sets arithmetic equations of typical and atypical problem structures. The purpose of this type of

activity is to intentionally provide students with opportunities to recognize connections between arithmetic operations more easily helping to establish algebraic reasoning. Examples of problems presented to students may resemble the following sets of equations:

$\square + 4 = 10$	$10 = \square + 4$	$10 - 4 = \square$	$10 - \square = 4$
$5 + \square = 12$	$12 = 7 - \square$	$16 = 2 \times \square$	$16 \div \square = 8$

More advanced, sophisticated:

$\square + \square + 4 = \square + \square + \square + 1$

Which One Does Not Belong. Activities of this type require students to make judgments about relationships between numbers written as expressions by circling the expression that does not express the same relationship, as the other expressions in a set.

a. ● ● ● ● ●	■ ■ ■ ■ ■ ■	♥♥♥♥♥
b. $6 + 2$	$4 + 4$	$5 + 4$
c. $10 - 4$	$12 - 5$	$7 - 0$
d. $20 - 8$	$6 + 6$	$7 + 5$
e. $3 + x$	$3x$	x x x

Equal or Not Equal. Activities of this type require students to make judgments about relationships between numbers using the = or ≠ symbols.

• • • ____ • • •	📖 📖 ____ 📖 📖 📖	4 ___ 4	5___4

Unequal Sets. Activities of this type are aimed at the development of relational thinking by comparing quantities with out calculating totals to determine if an equivalence relationship

exists between the two quantities. Students must provide verbal or written explanation to justify their reasoning.

a. $10 - 4 = 11 - 5$

b. $3 + 5 = 5 + 3$

c. $6\ (2 + 7) = (2 + 7)\ 6$

d. $2\ (5 + 2) = 2\ (\ 5 + 3)$

e. $(6 + 5)\ (2 + 7) = (2 + 7)\ (6 + 5)$

f. $3(x + 2) = 3x + 2$

<u>Use of manipulatives.</u> Activity of this type require student to explore the concept of equality by using a balance scale to reason abstractly about relationships between quantities on either side of the scale while using concrete manipulatives.

If one circle represents 4, one square represents ____.

If one circle is removed from the left side of the balance scale and one square is removed from the right side does the scale remain balanced? Explain.

<u>Number fluency.</u> The following types of activities focus on relational thinking by composing and decomposing 10, and using of place value to construct number bonds of equivalent expressions.

Breaking up 10:

$1 + 9 = 7 + 3$ (81 possibilities)

1 + 9 = 2 + 8	1 + 9 = 3 + 7	1 + 9 = 4 + 6	1 + 9 = 5 + 5	1 + 9 = 6 + 4	1 + 9 = 7 + 3	1 + 9 = 8 + 2	1 + 9 = 9 + 1
2 + 8 = 1 + 9	2 + 8 = 3 + 7	2 + 8 = 4 + 6	2 + 8 = 5 + 5	2 + 8 = 6 + 4	2 + 8 = 7 + 3	2 + 8 = 8 + 2	2 + 8 = 9 + 1
3 + 7 = 1 + 9	3 + 7 = 2 + 8	3 + 7 = 4 + 6	3 + 7 = 5 + 5	3 + 7 = 6 + 4	3 + 7 = 7 + 3	3 + 7 = 8 + 2	3 + 7 = 9 + 1
4 + 6 = 1 + 9	4 + 6 = 2 + 8	4 + 6 = 3 + 7	4 + 6 = 5 + 5	4 + 6 = 6 + 4	4 + 6 = 7 + 3	4 + 6 = 8 + 2	4 + 6 = 9 + 1
5 + 5 = 1 + 9	5 + 5 = 2 + 8	5 + 5 = 3 + 7	5 + 5 = 4 + 6	5 + 5 = 6 + 4	5 + 5 = 7 + 3	5 + 5 = 8 + 2	5 + 5 = 9 + 1

Meaning of digits

$300 = 3 \times 100$

$365 = 3 \times 100 + 6 \times 10 + 5$

Using meaning of digits to compute

$186 + 238 = 300 + 110 + 14 = 424$

Taking into consideration the ideas and results in this paper, it is clear that learning equality as a relationship between two quantities critical understanding and has many implications in learning mathematics. If students can achieve this level of understanding, they will possess a foundation of critical skills that are essential for building number sense, understanding number relations, and the ability to reason abstractly.

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ACTION ON EXEMPTION APPROVAL REQUEST

LSU

Institutional Review Board
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TO: Christy Duncan
Interdisciplinary Program

FROM: Dennis Landin
Chair, Institutional Review Board

DATE: June 26, 2015

RE: **IRB#** E9391

TITLE: Misconceptions of the Equal Sign

New Protocol/Modification/Continuation: New Protocol

Review Date: 6/25/2015

Approved _____X_____ **Disapproved**__________

Approval Date: 6/29/2015 **Approval Expiration Date:** 6/28/2018

Exemption Category/Paragraph: 1,2b,c

Signed Consent Waived?: No

Re-review frequency: (three years unless otherwise stated)

LSU Proposal Number (if applicable):

Protocol Matches Scope of Work in Grant proposal: (if applicable)

By: Dennis Landin, Chairman DLandin

PRINCIPAL INVESTIGATOR: PLEASE READ THE FOLLOWING –
Continuing approval is CONDITIONAL on:

1. Adherence to the approved protocol, familiarity with, and adherence to the ethical standards of the Belmont Report, and LSU's Assurance of Compliance with DHHS regulations for the protection of human subjects*
2. Prior approval of a change in protocol, including revision of the consent documents or an increase in the number of subjects over that approved.
3. Obtaining renewed approval (or submittal of a termination report), prior to the approval expiration date, upon request by the IRB office (irrespective of when the project actually begins); notification of project termination.
4. Retention of documentation of informed consent and study records for at least 3 years after the study ends.
5. Continuing attention to the physical and psychological well-being and informed consent of the individual participants, including notification of new information that might affect consent.
6. A prompt report to the IRB of any adverse event affecting a participant potentially arising from the study.
7. Notification of the IRB of a serious compliance failure.
8. **SPECIAL NOTE:**

**All investigators and support staff have access to copies of the Belmont Report, LSU's Assurance with DHHS, DHHS (45 CFR 46) and FDA regulations governing use of human subjects, and other relevant documents in print in this office or on our World Wide Web site at http://www.lsu.edu/irb*

Vita

Christy De'sha Duncan is a native of Baton Rouge, Louisiana. She attended Southern University and received a Bachelor of Science in Rehabilitation Counseling in 2007. The desire to improve her community along with her passion to be considered "one of the best" math and science educators in her field propelled her to further her education at Louisiana State University through the Louisiana Math and Science Teacher Institute. She taught math and science in the East Baton Rouge for six years before continuing her teaching career as an educator in Iberville Parish.