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KRYLOV-VERETENNIKOV FORMULA FOR FUNCTIONALS FROM THE STOPPED WIENER PROCESS

GEORGII V. RIABOV*

Abstract. We consider a class of measures absolutely continuous with respect to the distribution of the stopped Wiener process $w(\cdot \land \tau)$. Multiple stochastic integrals, that lead to the analogue of the Itô-Wiener expansions for such measures, are described. An analogue of the Krylov-Veretennikov formula for functionals $f = \varphi(w(\tau))$ is obtained.

1. Introduction

Let $\{w(t)\}_{t \geq 0}$ be a standard Wiener process in $\mathbb{R}^d$, starting from the point $u \in \mathbb{R}^d$. Consider an open connected set $G \ni u$, the exit time

$$\tau = \inf\{t > 0 : w(t) \notin G\},$$

and a Borel function $\rho : \mathbb{R}^d \to (0, 1)$.

The main object of the investigation in the present paper is the orthogonal structure of the space $L^2(\Omega, \sigma(w(\cdot \land \tau)), Q)$, where the measure $Q$ is given by the density

$$\frac{dQ}{dP} = \frac{\mathbb{1}_{\tau < \infty} \rho(w(\tau))}{\mathbb{E}\mathbb{1}_{\tau < \infty} \rho(w(\tau))}.$$

In [11, L. 2.4] it was proved that the space $L^2(\Omega, \sigma(w(\cdot \land \tau)), Q)$ possesses an orthogonal structure similar to the Itô-Wiener decomposition in the Gaussian case [1, 5, 8]. Namely, consider functions

$$\beta(v) = \mathbb{E}\mathbb{1}_{\tau(w-u+v) < \infty} \rho(w(\tau(w-u+v) - u + v)), \ v \in G,$n

$$\alpha(s, v) = \beta^{-1}(v) \mathbb{E}\mathbb{1}_{s < \tau(w-u+v) < \infty} \rho(w(\tau(w-u+v) - u + v)), \ s > 0, v \in G,$

and processes

$$\dot{w}(s) = w(s \land \tau) - \int_0^{s \land \tau} \nabla \log \beta(w(r))dr, \ s \geq 0;$$

$$\dot{w}_t(s) = \dot{w}(s) - \int_0^{s \land \tau} \nabla \log \alpha(t-r, w(r))dr, \ 0 \leq s \leq t.$$
Theorem 1.1. [11, L. 2.4] Each random variable \( f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q) \) can be uniquely represented as a series of pairwise orthogonal stochastic integrals

\[
    f = \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n<\tau} \cdot \cdot \cdot \cdot \int a_n(t_1, \ldots, t_n) d\hat{w}_{t_1}(t_1) \ldots d\hat{w}_{t_n}(t_{n-1}) d\hat{w}(t_n). \tag{1.1}
\]

Conversely, given a sequence of Borel functions \( a_n : (0, \infty)^n \to \mathbb{R}^{d^n}, n \geq 0, \) such that

\[
    \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n} \cdot \cdot \cdot \cdot \int \alpha(t_n, u) |a_n(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_n < \infty,
\]

the series in the right-hand side of (1.1) converges in \( L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q) \), and its sum \( f \) satisfies

\[
    \mathbb{E} f^2 = \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n} \cdot \cdot \cdot \cdot \int \alpha(t_n, u) |a_n(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_n.
\]

In this paper we derive the explicit form of the expansion (1.1) for random variables of the kind \( f = \varphi(w(\tau)) \). The resulting formula is similar to the well-known Krylov-Veretennikov formula [6]. It is written in terms of the transition semigroup \( \{T^k_t\}_{t \geq 0} \) of a certain diffusion, killed at the boundary of \( G \). Indeed, the process \( \hat{w} \) is a stopped Wiener process relatively to the measure \( Q \) [11, L. 2.4]. Respectively, the initial Wiener process \( w \) is a diffusion process relatively to the measure \( Q \). Then \( \{T^k_t\}_{t \geq 0} \) is the transition semigroup of the process \( w \) killed at the boundary of \( G \). Let \( T \) denote the integration with respect to the exit distribution of \( w \) from \( G \) (precise expressions for these operators are given in the section 2). The main result of the present paper is the following formula, proved in the theorem 2.1:

for every random variable \( \varphi(w(\tau)) \in L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q) \) the expansion (1.1) has the form

\[
    \varphi(w(\tau)) = \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n} \cdot \cdot \cdot \cdot \int \alpha(t_n, u) \left( T^k_t \alpha(t_2-t_1, \cdot) \nabla \left( \alpha(t_2-t_1, \cdot)^{-1} T^k_{t_2-t_1} \right) \right) \cdot \cdot \cdot \alpha(t_{n-1}, u) \nabla \left( \alpha(t_{n-1}, \cdot)^{-1} T^k_{t_{n-1}-t_{n-2}} \right) \nabla T^k_{t_{n-2}} \varphi \right) (u) d\hat{w}_{t_1}(t_1) \ldots d\hat{w}_{t_n}(t_{n-1}) d\hat{w}(t_n). \tag{1.2}
\]

Expansions of the kind (1.1) appeared in [4] in connection with the problem of studying the behaviour of Gaussian measures under nonlinear transformations. Such expansions have two main features:

1. the summands in (1.1) are pairwise orthogonal;
2. the summands in (1.1) are \( \sigma(w(\cdot \wedge \tau)) \)-measurable.

Of course, there are other possibilities to organize series expansions for random variables from \( L^2(\Omega, \sigma(w(\cdot \wedge \tau)), Q) \). For simplicity, consider the case \( Q = \mathbb{F} \). The most straightforward approach comes from the obvious inclusion \( \sigma(w(\cdot \wedge \tau)) \subset \)
It means that each random variable \( f \in L^2(\Omega, \sigma(w(\cdot \wedge \tau))), \mathbb{P} \) possesses an Itô-Wiener expansion with respect to the Wiener process \( w \):

\[
f = \sum_{n=0}^{\infty} \int_{0 \leq t_1 < \ldots < t_n} b_n(t_1, \ldots, t_n) dw(t_1) \ldots dw(t_n). \tag{1.3}
\]

The summands in the expansion are not \( \sigma(w(\cdot \wedge \tau)) \)-measurable. While the left-hand side of (1.3) is \( \sigma(w(\cdot \wedge \tau)) \)-measurable, one can condition (1.3) with respect to \( w(\cdot \wedge \tau) \) and get another expansion

\[
f = \sum_{n=0}^{\infty} \int_{0 \leq t_1 < \ldots < t_n < \tau} b_n(t_1, \ldots, t_n) dw(t_1) \ldots dw(t_n). \tag{1.4}
\]

Now the stochastic integrals of different degree are not orthogonal. This causes known inconveniences: the expansion (1.4) is not unique (an example is given in [4]); the conditions for the expression in the right-hand side of (1.4) to converge are complicated. An application of the Gram-Shmidt orthogonalization procedure to expansions (1.4) was considered in [2]. However, in our framework it seems to be too complicated either to obtain the orthogonalized form of (1.4), or to find the orthogonalized expansion (1.4) for a concrete random variable \( f \). The expansion (1.1) overcomes all these problems.

Motivation for the \( \sigma(w(\cdot \wedge \tau)) \)-measurability of the summands in (1.1) comes from B. S. Tsirelson’s theory of black noise. It is well-known that Brownian coalescing flows produce filtrations with trivial Gaussian parts [12, 7]. So, to get a unified description of functionals measurable with respect to such flows, it is reasonable to use the noise generated by the flow itself. The results from [4, 11] show that this idea works: in [4] an orthogonal expansion of the kind (1.1) was obtained for the stopped Brownian motion; in [11] the same was done for the \( n \)-point motions of the Arratia flow. We refer to [4, 11] for the detailed discussion of this and related questions.

Generalization of the Krylov-Veretennikov formula to the wide class of dynamical systems driven by the additive Gaussian noise was obtained in [3]. Our formula (2.3) is similar to the one obtained in [3] despite the additional multipliers \( \alpha \). They occur to normalize operators \( T_t^k \), as \( T_t^k 1 = \alpha(t, \cdot) \).

The article is organized in the following way. In the section 2 we introduce all the needed notions and constructions. Also, it contains the reduction of the main theorem 2.1 to lemmata 2.2 and 2.3. Sections 3 and 4 are devoted to the proof of these auxiliary results.

2. Notations and Main Results

To formulate our results, we will use the following notations.

\( \{w(t)\}_{t \geq 0} \) is the Wiener process in \( \mathbb{R}^d \). Without loss of generality, we will assume that \( w \) is constructed in a canonical way:

\( \Omega = C([0, \infty), \mathbb{R}^d) \) is a space of continuous functions equipped with a metric of uniform convergence on compacts;

\( \mathcal{F} \) is the Borel \( \sigma \)-field on \( \Omega \).
\( w(t, \omega) = \omega(t) \) is the canonical process on \((\Omega, \mathcal{F}), \mathcal{F}_t = \sigma(w(s) : 0 \leq s \leq t) \) is the natural filtration of \( w; \)

\((\mathbb{P}_v)_{v \in \mathbb{R}^d}\) is a family of probability measures on \((\Omega, \mathcal{F})\), such that relatively to \(\mathbb{P}_v\), \( w \) is a \(d\)-dimensional Wiener process starting from \( v \). The expectation with respect to certain probability measure \( Q \) on \((\Omega, \mathcal{F})\) will be denoted by \( \mathbb{E}_Q \). \( \mathbb{E}_{\mathbb{P}_v} \) will be abbreviated to \( \mathbb{E}_v \).

Let \( G \subset \mathbb{R}^d \) be an open connected set, \( \tau \) be the exit time of \( w \) from the set \( G : \)

\[ \tau = \inf \{ t > 0 : w(t) \notin G \}. \]

We will assume that for all \( v \in G \), \( \mathbb{P}_v(\tau < \infty) > 0 \). Fix a Borel function \( \rho : \mathbb{R}^d \to (0, 1) \) and consider the function

\[ \beta(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)), \ v \in G. \]

It is a harmonic function in \( G \) [9, Ch. 4, Prop. 2.1]:

\[ \Delta_\nu \beta(v) = 0, \ v \in G. \]

Denote \( Q_u \) the probability measure on \((\Omega, \mathcal{F})\), defined via the density

\[ \frac{dQ_u}{d\mathbb{P}_u} = \beta(u)^{-1} 1_{\tau < \infty} \rho(w(\tau)). \]

We will need another probability measure corresponding to the process \( w \) killed at the moment \( \tau \). Consider the function

\[ \alpha(s, v) = Q_v(\tau > s), \ s > 0, v \in G. \]

In the section 1 following processes were introduced.

\[
\begin{align*}
\tilde{w}(s) &= w(s \wedge \tau) - \int_0^{s \wedge \tau} \nabla_v \beta(w(r)) dr, \ s \geq 0; \\
\tilde{w}_t(s) &= \tilde{w}(s) - \int_0^{s \wedge \tau} \nabla_v \alpha(t - r, w(r)) dr, \ 0 \leq s \leq t.
\end{align*}
\]

Throughout the paper derivatives will be taken in \( v \in G \), so we will omit the index \( v \) in the derivatives’ notation.

Consider a probability measure \( Q_{t,u} \) on \((\Omega, \mathcal{F}_t)\), defined via the density

\[ \frac{dQ_{t,u}}{dQ_u} = \alpha(t, u)^{-1} 1_{\tau > t}. \]

The key observation leading to the theorem 1.1 is that on the probability space \((\Omega, \mathcal{F}_t, Q_{t,u})\) the process \( \tilde{w}_t \) is a Wiener process [10, Ch. VIII, Th. (1.4)].

Introduce following operators:

1. \( T \psi(v) = \mathbb{E}_v 1_{\tau < \infty} \rho(w(\tau)) \psi(w(\tau)), \ v \in G. \)

Denote \( \mu_v \) the distribution of \( w(\tau) \) relatively to the measure \( 1_{\tau < \infty} d\mathbb{P}_v \). Then the action of the operator \( T \) reduces to the integration with respect to \( \mu_v : \)

\[ T \psi(v) = \int \psi(x) \mu_v(dx). \]
Lemma 2.3. \( \bar{T}_v^w = \beta(v)^{-1}T^w \), \( v \in G \).

The operator \( \bar{T} \) is the expectation relatively to the probability measure \( Q_v : \)
\[ \bar{T}_v^w = E_{Q_v}(w(\tau)). \]

(3) \( T^w_s \psi(v) = E_{Q_s}, 1_{T \geq s}(w(s)), \) \( s > 0, v \in G. \)

From equations (2.1), (2.2) it follows that
\[ dw(s) = (\nabla \log \alpha(t - s, w(s)) + \nabla \log \beta(w(s)))ds + d\hat{w}(s), \]
where \( \hat{w} \) is a Wiener process on \((\Omega, F_t, Q_{1,v})\). So, relatively to the measure \( Q_{1,v} \) the process \( w \) satisfies (degenerate) SDE. Respectively, \( \{T^w_t\}_{t \geq 0} \) is the transition semigroup of a killed diffusion process \( w \). Denote \( \mu_{s,v} \) the distribution of \( w(s) \) relatively to the measure \( 1_{T \geq s}dQ_v \). Then the action of the operator \( T^w_s \) reduces to the integration with respect to \( \mu_{s,v} : \)
\[ T^w_s \psi(v) = \int \psi(x)\mu_{s,v}(dx). \]

(4) \( \bar{T}_s^w \psi(v) = \alpha(s, v)^{-1}T^w_s \psi(v), \) \( s > 0, v \in G. \)

The operator \( \bar{T}_s^w \) is the expectation relatively to the probability measure \( Q_{s,v} : \)
\[ \bar{T}_s^w \psi(v) = E_{Q_{s,v}}(w(s)). \]

The following theorem is the main result of the paper.

**Theorem 2.1.** For every \( \varphi \in L^2(\rho d\mu_v) \) the expansion (1.1) has the form
\[ \varphi(w(\tau)) = \sum_{n=0}^{\infty} \int \ldots \int \alpha(t_n, u)^{-1} \left( \alpha(t_1, \cdot)T^k_{t_1} \alpha(t_2 - t_1, \cdot) \nabla T^k_{t_2 - t_1} \ldots \alpha(t_n - t_{n-1}, \cdot) \nabla T^k_{t_n - t_{n-1}} \nabla \varphi \right)(u)d\hat{w}_{t_n}(t_1) \ldots d\hat{w}_{t_n}(t_{n-1})d\hat{w}(t_n). \]

The proof is divided into two lemmas, which are proved in the next sections. At first we derive the Clark representation for \( \varphi(w(\tau)) \) with respect to the stopped Wiener process \( \hat{w} \) [10, Ch. V, Th. (3.5)].

**Lemma 2.2.** For every \( \varphi \in L^2(\rho d\mu_v) \), one has the representation
\[ \varphi(w(\tau)) = \bar{T}_\tau \varphi(u) + \int_0^\tau \nabla \bar{T}_\tau \varphi(w(t))d\hat{w}(t), \] \( Q_u - a.s. \)

Subsequently, we find the Itô-Wiener expansion for the random variable \( \psi(w(t)) \) with respect to the Wiener process \( \hat{w} \).

**Lemma 2.3.** For every \( \psi \in L^2(\mu_{t,v}) \) the Itô-Wiener expansion of \( \psi(w(t)) \) has the form
\[ \psi(w(t)) = \sum_{n=0}^{\infty} \int \ldots \int \alpha(t, u)^{-1} \left( \alpha(t_1, \cdot)T^k_{t_1} \alpha(t_2 - t_1, \cdot) \nabla T^k_{t_2 - t_1} \ldots \alpha(t_n - t_{n-1}, \cdot) \nabla T^k_{t_n - t_{n-1}} \varphi \right)(u)d\hat{w}_{t_n}(t_1) \ldots d\hat{w}_{t_n}(t_n). \]
The theorem 2.1 follows by substituting $\psi = \nabla \tilde{T} \varphi$ in (2.5) and inserting the right-hand side of (2.5) into (2.4).

3. Clark Representation Formula: Proof of Lemma 2.2

Proof. 1) At first we will prove that the function $\tilde{T} \varphi$ is smooth and satisfies the equation

$$(\nabla \tilde{T} \varphi, \nabla \log \beta) + \frac{1}{2} \Delta \tilde{T} \varphi = 0 \quad (3.1)$$

in $G$. Indeed,

$$\tilde{T} \varphi(u) = \frac{E_u 1_{\tau < \infty} \rho(w(\tau)) \varphi(w(\tau))}{\beta(u)} \quad (3.2)$$

is the ratio of two harmonic functions [9, Ch. 4, Th. 3.7] (for the numerator the condition $\varphi \in L^2(\rho d\mu_u)$ is used). The equation (3.1) is checked by straightforward calculation.

2) We will prove the relation (2.4) for bounded and continuous functions $\varphi$ and $\rho$, the other cases being covered by the usual limiting procedure. Let $\{G_n\}_{n \geq 1}$ be a sequence of open relatively compact sets, such that $G_n \subset G$ and $G = \bigcup_{n=1}^{\infty} G_n$. Denote $\tau_n$ be the exit time from $G_n$:

$$\tau_n = \inf\{t \geq 0 : w(t) \notin G_n\}.$$ 

The convergence $\tau_n \to \tau$, $n \to \infty$, holds.

From the relation (2.1) it follows that the stopped process $w(\cdot \wedge \tau_n)$ satisfies the SDE

$$dw(s) = \nabla \log \beta(w(s)) ds + d\tilde{w}(s), \quad 0 \leq s \leq \tau_n.$$ 

Applying the Itô formula to the function $\tilde{T} \varphi$ and the process $w(\cdot \wedge \tau_n)$, and using (3.1), one gets the representation

$$\tilde{T} \varphi(w(\tau_n)) = \tilde{T} \varphi(u) + \int_0^{\tau_n} \nabla \tilde{T} \varphi(w(s)) d\tilde{w}(s), \quad Q_u - a.s.$$ 

It remains to check that $\tilde{T} \varphi(w(\tau_n)) \to \tilde{T} \varphi(w(\tau))$. As the function $\varphi$ is bounded, one has

$$\sup_{n \geq 1} \int_0^\infty E_u 1_{\tau_n > s} (\nabla \tilde{T} \varphi(w(s)))^2 ds < \infty.$$ 

Now, the convergence $\tau_n \to \tau$, $n \to \infty$, implies the convergence

$$\int_0^{\tau_n} \nabla \tilde{T} \varphi(w(s)) d\tilde{w}(s) \xrightarrow{L^2(Q_u)} \int_0^\tau \nabla \tilde{T} \varphi(w(s)) d\tilde{w}(s), \quad n \to \infty.$$ 

It remains to check that $\tilde{T} \varphi(w(\tau_n)) \to \varphi(w(\tau))$, $n \to \infty$. By [9, Ch. 4, Th. 2.3] the point $w(\tau)$ is the regular point for the Dirichlet problem on $G$. The needed convergence follows from the representation (3.2). \qed
4. The Krylov-Veretennikov Formula: Proof of Lemma 2.3

**Proof.** The kernels $a_n$ in the expansion

$$
\psi(w(t)) = \sum_{n=0}^{\infty} \int_{0 < t_1 < \ldots < t_n < t} a_n(t_1, \ldots, t_n) d\hat{w}_t(t_1) \ldots d\hat{w}_t(t_n)
$$

will be recovered from the expression

$$
E_{Q_t,u}[\psi(w(t))] = \int_{0 < t_1 < \ldots < t_n < t} b_n(t_1, \ldots, t_n) d\hat{w}_t(t_1) \ldots d\hat{w}_t(t_n)
$$

in which $b_n$ is a deterministic square integrable function. By induction, it is enough to check that for any square integrable $\hat{w}$–adapted process $\{g(s)\}_{0 \leq s \leq t}$, one has

$$
E_{Q_{t,u}}[\psi(w(t))] \int_0^t g(s) \, d\hat{w}_t(s) = \int_0^t \frac{\alpha(s,u)}{\alpha(t,u)} E_{Q_{s,u}} [\alpha(t-s, w(s)) \nabla T_{t-s}^k \psi(w(s)) g(s) \, ds].
$$

(4.1)

To do it note the equalities, which follow from (2.2) and lemma 2.2

$$
\int_0^t g(s) \, d\hat{w}_t(s) = \int_0^t g(s) \, d\hat{w}_t(s) - \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) \, ds, \quad Q_{t,u} \text{ a.s.}
$$

$$
1_{\tau > t} \psi(w(t)) = T_t^k \psi(u) + \int_0^{t \wedge \tau} \nabla T_{t-s}^k \psi(w(s)) d\hat{w}_t(s), \quad Q_u \text{ a.s.}
$$

Consequently,

$$
E_{Q_{t,u}}[\psi(w(t))] \int_0^t g(s) \, d\hat{w}_t(s)
$$

$$
= E_{Q_{t,u}}[\psi(w(t))] \int_0^t g(s) \, d\hat{w}_t(s) - E_{Q_{t,u}}[\psi(w(t))] \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) \, ds
$$

$$
= \alpha(t,u)^{-1} \left( E_{Q_u} [1_{\tau > t} \psi(w(t))] \int_0^t g(s) \, d\hat{w}_t(s)
$$

$$
- E_{Q_u}[1_{\tau > t} \psi(w(t))] \int_0^t g(s) \nabla \log \alpha(t-s, w(s)) \, ds \right)
$$

$$
= \alpha(t,u)^{-1} \left( \int_0^t E_{Q_u} [1_{\tau > s} \nabla T_{t-s}^k \psi(w(s)) g(s) \, ds
$$

$$
- \int_0^t E_{Q_u} [1_{\tau > s} T_{t-s}^k \psi(w(s)) g(s) \nabla \log \alpha(t-s, w(s)) \, ds \right) \right)
\[
\begin{align*}
\alpha(t, u)^{-1} & \int_0^t \mathbb{E}_{Q_s} 1_{T^k_{t-s}} \left( \nabla T^k_{t-s} \psi(w(s)) \right) \\
& - T^k_{t-s} \psi(w(s)) \nabla \log \alpha(t - s, w(s)) g(s) \, ds \\
& = \int_0^t \frac{\alpha(s, u)}{\alpha(t, u)} \mathbb{E}_{Q_s} \alpha(t - s, w(s)) \nabla T^k_{t-s} \psi(w(s)) g(s) \, ds.
\end{align*}
\]

The equality (4.1) is proved. \qed 

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References


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