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Transforms of pseudo-Boolean random variables[☆]

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ABSTRACT

As in earlier works, we consider $\{0, 1\}^n$ as a sample space with a probability measure on it, thus making pseudo-Boolean functions into random variables. Under the assumption that the coordinate random variables are independent, we show it is very easy to give an orthonormal basis for the space of pseudo-Boolean random variables of degree at most k. We use this orthonormal basis to find the transform of a given pseudo-Boolean random variable and to answer various least squares minimization questions.

1. Introduction

A pseudo-Boolean function of n variables is a function from $\{0, 1\}^n$ to the real numbers. Such functions are used in 0–1 optimization problems, cooperative game theory, multicriteria decision making, and as fitness functions. Such a function $f(x_1, \ldots, x_n)$ has a unique expression as a multilinear polynomial

$$f(x_1, \ldots, x_n) = \sum_{T \subseteq N} \left[a_T \prod_{i \in T} x_i \right], \tag{1}$$

where $N = \{1, \ldots, n\}$ and the a_T are real numbers [7, p. 22]. By the *degree* of a pseudo-Boolean function, we mean the degree of its multilinear polynomial representation.

Several authors have considered the problem of finding the best pseudo-Boolean function of degree $\leq k$ approximating a given pseudo-Boolean function f, where "best" means a least squares criterion. Hammer and Holzman [6] derived a system of equations for finding such a best degree $\leq k$ approximation, and gave explicit solutions when $k = 1$ and $k = 2$. They proved that such an approximation is characterized as the unique function of degree $\leq k$ that agrees with f in all average mth-order derivatives for $m = 0, 1, \ldots, k$, in analogy with the Taylor polynomials from calculus. Grabisch, Marichal, and Roubens [5] solved the system of equations derived by Hammer and Holzman, and gave explicit formulas for the coefficients of the best degree $\leq k$ function. Zhang and Rowe [12] used linear algebra to find the best approximation that lies in a linear

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subspace of the space of pseudo-Boolean functions; for example, these methods can be used to find the best approximation of degree $\leq k$.

Here, as in [10,3], instead of simply viewing the domain of a pseudo-Boolean function as the set $\{0, 1\}^n$, we consider $\{0, 1\}^n$ as a discrete sample space and introduce a probability measure on this space. Thus, a pseudo-Boolean function will be a random variable on this sample space. (Viewing $\{0, 1\}^n$ simply as a set corresponds to viewing all of its elements as equally likely outcomes.) Given a pseudo-Boolean random variable f, a best approximation random variable to f, which takes into account the weighting of the elements of $\{0, 1\}^n$, will then be close to f at the "most likely" n-tuples, and may not be so close to f at the "least likely" n-tuples. In [3], we gave a closed formula, using the coefficients in the multilinear polynomial, for the best linear approximation in this more general setting. Also, if the probability measure was a product probability measure, then we gave a closed formula for the best degree $\leq k$ approximation, for all k, thus generalizing the formulas in [5].

Under the assumption that the coordinate functions are independent random variables, we show that it is quite simple to give an orthonormal basis for the space of pseudo-Boolean random variables by "standardizing" the coordinate random variables. Indeed, the functions in our basis of degree k are simply the product of k linear functions in our basis. These functions may be viewed as the generalization of the well-known Walsh functions to our setting. We then define the transform of the given pseudo-Boolean function in terms of this orthonormal basis. This allows one to find the best approximation of a given degree if one starts with the values vector of the function, whereas in [3] we derived formulas for the best approximation starting with the multilinear representation of a function. In the final section, we start with the multilinear representation of a function and describe how the transform may be obtained very easily. We use this to give a simpler proof of Theorem 18 from [3], and we generalize the best linear "faithful" approximation from [6] to obtain the best higher-order faithful approximation in the case of a binomial distribution on $\{0, 1\}^n$.

2. Preliminaries

Put $B = \{0, 1\}$. Let $\mathcal{F}$ denote the space of all pseudo-Boolean functions in n variables; i.e.,

$$\mathcal{F} = \{f : B^n \to \mathbb{R}\}.$$

Then $\mathcal{F}$ has the structure of a 2^n-dimensional real vector space. A basis for this vector space is $\{\prod_{i\in T} x_i \,:\, T \subseteq N\}$.

We will have occasion to form matrices indexed by the elements in B^n, so we need to fix an ordering of these elements. We will order the n-tuples in B^n by using the following degree lexicographic ordering: $(i_1, i_2, \ldots, i_n) < (j_1, j_2, \ldots, j_n)$ if and only if $\sum_{k=1}^n i_k < \sum_{k=1}^n j_k$ or $\sum_{k=1}^n i_k = \sum_{k=1}^n j_k$ and the first coordinates from the left in which these two n-tuples differ satisfy $i_k > j_k$. For example, when $n = 3$, we will list the elements in B^3 as: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$. We will use an analogous ordering when we order multilinear monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where $(i_1, i_2, \ldots, i_n) \in B^n$.

Given a pseudo-Boolean function f, let $\nu(f) = (f(\mathbf{x}))^t$, for $\mathbf{x} \in B^n$, denote the $2^n \times 1$ column of values of f on the n-tuples in B^n. We will refer to $\nu(f)$ as the values vector of f. In [12], a pseudo-Boolean function is called a fitness function and the vector $\nu(f)$ is called a fitness vector. It is easy to see that the mapping $\nu : \mathcal{F} \to \mathbb{R}^{2^n}$ that takes each pseudo-Boolean function to its values vector is a vector space isomorphism.

As in [10,3], we wish to allow a weighting on the elements of B^n. By scaling, we may assume this weighting defines a probability measure $\mu(\mathbf{x})$ on B^n. As in [3], define a pseudo-inner product $\langle\ ,\ \rangle_\mu$ on $\mathcal{F}$ by

$$\langle f, g\rangle_\mu = \sum_{\mathbf{x}\in B^n} f(\mathbf{x})g(\mathbf{x})\mu(\mathbf{x}).$$

This is a "pseudo" (or semidefinite) inner product because we may have $\langle f, g\rangle_\mu = 0$ for all g without f being identically zero. Indeed, if $\mu(\mathbf{x}) = 0$ and if f satisfies $f(\mathbf{x}) = 1$ and $f(\mathbf{y}) = 0$ for all $\mathbf{y} \neq \mathbf{x}$, then $\langle f, g\rangle_\mu = 0$ for all g. On the other hand, it is easy to see that if $\mu(\mathbf{x}) > 0$ for all $\mathbf{x} \in B^n$, then this pseudo-inner product will be an inner product. For the remainder of this work, we assume that $\mu(\mathbf{x}) > 0$ for all $\mathbf{x} \in B^n$. This is not a serious practical restriction, since if one would like some n-tuples to have zero weight, then those n-tuples could be assigned an extremely small positive weight.

We note that $\langle f, g\rangle_\mu$ is the expected value $E_\mu(fg)$ of the random variable fg. This will be an important point of view here. Put $\|f\|_\mu = \sqrt{\langle f, f\rangle_\mu}$. Then $\|\ \|_\mu$ is a norm, under our positivity assumption above.

Now let $\mathcal{L} \subseteq \mathcal{F}$ be an affine space (a translation of a subspace; also known as a linear variety). For example, $\mathcal{L}$ might be the subspace of all pseudo-Boolean functions of degree at most k, for some fixed k. Given $f \in \mathcal{F}$, a "best approximation" to f by functions in $\mathcal{L}$ is a function $f^* \in \mathcal{L}$ that minimizes

$$\|f - g\|_\mu = \sqrt{\sum_{\mathbf{x}\in B^n} (f(\mathbf{x}) - g(\mathbf{x}))^2 \mu(\mathbf{x})}$$

over all $g \in \mathcal{L}$. Notice that if we take the uniform distribution on B^n, so that $\mu(\mathbf{x}) = (1/2)^n$ for all $\mathbf{x} \in B^n$, then the best approximation to f in $\mathcal{L}$ is the function $f^* \in \mathcal{L}$ that also minimizes $\sum_{\mathbf{x}\in B^n} (f(\mathbf{x}) - g(\mathbf{x}))^2$, over all $g \in \mathcal{L}$. This is the usual "least squares" condition used in [6,5,12], and in this case one may simply use the usual Euclidean inner product in $\mathbb{R}^{2^n}$.

3. Orthonormal basis for $\mathcal{F}$

In this section, we show that it is a simple matter to give an orthonormal basis for $\mathcal{F}$, with respect to $\langle\ ,\ \rangle_\mu$, when the coordinate functions are independent random variables. Since we are now thinking of the coordinate functions as random variables, we will denote them using capital letters.

We may view each coordinate function X_i, $i = 1, 2, \ldots, n$, as a Bernoulli random variable. Let C_i (resp., D_i) denote the set of all $\mathbf{x} \in B^n$ such that the ith coordinate of $\mathbf{x}$ equals 0 (resp., 1). Then $P[X_i = 0] = \sum_{\mathbf{x}\in C_i} \mu(\mathbf{x})$ and $P[X_i = 1] = \sum_{\mathbf{x}\in D_i} \mu(\mathbf{x})$. Put $p_i = P[X_i = 1]$ and $q_i = 1 - p_i = P[X_i = 0]$ for $i = 1, 2, \ldots, n$. Then p_i is the expected value of X_i and p_iq_i is the variance of X_i. Let

$$Z_i = \frac{X_i - p_i}{\sqrt{p_iq_i}}, \quad i = 1, 2, \ldots, n,$$

be the associated "standardized" random variables. Then each Z_i has expected value 0 and variance 1. So, we have

$$E_\mu[Z_i] = 0, \qquad E_\mu[Z_i^2] = 1 \quad \text{for } i = 1, \ldots, n.$$

Recall that random variables $Y_1, \ldots, Y_n$ are (mutually or jointly) *independent* if for any real numbers $c_1, \ldots, c_n$, we have

$$P(Y_1 = c_1, \ldots, Y_n = c_n) = P(Y_1 = c_1) \cdots P(Y_n = c_n).$$

From [2], it follows that if $Y_1, \ldots, Y_n$ are independent, then every subset of these random variables is also a set of independent random variables, and if $\varphi_1, \ldots, \varphi_n$ are real-valued functions, then the random variables

$$\varphi_1(Y_1), \ldots, \varphi_n(Y_n)$$

are also independent.

We will need the following elementary, but evidently not well-known, result about uncorrelated and independent random variables. If $Y_1, \ldots, Y_n$ are random variables such that $E[Y_{i_1} \cdots Y_{i_m}] = E[Y_{i_1}] \cdots E[Y_{i_m}]$ for all subsets

$$\{Y_{i_1}, \ldots, Y_{i_m}\} \subseteq \{Y_1, \ldots, Y_n\},$$

then one cannot in general infer that $Y_1, \ldots, Y_n$ are independent. However, one does have independence in the following special case.

Lemma 1. *Let $Y_1, \ldots, Y_n$ be random variables that each take on exactly two values. If $E[Y_{i_1} \cdots Y_{i_m}] = E[Y_{i_1}] \cdots E[Y_{i_m}]$ for all subsets $\{Y_{i_1}, \ldots, Y_{i_m}\} \subseteq \{Y_1, \ldots, Y_n\}$, then $Y_1, \ldots, Y_n$ are independent.*

Proof. Suppose the two values taken on by Y_i are a_i and b_i for $i = 1, 2, \ldots, n$. It suffices to prove the lemma when $Y_1, \ldots, Y_n$ are Bernoulli random variables, since we may replace each Y_i by $c_iY_i - d_i$, where $c_i = 1/(b_i - a_i)$ and $d_i = a_i/(b_i - a_i)$.

When $n = 2$, we have $P[Y_1 = 1, Y_2 = 1] = E[Y_1Y_2] = E[Y_1]E[Y_2] = P[Y_1 = 1]P[Y_2 = 1]$. Also,

$$\begin{aligned} P[Y_1 = 1, Y_2 = 0] &= P[Y_1 = 1, 1 - Y_2 = 1] = E[Y_1(1 - Y_2)] = E[Y_1 - Y_1Y_2] \\ &= E[Y_1] - E[Y_1Y_2] = E[Y_1] - E[Y_1]E[Y_2] \\ &= E[Y_1](1 - E[Y_2]) = P[Y_1 = 1](1 - P[Y_2 = 1]) \\ &= P[Y_1 = 1]P[Y_2 = 0]. \end{aligned}$$

Similarly, one can show that $P[Y_1 = 0, Y_2 = 1] = P[Y_1 = 0]P[Y_2 = 1]$ and $P[Y_1 = 0, Y_2 = 0] = P[Y_1 = 0]P[Y_2 = 0]$.

The proof in the general case is similar. If we wish to show that

$$P(Y_1 = r_1, \ldots, Y_n = r_n) = P(Y_1 = r_1) \cdots P(Y_n = r_n),$$

then we write $P(Y_1 = r_1, \ldots, Y_n = r_n)$ as the expected value of a product where the variable Y_i appears in the product if $r_i = 1$ and $1 - Y_i$ appears in the product $r_i = 0$. By expanding this product and using the linearity of expected value and the hypothesis in the lemma, the result follows. □

Our goal in this section is the following result.

Theorem 2. *The 2^n functions $Z_1^{r_1}Z_2^{r_2} \cdots Z_n^{r_n}$, where each r_i equals 0 or 1, form an orthonormal basis for $\mathcal{F}$, with respect to $\langle\ ,\ \rangle_\mu$, if and only if the random variables $X_1, \ldots, X_n$ are independent.*

Proof. Assume that $X_1, \ldots, X_n$ are independent. Then $Z_1, \ldots, Z_n$ are also independent. Clearly, the constant function 1 has expected value 1, and the inner product of 1 with any other function is just the expected value of that function. Given two "monomials" $Z_{i_1} \cdots Z_{i_q}$ and $Z_{j_1} \cdots Z_{j_r}$, their inner product will be the expected value of their product. This product will be of the form $Z_{k_1}^2 \cdots Z_{k_s}^2 Z_{m_1} \cdots Z_{m_t}$. By independence, the expected value of this random variable will be the product

$$E_\mu[Z_{k_1}^2] \cdots E_\mu[Z_{k_s}^2] E_\mu[Z_{m_1}] \cdots E_\mu[Z_{m_t}].$$

If the two monomials are distinct, then some Z_i in this product will appear to the first power and the product of the expected values will be 0. (This also shows that 1 is orthogonal to each monomial $Z_{i_1} \cdots Z_{i_q}$.) If the two monomials are the same, then all the Z_i's in this product will have exponent 2 and the product of the expected values will be 1. Thus the 2^n functions in the statement of the theorem form an orthonormal basis for $\mathcal{F}$.

Conversely, assume that the 2^n functions $Z_1^{r_1} Z_2^{r_2} \cdots Z_n^{r_n}$ form an orthonormal basis for $\mathcal{F}$. It suffices to show that $Z_1, \ldots, Z_n$ are independent. By orthonormality, the expected value of the product of any subset of these random variables will be the product of the expected values. Applying the above lemma, we see that $Z_1, \ldots, Z_n$ are independent. It follows that $X_1, \ldots, X_n$ are independent, and the proof of the theorem is complete. □

From now on, we assume that $X_1, \ldots, X_n$ are independent random variables. We will order the functions $Z_{i_1} \cdots Z_{i_n}$ in the orthonormal basis above according to the degree lexicographic order. (So we order these basis functions as: $1, Z_1, Z_2, \ldots, Z_n, Z_1Z_2, \ldots, Z_1Z_n, Z_1Z_2Z_3, \ldots, Z_1Z_2 \cdots Z_n$.)

Let $M_{\mathcal{Z}}$ be the $2^n \times 2^n$ matrix whose rows are the values of the elements in this orthonormal basis on the n-tuples in B^n; i.e., the rows of $M_{\mathcal{Z}}$ are the transposes of the values vectors of the $Z_{i_1} \cdots Z_{i_n}$. (There are good reasons for choosing $M_{\mathcal{Z}}$ to be either this matrix or the transpose of this matrix. One reason for our choice is to avoid a transpose occurring in the definition of the transform in the next section.) So, $M_{\mathcal{Z}} =$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ Z_1(0,\ldots,0) & Z_1(1,0,\ldots,0) & \cdots & Z_1(1,1,\ldots,1) \\ Z_2(0,\ldots,0) & Z_2(1,0,\ldots,0) & \cdots & Z_2(1,1,\ldots,1) \\ \vdots & \vdots & \cdots & \vdots \\ Z_1 \cdots Z_n(0,\ldots,0) & Z_1 \cdots Z_n(1,0,\ldots,0) & \cdots & Z_1 \cdots Z_n(1,1,\ldots,1) \end{pmatrix}.$$

Notice that once we know rows 2 through $n+1$ of this matrix, then later rows are obtained simply by multiplying some of these rows.

Let $W = (w_{ij})$ denote the $2^n \times 2^n$ diagonal matrix given by $w_{ii} = \mu(i-1)$; i.e., W is the diagonal matrix whose diagonal elements are the weights assigned by μ to the n-tuples $00 \cdots 0, 10 \cdots 0, \ldots, 11 \cdots 1$. The fact that the functions $Z_1^{r_1} Z_2^{r_2} \cdots Z_n^{r_n}$ form an orthonormal basis amounts to the matrix equation

$$M_{\mathcal{Z}} W M_{\mathcal{Z}}^t = I_{2n}.$$

To close this section, we will show that the orthonormal basis consisting of the functions $Z_1^{r_1} Z_2^{r_2} \cdots Z_n^{r_n}$ is actually the result of applying the Gram–Schmidt process to the basis $\mathcal{X}$ of $\mathcal{F}$ consisting of the functions $X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}$. Thus, our theorem above gives the result of the Gram–Schmidt process without having to go through all the computations.

Theorem 3. *Assume the random variables $X_1, \ldots, X_n$ are independent. Then the orthonormal basis $\{Z_1^{r_1} Z_2^{r_2} \cdots Z_n^{r_n}\}$ is the result of applying the Gram–Schmidt process to the basis $\{X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}\}$ of $\mathcal{F}$.*

Proof. This result follows from a uniqueness property of the Gram–Schmidt process; but, since this property may not be well-known, we will give a brief argument.

Let $\mathcal{Y}$ denote the orthonormal basis of $\mathcal{F}$ (with inner product $\langle\ ,\ \rangle_\mu$) obtained by applying the Gram–Schmidt process to the basis $\mathcal{X}$. Let $M_{\mathcal{X}}$ (resp. $M_{\mathcal{Y}}$) denote the $2^n \times 2^n$ matrix whose rows are the values of the elements in $\mathcal{X}$ (resp. $\mathcal{Y}$) on the n-tuples in B^n. It is easy to see from the definition of the Z_i that we have $M_{\mathcal{Z}} = L_1 M_{\mathcal{X}}$ for a lower triangular matrix L_1 with positive entries on the diagonal. Also, it follows from the definition of the Gram–Schmidt process that we have $M_{\mathcal{Y}} = L_2 M_{\mathcal{X}}$ for a lower triangular matrix L_2 with positive entries on the diagonal. Since $\mathcal{Y}$ is an orthonormal basis, we also have $M_{\mathcal{Y}} W M_{\mathcal{Y}}^t = I_{2n}$.

Now, we have

$$\begin{aligned} M_{\mathcal{X}} W M_{\mathcal{X}}^t &= L_1^{-1} M_{\mathcal{Z}} W (L_1^{-1} M_{\mathcal{Z}})^t \\ &= L_1^{-1} M_{\mathcal{Z}} W M_{\mathcal{Z}}^t (L_1^{-1})^t = L_1^{-1} (L_1^{-1})^t \end{aligned}$$

and also

$$\begin{aligned} M_{\mathcal{X}} W M_{\mathcal{X}}^t &= L_2^{-1} M_{\mathcal{Y}} W (L_2^{-1} M_{\mathcal{Y}})^t \\ &= L_2^{-1} M_{\mathcal{Y}} W M_{\mathcal{Y}}^t (L_2^{-1})^t = L_2^{-1} (L_2^{-1})^t. \end{aligned}$$

The matrix $M_{\mathcal{X}} W M_{\mathcal{X}}^t$ is a positive definite symmetric matrix. There are several ways to see this — one way is that W is positive definite and $M_{\mathcal{X}}$ is nonsingular. The above equations give two Cholesky decompositions of this matrix. By the uniqueness of Cholesky decomposition for positive definite symmetric matrices [4, Theorem 4.2.5], we conclude that $L_1 = L_2$. Therefore, $M_{\mathcal{Y}} = M_{\mathcal{Z}}$, and it follows that the basis $\mathcal{Y}$ is identical to the basis $\mathcal{Z}$. □

4. Transforms

The transform we will define in this section operates on a values vector and yields a vector of coordinates with respect to the orthonormal basis $\{Z_1^{r_1} Z_2^{r_2} \cdots Z_n^{r_n}\}$. We define $T(f) = T_\mu(f)$, the transform of f with respect to μ, by

$$T(f) = M_{\mathcal{Z}} W \nu(f).$$

Let $\alpha_0(f)$ denote the component of $T(f)$ corresponding to the basis function 1 and let $\alpha_{i_1 \ldots i_m}(f)$ denote the component of $T(f)$ corresponding to the basis function $Z_{i_1} \cdots Z_{i_m}$. From standard results about orthonormal bases (see [8]), we have the following result.

Theorem 4. *Let $\mathcal{F}_k$ denote the subspace of $\mathcal{F}$ of pseudo-Boolean functions of degree at most k. Given $f \in \mathcal{F}$, the function $g \in \mathcal{F}_k$ that minimizes $\|f - g\|_\mu$ is*

$$\begin{aligned} g &= \langle f, 1 \rangle_\mu + \sum_{i=1}^{n} \langle f, Z_i \rangle_\mu Z_i + \cdots + \sum_{i_1 < \cdots < i_k} \langle f, Z_{i_1} \cdots Z_{i_k} \rangle_\mu Z_{i_1} \cdots Z_{i_k} \\ &= \alpha_0(f) + \sum_{i=1}^{n} \alpha_i(f) Z_i + \cdots + \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k}(f) Z_{i_1} \cdots Z_{i_k}. \end{aligned}$$

Example 5. We first consider the uniform distribution case. Here, each n-tuple in B^n is equally likely, so we have $\mu(\mathbf{x}) = 1/2^n$ for all $\mathbf{x} \in B^n$. (So, in this case, the matrix W above is simply $(1/2^n) I_{2^n}$, where I_{2^n} denotes the $2^n \times 2^n$ identity matrix.) Each coordinate function X_i is then a Bernoulli random variable with $p = q = 1/2$. It follows that

$$Z_i = \frac{X_i - \frac{1}{2}}{\sqrt{\frac{1}{4}}} = 2X_i - 1 \quad \text{for } i = 1, \ldots, n.$$

These n functions, which take only the values -1 and 1, are orthogonal and have norm 1 and their products may be formed to fill out an orthonormal basis for $\mathcal{F}$. They are basically the well-known discrete Walsh functions (cf. [9]), but we have ordered them differently from the usual order. (Notice that the pseudo-Boolean function $1 - 2X_i$ could also be written as the function $(-1)^{X_i}$.) In the case when $n = 3$, the matrix $M_{\mathcal{Z}}$ above is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

With a suitable rearrangement of the columns (which amounts to ordering the elements of B^n according to the integers they represent in base 2 from 2^{n-1} down to 0), the matrix $M_{\mathcal{Z}}$ would become the matrix of the Rademacher–Walsh transform (cf. [9]) used in transforming Boolean functions. For example, when $n = 3$, we would then have the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

Example 6. We return to an example that was considered in [10]. Suppose that a "1" is twice as likely as a "0." Then each X_i is a Bernoulli random variable with $p = 2/3$, $q = 1/3$, and $X_1, \ldots, X_n$ are independent. We have

$$Z_i = \frac{X_i - \frac{2}{3}}{\sqrt{\frac{2}{9}}} = \frac{3X_i - 2}{\sqrt{2}}.$$

In the case when $n = 3$, the matrix $M_{\mathcal{Z}}$ is

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-\sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} & -\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \\
-\sqrt{2} & -\sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
2 & -1 & -1 & 2 & \frac{1}{2} & -1 & -1 & \frac{1}{2} \\
2 & -1 & 2 & -1 & -1 & \frac{1}{2} & -1 & \frac{1}{2} \\
2 & 2 & -1 & -1 & -1 & -1 & \frac{1}{2} & \frac{1}{2} \\
-2\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}}
\end{pmatrix}$$

and the matrix W is the diagonal matrix with diagonal entries

$$1/27, 2/27, 2/27, 2/27, 4/27, 4/27, 4/27, 8/27.$$

Let $f(X_1, X_2, X_3) = 5X_1 + 13X_3 + 9X_1X_2 - 4X_1X_3 - 4X_2X_3 + 4X_1X_2X_3$. Then the transpose of the values vector of f is $(0, 5, 0, 13, 14, 14, 9, 23)$ and the transform of f is

$$T(f)^t = \frac{1}{27}\left(368, \quad 91\sqrt{2}, \quad 46\sqrt{2}, \quad 85\sqrt{2}, \quad 70, \quad -8, \quad -8, \quad 8\sqrt{2}\right).$$

The best linear approximation to f (with respect to μ) is

$$\begin{aligned} g_1 &= \frac{1}{27}(368 + 91\sqrt{2}Z_1 + 46\sqrt{2}Z_2 + 85\sqrt{2}Z_3) \\ &= \frac{1}{27}(-76 + 273X_1 + 138X_2 + 255X_3), \end{aligned}$$

agreeing with the result in [10]. The best quadratic approximation to f is

$$\begin{aligned} g_2 &= g_1 + \frac{1}{27}(70Z_1Z_2 - 8Z_1Z_3 - 8Z_2Z_3) \\ &= \frac{1}{27}(32 + 87X_1 - 48X_2 + 303X_3 + 315X_1X_2 - 36X_1X_3 - 36X_2X_3). \end{aligned}$$

Notice that when the best quadratic approximation is expressed in terms of the orthonormal basis $\{Z_1^{r_1}Z_2^{r_2}Z_3^{r_3}\}$, then the terms of degree less than two in that expression give the best linear approximation. This property holds whenever one uses an orthonormal basis, but does not hold, in general, when one uses a basis that is not orthonormal.

5. Transforms of multilinear representations and approximation problems

In [3], we considered two types of approximation problems. Given $f \in \mathcal{F}$, we were interested in finding (1) the closest (with respect to $\| \ \|_\mu$) function $g \in \mathcal{F}_k$ to f and (2) the closest function $h \in \mathcal{F}_1$ among functions in $\mathcal{F}_1$ that also satisfy the constraints that $f(0, 0, \dots, 0) = h(0, 0, \dots, 0)$ and $f(1, 1, \dots, 1) = h(1, 1, \dots, 1)$. To answer these questions, we needed to assume special properties of the measure μ. In particular, we needed to assume that μ was permutation invariant, meaning that the weight assigned to an n-tuple only depends on the number of 1's in that n-tuple. In addition, to answer the first of these questions we needed to assume that μ is a binomial distribution, meaning that there exists p such that if $\mathbf{x} \in B^n$ has precisely r 1's, then $\mu(\mathbf{x}) = p^r(1-p)^{n-r}$.

Now suppose that μ is permutation invariant, and that the X_i are independent. By the permutation invariance, the X_i's are identically distributed. It follows then from the independence that μ is a binomial distribution. Conversely, it is clear that if μ is a binomial distribution, then it is permutation invariant and the X_i's are independent. In this section, we will assume the distribution is binomial and we will apply our orthonormal basis to these approximation problems.

First, we give a conceptually simpler proof of Theorem 18 from [3]. This new proof is much closer in form and spirit to the proof given in the unweighted (or uniform distribution) case by Grabisch et al. [5].

If we are given a pseudo-Boolean function f written as a multilinear polynomial in terms of the coordinate functions X_i, then we can find the transform and the best approximation of degree at most k by the following procedure, which will be used in our improved proof.

(1) Substitute $\sqrt{pq}Z_i + p$ for X_i, $i = 1, \dots, n$, and expand the resulting expression to get a multilinear polynomial in the Z_i's. Notice that the coefficients in the resulting expression are the components of the transform of f.
(2) Since the products of the Z_i's form an orthonormal basis, the best approximation of degree at most k is obtained by simply truncating this multilinear polynomial by deleting all terms of degree greater than k.
(3) Now substitute $(X_i - p)/\sqrt{pq}$ for Z_i, $i = 1, \dots, n$, and expand the resulting expression to get the best approximation of degree at most k as a multilinear polynomial in the X_i's.

If R, S, and T are subsets of N, we will put $r = |R|$, $s = |S|$, and $t = |T|$. As usual, let

$$f(X_1, \dots, X_n) = \sum_{T \subseteq N} \left[a_T \prod_{i \in T} X_i \right].$$

For $k \in N$, let $\mathcal{N}_k = \{T \subseteq N : t \le k\}$. For each vector $b = (b_S : S \in \mathcal{N}_k)$, let

$$f_b(X_1, \dots, X_n) = \sum_{S \in \mathcal{N}_k} \left[b_S \prod_{i \in S} X_i \right].$$

Theorem 7. *Assume μ is a binomial distribution. If f_{b^*} is the best degree $\le k$ approximation to f, then*

$$b_S^* = a_S + (-1)^{k-s} \sum_{\substack{T \supset S \\ t > k}} p^{t-s} \binom{t-s-1}{k-s} a_T, \quad \text{for all } S \in \mathcal{N}_k.$$

Proof. By substituting $X_i = \sqrt{pq}Z_i + p$ into the multinomial expansion for f, we obtain

$$\begin{aligned} f(Z_1, \dots, Z_n) &= \sum_{T \subseteq N} \left[a_T \prod_{i \in T} (\sqrt{pq}Z_i + p) \right] \\ &= \sum_{T \subseteq N} a_T \left(\sum_{R \subseteq T} p^{t-r} (\sqrt{pq})^r \prod_{i \in R} Z_i \right). \end{aligned}$$

By the orthonormality of the products of the Z_i's, the best approximation of degree k to f is obtained by truncating the above expression by eliminating terms of degree greater than k. So,

$$f_{b^*}(Z_1, \dots, Z_n) = \sum_{T \subseteq N} a_T \left(\sum_{\substack{R \subseteq T \\ r \le k}} p^{t-r} (\sqrt{pq})^r \prod_{i \in R} Z_i \right).$$

By substituting $Z_i = (X_i - p)/\sqrt{pq}$ into this expression, we obtain

$$\begin{aligned} f_{b^*}(X_1, \dots, X_n) &= \sum_{T \subseteq N} a_T \left(\sum_{\substack{R \subseteq T \\ r \le k}} p^{t-r} (\sqrt{pq})^r \prod_{i \in R} \frac{X_i - p}{\sqrt{pq}} \right) \\ &= \sum_{T \subseteq N} a_T \left(\sum_{\substack{R \subseteq T \\ r \le k}} p^{t-r} \prod_{i \in R} (X_i - p) \right) \\ &= \sum_{T \subseteq N} a_T \left(\sum_{\substack{R \subseteq T \\ r \le k}} p^{t-r} \left\{ \sum_{S \subseteq R} (-p)^{r-s} \prod_{i \in S} X_i \right\} \right) \\ &= \sum_{T \subseteq N} a_T \left(\sum_{\substack{R \subseteq T \\ r \le k}} \sum_{S \subseteq R} \left\{ (-1)^{r-s} p^{t-s} \prod_{i \in S} X_i \right\} \right) \\ &= \sum_{T \subseteq N} a_T \left(\sum_{\substack{S \subseteq T \\ s \le k}} \sum_{\substack{S \subseteq R \subseteq T \\ r \le k}} \left\{ (-1)^{r-s} p^{t-s} \prod_{i \in S} X_i \right\} \right) \end{aligned}$$

$$= \sum_{T\subseteq N} a_T \left(\sum_{\substack{S\subseteq T\\ s\le k}} \left\{ p^{t-s} \prod_{i\in S} X_i \sum_{\substack{S\subseteq R\subseteq T\\ r\le k}} (-1)^{r-s} \right\} \right)$$

$$= \sum_{\substack{S\subseteq N\\ s\le k}} \left(\sum_{S\subseteq T\subseteq N} a_T \left\{ p^{t-s} \sum_{\substack{S\subseteq R\subseteq T\\ r\le k}} (-1)^{r-s} \right\} \right) \prod_{i\in S} X_i.$$

Now,

$$\sum_{\substack{S\subseteq R\subseteq T\\ r\le k}} (-1)^{r-s} = \sum_{j=0}^{k-s} (-1)^j \quad \text{(the number of subsets of } T\setminus S \text{ of cardinality } j).$$

Hence, we have

$$\sum_{\substack{S\subseteq R\subseteq T\\ r\le k}} (-1)^{r-s} = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } t \le k \\ \sum_{j=0}^{k-s} (-1)^j \binom{t-s}{j} & \text{if } t > k. \end{cases}$$

Therefore,

$$f_{b^*}(X_1,\ldots,X_n) = \sum_{\substack{S\subseteq N\\ s\le k}} \left(a_S + \sum_{\substack{T\supset S\\ t>k}} \left(\sum_{j=0}^{k-s} (-1)^j \binom{t-s}{j} p^{t-s} \right) a_T \right) \prod_{i\in S} X_i$$

$$= \sum_{\substack{S\subseteq N\\ s\le k}} \left(a_S + \sum_{\substack{T\supset S\\ t>k}} (-1)^{k-s} \binom{t-s-1}{k-s} p^{t-s} a_T \right) \prod_{i\in S} X_i,$$

where we have used the combinatorial identity $\sum_{j=0}^{u} (-1)^j \binom{m}{j} = (-1)^u \binom{m-1}{u}$ (which follows by an easy induction argument from the fundamental identity $\binom{m}{u} = \binom{m-1}{u-1} + \binom{m-1}{u}$). □

Next, we consider a constrained approximation problem. Motivated by applications in game theory and the mathematical theory of evidence, Hammer and Holzman [6] defined the notion of a faithful linear approximation of a pseudo-Boolean function f. Put $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Let

$$\mathcal{F}_k^f = \{g \in \mathcal{F}_k \mid g(\mathbf{0}) = f(\mathbf{0}) \text{ and } g(\mathbf{1}) = f(\mathbf{1})\}.$$

Then $\mathcal{F}_k^f$ is an affine space, but not, in general, a vector subspace of $\mathcal{F}_k$. A best faithful linear random variable approximation to f is a function in $\mathcal{F}_1^f$ that minimizes $\|g - f\|_\mu$ over all $g \in \mathcal{F}_1^f$. Hammer and Holzman solved this problem in the unweighted (or uniform distribution) case. We generalized their result (and [1, Theorem 4]) in [3] under the assumption that μ is permutation invariant. Here, we will consider the problem of approximating f by a higher-order function with the same values as f at $\mathbf{0}$ and at $\mathbf{1}$.

Given $f \in \mathcal{F}$, let

$$f(Z_1,\ldots,Z_n) = \sum_{T\subseteq N} \left[b_T \prod_{i\in T} Z_i \right]$$

be the multilinear expression for f in terms of our orthonormal basis. Now, by an argument similar to the one used in the linear case in [3], we can find a best faithful approximation of degree k to f by first finding f_k, a best degree k approximation to f, and then finding a best approximation to f_k in $\mathcal{F}_k^f$. By orthonormality, we have

$$f_k(Z_1,\ldots,Z_n) = \sum_{T\in\mathcal{N}_k} \left[b_T \prod_{i\in T} Z_i \right].$$

Let $g(Z_1,\ldots,Z_n) = \sum_{T\in\mathcal{N}_k} \left[u_T \prod_{i\in T} Z_i \right] \in \mathcal{F}_k$. We now need to find $g^* \in \mathcal{F}_k^f$ that minimizes $\|g - f_k\|_\mu$ among all $g \in \mathcal{F}_k^f$. We will call such a function g^* the best faithful approximation to f in $\mathcal{F}_k$. (The uniqueness of g^* follows from [11, Theorem (4.2.4)].) Notice that, by orthonormality, we have $\|g - f_k\|_\mu = \sum_{T\in\mathcal{N}_k} (u_T - b_T)^2$. This means that if we work with the expansions in terms of the products of the Z_i's, then our minimization problem is an unweighted least squares problem as g ranges over the elements of an affine subspace. This is a key feature of our orthonormal basis.

We will introduce the following notation. Put

$$D_0(n,k) = \sum_{\substack{0\le j\le k \\ j \text{ even}}} \binom{n}{j}$$

$$D_1(n,k) = \sum_{\substack{1\le j\le k \\ j \text{ odd}}} \binom{n}{j}$$

$$D(n,k) = D_0(n,k) + D_1(n,k).$$

Let $\mathbf{b}_k$ denote the $D(n,k)\times 1$ column matrix with entries b_T for $T \in \mathcal{N}_k$ (these are the coefficients in the expansion of f_k in terms of our orthonormal basis) and let $\mathbf{u}$ denote the $D(n,k)\times 1$ column matrix with entries u_T for $T \in \mathcal{N}_k$.

First, we will solve this question in the case of the uniform distribution. Then $Z_i = 2X_1 - 1$ and $X_i = (Z_i+1)/2$. Notice that $X_i = 0$ corresponds to $Z_i = -1$ and $X_i = 1$ corresponds to $Z_i = 1$. Hence, we have $f(\mathbf{0}) = \sum_{T\subseteq N}(-1)^t b_T$ and $f(\mathbf{1}) = \sum_{T\subseteq N} b_T$. Also, $g \in \mathcal{F}_k^f$ if and only if $\sum_{T\in\mathcal{N}_k}(-1)^t u_T = f(\mathbf{0})$ and $\sum_{T\in\mathcal{N}_k} u_T = f(\mathbf{1})$. These conditions are equivalent to

$$\sum_{\substack{T\in\mathcal{N}_k \\ t \text{ even}}} u_T = \frac{f(\mathbf{1})+f(\mathbf{0})}{2} \quad\text{and}\quad \sum_{\substack{T\in\mathcal{N}_k \\ t \text{ odd}}} u_T = \frac{f(\mathbf{1})-f(\mathbf{0})}{2}. \tag{2}$$

Let $M = [m_{i,j}]$ be the $2\times D(n,k)$ matrix defined by

$$m_{1j} = \begin{cases} 1 & \text{if } D(n,l) < j \le D(n,l+1) \text{ for some even } l,\ 0\le l<k \\ 0 & \text{otherwise} \end{cases}$$

$$m_{2j} = \begin{cases} 1 & \text{if } j=1 \text{ or } D(n,l) < j \le D(n,l+1) \text{ for some odd } l,\ 1\le l<k \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$MM^T = \begin{bmatrix} D_1(n,k) & 0 \\ 0 & D_0(n,k) \end{bmatrix}. \tag{3}$$

Then by (2), $g \in \mathcal{F}_k^f$ if and only if

$$M\mathbf{u} = \begin{bmatrix} \dfrac{f(\mathbf{1})-f(\mathbf{0})}{2} \\ \dfrac{f(\mathbf{1})+f(\mathbf{0})}{2} \end{bmatrix}.$$

We also have

$$M\mathbf{b}_k = \begin{bmatrix} \displaystyle\sum_{\substack{T\in\mathcal{N}_k \\ t \text{ odd}}} b_T \\ \displaystyle\sum_{\substack{T\in\mathcal{N}_k \\ t \text{ even}}} b_T \end{bmatrix}.$$

Our minimization problem then amounts to finding g^* such that $\mathbf{u}^* - \mathbf{b}_k$ is the minimum norm solution to the system of equations

$$M(\mathbf{u}-\mathbf{b}_k) = \begin{bmatrix} \dfrac{f(\mathbf{1})-f(\mathbf{0})}{2} - \displaystyle\sum_{\substack{T\in\mathcal{N}_k \\ t \text{ odd}}} b_T \\ \dfrac{f(\mathbf{1})+f(\mathbf{0})}{2} - \displaystyle\sum_{\substack{T\in\mathcal{N}_k \\ t \text{ even}}} b_T \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{\substack{T\subseteq N \\ t>k,\, t \text{ odd}}} b_T \\ \displaystyle\sum_{\substack{T\subseteq N \\ t>k,\, t \text{ even}}} b_T \end{bmatrix}.$$

By standard results (cf. Section 4.3 of [11]), the minimum norm solution to this problem is

$$\mathbf{u}^* - \mathbf{b}_k = M^T(MM^T)^{-1} \begin{bmatrix} \displaystyle\sum_{\substack{T\subseteq N \\ t>k,\, t \text{ odd}}} b_T \\ \displaystyle\sum_{\substack{T\subseteq N \\ t>k,\, t \text{ even}}} b_T \end{bmatrix}.$$

Now, using (3), we have that $M^T(MM^T)^{-1} = [c_{ij}]$, where

$$c_{i1} = \begin{cases} \dfrac{1}{D_1(n,k)} & \text{if } D(n,l) < i \le D(n,l+1) \text{ for some even } l,\ 0 \le l < k \\ 0 & \text{otherwise} \end{cases}$$

$$c_{i2} = \begin{cases} \dfrac{1}{D_2(n,k)} & \text{if } i = 1 \text{ or } D(n,l) < i \le D(n,l+1) \text{ for some odd } l,\ 1 \le l < k \\ 0 & \text{otherwise.} \end{cases}$$

This yields the following result.

Theorem 8. *Let μ be the uniform distribution. Let*

$$f(Z_1,\ldots,Z_n) = \sum_{T\subseteq N}\left[b_T \prod_{i\in T} Z_i\right].$$

Then the best faithful approximation to f in $\mathcal{F}_k$ is the function $g^*(Z_1,\ldots,Z_n) = \sum_{S\in\mathcal{N}_k}\left[u_S^* \prod_{i\in T} Z_i\right]$, where

$$u_S^* = \begin{cases} b_S + \dfrac{\sum_{\substack{T\subseteq N\\ t>k,\, t \text{ even}}} b_T}{D_0(n,k)} & \text{if } s \text{ is even and } s \le k \\ b_S + \dfrac{\sum_{\substack{T\subseteq N\\ t>k,\, t \text{ odd}}} b_T}{D_1(n,k)} & \text{if } s \text{ is odd and } s \le k. \end{cases}$$

In the linear case, we recover the following result from [6].

Corollary 9. *Let μ be the uniform distribution. Let*

$$f(X_1,\ldots,X_n) = \sum_{T\subseteq N}\left[a_T \prod_{i\in T} X_i\right].$$

Then the best faithful linear approximation to f is the function $g^*(X_1,\ldots,X_n) = f(\mathbf{0}) + \sum_{i=1}^n v_i X_i$, where

$$v_i = \sum_{i\in T}\frac{a_T}{2^{t-1}} + \frac{1}{n}\left(f(\mathbf{1}) - f(\mathbf{0}) - \sum_{T\subseteq N}\frac{t a_T}{2^{t-1}}\right).$$

Proof. Substituting $(Z_i + 1)/2$ for X_i in the multilinear expression for f and expanding, we obtain

$$f(Z_1,\ldots,Z_n) = \sum_{T\subseteq N}\left[b_T \prod_{i\in T} Z_i\right],$$

and it is easy to see that

$$b_{\{i\}} = \sum_{i\in T}\frac{a_T}{2^t}.$$

Notice that $D_0(n,1) = 1$ and $D_1(n,1) = n$.

Let g^* denote the best faithful linear approximation to f. Then, from the Theorem, we have that $g^*(Z_1,\ldots,Z_n) = u_0^* + \sum_{i=1}^n u_i^* Z_i$, where

$$u_i^* = \sum_{i\in T}\frac{a_T}{2^t} + \frac{1}{n}\sum_{\substack{T\subseteq N\\ t>1\\ t \text{ odd}}} b_T \quad \text{for } i = 1,\ldots,n.$$

Put

$$\gamma = \frac{1}{n}\sum_{\substack{T\subseteq N\\ t>1\\ t \text{ odd}}} b_T,$$

and note that γ does not depend on i.

Now, substitute $2X_i - 1$ for Z_i in the above expression for g^* and expand to get $g^*(X_1, \ldots, X_n) = v_0 + \sum_{i=1}^n v_i X_i$, where

$$v_i = \sum_{i \in T} \frac{a_T}{2^{t-1}} + 2\gamma \quad \text{for } i = 1, \ldots, n.$$

Since g^* is faithful, we must have $v_0 = f(\mathbf{0})$ and

$$f(\mathbf{0}) + \sum_{j=1}^{n} \sum_{j \in T} \frac{a_T}{2^{t-1}} + 2n\gamma = f(\mathbf{1}).$$

Therefore,

$$\begin{aligned} 2\gamma &= \frac{1}{n}\left(f(\mathbf{1}) - f(\mathbf{0}) - \sum_{j=1}^{n}\sum_{j \in T} \frac{a_T}{2^{t-1}}\right) \\ &= \frac{1}{n}\left(f(\mathbf{1}) - f(\mathbf{0}) - \sum_{T \subseteq N} \frac{t a_T}{2^{t-1}}\right). \quad \square \end{aligned}$$

Finally, we consider the case of a binomial distribution where the probability of a 1 is p and the probability of a 0 is $q = 1 - p$. Here, $Z_i = (X_i - p)/\sqrt{pq}$ and $X_i = \sqrt{pq} Z_i + p$. Thus, $X_i = 0$ corresponds to $Z_i = -\sqrt{p/q}$ and $X_i = 1$ corresponds to $Z_i = \sqrt{q/p}$. If $f(Z_1, \ldots, Z_n) = \sum_{T \subseteq N} \left[b_T \prod_{i \in T} Z_i\right]$ is the multilinear expression for f in terms of our orthonormal basis, then we have

$$f(\mathbf{0}) = \sum_{T \subseteq N} (-1)^t (p/q)^{\frac{t}{2}} b_T \quad \text{and} \quad f(\mathbf{1}) = \sum_{T \subseteq N} (q/p)^{\frac{t}{2}} b_T.$$

(In the binomial, but not uniform, case, we are not able to find a simpler equivalent set of equations as in (2). This equivalent system led to the diagonal structure in (3).)

Let $\widetilde{M} = [\tilde{m}_{i,j}]$ be the $2 \times D(n, k)$ matrix defined by $\tilde{m}_{11} = \tilde{m}_{21} = 1$, and for $j > 1$ we have

$$\tilde{m}_{1j} = (-1)^l (p/q)^{\frac{l}{2}} \quad \text{if } D(n, l-1) < j \le D(n, l),\ 1 \le l \le k$$

$$\tilde{m}_{2j} = (q/p)^{\frac{l}{2}} \quad \text{if } D(n, l-1) < j \le D(n, l),\ 1 \le l \le k.$$

Then a function $g(Z_1, \ldots, Z_n) = \sum_{T \in \mathcal{N}_k} \left[u_T \prod_{i \in T} Z_i\right] \in \mathcal{F}_k$ is in $\mathcal{F}_k^f$ if and only if

$$\widetilde{M}\mathbf{u} = \begin{bmatrix} f(\mathbf{0}) \\ f(\mathbf{1}) \end{bmatrix}.$$

Also, we have

$$\widetilde{M}\mathbf{b}_k = \begin{bmatrix} \sum_{T \subseteq N_k} (-1)^t (p/q)^{\frac{t}{2}} b_T \\ \sum_{T \subseteq N_k} (q/p)^{\frac{t}{2}} b_T \end{bmatrix}$$

and

$$\widetilde{M}\widetilde{M}^T = \begin{bmatrix} \sum_{j=0}^{k} \binom{n}{j} \left(\frac{p}{q}\right)^j & \sum_{j=0}^{k} (-1)^j \binom{n}{j} \\ \sum_{j=0}^{k} (-1)^j \binom{n}{j} & \sum_{j=0}^{k} \binom{n}{j} \left(\frac{q}{p}\right)^j \end{bmatrix}.$$

Using the same reasoning as before, we get the following result.

Theorem 10. *Let μ be the above binomial distribution. With notation as above, the best faithful approximation to f in $\mathcal{F}_k$ is the function $g^*(Z_1, \ldots, Z_n) = \sum_{S \in \mathcal{N}_k} \left[u_S^* \prod_{i \in T} Z_i\right]$, where*

$$\mathbf{u}^* = \mathbf{b}_k + \widetilde{M}^T (\widetilde{M}\widetilde{M}^T)^{-1} \begin{bmatrix} \sum_{\substack{T \subseteq N \\ t > k}} (-1)^t (p/q)^{\frac{t}{2}} b_T \\ \sum_{\substack{T \subseteq N \\ t > k}} (q/p)^{\frac{t}{2}} b_T \end{bmatrix}.$$

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