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EXCLUDED MINORS FOR APEX CLASSES OF GRAPHS

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EXCLUDED MINORS FOR APEX CLASSES OF GRAPHS

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"Mathematics is the language with which God has written the universe."
-Galileo Galilei

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This dissertation is dedicated to my parents, Dennis and Diane Derbins, for their unyielding support and encouragement of my education.

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Abstract

We investigate a particular type of minor-closed classes of graphs that arise from simple and well understood classes of graphs by adding apex vertices to the classes' members. We give excluded minor characterizations of several such classes, specifically: one and two apex edgeless graphs, one apex forests and one apex linear forests.

Chapter 1
Introduction

1.1 Overview

One of the most familiar works in graph theory is that of Polish topologist Kazimierz Kuratowski. In 1929, Kuratowski announced his idea that planar graphs, which form an infinite class, could be characterized by just two specific, nonplanar graphs, K_5 and $K_{3,3}$. The former, K_5, is the graph with five vertices every pair of which is connected by one edge; the latter, $K_{3,3}$, is the graph with six vertices partitioned into two sets of three vertices with every vertex in the first set connected to every vertex of the second set, and no other edges.

Just one year later, 1930, Kuratowski provided the proof for what has come to be called Kuratowski's Theorem [1].

Theorem 1.1. (Kuratowski's Theorem) *A graph is planar if and only if it contains no subgraph that is a subdivision of* K_5 *or* $K_{3,3}$.

A graph's *minor* can be obtained by deleting vertices, deleting edges and/or contracting edges from the original graph. Note that graph minors, as well as other terms introduced here in the overview, are described in detail in the following sections. A class of graphs $\mathcal{G}$ is said to be *minor-closed* if, for every $G \in \mathcal{G}$, all minors of G are also in $\mathcal{G}$ [1].

German mathematician Klaus Wagner noticed that Kuratowski's Theorem could be written in the context of graph minors. In 1937, Wagner established what is now referred to as Wagner's Theorem, which is stated below as on page 243 of [1].

Theorem 1.2. (Wagner's Theorem) *A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.*

Wagner's Theorem describes the class of planar graphs, which is minor-closed, in terms of two specific graphs, K_5 and $K_{3,3}$, as excluded minors. This theorem led Wagner to conjecture that all minor-closed classes of graphs can be characterized in a similar way: by a finite list of excluded minors. Wagner discussed his conjecture with his students in the 1960s, but their discussions never resulted in a formal proof. Neil Robertson and Paul Seymour proved Wagner's Conjecture between 1986 and 2004 over a series of 20 papers [2]. Although sometimes still referred to as Wagner's Conjecture, this result has now come to be known as the Graph Minors Theorem, as stated below from page 307 of [1].

Theorem 1.3. (Graph Minors Theorem) *Every infinite set of graphs contains two graphs where one is (isomorphic to) a minor of the other.*

The Graph Minors Theorem implies that for every minor-closed class of graphs, the list of minor-minimal graphs not in the class must be finite. If the list were infinite, one element of it would be a minor of the other, and thus the latter would not be minimal. Wagner recognized this fact, and therefore knew that his conjecture, if true, would imply the following:

Theorem 1.4. *Let $\mathcal{G}$ be a minor-closed class of graphs. Then there exists a finite set $\mathcal{M}$ of graphs such that $G \in \mathcal{G}$ if and only if no graph in $\mathcal{M}$ is a minor of G.*

Not only is the Graph Minors Theorem important for well-known classes of graphs and the investigation of their respective excluded minors, but it also provides hope for future characterizations. Many, if not most, important classes of graphs are minor-closed. Even though no good characterization is known for most of such classes, Graph Minors Theorem asserts that each of them has a charac-

terization in terms of excluding a finite list of minors, which is one day likely to be found. Further, this theorem implies that a problem that is topological in nature can be made discrete. For example, testing whether a graph is planar, that is testing if it can be drawn in the plane without crossing edges, is topological in nature. Writing a computer program to perform such a task seems impractical, if not impossible, based on this description. However, the excluded minor characterization of planar graphs due to Wagner allows for writing such a computer program. Many other classes of graphs described through excluded minors allow for such discretization.

One of the key constructions in proving the Graph Minors Theorem is extending the class of graphs embeddable on a particular surface by adding a bounded number of vertices that do not lie on the surface, called *apex vertices*, to the graphs. Given a minor-closed class of graphs $\mathcal{G}$, the *apex class* of $\mathcal{G}$ consists of the empty graph and all graphs G for which the deletion of some vertex v results in a graph in $\mathcal{G}$.

Recall from above that minors can be created by deleting vertices, deleting edges and/or contracting edges. Suppose we have a graph G and a minor-closed class of graphs $\mathcal{G}$, and G is in an apex class of $\mathcal{G}$. Because G is in the apex class of $\mathcal{G}$, there is a vertex v in G such that $G - v$ is in $\mathcal{G}$. It is clear that the deletion of any vertex or any edge of G results in a graph that is still in the apex class of $\mathcal{G}$. Also, if an edge that is not incident with v is contracted, the resulting graph is still in the apex class of $\mathcal{G}$. If an edge incident with v is contracted, then the new vertex u resulting from the contraction is such that $G - u$ is in $\mathcal{G}$, therefore the resulting graph is in the apex class of $\mathcal{G}$. As a result of these observations, we know that if the original class of graphs is minor-closed, then the respective apex class is also minor-closed.

Very few lists of excluded minors for apex classes of known classes of graphs exist. One of the best known results combining apex graphs and excluded minors is the work of Stanislaw Dziobiak [3], who started with the class of outerplanar graphs. The list of excluded minors for outerplanar graphs consists of K_4 and $K_{2,3}$.

Dziobiak found the complete list of 57 excluded minors for the class of apex outerplanar graphs. It is important to note that the list of excluded minors for apex planar graphs still remains unknown.

The work of Kuratowski, Wagner, Robertson, Seymour and Dziobiak form the basis behind this work. The motivation comes from a question posed by Bogdan Oporowski, who asked the following: is there a function f such that if a class of graphs has n excluded minors, then the respective apex class has at most $f(n)$ excluded minors? We see that, for outerplanar graphs, the list goes from 2 excluded minors to 57 by the simple addition of just one vertex.

In this project, we study the excluded minors for basic classes of graphs and their respective apex classes in the hope that it would provide insight into how the structure of the excluded minors for a class of graphs affects the excluded minors for the respective apex class.

1.2 Preliminaries

We must first introduce some common terminology associated with graph theory that will be used within this paper. Most of the terminology will follow *Graphs and Digraphs* [1]. Here, however, we call *graphs* what *Graphs and Digraphs* calls *multigraphs*.

A *graph* G is $(V, E, \mathcal{J})$ where V is a finite set of elements called *vertices*, E is a finite set of elements disjoint from V called *edges*, and $\mathcal{J}$ is a subset of $V \times E$ such that every edge is incident with exactly one or two vertices. When it does not

lead to ambiguity, we denote an edge between vertices v and u by uv. An edge uv is *incident* to its endpoints, u and v. Edges that are incident with one vertex are called *loops*. Two edges who have the same set of endpoints are *parallel edges*.

The *degree* of a vertex v is the number of edges incident to v, with loops counting twice, and it will be denoted $\deg(v)$. Distinct vertices u and v are *adjacent* if there is an edge incident to both of them. Adjacent vertices are called *neighbors* of one another. The *neighborhood* of a vertex v is the set of neighbors of v.

Two graphs G and H are *isomorphic* if there are two bijective functions $\varphi : V(G) \to V(H)$ and $\psi : E(G) \to E(H)$ such that a vertex v and an edge e are incident in G if and only if $\varphi(v)$ and $\psi(e)$ are incident in H.

A graph is *planar* if it can be drawn in the plane without crossing edges. A graph is *outerplanar* if it can be drawn in the plane without crossing edges so that all of its vertices lie on the boundary of the exterior region.

As stated above, this thesis makes use of the idea of graph minors. A graph H is a *minor* of a graph G if H can be obtained from G by a succession (perhaps empty) of operations, each of which is an edge contraction, edge deletion or vertex deletion. Given a minor-closed class of graphs $\mathcal{G}$, an excluded minor for $\mathcal{G}$ is a graph that is not in $\mathcal{G}$, but whose every proper minor is in $\mathcal{G}$. Whenever we describe a graph as minimal, it will be understood in the context of the graph minors relation.

We use a specific notation to denote deletions. If we delete the edge f from the graph G, we write $G\backslash f$. If we delete a vertex v from the graph G, we write $G - v$. This notation for deletion may be extended to sets of edges, sets of vertices and entire subgraphs. When we delete an edge from a graph, the edge's vertices remain. When we delete vertices from a graph, the edges incident to the vertices are also deleted. When we delete a subgraph H, we delete the edges of H and those vertices of H incident only to the edges of H.

Recall from above the definition of an apex class: given a minor-closed class of graphs $\mathcal{G}$, the *apex class* of $\mathcal{G}$ consists of the empty graph and all graphs G for which the deletion of some vertex results in a graph in $\mathcal{G}$. Such a vertex will be called an apex vertex of G. Recall that K_4 is planar and K_5 is not. So for example, every vertex of K_5 is an apex vertex for the class of planar graphs.

The class $\mathcal{G}$ may also be noted $\mathcal{A}^0(\mathcal{G})$. Inductively, $\mathcal{A}^n(\mathcal{G})$ is the apex class of $\mathcal{A}^{n-1}(\mathcal{G})$, where n is a positive integer.

Finally, we must be familiar with the idea of bridges. Consider a graph G and a fixed subgraph H of G. An H-bridge in G is a subgraph B of G such that either B is an edge not in H, together with its ends, both of which are in H, or B is obtained from a component K of $G - V(H)$ by adding to K all the edges from vertices in K to vertices in H, along with their ends in H. The vertices common to B and H are called the vertices of attachment of B [4].

Chapter 2
Forests and Linear Forests

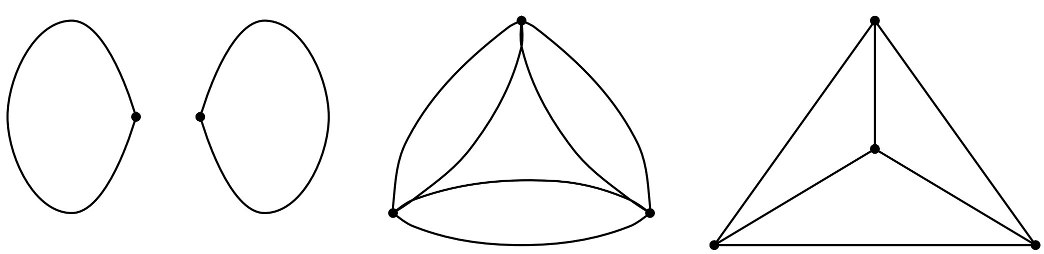
To get a better sense of how apex vertices affect the excluded minors of familiar classes, we first investigate forests and linear forests. A graph is a *forest* if it contains no cycles [1]. A *linear forest* is a forest in which the degree of every vertex is no greater than two.

Theorem 2.1. *Let $\mathcal{F}$ be the class of forests. A graph is in $\mathcal{A}^1(\mathcal{F})$ if and only if it contains no minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$ as indicated in Figure 2.1.*

Proof. For the necessity condition, it is immediate to check that none of the graphs in $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$ are apex forest. Therefore no graph containing a minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$ is an apex forest.

For the sufficiency condition, suppose G is a minimal non-apex forest; we will show G is isomorphic to an element of $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$. Let us pick a cycle C of G and consider the possibilities for the bridges of C.

We will show that if B has zero vertices of attachment, $G \cong X_1[\mathcal{A}^1(\mathcal{F})]$; there is no case for which B has exactly one vertex of attachment; and if B has three or



$X_1[\mathcal{A}^1(\mathcal{F})]$ $X_2[\mathcal{A}^1(\mathcal{F})]$ $X_3[\mathcal{A}^1(\mathcal{F})]$

FIGURE 2.1. $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$

more vertices of attachment, $G \cong X_3[\mathcal{A}^1(\mathcal{F})]$. We leave the case when every bridge B of C has exactly two vertices of attachment for the end.

1. Consider first the case when there is a bridge B of C with no vertices of attachment.

   Suppose first that B is acyclic. We now wish to show that B indeed contains a cycle. Because G is a minimal non-apex forest, $G - B$ is an apex forest, implying that $(G-B)-v$ is in fact a forest. If we take the union of $(G-B)-v$, which is a forest, and B, which is an acyclic graph (a forest), the resulting graph $G - v$ is a forest, meaning that G is an apex forest; a contradiction. Therefore, B must contain a cycle so G contains two disjoint cycles, and G is isomorphic to $X_1[\mathcal{A}^1(\mathcal{F})]$.

2. Suppose there is a bridge B with exactly one vertex of attachment.

   There are three possibilities for the structure of the bridge B. We reach a contradiction for each of the possibilities, implying that no such bridge exists. Let v be the vertex of attachment of B.

   (a) The bridge B has a cycle avoiding v. In this case, G has two disjoint cycles and we observe that $X_1[\mathcal{A}^1(\mathcal{F})]$ is a proper minor of G; a contradiction.

   (b) The bridge B contains a cycle D_1 that contains v. Deletion of v leaves at least one cycle D_2, because G is non-apex forest. Therefore G has two disjoint cycles D_1 and D_2 and again, $X_1[\mathcal{A}^1(\mathcal{F})]$ is a proper minor of G; a contradiction.

   (c) The bridge B is a tree T. Because G is minimal non apex forest, we have that $(G - T)$ is an apex forest, meaning $(G - T) - v$ is a forest.

Following the same reasoning as earlier, if we take the union of T, which is a tree, and $(G-T)-v$, which is a forest, the resulting graph $G-v$ is a forest, implying that G is an apex forest; a contradiction.

3. Consider next when there is a bridge B with three or more vertices of attachment.

   Because B is connected, B has a spanning tree T whose leaves are the vertices of attachments of B. Then $C \cup T$ contains a subdivision of K_4, which itself is an excluded minor for apex forests, so $K_4 \cong G \cong X_3[\mathcal{A}^1(\mathcal{F})]$.

4. It remains to consider the case when every bridge has exactly two vertices of attachment. Suppose the bridge B_1 has vertices of attachment u_1, v_1, and the bridge B_2 has vertices of attachment u_2, v_2. The bridges *cross* if u_1, u_2, v_1, v_2 appear on C in the order listed. If B_1 and B_2 of C cross in G, then $K_4 \cong G$. So we may assume that every bridge has two vertices of attachment, and no two bridges cross.

   If there are two bridges with disjoint vertices of attachment, and the bridges never cross, G has two disjoint cycles and G properly contains $X_1[\mathcal{A}^1(\mathcal{F})]$ as a minor; a contradiction.

   Because every bridge has exactly two vertices of attachment and no two bridges cross, every two bridges must share at least one vertex of attachment. There are two cases to consider.

   (a) There is a single vertex v that meets all bridges of C in G. One bridge B_1 must contain a cycle D_v avoiding v. The cycle must be incident to B_1's other vertex of attachment w. This implies that all other bridges have w as a vertex of attachment, else G has two disjoint cycles. The

vertex w meets C, D_v and all other bridges of C. So one of the bridges of C must contain a cycle D_w avoiding w. Clearly, D_w must contain v. If D_w and D_v are in different bridges, the graph has two disjoint cycles; a contradiction.

So B_1 contains two cycles D_w and D_v, the former of which meets vertex v, and the latter of which meets vertex w. The cycles must have at least one vertex in common or else G has two disjoint cycles. The cycles do not have more than one vertex in common because then $G \cong K_4$. This means the cycles share exactly one vertex, and so $G \cong X_2[\mathcal{A}^1(\mathcal{F})]$.

(b) No vertex is common to all bridges. Because every two bridges share a vertex of attachment, we know there are at least three bridges of C in G. With every two bridges sharing a vertex and at least three bridges, we arrive at $G \cong X_2[\mathcal{A}^1(\mathcal{F})]$.

□

Theorem 2.2. *Let $\mathcal{L}$ be the class of linear forests. A graph is in $\mathcal{A}^1(\mathcal{L})$ if and only if it contains no minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{L})]$ as indicated in Figure 2.2.*

Proof. For the necessity condition, it is immediate to check that none of the graphs in $\mathcal{X}[\mathcal{A}^1(\mathcal{L})]$ are apex linear forests. Therefore no graph containing a minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{L})]$ is an apex linear forest.

For the sufficiency condition, suppose G is not an apex linear forest, but each of its proper minors is. We will show that G is one of the graphs in $\mathcal{X}[\mathcal{A}^1(\mathcal{L})]$. Because G is not an apex linear forest, there are three possibilities for G:

1. The graph G is not an apex forest. In this case, Theorem 2.1 implies that G has a minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$. It is easy to verify that any proper minor of each

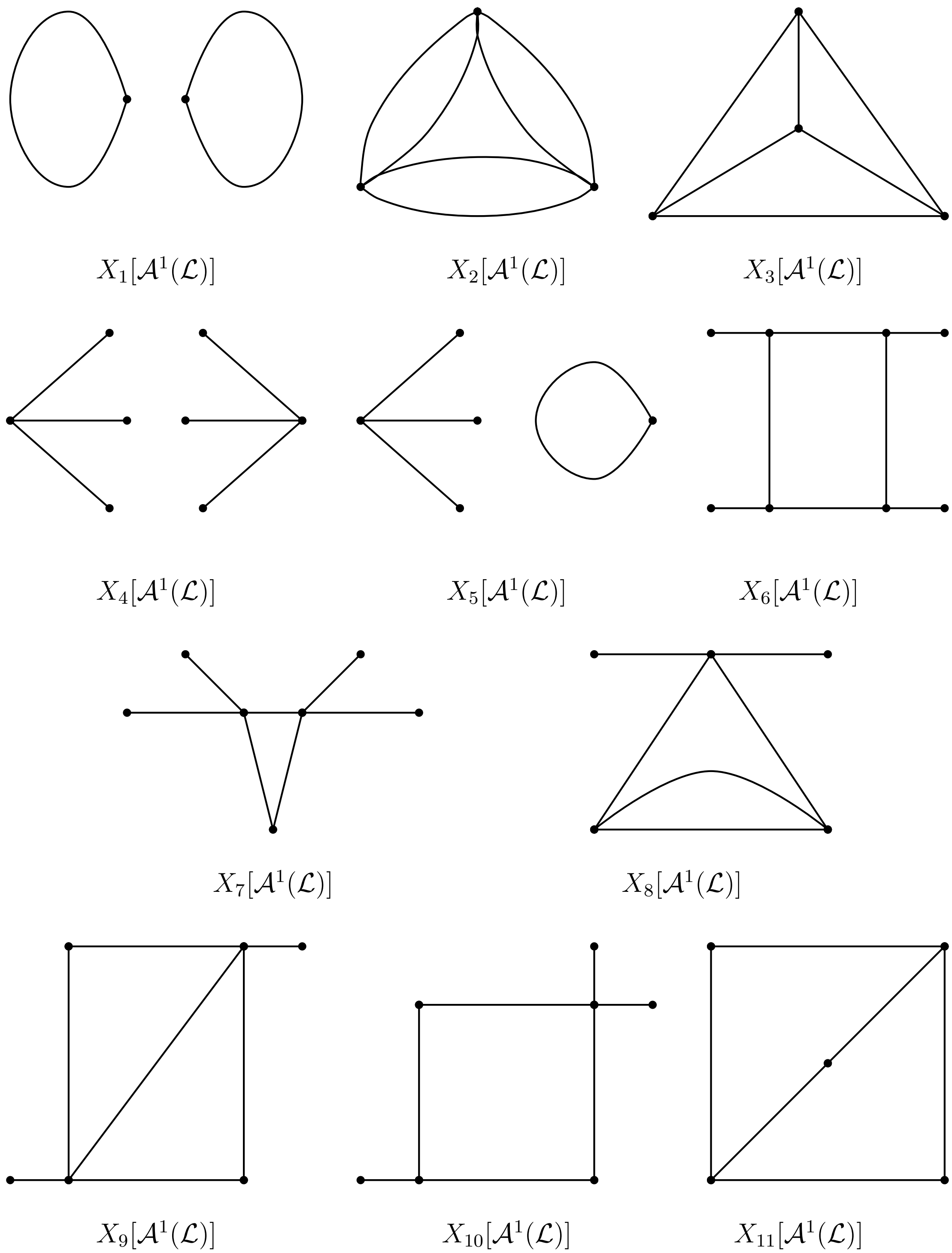


FIGURE 2.2. $\mathcal{X}[\mathcal{A}^1(\mathcal{L})]$

element in $\mathcal{X}[\mathcal{A}^1(\mathcal{F})]$ is an apex linear forest. So $\mathcal{X}[\mathcal{A}^1(\mathcal{F})] \subseteq \mathcal{X}[\mathcal{A}^1(\mathcal{L})]$, implying that $X_1[\mathcal{A}^1(\mathcal{F})]$, $X_2[\mathcal{A}^1(\mathcal{F})]$ and $X_3[\mathcal{A}^1(\mathcal{F})]$ are excluded minors for the class of apex linear forests, too. To help avoid confusion, we refer to $X_1[\mathcal{A}^1(\mathcal{F})]$, $X_2[\mathcal{A}^1(\mathcal{F})]$, and $X_3[\mathcal{A}^1(\mathcal{F})]$ as $X_1[\mathcal{A}^1(\mathcal{L})]$, $X_2[\mathcal{A}^1(\mathcal{L})]$ and $X_3[\mathcal{A}^1(\mathcal{L})]$, respectively, in the context of linear forests.

2. The graph G is an apex forest, but not a forest.

The graph G is an apex forest, but not a forest. It follows that G has a cycle C. We begin by proving a fact that will be useful in several cases of the analysis. Specifically, we will show that if the *girth* of G, that is, the length of the shortest cycle in G, is at least four, then

(A) no two vertices adjacent in C have degree two in G.

Suppose, to the contrary, that t and u are such vertices and G is as described above. Let s denote the neighbor of t other than u, and let v denote the neighbor of u other than t. By the assumption on the girth of G, all of s, t, u, and v are distinct. Let H denote the graph obtained from G by contracting the edge tu to form a new vertex w. By the minimality of G, there is a vertex x in H such that $H-x$ is a linear forest. If x is distinct from w, then consider $G-x$. Each component of $G-x$ other than that containing t and u is also a component of $H-x$, and hence a path. The component of $G-x$ containing t and u must also be a path, since it is obtained from the component of $H-x$ containing w, which is a path, by splitting w into two vertices connected by an edge. This contradicts the assumption that G is not apex linear forest, and leads us to conclude that $x = w$.

Consider now the graph J obtained from G by deleting the edge joining t and u. By the minimality of G, there is a vertex y such that $J-y$ is a linear forest. Since $G-y$ is not a linear forest, t and u must be in the same component of $J-y$, which is a path containing also s and v. Suppose that at least one of s and v, say, s, is not adjacent to y in G. Then the degree of s in G is two, and so its degree in $H-x$ is one. It follows that $G-u$ is a linear forest, since its only component that is not a component of $H-x$ is obtained by extending the path of $H-x$ containing s by a new edge joining s to t; a contradiction, which implies that both s and v are adjacent to y in G. But then the edges sy and vy together with the path of C from s to v avoiding t and u form a cycle in $H-x$; a contradiction, which implies that (A) holds.

We proceed to examine G based on its girth.

- The girth of G is one. So G contains a loop with a single vertex v such that $G-v$ is a minimal non-linear forest. We arrive at $X_5[\mathcal{A}^1(\mathcal{L})]$.

- The girth of G is two. This implies that G contains parallel edges. Let u and w be vertices incident with a pair of parallel edges. We split the argument further into cases based on how many vertices among u and w have three neighbors in $G-w$ and $G-u$, respectively; (a) both do; (b) exactly one, say w, does; (c) neither does. The cases (a), (b) and (c) are illustrated in Figures 2.3, 2.4, 2.5, respectively, with possible identifications between some of the vertices.

  (a) If all of a, b, c, d, e, f are distinct, we reach a contradiction; G properly contains $X_4[\mathcal{A}^1(\mathcal{L})]$. If u and w have exactly one neighbor in common, then G properly contains $X_7[\mathcal{A}^1(\mathcal{L})]$. If u and w have exactly two neighbors in common, then G properly contains $X_9[\mathcal{A}^1(\mathcal{L})]$. If u and

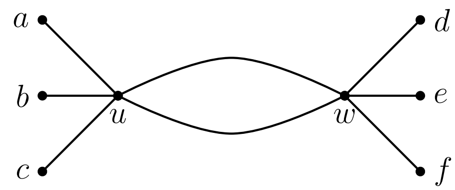


FIGURE 2.3.

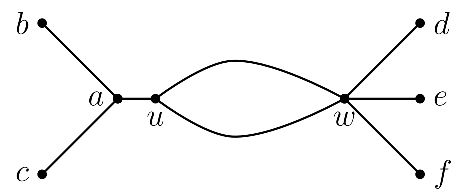


FIGURE 2.4.

w have exactly three neighbors in common, then G properly contains $X_{11}[\mathcal{A}^1(\mathcal{L})]$.

(b) If all of a, b, c, d, e, f are distinct, we reach a contradiction; G properly contains $X_4[\mathcal{A}^1(\mathcal{L})]$. If a is in $\{d, e, f\}$, then G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$. If exactly one member of $\{b, c\}$ is in $\{d, e, f\}$ is in $\{d, e, f\}$, then G properly contains $X_{10}[\mathcal{A}^1(\mathcal{L})]$.

(c) If all of a, b, c, d, e, f are distinct, we reach a contradiction; G properly contains $X_4[\mathcal{A}^1(\mathcal{L})]$. We also reach contradictions in the following scenarios. If $a = d$, then G properly contains $X_7[\mathcal{A}^1(\mathcal{L})]$. If a is in $\{e, f\}$,

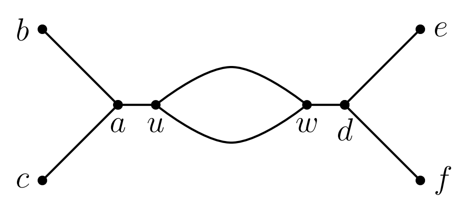


FIGURE 2.5.

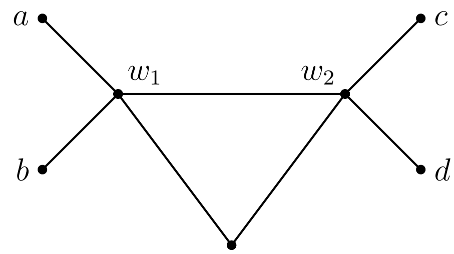


FIGURE 2.6.

then G properly contains $X_{10}[\mathcal{A}^1(\mathcal{L})]$. If exactly one member of $\{b, c\}$ is in $\{e, f\}$, then G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$. If both a and d have exactly two neighbors in common, then G properly contains $X_{11}[\mathcal{A}^1(\mathcal{L})]$.

This concludes the case when the girth of G is two.

- The girth of G is three. Let C be the shortest cycle in G. There must be a vertex v_1 whose degree in G is more than two. Then $G - v_1$ contains a vertex w_2 with at least three neighbors in G. The vertex w_2 may be on C or be adjacent to a vertex on C. Likewise, $G - v_2$ contains a vertex w_1 with at least three neighbors in G. The vertex w_1, which, without loss of generality, is v_1 or is adjacent to v_1.

  First, we consider the case where both w_1 and w_2 are on C as in Figure 2.6.

  If w_1 and w_2 have only one neighbor in common, which must lie on C, then $G \cong X_7[\mathcal{A}^1(\mathcal{L})]$. If the sets $\{a, b\}$ and $\{c, d\}$ have exactly one element in common, then $G \cong X_9[\mathcal{A}^1(\mathcal{L})]$. If the sets $\{a, b\}$ and $\{c, d\}$ have exactly two elements in common, then $G \cong X_{11}[\mathcal{A}^1(\mathcal{L})]$.

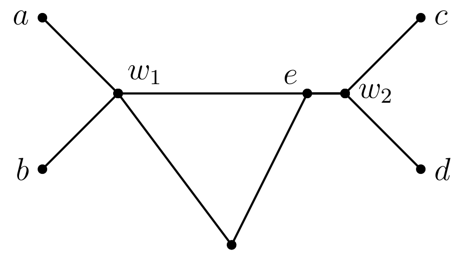


FIGURE 2.7.

Next, consider the case where w_1 is on C and w_2 is adjacent to a vertex on C as in Figure 2.7.

If all of a, b, c, d, w_1, w_2 are distinct, we reach a contradiction; G properly contains $X_4[\mathcal{A}^1(\mathcal{L})]$. If exactly one member of $\{a, b\}$, say a, is in $\{w_2, c, d\}$, then we again reach a contradiction by concluding G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$ or $X_{10}[\mathcal{A}^1(\mathcal{L})]$, depending whether a is in $\{c, d\}$ or not.

The remaining case is where both a and b are in $\{w_2, c, d\}$. If w_2 is in $\{a, b\}$, then G properly contains $X_{10}[\mathcal{A}^1(\mathcal{L})]$. If not, then G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$.

Next, consider where neither w_1 nor w_2 are on C as in Figure 2.8.

If all of a, b, c, d, w_1, w_2 are distinct, we reach a contradiction; G properly contains $X_4[\mathcal{A}^1(\mathcal{L})]$. We also reach contradictions in the following scenarios. If e is in $\{a, b\}$, then G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$. If exactly one member of $\{a, b\}$ is in $\{w_2, c, d\}$, then G properly contains $X_8[\mathcal{A}^1(\mathcal{L})]$. If both a and b are in $\{w_2, c, d\}$, then G properly contains

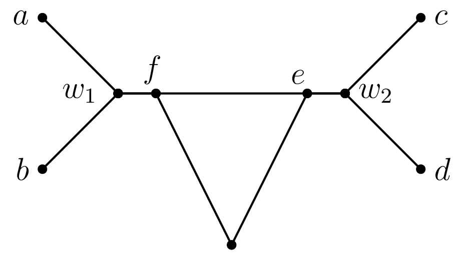


FIGURE 2.8.

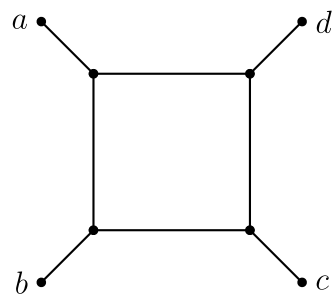


FIGURE 2.9.

$X_1[\mathcal{A}^1(\mathcal{L})]$. If $\{a, b\}$ and $\{c, d\}$ have exactly two elements in common, then G properly contains $X_{11}[\mathcal{A}^1(\mathcal{L})]$.

- The girth of G is four.

  Let C be the shortest cycle in G. Because of (A), we know G contains at least two vertices on C that have at least one neighbor that does not lie on C. Either C has a vertex of degree two in G or it does not.

  We consider the case where C does not have a vertex of degree two, as in Figure 2.9, with possible identifications between the vertices. If all of a, b, c, d are distinct, then $G \cong X_6[\mathcal{A}^1(\mathcal{L})]$. If one member of $\{a, c\}$ is in $\{b, d\}$, then the girth of G is three; a contradiction. If $a = c$ or $b = d$, then G properly contains $X_{11}[\mathcal{A}^1(\mathcal{L})]$.

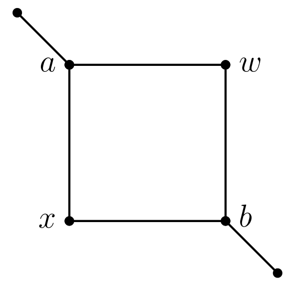


FIGURE 2.10.

Next we consider the case where C has at least one vertex w whose degree in G is two as in Figure 2.10. Let the vertices of C be a, w, b, x. Recall that C contains at least two vertices, say a, b, each of which has at least one neighbor that does not lie on C. In fact, no neighbor of a or b, other than w and x, lies on C since the girth of G is four. If any neighbor of a is equal to a neighbor of b, then G contains $X_{11}[\mathcal{A}^1(\mathcal{L})]$.

If the neighbors of a and b are distinct, then there must be a vertex of degree three in $G - x$. We are considering the case when w has degree two, so either one member of $\{a, b\}$ has degree three, or a neighbor of a or b has degree three. If one member of $\{a, b\}$ has degree three, then $G \cong X_{10}[\mathcal{A}^1(\mathcal{L})]$. If a neighbor of a or b, say the neighbor y of a, has degree three, we can contract the edge ya, so G properly contains $X_{10}[\mathcal{A}^1(\mathcal{L})]$.

- The girth of G is greater than four. We will show this to be impossible. Suppose C is the shortest cycle of G, and the length of C is at least five. Statement (A) implies that C has three vertices a, b, and c whose degree in G exceeds two, and such that a is adjacent to neither b nor c. Since C is the shortest cycle, each of a, b, and c has a neighbor not on C and since the girth exceeds four, no two of such neighbors can coincide. It follows that $X_{10}[\mathcal{A}^1(\mathcal{L})]$ is a proper minor of G; a contradiction.

3. The graph G is a forest. If G is a forest that is not an apex linear forest, there are two vertices with degree three or greater. If these two vertices are nonadjacent in G, we have $X_4[\mathcal{A}^1(\mathcal{L})]$ as a minor of G.

   We now consider the case where every two vertices with degree three or greater are adjacent. We show there are exactly two such vertices. If there are three or more vertices of degree exceeding two, and every two of those vertices are adjacent, then G contains a triangle; which is impossible because G is a forest. Therefore there are exactly two such vertices u and v whose degree exceeds two.

   If one of u and v has degree exactly three, then deletion of the other from G results in a linear forest, as u and v are adjacent; which is impossible. Therefore both vertices have degree four or more, and they are adjacent, so we have $X_4[\mathcal{A}^1(\mathcal{L})]$ as a subgraph.

$\square$

Chapter 3
Edgeless Graphs

So far we have taken two classes of graphs and added a single apex vertex. Now we investigate the class of edgeless graphs $\mathcal{E}$ by adding up to two apex vertices.

Membership of a graph in $\mathcal{A}^n(\mathcal{E})$ can be expressed also in terms of vertex covers. A *vertex cover* of a graph G is a set S of vertices of G such that every edge in G has at least one member of S as an endpoint [5]. We say in this case that G is coverable by the members of S. For example, K_3 has a vertex cover consisting of two vertices, as does a cycle of length four. Likewise, K_3 and C_4 are coverable by two vertices.

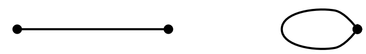
We begin with two easy lemmas about loops and parallel edges.

Lemma 3.1. *Suppose $\mathcal{E}$ is the class of edgeless graphs and n is a non-negative integer. Every minor-minimal graph not in $\mathcal{A}^n(\mathcal{E})$ does not contain a loop adjacent to another edge.*

Proof. Suppose G is a minor-minimal graph not in $\mathcal{A}^n(\mathcal{E})$, with a loop e adjacent to another edge f at a vertex v. The graph $G \setminus e$, by the minimality of G and the fact that $\mathcal{E}$ is the class of edgeless graphs, is coverable by at most n vertices. Because the edge e is incident to an endpoint of f, the graph G is also coverable by the same vertices as $G - f$. This means G is in $\mathcal{A}^n(\mathcal{E})$; a contradiction. □

Lemma 3.2. *Suppose $\mathcal{E}$ is the class of edgeless graphs and n is a non-negative integer. Every minor-minimal graph not in $\mathcal{A}^n(\mathcal{E})$ does not contain parallel edges.*

Proof. Suppose G is a minor-minimal graph not in $\mathcal{A}^n(\mathcal{E})$, and G has parallel edges e and f, both incident to vertices u and v. The graph $G \setminus e$, by the minimality



$X_1[\mathcal{A}^0(\mathcal{E})]$ $X_2[\mathcal{A}^0(\mathcal{E})]$

FIGURE 3.1. $\mathcal{X}[\mathcal{A}^0(\mathcal{E})]$

of G and the fact that $\mathcal{E}$ is the class of edgeless graphs, is coverable by at most n vertices. Because the edge e is incident to both endpoints of f, the graph G is also coverable by the same vertices as $G - f$. This means G is in $\mathcal{A}^n(\mathcal{E})$; a contradiction.

□

Theorem 3.3. *Let $\mathcal{E}$ be the class of edgeless graphs. A graph is in $\mathcal{A}^0(\mathcal{E})$ if and only if it contains no minor in $\mathcal{X}[\mathcal{A}^0(\mathcal{E})]$ as indicated in Figure 3.1.*

Proof. The necessity condition is obvious.

For the sufficiency condition, suppose G is not an edgeless forest. This means that G contains an edge, which may either be a loop or a non-loop edge, so $X_1[\mathcal{A}^1(\mathcal{E})]$ and $X_1[\mathcal{A}^1(\mathcal{E})]$ are minors of G. □

Theorem 3.4. *Let $\mathcal{E}$ be the class of edgeless graphs. A graph is in $\mathcal{A}^1(\mathcal{E})$ if and only if it contains no minor in $\mathcal{X}[\mathcal{A}^1(\mathcal{E})]$ as indicated in Figure 3.2.*

Proof. The necessity condition follows from a simple check that none of the graphs in $\mathcal{X}[\mathcal{A}^1(\mathcal{E})]$ are in $\mathcal{A}^1(\mathcal{E})$.

For the sufficiency condition, suppose G is not an apex edgeless graph. Then G contains at least two edges. If there are two nonadjacent edges (edges possibly being loops), G contains $X_1[\mathcal{A}^1(\mathcal{E})]$, $X_2[\mathcal{A}^1(\mathcal{E})]$ or $X_4[\mathcal{A}^1(\mathcal{E})]$ as a minor.

It remains to consider the case when G has at least two edges and each two are adjacent. Let e and f be two edges in G, both incident with a vertex v. Lemma 3.1 implies neither e nor f is a loop, and Lemma 3.2 implies e and f are not parallel.

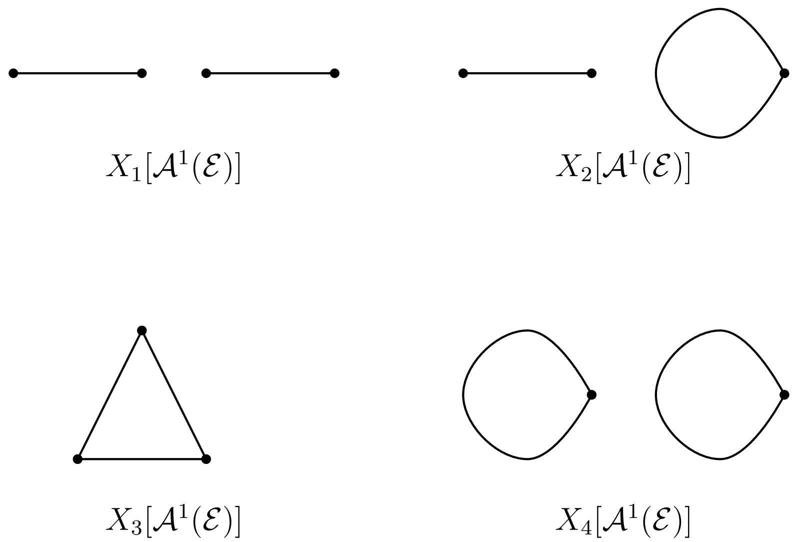


FIGURE 3.2. $\mathcal{X}[\mathcal{A}^1(\mathcal{E})]$

So each of e and f has a vertex u and w, respectively, not incident with the other. The graph G must have an edge h not incident with v. Because every two edges of G are adjacent, h must be incident with u and w, thereby forming K_3, which is isomorphic to $X_3[\mathcal{A}^1(\mathcal{E})]$. □

Theorem 3.5. *Let $\mathcal{E}$ be the class of edgeless graphs. A graph is in $\mathcal{A}^2(\mathcal{E})$ if and only if it contains no minor in $\mathcal{X}[\mathcal{A}^2(\mathcal{E})]$ as indicated in Figure 3.3.*

Proof. Just like in previous proofs, the necessity condition is easy to verify.

For the sufficiency condition, suppose G is a minor-minimal graph not in $\mathcal{A}^2(\mathcal{E})$. Suppose G has a vertex v whose degree exceeds two. We shall show that G must be isomorphic to $X_8[\mathcal{A}^2(\mathcal{E})]$.

Lemmas 3.1 and 3.2 imply that v has at least three neighbors. Let $U = \{u_1, u_2, u_3\}$, where u_1, u_2, u_3 are three neighbors of v. We shall show that each edge other than vu_1, vu_2, vu_3 must be incident to two vertices in U. Suppose not. Then G has an

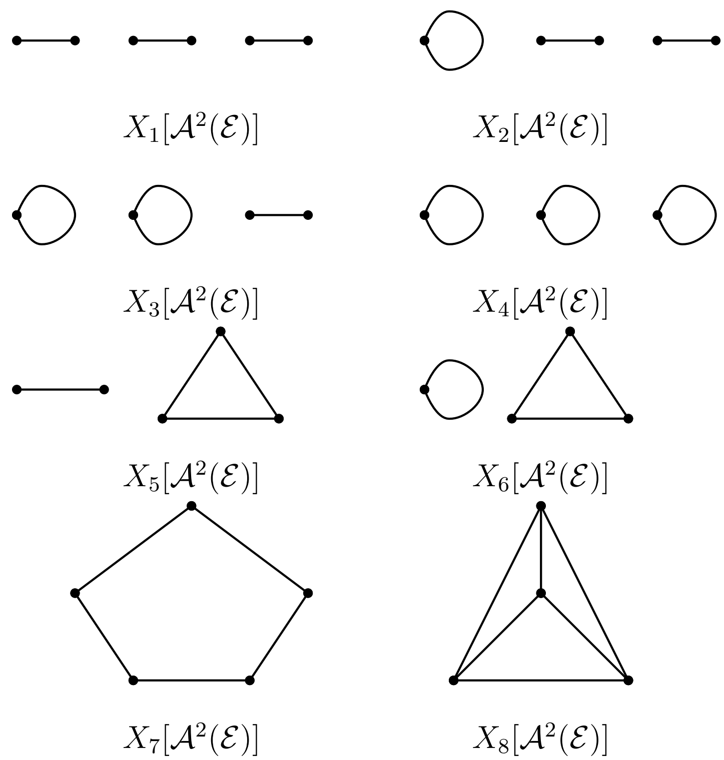


FIGURE 3.3. $\mathcal{X}[\mathcal{A}^2(\mathcal{E})]$

edge h incident to at most one vertex from U. Without loss of generality, let h be an edge of G incident to neither u_1 nor u_2. Let the edge vu_3 be called f. Because G is minor-minimal, all edges of $G\backslash f$ are covered by a two-element vertex set, W. The vertex v is not in W because then W would also cover G. This implies $W = \{u_1, u_2\}$, which leaves h uncovered; a contradiction. So we have shown that every edge in G other than vu_1, vu_2, vu_3 must be incident to two vertices in U.

Next we show the vertices in U induce a complete subgraph. Without loss of generality, suppose u_1 and u_2 are nonadjacent in G. In this case, the vertex set $\{v, u_3\}$ covers all edges in G; a contradiction. We arrive at the conclusion that if G has a vertex v whose degree exceeds two, then G must be isomorphic to $X_8[\mathcal{A}^2(\mathcal{E})]$, which is isomorphic to K_4.

Now we consider the case when all vertices in G have degree less than three. Then every component of G is a cycle or a path with at least one edge. Clearly,

G has at most three components so it must be one of the following: $X_1[\mathcal{A}^2(\mathcal{E})]$, $X_2[\mathcal{A}^2(\mathcal{E})]$, $X_3[\mathcal{A}^2(\mathcal{E})]$, $X_4[\mathcal{A}^2(\mathcal{E})]$.

If G has two components, and it is not coverable by two vertices, its minimality implies that one component is coverable by two vertices and the other is coverable by a single vertex. Thus, one component of G is a connected graph in $\mathcal{X}[\mathcal{A}^0(\mathcal{E})]$, and the other component is a connected graph in $\mathcal{X}[\mathcal{A}^1(\mathcal{E})]$. Theorems 3.3 and 3.4 imply that G must be either $X_5[\mathcal{A}^2(\mathcal{E})]$ or $X_6[\mathcal{A}^2(\mathcal{E})]$.

It remains to consider the case when G consists of only one component, which is a cycle or a path. If G is a path, its length must be at least five. Otherwise its edges can be covered by two vertices. However, if G is a path of length at least five, G properly contains $X_1[\mathcal{A}^2(\mathcal{E})]$ as a minor; a contradiction. So G is a cycle. Each cycle of length less than five can be covered by at most two vertices. The graph G is not a cycle of length greater than five, otherwise G properly contains $X_1[\mathcal{A}^2(\mathcal{E})]$ as a minor; a contradiction. This implies G is isomorphic to C_5, which is isomorphic to $X_7[\mathcal{A}^2(\mathcal{E})]$.

$\square$

Chapter 4
Future Work

With more time, this work would continue by investigating the list of excluded minors for $\mathcal{A}^3(\mathcal{E})$. We develop an estimate for the bound of the number of excluded minors as follows. An excluded minor G for $\mathcal{A}^3(\mathcal{E})$ may have at most four components. If G has four components, each component is a path of length one or a loop. This creates five excluded minors for $\mathcal{A}^3(\mathcal{E})$.

If G has three components, then it is comprised of the two-component members of $\mathcal{X}[\mathcal{A}^2(\mathcal{E})]$ and the connected members of $\mathcal{X}[\mathcal{A}^0(\mathcal{E})]$. This creates three excluded minors for $\mathcal{A}^3(\mathcal{E})$.

If G has two components, then either both of them are isomorphic to a connected member of $\mathcal{X}[\mathcal{A}^1(\mathcal{E})]$, or one is isomorphic to a connected member of $\mathcal{X}[\mathcal{A}^2(\mathcal{E})]$ and the other is isomorphic to a connected member of $\mathcal{X}[\mathcal{A}^0(\mathcal{E})]$. This creates five excluded minors for $\mathcal{A}^3(\mathcal{E})$.

The more difficult case is when G has one component. If G has one component, it is isomorphic to C_7 or K_5 or possibly another graph. This creates at least two excluded minors for $\mathcal{A}^3(\mathcal{G})$.

So far we have seen the possibility of at least fifteen excluded minors for $\mathcal{A}^3(\mathcal{G})$, and, more importantly, we see how we arrived at those possibilities.

One of the main goals of this work, which we discussed before, was to determine the existence of a function f such that if $\mathcal{G}$ is a minor-closed class of graphs for which $\mathcal{X}[(\mathcal{G})]$ has n members, then $\mathcal{X}[\mathcal{A}(\mathcal{E})]$ has at most n members.

This analysis of excluded minors for $\mathcal{A}^3(\mathcal{E})$ hints at the possibility of determining the cardinality of the excluded minors for $\mathcal{A}^n(\mathcal{E})$ inductively, which gives a small but valuable indication that indeed a function f as described before does exist.

In my investigations, I found the open-source software Sage very useful. However, Sage does not have built in functions to test membership in apex classes of well-known classes of graphs. I hope to contribute significant portions of code that will facilitate future work in this field.

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