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STOCHASTIC INTEGRAL REPRESENTATIONS OF $\mathcal{F}$ -SELFDECOMPOSABLE AND $\mathcal{F}$ -SEMI-SELFDECOMPOSABLE DISTRIBUTIONS

NADJIB BOUZAR

ABSTRACT. We study two specific mappings defined on the class of infinitely divisible distributions on $\mathbb{Z}_+$ with finite log-moments. As a consequence, we re-derive some known stochastic integral representations for $\mathcal{F}$ -selfdecomposable distributions and obtain some new ones for $\mathcal{F}$ -semi-selfdecomposable distributions. Stochastic integral representations for discrete distributions in nested subclasses of $\mathcal{F}$ -selfdecomposable and $\mathcal{F}$ -semi-selfdecomposable distributions are established via the iterates of the aforementioned mappings.

1. Introduction

Let $\mathcal{I}(\mathbb{R}^d)$ denote the class of infinitely divisible distributions on $\mathbb{R}^d$. Stochastic integral representations for distributions in sub-classes of $\mathcal{I}(\mathbb{R}^d)$ have been the object of numerous articles over the last three decades. The subclasses $\mathcal{SD}(\mathbb{R}^d)$ of selfdecomposable distributions (see Sato [14], for e.g.) and $\mathcal{SSD}_\alpha(\mathbb{R}^d)$, $\alpha \in (0, 1)$, of semi-selfdecomposable distributions (Maejima and Naito [11]) have been paid particular attention. We cite two important characterization results. For that we will need some notation first. We denote by $\mathcal{L}(Y)$ the probability law of a random variable Y and by $\lfloor t \rfloor$ the greatest integer function at $t \in \mathbb{R}$. For a probability law $\mu \in \mathcal{I}(\mathbb{R}^d)$, $\{X_t^{(\mu)}\}$ will designate a Lévy process such that $\mathcal{L}(X_1^{(\mu)}) = \mu$. We define the subclass $\mathcal{I}_{\log}(\mathbb{R}^d)$ by

$$\mathcal{I}_{\log}(\mathbb{R}^d) = \left\{ \mu \in \mathcal{I}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \ln^+ |x| \mu(dx) < \infty \right\},$$

where $\ln^+ |x| = \max(\ln |x|, 0)$.

A random variable X on $\mathbb{R}^d$ has a selfdecomposable distribution if and only if there exists a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty e^{-t} dX_t^{(\mu)}, \quad (1.1)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

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A random variable X on $\mathbb{R}^d$ has a semi-selfdecomposable distribution with order (span) $\alpha \in (0, 1)$ if and only if there exists a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty \alpha^{[t]} dX_t^{(\mu)}, \quad (1.2)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

The representation (1.1) was obtained by Wolfe [19] in the case $d = 1$ and generalized to the case $d > 1$ by Sato and Yamazato [15], and to Banach space-valued random variables by Jurek and Vervaat [9]. The result was extended to operator selfdecomposable distributions by Sato and Yamazato [16]. Versions of (1.1) for distributions in nested subclasses of $\mathcal{SD}(\mathbb{R}^d)$ were given in [9], [15], and Barndorff-Nielsen et al. [2].

Maejima and Ueda [12] established the representation (1.2) for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ and extended it to distributions in nested subclasses of $\mathcal{SSD}_\alpha(\mathbb{R}^d)$. Maejima and Miura [10] obtained a stochastic integral representation for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ similar to (1.1) but where the integration was with respect to a semi-Lévy process.

For more on the stochastic integrals with respect to Lévy processes, additive processes, and semi-Lévy processes, we refer to Rocha-Arteaga and Sato [13] and to references therein.

van Harn et al. [7] proposed discrete analogues of self-decomposability and stability for distributions on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. We recall below some important facts on the topic and refer without further mention to [7] and to Steutel and van Harn [18], Chapter 5, Section 8, for more details.

The $\mathbb{Z}_+$ -valued multiple $\alpha \odot_{\mathcal{F}} X$ for a $\mathbb{Z}_+$ -valued random variable X and $0 < \alpha < 1$ is defined as follows:

$$\alpha \odot_{\mathcal{F}} X = \sum_{k=1}^X Y_k(t) := Z_X(t) \quad (t = -\ln \alpha), \quad (1.3)$$

where $Y_1(\cdot), Y_2(\cdot), \dots$ are independent copies of a continuous-time Markov branching process, independent of X , such that for every $k \geq 1$, $P(Y_k(0) = 1) = 1$. The processes $(Y_k(\cdot), k \geq 1)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geq 0)$:

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leq 1; s, t \geq 0). \quad (1.4)$$

The process $Z_X(\cdot)$ of (1.3) is itself a Markov branching process driven by $\mathcal{F}$ and starting with X individuals ($Z_X(0) = X$).

Let $P(z)$ be the pgf of X . Then the pgf $P_{\alpha \odot_{\mathcal{F}} X}(z)$ of $\alpha \odot_{\mathcal{F}} X$ is given by

$$P_{\alpha \odot_{\mathcal{F}} X}(z) = P(F_t(z)) \quad (t = -\ln \alpha; 0 \leq z \leq 1). \quad (1.5)$$

As an analogue of scalar multiplication, the operation $\odot_{\mathcal{F}}$ must satisfy some minimal conditions. In particular, the following regularity conditions are imposed on the composition semigroup $\mathcal{F}$:

$$\lim_{t \downarrow 0} F_t(z) = F_0(z) = z, \quad \lim_{t \rightarrow \infty} F_t(z) = 1. \quad (1.6)$$

The first part of (1.6) implies the continuity of the semigroup $\mathcal{F}$ (by way of (1.4)) and the second part is equivalent to assuming that $m = F'_1(1) \leq 1$, which implies the (sub-)criticality of the continuous-time Markov branching process $Y_k(\cdot)$ in (1.3).

We will restrict ourselves to the subcritical case ($m < 1$) and we will assume without loss of generality that $m = e^{-1}$.

A $\mathbb{Z}_+$ -valued random variable X , or its distribution, is said to have an $\mathcal{F}$ -self-decomposable distribution if for every $t > 0$,

$$X \stackrel{d}{=} e^{-t} \odot_{\mathcal{F}} X + X_t, \quad (1.7)$$

where X_t is $\mathbb{Z}_+$ -valued and X and X_t are independent (see [7]).

Equivalently, by (1.5) and (1.7), a distribution on $\mathbb{Z}_+$ with pgf $P(z)$ is $\mathcal{F}$ -self-decomposable if for every $t > 0$, there exists a pgf $P_t(z)$ such that

$$P(z) = P(F_t(z))P_t(z) \quad (0 \leq z \leq 1), \quad (1.8)$$

where P and P_t are the pgf's of X and X_t , respectively, in (1.7).

$\mathcal{F}$ -self-decomposable distributions are infinitely divisible. Moreover the distribution of X_t is infinitely divisible for every $t > 0$.

Let $0 < \alpha < 1$. A $\mathbb{Z}_+$ -valued random variable X with a nondegenerate distribution is said to be $\mathcal{F}$ -semi-selfdecomposable of order α if X satisfies (1.7) for some infinitely divisible $\mathbb{Z}_+$ -valued random variable X_t independent of X and for $t = -\ln \alpha$ (Bouzar [4]). $\mathcal{F}$ -semi-selfdecomposable distributions are infinitely divisible. $\mathcal{F}$ -semistable and $\mathcal{F}$ -geometric semistable distributions are $\mathcal{F}$ -semi-selfdecomposable (see [4]).

Steutel et al. [17] introduced a stochastic integral with respect to a $\mathbb{Z}_+$ -valued Lévy process and used it to give an integral representation of (continuous time) sub-critical branching processes with immigration. They showed that $\mathcal{F}$ -selfdecomposable distributions arise as the weak limit for such processes (as $t \rightarrow \infty$). Although not explicitly noted by the authors, but implicit in the proof of their main result (Theorem 3.2), is an integral representation for $\mathcal{F}$ -selfdecomposable distributions.

The aim of this paper is to obtain the discrete analogues of the representations (1.1) and (1.2) for $\mathcal{F}$ -selfdecomposable and $\mathcal{F}$ -semi-selfdecomposable distributions and to extend Theorem 3.2 in [17] to $\mathcal{F}$ -semi-selfdecomposable distributions. We parallel the treatment for the continuous case, as covered in [12]. In Section 2, we recall some facts on the stochastic integral with respect to a $\mathbb{Z}_+$ -valued Lévy process and formally describe the associated improper integral. In Section 3, we define the mapping Ψ_α ($\alpha \in (0, 1)$) on the class of infinitely divisible distributions on $\mathbb{Z}_+$ and identify its domain and range. As a corollary, we re-derive a stochastic integral representation for $\mathcal{F}$ -selfdecomposable distributions obtained implicitly in [17]. In addition, we describe a fixed-point property for Ψ_α in terms of $\mathcal{F}$ -stable distributions. In Section 4, we introduce and study the operator Φ_α and study its properties. We obtain a stochastic integral representation for $\mathcal{F}$ semi-selfdecomposable distributions with order $\alpha \in (0, 1)$ and we show that these distributions arise as weak limits to a specific class of $\mathbb{Z}_+$ -valued stochastic processes. Finally, in Section 5, stochastic integral representations for distributions in

nested subclasses of $\mathcal{F}$ -decomposable and $\mathcal{F}$ -semi-selfdecomposable distributions are derived via the iterates of Ψ_α and Φ_α , respectively.

In the rest of this section we recall some additional definitions and results about the continuous composition semigroup of pgf's $\mathcal{F} := (F_t, t \geq 0)$.

The following characterization of $\mathcal{F}$ -self-decomposable distributions (see [18], Chapter V, Theorem 8.3) plays a key role throughout the paper.

Theorem 1.1. *A function $P(z)$ on $[0, 1]$ is the pgf of an $\mathcal{F}$ -selfdecomposable on $\mathbb{Z}_+$ if and only if it has the form*

$$\ln P(z) = \int_z^1 \frac{\ln Q(x)}{U(x)} dx = \int_0^\infty \ln Q(F_t(z)) dt \quad (0 \leq z \leq 1), \quad (1.9)$$

where $U(z)$ is the infinitesimal generator of the semigroup $\mathcal{F}$ (see definition below) and $Q(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ such that

$$\int_0^1 \frac{[-\ln Q(x)]}{U(x)} dx < \infty. \quad (1.10)$$

The infinitesimal generator U of the semigroup $\mathcal{F}$ is defined by

$$U(z) = \lim_{t \downarrow 0} (F_t(z) - z)/t \quad (|z| \leq 1), \quad (1.11)$$

and satisfies $U(z) > 0$ for $0 \leq z < 1$. Moreover,

$$U(x) \sim 1 - x \quad (x \uparrow 1). \quad (1.12)$$

The related A -function is defined by

$$A(z) = \exp \left\{ - \int_0^z (U(x))^{-1} dx \right\} \quad (0 \leq z < 1). \quad (1.13)$$

The functions $U(z)$ and $A(z)$ satisfy for any $t > 0$,

$$\frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z) F_t'(z) \quad (|z| \leq 1), \quad (1.14a)$$

and

$$A(F_t(z)) = e^{-t} A(z) \quad (0 \leq z < 1). \quad (1.14b)$$

Following standard terminology, we will refer to any $\mathbb{Z}_+$ -valued Lévy process as a $\mathbb{Z}_+$ -valued subordinator which we denote by $\{X_t^{(\mu)}\}$, with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We will be referring quite frequently to the following sets:

- $\mathcal{I}(\mathbb{Z}_+)$: the set of infinitely divisible distributions on $\mathbb{Z}_+$.
- $\mathcal{I}_{\log}(\mathbb{Z}_+)$: the subset of distributions $\mu = (q_n, n \geq 0)$ in $\mathcal{I}(\mathbb{Z}_+)$ such that $\sum_{n=0}^\infty q_n \ln(n+1) < \infty$.
- $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$: the subset of $\mathcal{F}$ -self-decomposable distributions in $\mathcal{I}(\mathbb{Z}_+)$.
- $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$: the subset of distributions in $\mathcal{I}(\mathbb{Z}_+)$ that are $\mathcal{F}$ -semi-selfdecomposable of order $\alpha \in (0, 1)$.

2. A Stochastic Integral With Respect to a $\mathbb{Z}_+$ -valued Subordinator

Let $(X_t^{(\mu)}, t \geq 0)$ be a $\mathbb{Z}_+$ -valued subordinator. We will assume without loss of generality that $X_0 = 0$. Since $\mathbb{Z}_+$ -valued subordinators are piecewise constant, it follows from Theorem 21.2, p. 135, in [14], that $\{X_t^{(\mu)}\}$ is a compound Poisson process, i.e., there exists a Poisson process $\{N_t\}$ and a sequence of $\mathbb{Z}_+$ -valued random variables $\{C_k\}$ with $C_0 = 0$, independent of $\{N_t\}$, such that

$$X_t^{(\mu)} = \sum_{k=1}^{N_t} C_k,$$

or, equivalently,

$$X_t^{(\mu)} = \sum_{\{k: 0 < T_k \leq t\}} C_k, \quad (2.1)$$

where $(T_k, k \geq 1)$ is the sequence of jump times of $\{X_t^{(\mu)}\}$.

We recall the definition of the stochastic integral with respect to $\{X_t^{(\mu)}\}$ introduced in [17]. Let $f(s)$ be a (Lebesgue) measurable function on an interval $[t_0, t_1]$ in $[0, \infty)$ and taking values in $(0, 1]$. We define

$$\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k: t_0 < T_k \leq t_1\}} f(T_k) \odot_{\mathcal{F}} C_k. \quad (2.2)$$

As noted in [17], the operation $A \odot_{\mathcal{F}} X$ for a random variable A taking values in $[0, 1]$ has pgf

$$P_{A \odot_{\mathcal{F}} X}(z) = \int_0^1 P_{a \odot_{\mathcal{F}} X}(z) dG_A(a),$$

where G_A is the distribution function of A .

We also recall a useful representation theorem obtained in [17].

Theorem 2.1. *Let $f(s)$ be a measurable function on $[t_0, t_1]$ with range in $(0, 1]$. The stochastic integral $\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ has an infinitely divisible distribution with pgf*

$$P(z) = \exp \left\{ \int_{t_0}^{t_1} \ln Q_{\mu}(F_{-\ln f(s)}(z)) ds \right\}, \quad (2.3)$$

where $Q_{\mu}(z)$ is the (infinitely divisible) pgf of $X_1^{(\mu)}$.

Next, we describe an improper integral.

Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator and $f(s)$ a measurable function on $[0, \infty)$ with values in $(0, 1]$. We define the $\mathbb{Z}_+$ -valued process $\{Y_t\}$ by

$$Y_t \stackrel{d}{=} \int_0^t f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k: 0 < T_k \leq t\}} f(T_k) \odot_{\mathcal{F}} C_k \quad (t \geq 0). \quad (2.4)$$

If $\{Y_t\}$ converges in distribution to a $\mathbb{Z}_+$ -valued random variable Y_{∞} as $t \rightarrow \infty$, then we set

$$\int_0^{\infty} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} Y_{\infty}. \quad (2.5)$$

Theorem 2.2. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator and $f(t)$ a measurable function on $[0, \infty)$ with values in $(0, 1]$. Assume that the process $\{Y_t\}$ of (2.4) converges weakly as $t \rightarrow \infty$. Then*

(i) *the distribution of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ is infinitely divisible and its pgf admits the representation*

$$P(z) = \exp \left\{ \int_0^\infty \ln Q_\mu(F_{-\ln f(s)}(z)) ds \right\}. \quad (2.6)$$

(ii) *$\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the following infinite series representation*

$$\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty f(T_k) \odot_{\mathcal{F}} C_k. \quad (2.7)$$

Proof. By Theorem 2.1, $L(\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)})$ is infinitely divisible as the weak limit of the infinitely divisible distributions $\{\mathcal{L}(Y_t)\}$, when $t \rightarrow \infty$. Since the pgf of Y_t is (by (2.3))

$$P_t(z) = \exp \left\{ \int_0^t \ln Q_\mu(F_{-\ln f(s)}(z)) ds \right\}, \quad (2.8)$$

it ensues that the pgf of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the representation (2.6) by letting $t \rightarrow \infty$ in (2.8). The infinite series representation (2.7) follows from the second equation in (2.4). $\square$

Remark 2.3. One can arrive at an equivalent definition of the stochastic integral (2.2) that is similar to the one constructed in the continuous case (see for e.g. [13]), by first defining it for step functions and then extending the definition to measurable functions via the standard limit process.

3. A Stochastic Integral Representation for $\mathcal{F}$ -selfdecomposable Distributions

Let $\alpha \in (0, 1)$. We define the mapping $\Psi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Psi_\alpha(\mu) = \mathcal{L} \left(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)} \right), \quad (3.1)$$

where $\{X_t^{(\mu)}\}$ is a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

We identify the domain $\mathcal{D}(\Psi_\alpha)$ and range $\mathcal{R}(\Psi_\alpha)$ of the mapping Ψ_α and as a consequence we obtain an integral representation for $\mathcal{F}$ -selfdecomposable distributions on $\mathbb{Z}_+$.

We start out with a useful lemma.

Lemma 3.1. *Assume $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z)$. The following assertions are equivalent.*

- (i) $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$.
- (ii) (1.10) holds for $Q = Q_\mu(z)$.
- (iii) $\int_0^\infty [-\ln Q_\mu(F_s(z))] ds < \infty$ for every $z \in [0, 1]$.

Proof. (i) $\Leftrightarrow$ (ii): By Proposition 4.2, Appendix A, in [18], $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ if and only if $\int_0^1 \frac{[-\ln Q_\mu(z)]}{1-x} dx < \infty$. The conclusion follows by applying (1.12). The statement (ii) $\Leftrightarrow$ (iii) follows essentially from the proof of Theorem 8.3, Chapter V, in [18]. $\square$

Next, we give the main result of the section. Parts of the statement of Theorem 3.2 below can be derived from the proof of Theorem 3.2 in [17]. We provide a somewhat abbreviated proof for the sake of completeness.

Theorem 3.2. $\mathcal{D}(\Psi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) = \mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$.

Proof. Assume $\mu \in \mathcal{D}(\Psi_\alpha)$. By applying (2.6) to $f(s) = \alpha^s$, one can show that the pgf $P_{\Psi_\alpha(\mu)}(z)$ of $\Psi_\alpha(\mu)$ admits the representation (1.9) with $Q(z) = Q_\mu^{-1/\ln \alpha}(z)$. Therefore, by Theorem 1.1, $\Psi_\alpha(\mu)$ is $\mathcal{F}$ -selfdecomposable and $Q_\mu(z)$ satisfies (1.10). It follows by Lemma 3.1[(ii) $\Rightarrow$ (i)] that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. We have thus shown that $\mathcal{D}(\Psi_\alpha) \subset \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) \subset \mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By Lemma 3.1[(i) $\Rightarrow$ (iii)] and Theorem 1.1, the function $P(z) = \exp\{\int_0^\infty \ln Q_\mu(F_{-s \ln \alpha}(z)) ds\}$ is the pgf of an $\mathcal{F}$ -selfdecomposable distribution. Letting $P_t(z)$ be the pgf of $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$ and applying Theorem 2.1, as well as (2.8), to $f(s) = \alpha^s$, we conclude that $\lim_{t \rightarrow \infty} P_t(z) = P(z)$. Therefore, $\mathcal{L}(\int_0^\infty \alpha^t \odot_{\mathcal{F}} dX_s^{(\mu)})$ exists as the weak limit of $\{Y_t\}$ as $t \rightarrow \infty$, or equivalently, $\mu \in \mathcal{D}(\Psi_\alpha)$. We have thus shown $\mathcal{I}_{\log}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha)$. It remains to prove that $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_\alpha)$. Let $P(z)$ be the pgf of an $\mathcal{F}$ -selfdecomposable distribution on $\mathbb{Z}_+$. Then $P(z)$ admits the canonical representation (1.9) for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{-\ln \alpha}(z)$. It follows that $P(z)$ satisfies (2.6) with $f(s) = \alpha^s$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and let $\{Y_t\}$ be as above. It is clear that $\{Y_t\}$ converges weakly and therefore $\mathcal{L}(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. $\square$

Corollary 3.3. Let $\alpha \in (0, 1)$. The mapping Ψ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$.

Proof. We only need prove Ψ_α is one-to-one. Let $\mu_1, \mu_2 \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ such that $\Psi_\alpha(\mu_1) = \Psi_\alpha(\mu_2)$. Denote by $P(z)$ the pgf of $\Psi_\alpha(\mu_1)$ (and thus of $\Psi_\alpha(\mu_2)$). It follows by (2.6) and (1.4) that

$$\ln \frac{P(z)}{P(F_{-t \ln \alpha}(z))} = \int_0^t \ln Q_{\mu_i}(F_{-s \ln \alpha}(z)) ds \quad (z \in [0, 1]; i = 1, 2).$$

Therefore, $Q_{\mu_1}(F_{-t \ln \alpha}(z)) = Q_{\mu_2}(F_{-t \ln \alpha}(z))$ for every $z \in [0, 1]$ and $t \geq 0$. Letting $t \downarrow 0$ and using (1.6) leads to $Q_{\mu_1}(z) = Q_{\mu_2}(z)$, or $\mu_1 = \mu_2$. $\square$

The corollary below follows straightforwardly from Theorem 3.2 and the infinite series representation (2.7).

Corollary 3.4. A $\mathbb{Z}_+$ -valued random variable X has an $\mathcal{F}$ -selfdecomposable distribution if and only if for some $\alpha \in (0, 1)$, and therefore for every $\alpha \in (0, 1)$, there

exists a $\mathbb{Z}_+$ -valued subordinator $\{X_t^{(\mu)}\}$, with $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ (depending on α), such that

$$X \stackrel{d}{=} \int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{T_k} \odot_{\mathcal{F}} C_k. \quad (3.2)$$

Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see (2.1)).

For the sake of completeness, and as a consequence of Theorem 3.2, we restate the convergence result obtained in [17] (Theorem 3.2, therein).

Corollary 3.5. *Let $\alpha \in (0, 1)$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator and $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some $\mathbb{Z}_+$ -valued random variable Y_∞ (as $t \rightarrow \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is $\mathcal{F}$ -selfdecomposable.*

Next, we derive a fixed point property for the mapping Ψ_α that is the analogue of a result obtained in [9] for distributions on Banach and Euclidean spaces.

First, we briefly recall some results on $\mathcal{F}$ -stable distributions (see Chapter V, Section 8, in [18]).

A $\mathbb{Z}_+$ -valued random variable X , or its distribution, is said to be $\mathcal{F}$ -stable with exponent $\gamma \in (0, 1]$ if for every $\alpha \in (0, 1)$,

$$X \stackrel{d}{=} \alpha \odot_{\mathcal{F}} X + (1 - \alpha^\gamma)^{1/\gamma} \odot_{\mathcal{F}} X', \quad (3.3)$$

where $X \stackrel{d}{=} X'$ and X and X' are independent.

An $\mathcal{F}$ stable distribution μ on $\mathbb{Z}_+$ is $\mathcal{F}$ -selfdecomposable and thus necessarily infinitely divisible. Moreover, its pgf P_μ admits the following canonical representation:

$$P_\mu(z) = \exp[-\lambda A(z)^\gamma] \quad (0 \leq z \leq 1), \quad (3.4)$$

where $A(z)$ is the A -function given in (1.13) and λ is a positive constant.

Theorem 3.6. *Let $\alpha \in (0, 1)$. A distribution μ on $\mathbb{Z}_+$ is $\mathcal{F}$ -stable if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and there exists $c > 0$ such that the pgf's P_μ and $P_{\Psi_\alpha(\mu)}$ of μ and $\Psi_\alpha(\mu)$, respectively, satisfy the equation*

$$\ln P_{\Psi_\alpha(\mu)}(z) = c \ln P_\mu(z), \quad (0 \leq z \leq 1), \quad (3.5)$$

in which case $\Psi_\alpha(\mu)$ is also $\mathcal{F}$ -stable with the same exponent γ as μ , $\gamma = \frac{-1}{c \ln \alpha} \in (0, 1]$.

Proof. The “only if” part: assume that μ is $\mathcal{F}$ -stable with exponent $\gamma \in (0, 1]$. We have by (3.4) and the fact that $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, $x \in (0, 1)$ (see (1.13)),

$$\int_0^1 \frac{[-\ln P_\mu(x)]}{U(x)} dx = \lambda \int_0^1 \frac{A(x)^\gamma}{U(x)} dx = \lambda \int_0^1 A(x)^{\gamma-1} A'(x) dx \quad (\lambda > 0).$$

A simple change of variable argument shows that the third integral in the above equation is finite, and thus $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By (2.6) and (3.4),

$$\ln P_{\Psi_\alpha(\mu)}(z) = \int_0^\infty \ln P_\mu(F_{-s \ln \alpha}(z)) ds = -\lambda \int_0^\infty [A(F_{-s \ln \alpha}(z))]^\gamma ds,$$

for some $\lambda > 0$ and $\gamma \in (0, 1]$. It follows by (1.14b) that

$$\ln P_{\Psi_\alpha(\mu)}(z) = -\lambda A(z)^\gamma \int_0^\infty \alpha^{\gamma s} ds = \frac{\lambda}{\gamma \ln \alpha} A(z)^\gamma.$$

Therefore (3.5) holds with $c = -1/\gamma \ln \alpha$ and $\Psi_\alpha(\mu)$ is $\mathcal{F}$ -stable with exponent γ (by (3.4)). We now prove the "if part". Assume that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and that (3.5) holds. We have via (2.6) and (1.14a)

$$\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln \alpha} \int_0^\infty \ln P_\mu(F_s(z)) ds = \frac{-1}{\ln \alpha} \int_0^\infty \frac{\ln P_\mu(F_s(z))}{U(F_s(z))} \frac{\partial}{\partial s} F_s(z) ds.$$

The change of variable $x = F_t(z)$ and (1.6) yield $\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln \alpha} \int_z^1 \frac{\ln P_\mu(x)}{U(x)} dx$, which, combined with (3.5), yields in turn the differential equation

$$c \frac{d}{dx} \ln P_\mu(x) = \frac{1}{\ln \alpha} \frac{\ln P_\mu(x)}{U(x)}.$$

Since $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, one easily deduces that the solution $P_\mu(z)$ takes the form (3.4) with $\lambda = -\ln P_\mu(0)$ and $\gamma = -1/(c \ln \alpha)$. The fact that $\gamma \in (0, 1]$ follows from a result in van Harn and Steutel [8] (see the proof of Lemma 4.2 therein). $\square$

4. A Stochastic Integral Representation for $\mathcal{F}$ -semi-selfdecomposable Distributions

Let $\alpha \in (0, 1)$. We define the mapping $\Phi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Phi_\alpha(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\right), \quad (4.1)$$

where $\{X_t^{(\mu)}\}$ is a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and, recall, $\lfloor s \rfloor$ denotes the greatest integer function at s .

The first two results identify the domain $\mathcal{D}(\Phi_\alpha)$ and range $\mathcal{R}(\Phi_\alpha)$ of the mapping Φ_α .

Theorem 4.1. $\mathcal{D}(\Phi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$.

Proof. Let $f(s) = \alpha^{\lfloor s \rfloor}$ for $s \in [0, t]$, and let $\{s_0, s_1, \dots, s_{\lfloor t \rfloor}, s_{\lfloor t \rfloor+1}\}$ be the subdivision of $[0, t]$ such that $s_j = j$, $j = 0, 1, \dots, \lfloor t \rfloor$, and $s_{\lfloor t \rfloor+1} = t$. The function $f(s)$ can be rewritten as

$$f(s) = \sum_{j=0}^{\lfloor t \rfloor} \alpha^j I_{[s_j, s_{j+1})}(s), \quad (s \in [0, t]).$$

A straightforward pgf argument, by way of Theorem 2.1 and equation (2.3), leads to

$$\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} = \sum_{j=0}^{\lfloor t \rfloor} \alpha^j \odot_F (X_{s_{j+1}}^{(\mu)} - X_{s_j}^{(\mu)}).$$

Moreover, the pgf $P_t(z)$ of $\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$ simplifies to

$$P_t(z) = \left\{ \prod_{j=0}^{\lfloor t \rfloor - 1} Q_{\mu}(F_{-j \ln \alpha}(z)) \right\} Q_{\mu}^{t - \lfloor t \rfloor}(F_{-\lfloor t \rfloor \ln \alpha}(z)). \quad (4.2)$$

Since $0 \leq t - \lfloor t \rfloor \leq 1$ and $\lim_{t \rightarrow \infty} F_{-\lfloor t \rfloor \ln \alpha}(z) = 1$, we have $\lim_{t \rightarrow \infty} Q_{\mu}^{t - \lfloor t \rfloor}(F_{-\lfloor t \rfloor \ln \alpha}(z)) = 1$. Therefore, $\lim_{t \rightarrow \infty} P_t(z)$ exists if and only if $\lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ is finite and

$$\lim_{t \rightarrow \infty} P_t(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z)). \quad (4.3)$$

Now $\prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ is the pgf of the size of the n -th generation of a Galton-Watson branching process with stationary immigration described by the pgf $Q_{\mu}(z)$ and an offspring distribution with pgf $F_{-\ln \alpha}(z)$ (see Athreya and Ney [1], Chapter 6). By Corollary 2 in Foster and Williamson [5], $\prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ converges to a pgf if and only if $\sum_{n=0}^{\infty} q_n \ln n < \infty$, where $\{q_n\}$ is the distribution with pgf Q_{μ} , or equivalently (since $\ln(n+1) \sim \ln n$ as $n \rightarrow \infty$), if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$. Therefore, $\Phi_{\alpha}(\mu)$ exists if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. $\square$

Theorem 4.2. $\mathcal{R}(\Phi_{\alpha}) = \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$.

Proof. Assume that $\Phi_{\alpha}(\mu)$ exists for some $\mu \in \mathcal{I}(\mathbb{Z}_+)$. Let $P_t(z)$ be the pgf of $\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then by (4.3), the pgf $P_{\Phi_{\alpha}(\mu)}(z)$ of $\Phi_{\alpha}(\mu)$ can be written as

$$P_{\Phi_{\alpha}(\mu)}(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z)) = \left[\lim_{n \rightarrow \infty} \prod_{j=1}^n Q_{\mu}(F_{-j \ln \alpha}(z)) \right] Q_{\mu}(z), \quad (4.4)$$

or,

$$\begin{aligned} P_{\Phi_{\alpha}(\mu)}(z) &= \left[\lim_{n \rightarrow \infty} \prod_{j=1}^n Q_{\mu}(F_{-(j-1) \ln \alpha}(F_{-\ln \alpha}(z))) \right] Q_{\mu}(z) \\ &= P(F_{-\ln \alpha}(z)) Q_{\mu}(z). \end{aligned} \quad (4.5)$$

Since $Q_{\mu}(z)$ is infinitely divisible, it follows by (1.8) (applied at $t = -\ln \alpha$) that $\Phi_{\alpha}(\mu) \in \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$ and thus $\mathcal{R}(\Phi_{\alpha}) \subset \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$. Assume now that $P(z)$ is the pgf of an $\mathcal{F}$ -semi-selfdecomposable distribution on $\mathbb{Z}_+$ with order α . We have by (1.8) that $P(z) = P(F_{-\ln \alpha}(z))Q(z)$ for some infinitely divisible pgf $Q(z)$. It follows by equation (2.2) in [4] that

$$P(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q(F_{-j \ln \alpha}(z)). \quad (4.6)$$

Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z) (= Q_\mu(z))$ and let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We conclude by (4.3) and (4.6) that $\mathcal{L}(\int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. We have thus shown $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha)$. $\square$

Corollaries 4.3 and 4.4 below are straightforward consequences of Theorems 4.1 and 4.2 and the infinite series representation (2.7).

Corollary 4.3. *A $\mathbb{Z}_+$ -valued random variable X with pgf $P(z)$ has an $\mathcal{F}$ -semi-selfdecomposable distribution with order $\alpha \in (0, 1)$ if and only if there exists a $\mathbb{Z}_+$ -valued subordinator $\{X_t^{(\mu)}\}$ such that*

$$X \stackrel{d}{=} \int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{\lfloor T_k \rfloor} \odot_{\mathcal{F}} C_k, \quad (4.7)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see (2.1)).

Corollary 4.4. *Let $\alpha \in (0, 1)$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator and $Y_t = \int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some $\mathbb{Z}_+$ -valued random variable Y_∞ (as $t \rightarrow \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is $\mathcal{F}$ -semi-selfdecomposable with order α .*

The proof of the next result is similar to that of Corollary 3.3 (by way of (4.5)).

Corollary 4.5. *Let $\alpha \in (0, 1)$. The mapping Φ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$.*

5. Integral Representations for Elements in Nested Subclasses of $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$

We define nested subclasses of $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$ and study the actions of the mappings Ψ_α and Φ_α , respectively, on their elements.

First, we formulate an equivalent definition of $\mathcal{F}$ -selfdecomposability and $\mathcal{F}$ -semi-selfdecomposability (see (1.7) and (1.8)). A distribution μ on $\mathbb{Z}_+$ belongs to $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ (resp. $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$ for some $\alpha \in (0, 1)$) if and only if for every $t > 0$ (resp. for $t = -\ln \alpha$),

$$\mu = (\mu \vee \nu_t) * \mu_t, \quad (5.1)$$

where ν_t is the distribution with pgf $F_t(z)$ and $\mu_t \in \mathcal{I}(\mathbb{Z}_+)$. The binary operation $\vee$ is the compounding operation and $*$ is the convolution operation.

We define $\mathcal{F}\text{-SD}_0(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ such that every $t > 0$, there exists $\mu_t \in \mathcal{F}\text{-SD}_m(\mathbb{Z}_+)$ such that (5.1) holds.

Likewise, we define for $\alpha \in (0, 1)$, $\mathcal{F}\text{-SSD}_{\alpha,0}(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-SSD}_{\alpha,m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ that satisfy (5.1) for $t = -\ln \alpha$ and some $\mu_t \in \mathcal{F}\text{-SSD}_{\alpha,m}(\mathbb{Z}_+)$.

For $m \geq 1$, we define

$$\mathcal{I}_{\log^m}(\mathbb{Z}_+) = \{\mu = (q_n, n \geq 0) \in \mathcal{I}(\mathbb{Z}_+) : \sum_{n=0}^{\infty} q_n (\ln(n+1))^m < \infty\}.$$

The sequences $\{\mathcal{I}_{\log^m}(\mathbb{Z}_+)\}$, $\{\mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)\}$, and $\{\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)\}$ are easily checked to be decreasing in $\mathcal{I}(\mathbb{Z}_+)$ by an induction argument.

Hansen [6] studied extensively the subclasses $\{\mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)\}$ (see also Berg and Forst [3]).

We start out by recalling the discrete versions of two important results obtained in [6].

Theorem 5.1. (Theorem 7.4 in [6]) Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $P(z)$ and $m \geq 0$. Then $\mu \in \mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$ if and only if

$$\ln P(z) = \int_0^{\infty} \ln Q(F_s(z)) s^m ds,$$

where $Q(z)$ is the pgf of a distribution in $\mathcal{I}(\mathbb{Z}_+)$.

Theorem 5.2. (Theorem 7.5 in [6]) Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$ and $m \geq 0$. The following assertions are equivalent.

- (i) $\int_0^{\infty} [-\ln Q(F_s(z))] s^m ds < \infty \quad (0 \leq z \leq 1)$
- (ii) The canonical sequence $\{r_k\}$ of μ defined by $\frac{Q'(z)}{Q(z)} = \sum_{k=0}^{\infty} r_k z^k$ satisfies

$$\sum_{k=0}^{\infty} (\ln(k+1))^{m+1} \frac{r_k}{k+1} < \infty \quad (r_k \geq 0).$$

Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu \in \mathcal{I}(\mathbb{Z}_+)$. For $\alpha \in (0, 1)$, we define the sequences of mappings $\{\Psi_{\alpha}^m(\mu)\}$ and $\{\Phi_{\alpha}^m(\mu)\}$ recursively as follows: $\Psi_{\alpha}^0(\mu) = \Phi_{\alpha}^0(\mu) = \mu$ and for every $m \geq 0$, $\Psi_{\alpha}^{m+1}(\mu) = \Psi_{\alpha}(\Psi_{\alpha}^m(\mu))$ and $\Phi_{\alpha}^{m+1}(\mu) = \Phi_{\alpha}(\Phi_{\alpha}^m(\mu))$ where Ψ_{α} and Φ_{α} are defined by (3.1) and (4.1), respectively. The aim of this section is to identify the domains and ranges of Ψ_{α}^m and Φ_{α}^m .

Next, we derive a useful lemma.

Lemma 5.3. Let $\alpha \in (0, 1)$ and m a positive integer. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$. Then the following assertions are equivalent.

- (i) $J_1(z) = \int_0^{\infty} [-\ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z))] \binom{\lfloor s \rfloor + m}{m} ds < \infty \quad (0 \leq z \leq 1).$
- (ii) $J_2(z) = \int_0^{\infty} [-\ln Q(F_s(z))] s^m ds < \infty \quad (0 \leq z \leq 1).$
- (iii) $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$

Proof. First, we note that for every $s > 0$, $\lfloor s \rfloor \leq s \leq \lfloor s \rfloor + 1$ and for every $z \in [0, 1]$, $F_s(z)$ is increasing as a function of $s \in (0, \infty)$ (see 1.14a). Moreover,

$\binom{\lfloor s \rfloor + m}{m} \sim \frac{s^m}{m!}$ as $s \uparrow \infty$. Rewriting $J_2(z)$ as

$$J_2(z) = t_0^{m+1} \int_0^\infty [-\ln Q(F_{st_0}(z))] s^m ds,$$

where $t_0 = -\ln \alpha$, and using a standard integration argument, one can easily show that (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) follows by Theorem 5.2, combined with the discrete versions of Theorem 25.3 and Proposition 25.4 in [14]. $\square$

The next result identifies the domain $\mathcal{D}(\Psi_\alpha^m)$ and the range $\mathcal{R}(\Psi_\alpha^m)$ of the mapping Ψ_α^m , $m \geq 1$.

Theorem 5.4. *For every positive integer m , $\mathcal{D}(\Psi_\alpha^m) = \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^m) = \mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)$. Moreover, if $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\ln P_{\Psi_\alpha^m(\mu)}(z) = \int_0^\infty \frac{s^{m-1}}{(m-1)!} \ln Q_\mu(F_{-s \ln \alpha}(z)) ds. \quad (5.2)$$

Proof. We prove the theorem by induction. The case $m = 1$ follows from Theorem 3.2 and equation (2.6) (applied to $f(s) = \alpha^s$). Assume that Theorem 5.4 holds for $m \geq 1$. Suppose that $\mu \in \mathcal{D}(\Psi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Then by (2.6), (5.2), and (1.4),

$$\begin{aligned} \ln P_{\Psi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Psi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z)) dt \\ &= \int_0^\infty \left(\int_0^\infty \frac{s^{m-1}}{(m-1)!} \ln Q_\mu(F_{-(s+t) \ln \alpha}(z)) ds \right) dt, \end{aligned}$$

which implies

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \left(\int_0^s \frac{(s-t)^{m-1}}{(m-1)!} dt \right) \ln Q_\mu(F_{-s \ln \alpha}(z)) ds,$$

which in turn implies that (5.2) holds for $m+1$. By making the change of variable $t = -s \ln \alpha$ in (5.2), with $m+1$ in place of m , we obtain

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = B_m \int_0^\infty t^m \ln Q_\mu(F_t(z)) dt, \quad (5.3)$$

where

$$B_m = \frac{(-\ln \alpha)^{-m-1}}{m!}. \quad (5.4)$$

Since the integral in (5.3) converges, we conclude by Lemma 5.3[(ii) $\Rightarrow$ (iii)] that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. Moreover, noting that $Q_\mu^{B_m}(z)$ is an infinitely divisible pgf, the representation (5.3) implies that $\Psi_\alpha^{m+1}(\mu) \in \mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$ by Theorem 5.1. We have thus shown that $\mathcal{D}(\Psi_\alpha^{m+1}) \subset \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^{m+1}) \subset \mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii) $\Rightarrow$ (ii)], the integral $J_2(z)$ converges for every $z \in [0, 1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Psi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty [-\ln P_{\Psi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z))] dt = B_m J_2(z) < \infty \quad (z \in [0, 1]),$$

where B_m is given by (5.4). It ensues by Lemma 3.1[(iii) $\Rightarrow$ (i)], along with an obvious change of variable argument, that $\Psi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Psi_\alpha^{m+1}(\mu) = \psi_\alpha(\Psi_\alpha^m(\mu))$ exists by Theorem 3.2 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha^{m+1})$. What remains to be shown is $\mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$. By Theorem 5.1, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty t^m \ln Q(F_t(z)) dt \quad (5.5)$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{1/B_m}(z)$, where B_m is as in (5.4). We have by (5.5) that the integral $J_2(z)$ of Lemma 5.3 converges for every $0 \leq z \leq 1$. By Lemma 5.3[(ii) $\Rightarrow$ (iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Psi_\alpha^{m+1}(\mu)$ exists. It is easily seen that

$$\ln P(z) = \int_0^\infty \frac{s^m}{m!} \ln Q_\mu(F_{-s \ln \alpha}(z)) ds,$$

or, $P(z) = P_{\Psi_\alpha^{m+1}(\mu)}(z)$. $\square$

The following corollary gives an integral representation of the iterate $\Psi_\alpha^m(\mu)$. We will need the following sequence of functions $\{f_m\}$ defined on $[0, \infty)$ as follows:

$$f_m(s) = ((m+1)!s)^{1/(m+1)} \quad (m \geq 0). \quad (5.6)$$

Corollary 5.5. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\Psi_\alpha^m(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\right). \quad (5.7)$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ ($m \geq 1$). We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}. \quad (5.8)$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\left\{\int_0^t \ln Q_\mu(F_{-f_{m-1}(s)}(z)) ds\right\}. \quad (5.9)$$

Making the change of variable $u = f_{m-1}(s)$ in (5.9) and letting $t \rightarrow \infty$ leads to

$$\lim_{t \rightarrow \infty} P_t(z) = \exp\left\{\int_0^\infty \frac{u^{m-1}}{(m-1)!} \ln Q_\mu(F_{-u \ln \alpha}(z)) du\right\},$$

or, by Theorem 5.3 and (5.2), $\lim_{t \rightarrow \infty} P_t(z) = P_{\Psi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}\left(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\right)$ is definable and has pgf $P_{\Psi_\alpha^m(\mu)}(z)$. $\square$

We now turn to the problem of characterizing the domain and range of the iterates $\{\Phi_\alpha^m(\mu)\}$. We denote by $\mathcal{D}(\Phi_\alpha^m)$ and $\mathcal{R}(\Phi_\alpha^m)$, the domain and range of Φ_α^m , respectively.

First, we give a useful characterization of the distributions in $\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$.

Theorem 5.6. *Let $\alpha \in (0, 1)$ and m a positive integer. A distribution on $\mathbb{Z}_+$ belongs to $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$ if and only if its pgf $P(z)$ admits the representation*

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds, \quad (5.10)$$

where $Q(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$.

Proof. The proof is by induction. First, we note that

$$\int_0^\infty \ln Q(F_{-\lfloor s \rfloor t_0}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds = \sum_{j=0}^\infty \ln Q(F_{-jt_0}(z)) \binom{j + m - 1}{m - 1}, \quad (5.11)$$

where $t_0 = -\ln \alpha$. Assume $m = 1$. The “only if” part follows from (4.6) and (5.11). For the “if part” we note that if $P(z)$ satisfies (4.6), then $P(F_{t_0}(z)) = \prod_{k=0}^\infty Q(F_{(k+1)t_0}(z))$, and thus (1.8) holds with $P_{t_0}(z) = Q(z)$. This implies that $P(z)$ is the pgf of an $\mathcal{F}$ -semi-selfdecomposable distribution. Assume now that the theorem holds for a positive integer m . If $P(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$, then $Q_1(z) = P(z)/P(F_{t_0}(z))$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$ and

$$\ln P(z) = \sum_{j=0}^\infty \ln Q_1(F_{jt_0}(z)). \quad (5.12)$$

We have by the induction hypothesis that $Q_1(z)$ satisfies (5.10), and thus (5.11), for some infinitely divisible pgf $Q(z)$. It follows by (5.12) and (1.4) that

$$\begin{aligned} \ln P(z) &= \sum_{j=0}^\infty \sum_{k=0}^\infty \ln Q(F_{(k+j)t_0}(z)) \binom{j + m - 1}{m - 1} \\ &= \sum_{k=0}^\infty \left(\sum_{j=0}^k \binom{k - j + m - 1}{m - 1} \right) \ln Q(F_{kt_0}(z)). \end{aligned}$$

Since

$$\sum_{j=0}^k \binom{k - j + m - 1}{m - 1} = \binom{k + m}{m}, \quad (5.13)$$

we conclude that $\ln P(z)$ satisfies (5.11), and thus (5.10), with m in place of $m - 1$. Assume now that $P(z)$ is a pgf that admits the representation (5.10), with m in place of $m - 1$, for some infinitely divisible pgf $Q(z)$. It follows by (5.11), (1.4), and the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, that

$$\ln P(z) = \ln P(F_{t_0}(z)) + \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k + m - 1}{m - 1}. \quad (5.14)$$

Define the function $Q_1(z)$ by the equation

$$\ln Q_1(z) = \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k + m - 1}{m - 1}.$$

It is clear that $Q_1(0) > 0$. Let $l_k = \binom{k+m-1}{m-1}$. The pgf $Q^{l_k}(z)$ is infinitely divisible. Therefore, by Theorem 4.2, Chapter II, in [18] and the properties of the semigroup $\mathcal{F}$, we have for each $k \geq 0$,

$$-\ln Q^{l_k}(F_{kt_0}(z)) = \int_{F_{kt_0}(z)}^1 R_k(x) dx = \int_z^1 R_k(F_{kt_0}(x)) F'_{kt_0}(x) dx \quad (z \in [0, 1]),$$

for some absolutely monotone function $R_k(x)$ on $[0, 1)$. It follows that $-\ln Q_1(z) = \int_z^1 R(x) dx$, where $R(x) = \sum_{k=0}^{\infty} R_k(F_{kt_0}(x)) F'_{kt_0}(x)$, $x \in [0, 1)$. Since $R(x)$ is absolutely monotone on $[0, 1)$ (as the limit of sums of absolutely monotone functions on $[0, 1)$), we have, again by Theorem 4.2, Chapter II, in [18], that $Q_1(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$. By the induction hypothesis and (5.11), applied at $m-1$, we can conclude that $Q_1(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$. By (5.14), $P(z) = P(F_{t_0}(z))Q_1(z)$. Therefore, $P(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. $\square$

Theorem 5.7. *For every positive integer m , $\mathcal{D}(\Phi_\alpha^m) = \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Phi_\alpha^m) = \mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$. Moreover,*

$$\ln P_{\Phi_\alpha^m(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds. \quad (5.15)$$

Proof. The proof is by induction. The case $m = 1$ follows from Theorems 4.1 and 4.2 and equation (4.4). Let's assume that Theorem 5.7 holds for $m \geq 1$. Suppose that $\mu \in \mathcal{D}(\Phi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Since $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$, we have by the induction hypothesis (or, (5.15)), (4.4), and (1.4) that

$$\begin{aligned} \ln P_{\Phi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Phi_\alpha^m(\mu)}(F_{-\lfloor t \rfloor \ln \alpha}(z)) dt \\ &= \int_0^\infty \left(\int_0^\infty \ln Q_\mu(F_{-(\lfloor s \rfloor + \lfloor t \rfloor) \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds \right) dt. \end{aligned}$$

By making the change of variable $u = s + \lfloor t \rfloor$ in the inner integral above, and noting that $\lfloor u \rfloor = \lfloor s \rfloor + \lfloor t \rfloor$, we have

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \left(\int_{\lfloor t \rfloor}^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} du \right) dt.$$

Since $u \geq \lfloor t \rfloor$ if and only if $t < \lfloor u \rfloor + 1$, it follows that

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \left(\int_0^{\lfloor u \rfloor + 1} \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} dt \right) du.$$

It is easily seen that

$$\int_0^{\lfloor u \rfloor + 1} \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} dt = \sum_{j=0}^{\lfloor u \rfloor} \binom{\lfloor u \rfloor - j + m - 1}{m - 1} = \binom{\lfloor u \rfloor + m}{m},$$

where the second equation follows from (5.13). We have thus shown that (5.15) holds for $m+1$. Since the integral in (5.15), with $m+1$ in place of m , converges, we conclude that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ (by Lemma 5.3[(i) $\Rightarrow$ (iii)]) and that $\Phi_\alpha^{m+1}(\mu) \in \mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$ (by Theorem 5.6). We have thus shown that

$\mathcal{D}(\Phi_\alpha^{m+1}) \subset \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and that $\mathcal{R}(\Phi_\alpha^{m+1}) \subset \mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii) $\Rightarrow$ (i)], the integral $J_1(z)$ converges for every $z \in [0, 1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Phi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty [-\ln P_{\Phi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z))] dt = J_1(z) < \infty \quad (z \in [0, 1]).$$

It follows by Lemma 3.1[(iii) $\Rightarrow$ (i)], and an obvious change of variable argument, that $\Phi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$ exists by Theorem 4.1 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Phi_\alpha^{m+1})$. We complete the proof by showing that $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. Then by Theorem 5.6, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m}{m} ds \quad (5.16)$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q(z)$. We have by (5.16) that the integral $J_1(z)$ of Lemma 5.3 converges for every $0 \leq z \leq 1$. By Lemma 5.3[(i) $\Rightarrow$ (iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Phi_\alpha^{m+1}(\mu)$ exists. It is clearly seen that $P(z) = P_{\Phi_\alpha^{m+1}(\mu)}(z)$. $\square$

We now deduce an integral representation of the iterate $\Phi_\alpha^m(\mu)$. We will need the sequence of functions $\{g_m\}$ defined by

$$g_m(s) = \int_0^s \binom{\lfloor u \rfloor + m}{m} du \quad (m \geq 0).$$

We denote by $\tilde{g}_m$ the inverse function of g_m .

Corollary 5.8. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\Phi_\alpha^m(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\right). \quad (5.18)$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ ($m \geq 1$). We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}.$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\left\{\int_0^t \ln Q_\mu(F_{-\lfloor \tilde{g}_{m-1}(s) \rfloor \ln \alpha}(z)) ds\right\}. \quad (5.19)$$

Making the change of variable $u = \tilde{g}_{m-1}(s)$ in (5.19) and letting $t \rightarrow \infty$ leads to

$$\lim_{t \rightarrow \infty} P_t(z) = \exp\left\{\int_0^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \binom{\lfloor u \rfloor + m - 1}{m - 1} du\right\}.$$

or, by Theorem 5.7 and (5.15), $\lim_{t \rightarrow \infty} P_t(z) = P_{\Phi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and has pgf $P_{\Phi_\alpha^m(\mu)}(z)$. $\square$

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