3-28-2009

The edge version of Hadwiger's conjecture

Guoli Ding
Louisiana State University

Recommended Citation
Ding, G. (2009). The edge version of Hadwiger's conjecture. *Discrete Mathematics, 309* (5), 1118-1122.
https://doi.org/10.1016/j.disc.2007.12.017

Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 1118–1122
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The edge version of Hadwiger's conjecture

Guoli Ding

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, United States

Received 10 March 2006; accepted 3 December 2007
Available online 18 January 2008

Abstract

A well known conjecture of Hadwiger asserts that K_{n+1} is the only minor minimal graph of chromatic number greater than n. In this paper, all minor minimal graphs of chromatic index greater than n are determined.

Keywords: Hadwiger's conjecture; Chromatic index; Minor minimal graps

0. Introduction

One of the well known conjectures in graph theory is Hadwiger's conjecture [4], which asserts that K_{n+1} is the only minor minimal graph of chromatic number greater than n. This is a long-standing conjecture (see [6] for a survey) which remains open except for n equal to four [1,9,14] and five [10] when the conjecture is equivalent to the four-color theorem, and for n less than four [3,6] when the conjecture is easy to prove. The purpose of this paper is to consider a similar problem, to determine all minor minimal graphs of chromatic index greater than n, for all n. Unlike chromatic number, chromatic index is affected by whether the graph under consideration is simple or not. We will consider multigraph first in Section 1 and then consider simple graphs in Section 2.

1. Multigraphs

In this section, we only consider loopless multigraphs. The corresponding results for graphs with loops can be derived from the result in this section very easily and thus they are omitted from the paper.

Let n be a nonnegative integer. Let $\mathcal{S}_n$ be the set of graphs for which there is a vertex meeting all its $n+1$ edges. Clearly, every graph in $\mathcal{S}_n$ has chromatic index greater than n. Moreover, every proper minor of such a graph either has a loop or has chromatic index at most n. In another words, all graphs in $\mathcal{S}_n$ are minor minimal loopless graphs of chromatic index greater than n. Now let $\mathcal{T}_n$ be the set of connected loopless graphs with three vertices and $n+1$ edges. It is clear that all members of $\mathcal{T}_n$ are also minor minimal loopless graphs of chromatic index greater than n. The next result states that these are all such graphs.

Theorem 1.1. *Let n be a nonnegative integer. If a loopless multigraph has chromatic index greater than n, then it contains a member of $\mathcal{S}_n \cup \mathcal{T}_n$ as a minor.*

E-mail address: ding@math.lsu.edu.

doi:10.1016/j.disc.2007.12.017

Proof. Let G be a loopless multigraph with chromatic index greater than n. Clearly, to prove the theorem, we may assume that G is minor minimal with these properties. That is, every proper minor of G either has a loop or has chromatic index at most n. Depending on the connectivity of G, we will consider three cases.

If G can be expressed as the disjoint union of two graphs G_1 and G_2 with $V(G_1) \neq \emptyset \neq V(G_2)$, then we have a contradiction since

$$n < \chi'(G) = \max\{\chi'(G_1), \chi'(G_2)\} \leq n.$$

It follows that G is connected. Next, if G can be expressed as the union of two edge disjoint graphs G_1 and G_2 with $E(G_1) \neq \emptyset \neq E(G_2)$ and $V(G_1) \cap V(G_2) = \{x\}$, then $n < \chi'(G) = \max\{\chi'(G_1), \chi'(G_2), d(x)\}$, where $d(x)$ is the number of edges of G that are incident with x. From the minimality of G we conclude that $d(x) > n$ and thus G contains a member of $\mathcal{S}_n$ as a minor. Finally, we consider the case when G is 2-connected. It is not difficult to show (see 6.8 in [7], for instance) that G has an edge e with end vertices x and y such that $G - \{x, y\}$ is still connected. Since $\chi'(G) > \chi'(G \backslash e)$, we deduce that G has at least $\chi'(G)$ edges incident with either x or y. Therefore, $G/E(G - \{x, y\})$, and thus G, contains a member of $\mathcal{T}_n$ as a minor. The proof is completed. □

2. Simple graphs

Let n be a nonnegative integer. Let $\mathcal{R}_n$ be the set of simple graphs with $2\lceil n/2\rceil + 1$ vertices, $n\lceil n/2\rceil + 1$ edges, and with maximum degree n. If G is a graph in $\mathcal{R}_n$, then each matching of G contains at most $\lceil n/2\rceil$ edges. It follows that n colors can color at most $n\lceil n/2\rceil$ edges of G and thus $\chi'(G) > n$. The next theorem, which is the main result of this paper, states that, except for $K_{1,n+1}$, graphs in $\mathcal{R}_n$ are precisely all minor minimal simple graphs of chromatic index greater than n.

Theorem 2.1. *Let n be a nonnegative integer. If a simple graph has chromatic index greater than n, then it contains a member of $\mathcal{R}_n \cup \{K_{1,n+1}\}$ as a minor.*

To prove the theorem, we need some preparations. First, we state two well known theorems of Vizing [11,12].

(2.2) Vizing's theorem. *If G is a simple graph with maximum degree n, then $\chi'(G) \leq n + 1$.*

Let us recall that $d(x)$, the degree of a vertex x, is the number of edges incident to x. We will denote the maximum degree of a graph G by $\Delta(G)$ or, sometimes, simply by Δ. A simple connected graph G is called *critical* if $\chi'(G \setminus e) < \chi'(G) = \Delta(G) + 1$ for all edges e of G.

(2.3) Vizing's adjacency lemma. *Let (u, v) be an edge of a critical graph G. Then u is adjacent to at least $\Delta - d(v) + 1$ vertices of degree Δ.*

The following two results will also be used. Their proofs are omitted since they are very easy (see [13,15], for instance).

(2.4) *If (u, v) is an edge of a critical graph, then $d(u) + d(v) \geq \Delta + 2$.*

(2.5) *Critical graphs are 2-connected.*

The rest of the paper consists of a proof of Theorem 2.1. This proof will be separated into the proofs of several lemmas. For a vertex v of a graph, let $N(v)$ denote the set of vertices that are adjacent to v.

Lemma 2.6. *Let G be a critical graph with $\Delta(G) = n \geq 3$. If G has no $K_{1,n+1}$ minor, and if v is a vertex of degree n, then $\{v\} \cup N(v)$ meets all edges of G, and $N(x) \cap N(y) = \emptyset$ for all distinct vertices x and y in $V(G) \setminus (\{v\} \cup N(v))$.*

Proof. Let $N = N(v)$ and $M = V(G) \setminus (\{v\} \cup N)$. If there is a vertex u in N adjacent to at least two vertices in M, then the new vertex in $G/(u, v)$ has more than n neighbors. This is a contradiction since G has no $K_{1,n+1}$ minor. It follows that

(a) every vertex in N is adjacent to at most one vertex in M.
   Similarly,

(b) if a vertex u in M is adjacent to a vertex in N, then u is adjacent to at most one vertex in M.
   For the same reason, if a vertex u in M is not adjacent to any vertex in N, then $d(u) \leq 2$. In fact, since G is 2-connected by (2.5), we must have

(c) $d(u) = 2$, for every $u \in M$ that is not adjacent to any vertex in N.

Next, we prove that every vertex in M is adjacent to a vertex in N. Suppose there is a vertex z in M that does not have this property. Then we conclude from (c) that $d(z) = 2$. Let $N(z) = \{x, y\}$. It follows from (2.4) that $d(x) = d(y) = n$. Since $n \geq 3$, we conclude from (c) again that both x and y are adjacent to some vertices in N. Now, by (b), we have

$$|N(x) \cap N| = n - 1 = |N(y) \cap N|,$$

and, by (a), we have

$$N(x) \cap N(y) \cap N = \emptyset.$$

It follows that

$$n = |N| \geq |N(x) \cap N| + |N(y) \cap N| = 2n - 2,$$

contradicting the assumption that $n \geq 3$. This contradiction proves that every vertex in M is adjacent to a vertex in N.

To finish the proof of 2.6, we only need to show that M is independent. Suppose there is an edge between two vertices x and y of M. We will produce a contradiction by showing that G has a $K_{1,n+1}$ minor.

We first show that $M = \{x, y\}$ and $d(u) = n$ for all vertices u in N. We consider two cases. If at least one of x and y, say x, has degree n, then, by (b), we have $|N(x) \cap N| = n - 1$. Thus we conclude that $M = \{x, y\}$ and then $d(y) = 2$ from (a) and the fact that every vertex in M is adjacent to a vertex in N. Let z be the other vertex adjacent to y. Then, by (2.4), we have $d(z) = n$. To see that all other vertices in N have degree n, apply Vizing's adjacency lemma to the edge (x, y). We conclude that x is adjacent to at least $n - d(y) + 1 = n - 1$ vertices of degree n. Since $d(y) \neq n$, all other neighbors of x, which are precisely those in $N \setminus \{z\}$, must have degree n, as required. Now we consider the second case when both $d(x)$ and $d(y)$ are less than n. By Vizing's adjacency lemma, x is adjacent to at least $n - (d(y) - 1)$ vertices of degree n. Since $d(y) \neq n$ and $(N(x) \setminus \{y\}) \cup (N(y) \setminus \{x\}) \subseteq N$ by (b), we have

$$|N(x) \cap N| \geq n - (d(y) - 1) = n - |N(y) \cap N|.$$

Thus, by (a), we deduce that $(N(x) \setminus \{y\}) \cup (N(y) \setminus \{x\}) = N$, and so, $M = \{x, y\}$ because every vertex in M is adjacent to a vertex in N. In addition, we also deduce that $|N(x) \cap N| = n - (d(y) - 1)$, which implies that all vertices in $N(x) \cap N$ have degree n. Similarly, all vertices in $N(y) \cap N$, and therefore all vertices in N, have degree n.

Now we show that G has a $K_{1,n+1}$ minor. Let e be an edge with one end in $N(x) \cap N$ and one end in $N(y) \cap N$. One way to see the existence of e is the following. Without loss of generality, we may assume that $d(x) \geq d(y)$, and thus, by (2.4), we have $|N(x) \cap N| \geq 2$. Let u be a vertex in $N(y) \cap N$. Then we conclude from $d(u) = n$ that there is a vertex u' in $N(x) \cap N$ so that $e = (u, u')$ is an edge, as required. It is clear that x, y, and v are all adjacent to the new vertex z of G/e. In addition, all the other $n - 2$ vertices of G/e are also adjacent to z since they have degree n in G. Therefore z has $n + 1$ neighbors and thus G has a $K_{1,n+1}$ minor. This contradiction finishes the proof of 2.6. □

Lemma 2.6 suggests the following definition. Let x be a vertex of a simple G and let $N(x)$ be partitioned into nonempty sets $N_1, N_2, \ldots, N_t$. Then we define a new graph with vertex set $(V(G) \setminus \{x\}) \cup \{N_1, N_2, \ldots, N_t\}$, and edge set $E(G - x) \cup \{(N_i, y) : y \in N_i,\ 1 \leq i \leq t\}$. We say that the new graph is obtained from G by splitting the vertex x.

Lemma 2.7. *Let G be a critical graph with $\Delta(G) = n \geq 3$. If G has more than $n + 2$ vertices and has no $K_{1,n+1}$ minor, then G is obtained by splitting a vertex of a graph on $n + 1$ vertices.*

Proof. Let v be a vertex of G with degree n. Let $N = N(v)$, $M = V(G) \setminus (\{v\} \cup N)$, and $L = \bigcup\{N(x) : x \in M\}$. To prove the lemma, it is clear from 2.6 that we only need to show that there is a vertex z in $N \setminus L$ such that $N(z) \cap L = \emptyset$.

From $|V(G)| \geq n+3$ we know that $|M| \geq 2$. Then from 2.6 we conclude that M has a vertex x with $d(x) \leq |L|/2$. By Vizing's adjacency lemma, x is adjacent to a vertex y of degree n. It follows from 2.6 again that y is adjacent to all vertices in $N \setminus \{y\}$, except for precisely one vertex, which we denote by z. We will show that z has the required properties.

By considering y as the vertex v in 2.6 we deduce that $z \notin L \setminus N(x)$ and $N(z) \cap (L \setminus N(x)) = \emptyset$. We first show that $z \in N \setminus L$. Suppose z is in L. Then $z \in N(x)$ and $N(z) \subseteq \{v, x\} \cup (N(x) \setminus \{y\}) \cup (N \setminus L)$. It follows that $d(z) \leq 2 + d(x) - 1 + n - |L|$. By (2.4), we also have $d(x) + d(z) \geq n + 2$. These two inequalities imply that $2d(x) \geq |L| + 1$, contradicting the choice of x. Thus $z \in N \setminus L$, as desired. To finish proving Lemma 2.7, it remains to show that $N(z) \cap N(x) = \emptyset$. Since $|M| \geq 2$, there is a vertex u in $M \setminus \{x\}$. From Vizing's adjacency lemma we know that u is adjacent to a vertex w of degree n. Notice that w is not adjacent to z. Thus, if we consider w as the vertex v in 2.6, we deduce that $N(z) \cap N(x) = \emptyset$, as desired. Therefore, Lemma 2.7 is proved. □

To prove Theorem 2.1, we also need the following two results. The first is a combination of two results from [2,8], while the second is due to Hakimi [5].

(2.8) *If G is a simple graph with $|V(G)| = 2\lceil n/2 \rceil + 1$, $|E(G)| = n\lceil n/2 \rceil$, and $\Delta(G) = n$, then $\chi'(G) = n$.*

(2.9) *Let $d_1, d_2, \ldots, d_t$ be nonnegative integers such that their sum D is even and $d_i \leq D/2$ for all i. Then there is a loopless graph with vertex set $\{1, 2, \ldots, t\}$ such that $d(i) = d_i$ for all i.*

Now we are ready to prove Theorem 2.1.

Proof of 2.1. Let G be a simple graph with $\chi'(G) > n$. The result is clear if $n < 3$ and thus let us assume $n \geq 3$. Without loss of generality, let us also assume that G is connected and $\chi'(G \setminus e) \leq n$ for all edges e of G. Let us assume further that G has no $K_{1,n+1}$ minor. Then we deduce from Vizing's theorem that $\Delta(G) = n$ and $\chi'(G) = n + 1$. Therefore, G is critical. What we need to show now is that G contains a member of $\mathcal{R}_n$ as a minor.

Let us first consider the case when n is odd. Obviously, as $\Delta(G) = n$, G has more than n vertices. Since every simple graph on $n + 1$ vertices has chromatic index at most n, every graph obtained by splitting a vertex of a graph on $n + 1$ vertices must also have chromatic index at most n. Now we deduce from 2.7 that G has exactly $n + 2$ vertices and thus the result follows from (2.8).

From now on we assume that n is even. We may also assume that $|V(G)| \geq n + 2$ because the result follows from (2.8) immediately if G has $n + 1$ vertices. Now we show that G has a subgraph with $n + 1$ vertices and more than $n^2/2$ edges. Such a subgraph must have maximum degree n, which shows that it is a member of $\mathcal{R}_n$ and thus 2.1 follows. Suppose that

(*) *every subgraph of G with $n + 1$ vertices has at most $n^2/2$ edges.*

Then we will produce a contradiction by showing that $\chi'(G) \leq n$. By 2.7, G is one of the following two types:

(i) $|V(G)| = n + 2$;
(ii) G is obtained by splitting a vertex of a graph on $n + 1$ vertices.

To prove $\chi'(G) \leq n$, we may assume that G is maximal with the above properties. That is, if x and y are nonadjacent vertices of G, then either $G + (x, y)$ does not have property (*) or $G + (x, y)$ is not of type (ii). We consider two cases. If every subgraph of G on $n + 1$ vertices has less than $n^2/2$ edges, then G must be of type (ii). Let G be obtained from H by splitting the vertex h and let $X = V(G) \setminus V(H)$. Then, by the maximality of G, the graph $H - h$ is a complete graph and the sum of $d(x)$, over all x in X, is n. These clearly imply that $H = K_{n+1}$. It follows that $d(x) < n/2$ for all vertices x in X. Therefore, by (2.9), the n edges of H that are incident with h can be paired in such a way that none of the paired edges are incident in G. Notice that H can be decomposed into $n/2$ Hamiltonian circuits, each containing a pair of edges as determined above. It follows that G can be decomposed into $n/2$ paths and thus $\chi'(G) \leq n$, as required.

Suppose next that G has a subgraph G' with $|V(G')| = n + 1$ and $|E(G')| = n^2/2$. If G is of type (ii) and it is obtained from a graph H on $n+1$ vertices by splitting a vertex h, then G' contains at most one vertex in $V(G) \setminus V(H)$. This is because otherwise, G' is obtained from a graph on n vertices by splitting a vertex and thus G' can have at most $|E(K_n)| < n^2/2$ edges, which is a contradiction. It follows that all vertices not in G' are in $V(G) \setminus V(H)$ and thus they do not have common neighbors. Therefore, without loss of generality, we may assume, by identifying these vertices if necessary, that there is only one such vertex. In another words, we only need to consider that case when G has $n + 2$ vertices. Now by (2.8), G' can be decomposed into matchings $M_1, M_2, \ldots, M_n$. It is not difficult to see that each M_i contains exactly $n/2$ edges, and thus misses exactly one vertex of G'. Let x be the vertex of G not in G' and let $x_1, x_2, \ldots, x_t$ be all vertices in $N(x)$. Since the degree of each x_i in G' is less than $\Delta(G)$, which is n, there must be a matching $M_{p(i)}$ misses x_i. Observe that all $p(i)$'s are distinct, because each of these matchings misses only one vertex. It follows that the n-coloring of G' can be extended into an n-coloring of G by coloring (x, x_i) the same as those in $M_{p(i)}$. Now we have shown that $\chi'(G) \leq n$ and thus we have completed the proof of 2.1 □

Acknowledgments

This research was partially supported by NSA grant H98230-05-1-0081, NSF grants DMS-0556091, and NSF grant ITR-0326387.

References

[1] K. Appel, W. Haken, Every planar map is four colorable, Bull. Amer. Math. Soc. 82 (1976) 711–712.
[2] A.G. Chetwynd, A.J.W. Hilton, Partial edge-colourings of complete graphs or of graphs which are nearly complete, in: B. Bollobás (Ed.), Graph Theory and Combinatorics, Academic Press, London, 1984, pp. 81–97.
[3] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[4] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrsch. Naturforsch. Ges. Zürich 88 (1943) 133–142.
[5] S.L. Hakimi, On realizability of a set of integers as degrees of the vertices of a linear graph, SIAM J. Appl. Math. 10 (1962) 496–506.
[6] T.R. Jensen, B. Toft, Graph Coloring Problems, John Wiley & Sons, 1995.
[7] L. Lovász, Combinatorial Problems and Exercises, Elsevier North-Holland, 1979.
[8] M. Plantholt, The chromatic index of graphs with a spanning star, J. Graph Theory 5 (1981) 5–13.
[9] N. Robertson, D.P. Sanders, P.D. Seymour, R. Thomas, The four color theorem, J. Combin. Theory B (70) (1997) 2–44.
[10] N. Robertson, P.D. Seymour, R. Thomas, Hadwiger's conjecture for K_6-free graphs, Combinatorica 13 (1993) 279–361.
[11] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Metody Diskret. Analiz. 3 (1964) 25–30 (in Russian).
[12] V.G. Vizing, Critical graphs with a given chromatic class, Metody Diskret. Analiz. 5 (1965) 9–17 (in Russian).
[13] V.G. Vizing, The chromatic class of a multigraph, Kibernetika (Kiev) 3 (1965) 29–39 (in Russian).
[14] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.
[15] H.P. Yap, Some topics in graph theory, London Math. Soc. Lecture Note Ser. 108 (1986).