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A POISSON SHOT-NOISE PROCESS OF PULSES AND ITS SCALING LIMITS

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Abstract. A shot-noise process on $\mathbb{R}$ is constructed by shifting and amplifying a deterministic pulse with random parameters generated by a Poisson random measure. It is motivated by applications where self-similarity and/or long-range dependence is indicated. Lipschitz continuity of the pulse is assumed, in order to obtain limit theorems under various scalings. In the limit, the centered and scaled Poisson shot-noise process approximates a fractional Brownian motion or a stable Lévy process depending on the type of scaling. An intermediate limit also emerges essentially due to a limiting form for the intensity of the Poisson random measure. We show that our scaling through the distributions involved is equivalent to time scaling used in other studies.

1. Introduction

We study the scaling limits of a Poisson shot-noise process which finds applications in various fields such as workload models, finance and medicine. It is constructed as a sum of pulses shifted and scaled according to a Poisson random measure. The limit is either a fractional Brownian motion (fBm) or a stable Lévy process, which are used to represent self-similarity and/or long-range dependence observed in data.

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. Let $\mathcal{B}_{\mathbb{R}}$ denote the Borel σ-algebra on $\mathbb{R}$. Let N be a Poisson random measure on $(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \mathcal{B}_{\mathbb{R}}\otimes\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{B}_{\mathbb{R}})$ with mean measure

$$\mu(ds, du, dr) = \lambda\, ds\, \nu(du)\, \gamma(dr), \tag{1.1}$$

where $\lambda > 0$, γ is the distribution of a random variable R and ν is an absolutely continuous probability measure on $\mathbb{R}_+$ that satisfies

$$\int_u^\infty \nu(dy) \sim h(u)\frac{u^{-\delta}}{\delta} \quad \text{as} \quad u \to \infty, \tag{1.2}$$

where $1 < \delta < 2$ and h is a slowly varying function at infinity, that is, h is such that for every $u > 0$

$$\lim_{x\to\infty} h(ux)/h(x) = 1\ . \tag{1.3}$$

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Each atom (S_j, U_j, R_j) of N is used to form the amplitude of pulse j at $t \geq 0$ by

$$R_j\, U_j\, f\left(\frac{t - S_j}{U_j}\right), \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x) = 0$ for $x < 0$. We call f a deterministic pulse, which is shifted to time S_j, scaled and amplified with U_j, and adjusted once more with the factor R_j. We construct a Poisson shot-noise process Z by aggregating the difference in the amplitudes at $t = 0$ and $t > 0$ of the randomized pulses with respect to N as

$$Z(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} ru\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right] N(ds, du, dr)\,. \tag{1.5}$$

Since N has atoms with $s \in (-\infty, 0)$ as well, Z has stationary increments and $Z(0) = 0$ by construction. The assumption that ν is a heavy-tailed probability distribution as in (1.2) implies long-range dependence observed in applications.

For each $n \in \mathbb{Z}_+$, let N_n denote the Poisson random measure with scaled mean measure μ_n that involves the scaled arrival rate and

$$\nu_n(du) := \nu(n\, du)\ . \tag{1.6}$$

Under certain assumptions, we prove In Theorem 4.1 that if nZ_n is the process formed as in the right hand side of (1.5) with N replaced by N_n, and if

$$\mu_n(ds, du, dr) = \frac{n^{2+\delta}}{h(n)}\,\lambda\, ds\, \nu_n(du)\, \gamma(dr)\ ,$$

then the process $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$ converges in distribution to an fBm with Hurst parameter $H = (3-\delta)/2 \in (1/2, 1)$. As a result, both long-range dependence and self-similarity are attained in the limit. On the other hand, if $n^{\alpha/\delta-1}Z_n$ is formed by N_n with mean measure

$$\mu_n(ds, du, dr) = \frac{n^{\alpha}}{h(n^{\alpha/\delta})}\,\lambda\, ds\, \nu_n(du)\, \gamma(dr)$$

for $0 < \alpha < \delta$, then the limit in distribution is a δ-stable Lévy process as shown in Theorem 5.2. In this case, the limiting process is self-similar with independent increments.

We also consider a σ-finite measure given by

$$\nu(du) = u^{-\delta-1}\, du \tag{1.7}$$

in (1.1) together with the compensated Poisson random measure $\tilde{N} = N - \mu$. A process $\tilde{Z}$ which is defined analogously to (1.5) with $\tilde{N}$ replacing N there, emerges as a scaling limit of the fluctuations of Z around its mean as shown in Theorem 3.1. The scaling for such a limit is an intermediate regime between fBm and Lévy scalings given in Theorems 4.1 and 5.2. Since $\tilde{Z}$ can be taken as an abstract model of the fluctuations in an application directly, we prove fBm and Lévy limits for this process as well. In Theorem 4.3, we show that if $n\tilde{Z}_n$ is the process formed as in the right hand side of (1.5) with N replaced by $\tilde{N}_n$ where

$$\mu_n(ds, du, dr) = n^2\lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr),$$

then the process $\{\tilde{Z}_n(t), t \geq 0\}$ converges in distribution to an fBm with Hurst parameter $H = (3 - \delta)/2$. In Theorem 5.5, we prove that if $n^{-1}\tilde{Z}_n$ is formed by $\tilde{N}_n$ with

$$\mu_n(ds, du, dr) = n^{-\delta}\lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr)\ ,$$

then the limit is a δ-stable Lévy process.

In [19, 23, 27], an increasing input with unit rate on $[0, 1]$ which remains constant thereafter is used for modeling workload. The present work uses the proof techniques of [19] to construct a Poisson shot-noise process approximating well-known self-similar processes, which is suitable for applications. The approach of [19] is elaborated in detail and its merits including applications to medicine are emphasized recently in [25, Chp.3]. In [22], a similar construction is considered for asset prices in finance with an fBm limit. For the same application and limiting result, [2] uses a semi-Markov process. Micropulses with compact support have been introduced in [12] for the aim of approximating fBm. In [26], a variation of the micropulses of [12] is considered to yield a fBm or a bifractional Brownian motion in the limit.

Micropulses are generalized as random ball models where overlapping balls are positioned according to a Poisson random measure (see e.g. [5]). With the help of a signed measure on $\mathbb{R}^d$, a variety of random fields are approximated on $\mathbb{R}^d$. We introduce the variable R for modeling both positive and negative pulses on $\mathbb{R}$ and require finiteness for only its lower moments considering real applications. The proof techniques, assumptions, and the generality achieved are different from random ball models as a result.

Scaled workload processes based on a Poisson random measure and their weak convergence have been studied also in [17, 21, 28] besides [19, 23, 27]. In all of these models, which are sometimes called infinite source Poisson, the limit is fBm or a Lévy process as a result of different scalings. We remark on their relevance and compare with the results and approach of the present work, as we prove the theorems. Our original motivation behind constructing Z comes from its use as a stock price process. For an arbitrage-free model of stock prices, it is sufficient that the pulse f has a jump at the start and it can possibly have positive or negative jump as made possible by the sign of R. The model is fit to real data in [10] demonstrating its applicability as a price process. It could be modified as a limit order book model in future work as in [13].

The paper is organized as follows. In Section 2, we outline the assumptions on the approximating process Z in relation to the proof techniques of the present paper and the related work. The scaling theorems for the intermediate regime, fBm, and stable Lévy motion are given in Sections 2, 3 and 4, respectively.

2. Preliminaries

In this section, we first outline our assumptions on the process Z and relate to the scaling proofs of the following sections. Then we elaborate on different scalings to show that they exhaust all possibilities.

2.1. Assumptions and Notation. We assume that the pulse $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+ := [0, \infty)$ with $f(x) = 0$ for all $x < 0$ and $f(x) = f(1)$

for all $x \geq 1$. We also assume that $\mathbb{E}|R| < \infty$. Note that for ν satisfying (1.2), we have $\int u\,\nu(du) < \infty$ since $1 < \delta < 2$. Under these assumptions, one can show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}} |r|u \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right| \lambda\, ds\, \nu(du)\, \gamma(dr) < \infty\,. \tag{2.1}$$

This implies $\mathbb{E}|Z(t)| < \infty$ for every $t \geq 0$. Then, from Campbell's theorem [20], the process Z of (1.5) is well-defined and its characteristic function $\mathbb{E}e^{i\xi Z(t)}$ is given by

$$\exp \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left\{ \exp\left[i\xi\, ru \left(f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right)\right] - 1 \right\} \lambda\, ds\, \nu(du)\, \gamma(dr)$$

for $\xi \in \mathbb{R}$. In view of (2.1), it follows from [15, Def.I.4.1] that Z is a semimartingale with respect to the filtration generated by the Poisson random measure N. Semimartingales defined in this way form a special sub-class and satisfy the more general definition [16, Def.I.4.21]. They have sample paths which are right-continuous with left-hand limits (càdlàg) [16, pg.43]. Therefore, Z takes values in $D(0,\infty)$. The process Z becomes a martingale if its mean is zero which would be the case for example when $\mathbb{E}R = 0$.

If ν is σ-finite as in (1.7), then $\tilde{Z}$ is defined as a stochastic integral with respect to compensated Poisson random measure $\tilde{N}$. When $\mathbb{E}R^2 < \infty$ and f' is bounded as before, its characteristic function is given by

$$\mathbb{E}e^{i\xi\tilde{Z}(t)} = \exp \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Phi\left(\xi\, ru \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\right) u^{-\delta-1}\lambda\, ds\, du\gamma(dr), \tag{2.2}$$

where $\Phi(x) := e^{ix} - 1 - ix$ is introduced for simplicity of notation. The derivation of (2.2) follows as in [12, Thm.2.1], similar to the analysis of Lévy processes [24, Thm.2.10]. The L^2-theory of integration with martingale-valued measures is well-known [1, 9, 16]. However, we refer to L^p-integration for $p < 2$ as a result of relaxing the second moment condition for R as $\mathbb{E}|R|^{1+\kappa} < \infty$, for $\kappa < 1$ in some of the theorems in this paper. The L^p-integrators, and in particular the compensated Poisson random measure, are studied in [4, Chp.II,III] to extend the integral from step functions to general integrands. This is performed through L^p-norm or a related seminorm called Daniell mean. The paths of the limit process are also càdlàg by Exer. 3.10.14 and Thm. 3.10.20 of [4].

For proving various scaling limits of the process Z or $\tilde{Z}$, we use integration by parts technique of [19] and this leads to the derivative f'. We assume that the pulse function f' is Lipschitz continuous, which seems to be a strong condition. However, this approach avoids requiring higher moments of R to be finite, but only $\mathbb{E}|R|^{1+\kappa} < \infty$ with $\kappa \in (0,1)$, and $\kappa = 1$ in the case of a Gaussian limit. This may be convenient in applications where higher moments may not exist. Note that differentiability of f and Lipschitz condition on both f and f' are assumed on a bounded open interval, and the latter could be replaced by a requirement of bounded second derivative. Clearly, f' exists a.e. if f is Lipschitz, and when f' is Lipschitz it exists everywhere.

If f does not have a discontinuity at 0, then using f' would be both convenient in terms of notation, and in alignment with recent work on constructions of random fields with random balls on $\mathbb{R}^d$ [5, 14]. Then, as in [14], one could call f' a *pulse function*, which is simply shifted to s and scaled by u to obtain a variety of pulses with their contribution over $[0, t]$ given by $\int_0^t \tau_{(s,u)} f'(y)dy \equiv u[f((t-s)/u) - f(-s/u)]$, where τ is defined as a shift and scale mapping acting on pulse functions by $\tau_{(s,u)} f'(\cdot) := f'((\cdot - s)/u)$. As f is assumed to have a possible discontinuity at 0 in the present work, we will use directly the shifted and scaled versions of the pulse f. In random ball models, a signed measure on $\mathbb{R}^d$ is used for constructing a shot-noise process. We use the parameter r which takes either positive or negative values and $r\,dy$ plays the role of a signed measure on $\mathbb{R}$.

As technical contributions in Theorems 4.1 and 5.2, we generalize the results of [19] to Lipschitz functions f. The pulse used in [19, 23, 27] appears as a special case, which is a continuous linear pulse increasing from 0 to 1 and its derivative is just a constant. There are more terms to be bounded for the generalization in the present work. The following lemma will be used in the sequel.

Lemma 2.1. *Suppose that ν is a probability measure with a regularly varying tail as given in (1.2), and the function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+$ with $f(x) = 0$ for all $x < 0$, and $f(x) = f(1)$ for all $x \geq 1$. Then*

$$\int_{-\infty}^{\infty} \int_0^{\infty} u^{1+\kappa} \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right|^{1+\kappa} u^{-\delta-1}\, du\, ds \ < \infty \tag{2.3}$$

for $1 < \delta < 3$ and $\kappa > 0$ such that $1 + \kappa > \delta$.

Proof. Let us call the integral (2.3) as I. In view of the assumptions on f, we have the upper bound

$$M^{1+\kappa} \left\{ \int_{-\infty}^{0} \int_{-s}^{t-s} (u+s)^{1+\kappa} u^{-\delta-1} du\, ds + 2^{1+\kappa} \int_0^t \int_0^{t-s} u^{1+\kappa} u^{-\delta-1} du\, ds \right.$$
$$\left. + 2^{1+\kappa} \int_0^t \int_{t-s}^{\infty} (t-s)^{1+\kappa} u^{-\delta-1} du\, ds + \int_{-\infty}^{0} \int_{t-s}^{\infty} t^{1+\kappa} u^{-\delta-1} du\, ds \right\}$$

for I, where $M > 0$ is a constant that is larger than both the Lipschitz constant of f and $|f(0)|$. Evaluating the above integrals, we find that

$$I \leq M^{1+\kappa}\, t^{2+\kappa-\delta} \left[\frac{1}{(2+\kappa)(2+\kappa-\delta)} + \frac{1}{(2+\kappa)\delta} + \frac{2^{1+\kappa}}{(2+\kappa-\delta)(1+\kappa-\delta)} \right.$$
$$\left. + \frac{2^{1+\kappa}}{\delta(2+\kappa-\delta)} + \frac{1}{\delta(\delta-1)} \right]$$

□

2.2. Various Scalings. In the present work, we scale the parameters of the distributions rather than speeding the time t, which is frequently the case in similar scaling theorems. These are all performed through the intensity of the Poisson random measure in order to obtain various limits.

Consider $\nu_n(du)$ as defined in (1.6). Although the parameters of the distribution ν is scaled, this is essentially a time scaling as the random variable U has the interpretation of duration in many applications. We look at the contribution of individual pulses over shorter time periods by scaling U as U/n, which now has distribution $\nu(n\,du)$. As an example, if ν is the Pareto distribution $\nu(du) = \delta b^{\delta} u^{-\delta-1}\,du$ for $u > b$, with parameters $\delta > 0$ and $b > 0$, we have $\nu_n(du) = \delta(b/n)^{\delta} u^{-\delta-1}\,du$ for $u > b/n$, which amounts to scaling the scale parameter b as b/n. This approximates an infinite measure on $\mathbb{R}_+$ as the cutoff parameter b decreases.

The intensity of N is further scaled through the arrival rate λ and the process itself is appropriately centered and scaled in space, in alignment with previous work. As outlined in Section 1, we have either an fBm or a stable Lévy process, or a third intermediate process in the limit depending on these scalings.

The equivalence of the scalings of previous work with the current scaling of the parameters is not obvious. We demonstrate one case through the continuous flow rate model as given in [19] by

$$Z(t) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} [(t-s)^{+} \wedge u - (-s)^{+} \wedge u]\, r\, N(ds, du, dr)\ . \tag{2.4}$$

In [19, Thm.2], the limit is studied when the speed of time increases in proportion to the intensity of Poisson arrivals. To balance the increasing intensity λ_n, time is speeded up by a factor n and the size is normalized by a factor $\lambda_n^{1/2} n^{(3-\delta)/2}$ provided that $\lambda_n/n^{\delta-1} \to \infty$. We can let $\lambda_n = n^{\varepsilon+\delta-1}$ with $\varepsilon > 0$. Taking $\varepsilon = 2$, that is $\lambda_n = n^{1+\delta}$, we show the equivalence of the scaling of [19, Thm.2] to the scaling in Theorem 4.1. The scaled and centered process has the form

$$\frac{Z(nt) - \mathbb{E}Z(nt)}{\lambda_n^{1/2} n^{(3-\delta)/2}} = \frac{1}{n^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} ru \left[f\left(\frac{nt-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(ds, du, dr),$$

where we have written a pulse function f' in general. Then we can substitute λ_n, make change of variables $s \to ns$ and $u \to nu$, and then simplify to get

$$\begin{aligned} &\frac{Z(nt) - \mathbb{E}Z(nt)}{n^2} = \\ &\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{r}{n}\, u \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(d(ns), d(nu), dr), \end{aligned} \tag{2.5}$$

where the mean measure of N is

$$\mu_n(d(ns), d(nu), dr) = \lambda_n\,(n\,ds)\,\nu(n\,du)\gamma(dr) = n^{2+\delta}\, ds\, \nu(n\,du)\gamma(dr)\ . \tag{2.6}$$

In Theorem 4.1, we start with the scaled process (2.5) essentially. This can be observed by the fact that

$$N_n(d(ns), d(nu), dr) \stackrel{d}{=} N'_n(ds, du, dr)$$

for a Poisson random measure N' with mean measure $\mu'_n(ds, du, dr)$ equal to (2.6) [18],[11, Def.V.2.2]. Clearly, equivalence of the scalings in Theorem 4.1 and [19, Thm.2] is in distributional sense.

It is shown in [19] that the asymptotic behavior of the ratio $\lambda_n/n^{\delta-1}$ determines the type of the limit process when time is speeded up by a factor n. For a choice of sequences λ_n and n, let $\sharp(\lambda_n, n)$ denote the number of active pulses at time n. Then we have

$$\mathbb{E}\ \sharp(\lambda_n, n) \sim \frac{\lambda_n}{n^{\delta-1}}$$

for large n. The limit is considered in the cases where this value tends to a finite positive constant, to infinity, or to zero as λ_n and n go to infinity. Indeed, the limits of finite constant, infinity, and zero correspond to Theorem 3.1, Theorem 4.1, and Theorem 5.2, respectively. They are called intermediate, fast and slow connection rates in view of telecommunication applications.

3. Intermediate Scaling

In this section, we prove the first scaling theorem which demonstrates the relationship of the probability measure (1.2) and the σ-finite measure (1.7).

Theorem 3.1. *Suppose that ν is a probability measure with a regularly varying tail as given in (1.2), the function $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+$ with $f(x)=0$ for all $x<0$, $f(x)=f(1)$ for all $x\geq 1$ and f' satisfying a Lipschitz condition a.e. on $(0,1)$, and $\mathbb{E}|R|^{1+\kappa}<\infty$ for some $0<\kappa\leq 1$ with $1+\kappa>\delta>1$. Let*

$$Z_n(t)=\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} r\,u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dr)$$

and

$$\mu_n(ds,du,dr)=\frac{n^\delta}{h(n)}\,\lambda\,ds\,\nu_n(du)\,\gamma(dr),$$

where $\nu_n(du)=\nu(n\,du)$. Then $\{Z_n(t)-\mathbb{E}Z_n(t), t\geq 0\}$ converges in the Skorohod topology on $D(0,\infty)$ to the process

$$\left\{\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} r\,u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\tilde{N}'(ds,du,dr)\ ,\ t\geq 0\right\}$$

as $n\to\infty$, where $\tilde{N}'=N'-\mu'$ for a Poisson random measure N' with mean measure $\mu'(ds,du,dr)=\lambda\,u^{-\delta-1}ds\,du\,\gamma(dr)$.

Proof. For the convergence of finite dimensional distributions of $\{Z_n(t)-\mathbb{E}Z_n(t), t\geq 0\}$, consider the characteristic function $\mathbb{E}\exp i\sum_{k=1}^m \xi_k[Z_n(t_k)-\mathbb{E}Z_n(t_k)]$ for $\xi_k\in\mathbb{R}$, $t_k\geq 0$ and $m\in\mathbb{N}$. It is given by

$$\exp\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\Phi\left(ru\sum_{k=1}^m\xi_k\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right)\frac{n^\delta}{h(n)}\lambda\,ds\,\nu_n(du)\,\gamma(dr), \tag{3.1}$$

where Φ is used for simplicity of notation as defined after (2.2). We first show that the exponent in (3.1) is bounded and then use bounded convergence theorem to take the limit. This theorem is a generalization of [19, Thm.1] with the general pulse f. Although we follow the same approach as in [19, Thm.1], there are more terms to bound in the present case. Let

$$g(s,u,r):=\Phi\left(ru\sum_{k=1}^m\xi_k\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right). \tag{3.2}$$

Using the random variable U, we denote the left hand side of (1.2) as $\mathbb{P}\{U\geq u\}$ below. By integration by parts, the exponent in (3.1) is equal to

$$\int\int\int\partial_u g(s,u,r)\,\mathbb{P}\{U>nu\}\,\frac{n^\delta}{h(n)}\,\lambda\,ds\,du\,\gamma(dr), \tag{3.3}$$

where ∂_u is $\partial/\partial u$ and the hypothesis that $\nu_n(du) = \nu(n\,du)$ is used.

a) Bound for the integrand of (3.3) for large values of u:

In view of Potter bounds [8], for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\mathbb{P}\{U > nu\}}{\mathbb{P}\{U > n\}} \leq 2u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon})$$

for all $n \geq n_0$ and $nu \geq n_0$, that is, $u \geq n_0/n$. Since $\lim_{n\to\infty} \mathbb{P}\{U > n\} n^{\delta}/h(n) = C$ for some $C > 0$, we have $\mathbb{P}\{U > n\} n^{\delta}/h(n) \leq (C + \epsilon)$ for all $n \geq n_0'$ for some $n_0' \in \mathbb{N}$. Note that $C = 1/\delta$ by (1.2). Assume $n_0' \leq n_0$ for simplicity of notation. Therefore, we get

$$\mathbb{P}\{U > nu\} \frac{n^{\delta}}{h(n)} \leq 2u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon})(C + \epsilon) \tag{3.4}$$

for all $n \geq n_0$ and $u \geq n_0/n$.

In (3.3), we explicitly have

$$\partial_u g(s, u, r) = i \left[e^{i \sum_{k=1}^{m} \xi_k r\, u \left[f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 \right] \partial_u S(s, u, r), \tag{3.5}$$

where $S(s, u, r) := \sum_{k=1}^{m} \xi_k r\, u \left[f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]$ and hence

$$\begin{aligned} \partial_u S(s, u, r) \quad &= \quad \sum_k \xi_k r \left[f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \\ &\qquad + \sum_k \xi_k r \left[-f'\left(\frac{t_k - s}{u}\right) \frac{t_k - s}{u} + f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right] \end{aligned}$$

Next, we can bound $|\partial_u S|$ using the Lipschitz property of f and f' on different regions for u and s.

Let $M > 1$ stand for a constant which is larger than the Lipschitz constants of f and f', as well as $|f(0)|$ and $|f'(0)|$, where $f'(0)$ is the right derivative at 0.

i) $s < 0$ and $0 < s + u < t_k$

Since $(t_k - s)/u > 1$ and $-s/u < 1$, we have

$$\left| f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right| = \left| f(1) - f\left(\frac{-s}{u}\right) \right| \leq M \left| 1 + \frac{s}{u} \right|$$

and

$$\left| -f'\left(\frac{t_k - s}{u}\right) \frac{t_k - s}{u} + f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right| = \left| 0 + f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right| \leq M \left| \frac{s}{u} \right|$$

due to the form of f and Lipschitz assumptions. Therefore, we get

$$|\,\partial_u S(s, u, r)| \leq \left(M \left| 1 + \frac{s}{u} \right| + M \left| \frac{s}{u} \right| \right) \sum_k \xi_k |r| = M \sum_k |\xi_k|\, |r|$$

since $1 + s/u > 0$ and $s/u < 0$ in this region.

ii) $s > 0$ and $s + u < t_k$

In this region, $f'((t_k-s)/u)$ and $f'(-s/u)$ vanish, $f((t_k-s)/u)=f(1)$ and $f(-s/u)=0$. Due to Lipschitz continuity of f on $(0,1)$, we have

$$\begin{aligned}\left|f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right| &\le \left|f\left(\frac{t_k-s}{u}\right)-f(0)\right|+\left|f(0)-f\left(\frac{-s}{u}\right)\right| \\ &= |f(1)-f(0)|+|f(0)| \\ &\le M+|f(0)| \;\le\; 2M\end{aligned}$$

as $M>|f(0)|$ is assumed. Therefore, we get

$$|\partial_u S(s,u,r)| \le 2M\sum_k |\xi_k|\,|r|\,.$$

iii) $0<s<t_k$ and $t_k<s+u$

In this region, $f(-s/u)=f'(-s/u)=0$ and we get

$$|\partial_u S(s,u,r)| \le 4M\sum_k |\xi_k|\,|r|$$

by similar arguments in ii), as $|(t_k-s)/u|<1$, $|f(0)|<M$ and $|f'(0)|<M$.

iv) $s<0$ and $t_k<s+u$

We have

$$\left|f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right| \le M\left|\frac{t_k}{u}\right|$$

and

$$\left|f'\left(\frac{t_k-s}{u}\right)\frac{t_k-s}{u}-f'\left(\frac{-s}{u}\right)\frac{-s}{u}\right| \le M\left|\frac{t_k}{u}\right|+M\left|\frac{st_k}{u^2}\right|\,.$$

The corresponding bound on $|\partial_u S(s,u,r)|$ follows.

Now, we can bound the remaining terms in (3.5) by

$$2^{1-\kappa}\sum_j |\xi_j|^\kappa |r|^\kappa u^\kappa \left|f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right|^\kappa \;\wedge\; 2 \tag{3.6}$$

using the inequalities $|e^{ix}-1|\le 2^{1-\kappa}|x|^\kappa$ and $\left(\sum_j |x_j|\right)^\kappa \le \sum_j |x_j|^\kappa$, $0<\kappa\le 1$ [19], and the fact that $|e^{ix}-1|\le 2$. The index k is replaced by j in order to distinguish the cross products of sums below. We further note that

$$\left|f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right|^\kappa \le (2M)^\kappa \le 2^\kappa M \tag{3.7}$$

since f is bounded and $M^\kappa\le M$, having assumed $M>1$ for simplicity of notation. Putting all terms together by (3.4), (3.6), (3.7) and i)-iv), we find that (3.3) is bounded as

$$\begin{aligned}&\int\int_{n_0/n}^{\infty}\int |\partial_u g(s,u,r)|\,\mathbb{P}\{U>nu\}\,\frac{n^\delta}{h(n)}\,\lambda\,ds\,du\,\gamma(dr) \\ &\quad\le 4M(C+\epsilon)\sum_k |\xi_k|\int\int_0^\infty\int |r|\,B(s,u,t_k)\max(u^{-\epsilon},u^{\epsilon})\,u^{-\delta}\,\lambda\,ds\,du\,\gamma(dr),\end{aligned} \tag{3.8}$$

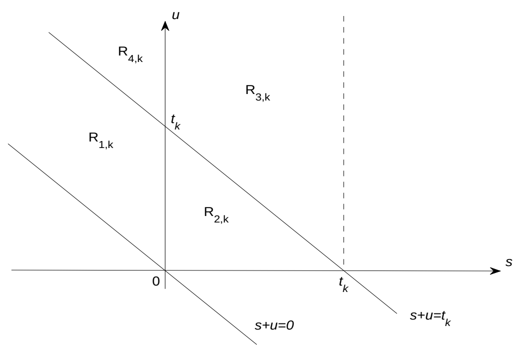

FIGURE 1. Subregions considered for (s, u)

where

$$B(s,u,t_k) = \left(1 \wedge \sum_j M|\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s\le t_j\}}\right)$$
$$\cdot \left[1_{R_{1,k}} + 2\,.\,1_{R_{2,k}} + 4\,.\,1_{R_{3,k}} + \left(2\frac{t_k}{u} + \frac{|s|t_k}{u^2}\right)1_{R_{4,k}}\right] \tag{3.9}$$

and $R_{1,k},\ldots,R_{4,k}$ denote the regions in i)-iv). Since $[1\wedge\sum_j M|\xi_j|^\kappa|r|^\kappa u^\kappa 1_{\{s\le t_j\}}]$ ≤ 1, and $(t_k-s)/u \le 1$ we can write

$$B(s,u,t_k) \le 1_{R_{1,k}} + 4\,\cdot\,1_{R_{3,k}} + \left(2\frac{t_k}{u} + \frac{|s|t_k}{u^2}\right)1_{R_{4,k}} \tag{3.10}$$
$$+\,2\left(1 \wedge \sum_j M|\xi_j|^\kappa|r|^\kappa u^\kappa 1_{\{s\le t_j\}}\right)1_{R_{2,k}}.$$

We keep the extra bounding term for $R_{2,k}$, as the integration in this region is more delicate. For fixed $k \in \{1,\ldots n\}$, $R_{1,k},\ldots,R_{4,k}$ are depicted in Fig.1. If we choose $\epsilon > 0$ such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 1 + \kappa\,, \tag{3.11}$$

then the right hand side of (3.8) is finite as shown next.

When the right hand side of (3.8) is splitted over different regions, checking the finiteness of the integrals over $R_{1,k}, R_{3,k}, R_{4,k}$ reduces to showing that

$$\int_{-\infty}^0\int_{-s}^{t_k-s} u^{-\delta}\max(u^{-\epsilon},u^\epsilon)\,du\,ds + \int_0^{t_k}\int_{t_k-s}^\infty u^{-\delta}\max(u^{-\epsilon},u^\epsilon)\,du\,ds$$
$$+\int_{-\infty}^0\int_{t_k-s}^\infty\left(\frac{1}{u}+\frac{|s|}{u^2}\right)u^{-\delta}\max(u^{-\epsilon},u^\epsilon)\,du\,ds$$

is finite. This is indeed true when we choose $\epsilon > 0$ such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 2 \,. \tag{3.12}$$

In region $R_{2,k}$, we have

$$I := \int \int_0^{t_k} \int_0^{t_k - s} |r| [1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \le t_j\}}] u^{-\delta} \max(u^{-\epsilon}, u^\epsilon) \, du \, ds \, \gamma(dr) \tag{3.13}$$

If $t_j > t_k$, it can be observed from Fig.1 that the integral reduces to that over region $R_{2,k}$. If $t_j < t_k$, then the integral over $R_{2,k}$ yields an upper bound. That is, we can replace $1_{\{s \le t_j\}}$ by the constant function 1 and get

$$I \le \mathbb{E}|R|^{1+\kappa} \sum_j |\xi_j|^\kappa \int_0^{\bar{u}} \int_0^{t_k} u^{\kappa - \delta} \max(u^{-\epsilon}, u^\epsilon) \, ds \, du \tag{3.14}$$
$$+ \mathbb{E}|R| \int_{\bar{u}}^{t_k} \int_0^{t_k} u^{-\delta} \max(u^{-\epsilon}, u^\epsilon) \, ds \, du,$$

where $\bar{u}$ denotes a cutoff value of u such that $\sum_j |\xi_j|^\kappa u^\kappa$ is too large in (3.14), and we use the fact that $t_k - u \le t_k$ for $u \ge 0$ after changing the order of integration for u and s in (3.13). Then the right hand side of (3.14) is finite if we choose $\epsilon > 0$ such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 1 + \kappa$$

which clearly satisfies (3.12) since $\kappa \le 1$.

b) Bound for the integrand of (3.3) for small values of u:

We now consider $u \le n_0/n \le 1$ as $n \ge n_0$. We use Markov inequality for $\mathbb{P}\{U \ge nu\}$, together with the bounds (3.6), (3.7) and i)-iv), and we get

$$|\partial_u g(s,u,r)| \, \mathbb{P}\{U > nu\} \, \frac{n^\delta}{h(n)} \le 2M \frac{\mathbb{E}U}{u} \frac{n^{\delta-1}}{h(n)} \, |r| \sum_k |\xi_k| B(s,u,t_k) \tag{3.15}$$

From (3.10), we can write

$$B(s,u,t_k) \le 1_{R_{1,k}} \; + \; 4 \,.\, 1_{R_{3,k}} \; + \; \left(2 \frac{t_k}{u} + \frac{|s| t_k}{u^2} \right) 1_{R_4,k}$$
$$+ \;\; 2 \,.\, 1_{R_{2,k}} \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \le t_j\}}$$

considering that u is bounded as $u \le 1$. Now, we have

$$1 = u^{\delta+\epsilon-1} u^{1-\delta-\epsilon} \le n_0^{\delta+\epsilon-1} n^{-\epsilon} n^{1-\delta} u^{1-\delta-\epsilon} \le n_0^{\delta+\epsilon-1} h(n) n^{1-\delta} u^{1-\delta-\epsilon} \tag{3.16}$$

since $u \le n_0/n$ and $n^{-\epsilon} \le h(n)$ for the slowly varying function h when n is sufficiently large [19]. Using (3.16) to increase the right hand side of (3.15) and in view of (3.10), we get

$$|\partial_u g(s,u,r)| \, \mathbb{P}\{U > nu\} \, \frac{n^\delta}{h(n)} \le 2M \mathbb{E}U \, n_0^{\delta+\epsilon-1} \, |r| u^{-\delta-\epsilon} \sum_k |\xi_k|$$
$$\cdot \left[1_{R_{1,k}} + 4 \,.\, 1_{R_{3,k}} + \left(2 \frac{t_k}{u} + \frac{|s| t_k}{u^2} \right) 1_{R_{4,k}} + 2 \,.\, 1_{R_{2,k}} \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \le t_j\}} \right]$$

which is integrable over $0 < u < 1$, as in part a).

As a result of a) and b), we can use dominated convergence theorem to find the limit in (3.3) as

$$\lim_{n\to\infty} \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} = \frac{u^{-\delta}}{\delta} \tag{3.17}$$

by (1.2) and (1.3), and then revert (3.3) by another integration by parts to get the limit of (3.1) as

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s,u,r)\, \lambda u^{-\delta-1} ds\, du\, \gamma(dr)\ .$$

It can be shown that the above characteristic function and the corresponding process are well defined since $|\Phi(x)|$ is bounded by $|x|^{1+\kappa}$. Hence, we have shown the convergence of finite dimensional distributions.

To prove weak convergence in the Skorohod topology on $D(0,\infty)$, we first observe that

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq \\ 2\mathbb{E}|R|^{1+\kappa} \int_0^\infty \int_{-\infty}^t u^{1+\kappa} \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right|^{1+\kappa} \frac{n^\delta}{h(n)} \lambda\, ds\, \nu_n(du) \tag{3.18}$$

by [19, Lemma 5]. By integration by parts and in view of Potter bounds as before, for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the part of the integral for $u \geq n_0/n$ on the right hand side of (3.18) is bounded from above by

$$2(C+\epsilon)\int_{n_0/n}^\infty \int_{-\infty}^t \left| \frac{\partial}{\partial u} \left[u^{1+\kappa} \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right|^{1+\kappa}\right]\right| \lambda\, u^{-\delta} \max\{u^{-\epsilon}, u^{\epsilon}\}\, ds\, du. \tag{3.19}$$

The absolute value term here is exceeded by

$$(1+\kappa) u^\kappa \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right|^{1+\kappa} \\ +(1+\kappa) u^\kappa \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right|^{\kappa} \left| -f'\left(\frac{t-s}{u}\right) \frac{t-s}{u} - f'\left(\frac{-s}{u}\right) \frac{-s}{u}\right|$$

for which an upper bound is

$$M^{1+\kappa}(1+\kappa)\left\{[u^{-1}(u+s)^{1+\kappa} + u^{-1}s(u+s)^\kappa]1_{R_1} + 2^{1+\kappa}\, u^\kappa 1_{R_2} \\ + 2^{1+\kappa}\, u^{-1}(t-s)^{1+\kappa} 1_{R_3} + u^{-1} t^{1+\kappa}(2+|s|) 1_{R_4}\right\} \tag{3.20}$$

by Lipschitz assumptions on f and f', where $R_1, \ldots, R_4$ are as in i) through iv) above, with $t_k \equiv t$. Substituting (3.20) in (3.19) and starting the lower limit for u

from 0, we have an upper bound for the integral in (3.19) given by

$$\int_{-\infty}^{0}\int_{-s}^{t_k-s}[(u+s)^{1+\kappa}+s(u+s)^{\kappa}]u^{-\delta-1}\max(u^{-\epsilon},u^{\epsilon})\,du\,ds \quad (3.21)$$
$$+2^{1+\kappa}\int_0^t\int_0^{t-s}u^{\kappa-\delta}\max(u^{-\epsilon},u^{\epsilon})\,du\,ds$$
$$+2^{1+\kappa}\int_0^{t_k}\int_{t-s}^{\infty}(t-s)^{1+\kappa}u^{-\delta-1}\max(u^{-\epsilon},u^{\epsilon})\,du\,ds$$
$$+\int_{-\infty}^{0}\int_{t-s}^{\infty}t^{1+\kappa}(2+|s|)\,u^{-\delta}\max(u^{-\epsilon},u^{\epsilon})\,du\,ds$$

which is finite when we choose ϵ as in (3.11). On the other hand, for $0<u<n_0/n$, we use Markov's inequality as before to get

$$\mathbb{P}\{U>nu\}\,\frac{n^{\delta}}{h(n)}\le \mathbb{E}U\,n_0^{\delta+\epsilon-1}u^{-\delta-\epsilon}.$$

Then the finiteness of the integrals in (3.21) is sufficient again for the integrability of a dominating function for $0<u<n_0/n<1$ which complements (3.19). It follows from dominated convergence theorem that the limit of the right hand side of (3.18) exists. Therefore, possibly for $n\ge n_1$ for some $n_1\in\mathbb{N}$, the upper bound in (3.18) is further bounded by a multiple of its limit given by

$$C_1\,\mathbb{E}|R|^{1+\kappa}\int_0^{\infty}\int_{-\infty}^{t}u^{1+\kappa}\left|f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right|^{1+\kappa}\lambda\,u^{-\delta-1}\,du\,ds \quad (3.22)$$

for some $C_1>2$. In view of the proof of Lemma 2.1, the integral in (3.22) is bounded by a constant multiple of $t^{2+\kappa-\delta}$ which clearly dominates $\mathbb{E}|Z_n(t)-\mathbb{E}Z_n(t)|^{1+\kappa}$ in (3.18) for sufficiently large n. Since the increments of $\{Z_n(t)-\mathbb{E}Z_n(t):t\ge 0\}$ are stationary, this implies that

$$\begin{aligned}\mathbb{E}\left[|Z_n(t_2)-\mathbb{E}Z_n(t)|^{\frac{1+\kappa}{2}}\,|Z_n(t)-\mathbb{E}Z_n(t_1)|^{\frac{1+\kappa}{2}}\right] &\le C_2\,(t_2-t)^{\frac{2+\kappa-\delta}{2}}(t-t_1)^{\frac{2+\kappa-\delta}{2}}\\ &\le C_2\,(t_2-t_1)^{2+\kappa-\delta} \quad (3.23)\end{aligned}$$

for $0<t_1<t<t_2$ and some $C_2>0$, by Cauchy-Schwarz inequality and the assumption that $\delta<1+\kappa$. This concludes the proof by [7, Thm.13.5 and Eqn.(13.14)] as $2+\kappa-\delta>1$. □

4. Fractional Brownian Motion Limit

In this section, we scale the shot-noise process as follows to approximate a fBm in the limit. Recall that fBm with Hurst parameter $0<H<1$ is a mean zero Gaussian process B^H on $\mathbb{R}_+$ with $B^H(0)=0$ and covariance

$$\mathrm{Cov}(B^H(t_1),B^H(t_2))=\frac{\sigma^2}{2}\left(|t_1|^{2H}+|t_2|^{2H}-|t_1-t_2|^{2H}\right)\qquad t_1,t_2\ge 0$$

for $t_1,t_2\ge 0$, $\sigma>0$ [29]. Let R be scaled as R/n which can be interpreted as a decrease in the effect of the pulse in absolute value as n increases. On the other hand, we will let the arrival rate λ increase with a factor which is a function of $n\in\mathbb{Z}_+$. In the following theorems, we prove convergence of $Z_n-\mathbb{E}Z_n$ to fBm with a properly scaled measure ν_n for a finite measure ν as in (1.2), and with ν as in (1.7) with no scaling.

Theorem 4.1. *Suppose that ν is a probability measure with a regularly varying tail as given in (1.2), the function $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+$ with $f(x)=0$ for all $x<0$, $f(x)=f(1)$ for all $x\geq 1$ and f' satisfying a Lipschitz condition a.e. on $(0,1)$, and $\mathbb{E}R^2<\infty$. Let*

$$Z_n(t)=\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{r}{n}\,u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\,N_n(ds,du,dr)$$

and

$$\mu_n(ds,du,dr)=\frac{n^{2+\delta}}{h(n)}\lambda\,ds\,\nu_n(du)\,\gamma(dr),$$

where $\nu_n(du)=\nu(n\,du)$ and $1<\delta<2$. Then the process $\{Z_n(t)-\mathbb{E}Z_n(t),t\geq 0\}$ converges in the Skorohod topology on $D(0,\infty)$ to an fBm with Hurst parameter $H=(3-\delta)/2$ and variance parameter

$$\sigma^2=\lambda\,\mathbb{E}R^2\int_{-\infty}^{\infty}\int_0^{\infty}\left[f\left(\frac{1-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2u^{1-\delta}du\,ds$$

as $n\to\infty$.

Proof. The same approach will be followed as in the proof of Theorem 3.1. The characteristic function $\mathbb{E}\exp i\sum_{k=1}^m\xi_k[Z_n(t_k)-\mathbb{E}Z_n(t_k)]$ for $\xi_k\in\mathbb{R}$, $t_k\geq 0$ and $m\in\mathbb{N}$ is given by

$$\exp\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{r}{n}u\sum_{k=1}^m\xi_k\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right)\cdot\frac{n^{2+\delta}}{h(n)}\,\lambda\,ds\,\nu_n(du)\,\gamma(dr)\;. \tag{4.1}$$

By integration by parts, we find that the exponent of (4.1) is given by

$$\int\int\int\partial_u g(s,u,r/n)\,\mathbb{P}\{U>nu\}\,\frac{n^{2+\delta}}{h(n)}\,\lambda\,ds\,du\,\gamma(dr), \tag{4.2}$$

where g is as in (3.2). Using Potter bounds [8] and Lipschitz conditions on f and f', we get an inequality similar to (3.8) for $u\geq n_0/n$ given by

$$\begin{aligned}\int\int_{n_0/n}^{\infty}\int|\partial_u g(s,u,r/n)|\,\mathbb{P}\{U>nu\}\,\frac{n^{2+\delta}}{h(n)}\,\lambda\,ds\,du\,\gamma(dr)\\ \leq 4M^2(C+\epsilon)\sum_k|\xi_k|\int\int_0^{\infty}\int|r|\,\tilde{B}(s,u,t_k)\max(u^{-\epsilon},u^{\epsilon})\,u^{-\delta}\,\lambda\,ds\,du\,\gamma(dr),\end{aligned} \tag{4.3}$$

where $\tilde{B}$ is similar to (3.9) but with $\kappa=1$ by hypothesis, and $\epsilon>0$ and $n_0\in\mathbb{N}$. If we choose $\epsilon>0$ such that $1<\delta-\epsilon<\delta<\delta+\epsilon<2$, then the right hand side of (4.3) is finite along the same lines of the proof of Theorem 3.1 with $\kappa=1$. On the other hand, we can bound (4.2) for $0<u\leq 1$ similarly. Therefore, we can use dominated convergence theorem. We have the limit in (3.17), and

$$\lim_{n\to\infty}n^2\partial_u g(s,u,r/n)=\partial_u\lim_{n\to\infty}n^2 g(s,u,r/n)$$

as g is bounded, hence, uniformly continuous. Then we get

$$\lim_{n\to\infty} n^2 g(s,u,r/n) = -\frac{r^2}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\xi_j\xi_k u^2\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right] \quad (4.4)$$

since $|e^{ix}-1-ix+x^2/2|$ is $o(x^3)$ [6, Eq.(26.4$_2$)]. We now revert (4.2) after taking the limits above, by another integration by parts, and get the limit of (4.1) as

$$\exp\Big\{-\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} r^2\sum_{k=1}^{m}\sum_{j=1}^{m}\xi_j\xi_k \left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\, u^2\lambda\, ds\, u^{-\delta-1}\, du\, \gamma(dr)\Big\}$$

which is the characteristic function of $\sum_{k=1}^m \xi_k Z(t_k)$, where $Z=(Z(t_1),\ldots,Z(t_m))$ is a Gaussian vector with zero mean and covariance

$$\lambda\mathbb{E}R^2\int_0^\infty\int_{-\infty}^{\infty}\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\, ds\, u^{1-\delta}\, du. \quad (4.5)$$

When (4.5) is evaluated at $t_j=t_k=1$, the variance coefficient σ^2 is found. Using the identity $2ab=-(a-b)^2+a^2+b^2$ for $a,b\in\mathbb{R}$ and making several change of variables, we find that the covariance of Z in (4.5) is given by

$$\mathrm{Cov}(Z(t_j),Z(t_k))=\frac{\sigma^2}{2}\,(t_j^{2H}+t_k^{2H}-|t_j-t_k|^{2H})$$

for $t_j,t_k\geq 0$ with $H=(3-\delta)/2$. By definition, Z has the characteristic function of an fBm.

Convergence in the Skorohod topology on $D(0,\infty)$ follows along the same lines of proof of Theorem 3.1. In this case, we have

$$\mathbb{E}|Z_n(t)-\mathbb{E}Z_n(t)|^2=\mathbb{E}R^2\int_0^\infty\int_{-\infty}^{t}u^2\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2\lambda\, ds\, \nu_n(du)$$

and (3.23) holds with $\kappa=1$. □

Remark 4.2. General self-similar Gaussian random fields on $\mathbb{R}^d$ are approximated in [25, Thm.2.1] in a similar fashion to Theorem 4.1. We do not assume finiteness of the third moment of R. Therefore, the proof in [25, pg.1121] does not work for our model and the integration by parts technique proves to be useful with Lipschitz continuity assumptions.

When f is continuous on $\mathbb{R}$, with no discontinuity at 0, the notation can be aligned with random balls of [5, Lem.2.3] and we can write

$$\mathrm{Cov}(Z(t_j),Z(t_k))=\lambda\mathbb{E}R^2\int_0^\infty u^{-\delta}V(1/u)\,du\int_0^{t_j}\int_0^{t_k}|y-y'|^{1-\delta}dy\,dy',$$

where $V(x)=\int_{-\infty}^{\infty}f'(s)f'(s+x)\,ds$.

The next theorem is a simpler version of Theorem 4.1 due to the form (1.7) of the measure ν. Note that (2.6) can be approximated as

$$\mu_n(d(ns), d(nu), dr) \sim n^{2+\delta}\, ds\, n^{-\delta} u^{-\delta-1} du\, \gamma(dr) = n^2 u^{-\delta-1}\, ds\, du\, \gamma(dr)$$

for large n. It can be interpreted as half way in taking the more involved limit of Theorem 4.1.

Theorem 4.3. *Let*

$$\tilde{Z}_n(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} \frac{r}{n}\, u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right] \tilde{N}_n(ds, du, dr),$$

where $\tilde{N}_n = N_n - \mu_n$ *and*

$$\mu_n(ds, du, dr) = n^2 \lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr).$$

Suppose that $\mathbb{E}R^2 < \infty$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous function on* $(0, \infty)$ *satisfying either of the following conditions*

(i) $f(x) = 0$ *for all* $x < 0$ *and* $f(x) = f(1)$ *for all* $x \geq 1$*, or*
(ii) f *has compact support.*

Then the process $\{\tilde{Z}_n(t), t \geq 0\}$*, for* $1 < \delta < 3$*, converges in the Skorohod topology on* $D(0, \infty)$ *to an fBm with Hurst parameter* $H = (3-\delta)/2$ *and variance parameter*

$$\sigma^2 = \lambda\, \mathbb{E}R^2 \int_{-\infty}^{\infty}\int_0^{\infty} \left[f\left(\frac{1-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]^2 u^{1-\delta} du\, ds$$

as $n \to \infty$.

Proof. For the convergence of finite dimensional distributions of $\{\tilde{Z}_n(t), t \geq 0\}$, consider the characteristic function $\mathbb{E}\exp i\sum_{k=1}^m \xi_k \tilde{Z}_n(t_k)$ for $\xi_k \in \mathbb{R}$, $t_k \geq 0$ and $m \in \mathbb{N}$. It is given by

$$\exp \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} g(s, u, r/n) n^2 \lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr), \tag{4.6}$$

where g is given in (3.2). Note that the characteristic function exists since $\tilde{Z}_n(t_k)$ are well defined in view of (2.3) which follows from Lemma 2.1 with $\kappa = 1$ under assumption i, and by (2.2) under assumption ii. As $n \to \infty$, we will show that the above characteristic function converges to

$$\exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} r^2 \sum_{k=1}^m\sum_{j=1}^m \xi_j\xi_k \right. \tag{4.7}$$
$$\left. \left[f\left(\frac{t_j-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right) - f\left(\frac{-s}{u}\right)\right] u^2\lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr)\right\}$$

Due to the inequality $|e^{ix} - 1 - ix| < \frac{1}{2}x^2$ for $x \in \mathbb{R}$, the integrand in (4.7) is an upper bound to $|g(s, u, r/n)|\, n^2$. Therefore, dominated convergence theorem allows us to take the limit inside the integral in (4.6). Then (4.6) converges to (4.7) as $n \to \infty$ by the continuity of the exponential function as in the proof of Theorem 4.1.

To complete the proof, we need to show convergence in $D(0,\infty)$ with Skorohod topology. This is straight forward since the variance of $\tilde{Z}_n(t)$ is already free of n and is bounded by a constant multiple of $t^{3-\delta}$ by the proof of Lemma 2.1. $\square$

Remark 4.4. Theorem 4.3 with condition ii. is [12, Thm.3.1] where it is noted that fBm with $H > 1/2$ can be approximated if the pulse is continuous and has compact support. Condition i. above considers a pulse which is continuous, but with no compact support as an alternative.

5. Lévy Process Limit

A process with stationary and independent increments is called a Lévy process [3, 24]. The results of this section concerns a particular class of Lévy processes, namely stable Lévy motion [29]. Let $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, and let $\delta \in (1,2)$, and $\beta \in [-1,1]$ be the index of stability and skewness parameter, respectively. Then a δ-stable Lévy motion L with mean 0 can be defined through its characteristic function

$$\mathbb{E}e^{i\xi L(t)} = \exp\{-t\,\sigma^\delta\,|\xi|^\delta[1 - i\beta(\operatorname{sign}\xi)\tan(\pi\delta/2)]\}$$

for $\xi \in \mathbb{R}$, where $\sigma \geq 0$ is a scale parameter. We prove that the limiting process is a δ-stable Lévy motion when we have a smaller arrival rate than those that yield an fBm. Theorem 5.2 considers a probability measure ν and Theorem 5.5 starts with its limiting form. For simplicity of notation, we take $f(1) = 1$ for the pulse f.

Lemma 5.1. *Let N be a Poisson random measure with mean measure*

$$\mu = \lambda\, ds\, u^{-\delta-1} du\, \gamma(dr)$$

and $\tilde{N} = N - \mu$. Then

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^t r\, u\, \tilde{N}(ds,du,dr) \stackrel{d}{=} (\lambda\, C_1)^{1/\delta}\, L_1(t) + (\lambda\, C_2)^{1/\delta}\, L_2(t),$$

where $C_1 = \int_0^\infty r^\delta \gamma(dr)$, $C_2 = \int_{-\infty}^0 |r|^\delta \gamma(dr)$, and L_1 and L_2 are independent δ-stable Lévy motions with mean 0, skewness parameter β equal to 1 and -1, respectively, and scale parameter

$$\sigma = \left[-\frac{2\Gamma(2-\delta)}{\delta(\delta-1)}\cos\frac{\pi\delta}{2}\right]^{1/\delta}.$$

Proof. Putting $u' = |r|u$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^t r\,u\,\tilde{N}(ds,du,dr) \\
&\qquad = \int_0^{\infty}\int_0^{\infty}\int_0^t u'\,[N(ds,d(u'/r),dr) - \lambda\,ds\,u'^{-\delta-1}r^{\delta}\,du'\,\gamma(dr)] \\
&\qquad\quad - \int_{-\infty}^{0}\int_0^{\infty}\int_0^t u'\,[N(ds,d(u'/|r|),dr) - \lambda\,ds\,u'^{-\delta-1}|r|^{\delta}\,du'\,\gamma(dr)] \\
&\qquad =: \int_0^{\infty}\int_0^t u'\,[N_1'(ds,du') - C_1\lambda\,ds\,u'^{-\delta-1}\,du'] \qquad (5.1)\\
&\qquad\quad - \int_0^{\infty}\int_0^t u'\,[N_2'(ds,du') - C_2\lambda\,ds\,u'^{-\delta-1}\,du'],
\end{aligned}$$

where $C_1 := \int_0^{\infty} r^{\delta}\gamma(dr)$, $C_2 := \int_{-\infty}^{0}|r|^{\delta}\gamma(dr)$, and N_1' and N_2' are defined as transformations of N over positive and negative half lines, respectively [11]. They are also Poisson random measures with means $C_i\lambda ds\,u'^{-\delta-1}\,du'$, $i=1,2$, and are independent as their domains are disjoint. Making another change of variable $u = u'/(\lambda C_i)^{1/\delta}$ for $i=1,2$ in respective integrals in (5.1), we get

$$\begin{aligned}
&(\lambda C_1)^{1/\delta}\int_0^{\infty}\int_0^t u\,[N_1'(ds,d((\lambda C_1)^{1/\delta}u)) - ds\,u^{-\delta-1}\,du] \\
&\qquad -(\lambda C_2)^{1/\delta}\int_0^{\infty}\int_0^t u\,[N_2'(ds,d((\lambda C_2)^{1/\delta}u)) - ds\,u^{-\delta-1}\,du] \\
&\qquad\qquad =: (\lambda C_1)^{1/\delta}L_1'(t) - (\lambda C_2)^{1/\delta}L_2'(t), \qquad (5.2)
\end{aligned}$$

where L_1' and L_2' are independent δ-stable Lévy motions with skewness parameter $\beta = 1$ and scale parameter σ [29, pg.s 5,156]. Now, we have

$$-L_2' \overset{d}{=} L_2 := \int_{-\infty}^{0}\int_0^t u\,[N_2'(ds,d(-(\lambda C_2)^{1/\delta}u)) - ds\,|u|^{-\delta-1}\,du],$$

where L_2 is also a δ-stable Lévy motion since $N_2''(ds,du) := N_2'(ds,d(-(\lambda C_2)^{1/\delta}u))$ is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}_-$ with mean measure $ds\,|u|^{-\delta-1}\,du$, but skewness parameter $\beta = -1$ [29, pg.5]. We take $L_1 = L_1'$ and the result follows. □

Theorem 5.2. *Suppose that ν is a probability measure with a regularly varying tail as given in (1.2), the function $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+$ with $f(x) = 0$ for all $x < 0$, $f(x) = f(1)$ for all $x \geq 1$ and f' satisfying a Lipschitz condition a.e. on $(0,1)$, and $\mathbb{E}|R|^{1+\kappa} < \infty$ for some $0 < \kappa \leq 1$ such that $1+\kappa > \delta > 1$. Let*

$$Z_n(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} n^{1-\alpha/\delta} r\,u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dr)$$

and

$$\mu_n(ds,du,dr) = \frac{n^{\alpha}}{h(n^{\alpha/\delta})}\lambda\,ds\,\nu_n(du)\,\gamma(dr),$$

where $\nu_n(du) = \nu(n\,du)$ *and* $0 < \alpha < \delta$. *Then the process* $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$, *for* $1 < \delta < 2$, *converges in the Skorohod topology on* $D(0, \infty)$ *to*

$$(\lambda\,\mathbb{E}R^\delta 1_{\{R>0\}})^{1/\delta}\,L_1(t) + (\lambda\,\mathbb{E}|R|^\delta 1_{\{R<0\}})^{1/\delta}\,L_2(t)$$

as $n \to \infty$, *where* L_1 *and* L_2 *are independent* δ*-stable Lévy motions with mean 0, and skewness parameter* 1 *and* -1, *respectively.*

Proof. The idea is the same as in earlier proofs, so we indicate only the differences in details. Applying integration by parts in the characteristic function of the scaled and centered process and making a change of variable u to $u/n^{1-\alpha/\delta}$, we get

$$\exp \int\int\int \frac{1}{n^{1-\alpha/\delta}}\,(\partial_u g)(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta}r)\,\mathbb{P}\{U > n^{\alpha/\delta}u\} \cdot \frac{n^\alpha}{h(n^{\alpha/\delta})}\,\lambda\,ds\,du\,\gamma(dr)\,. \tag{5.3}$$

Using Potter bounds and Lipschitz conditions on f and f', we get an inequality similar to (3.8). We can bound $\mathbb{P}\{U > n^{\alpha/\delta}u\}\,n^\alpha/h(n^{\alpha/\delta})$ as in (3.4) and consider $|(\partial_u g)(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta}r)|/n^{1-\alpha/\delta}$ separately. As a result, for fixed $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all n with $n^{\alpha/\delta} \geq n_0$, we have the following upper bound for the absolute value of (5.3) when evaluated over $u \geq n_0/n^{\alpha/\delta}$

$$4M(C+\epsilon)\sum_k |\xi_k| \int\int_0^\infty\int |r|\,B'(s,u,t_k,n)\max(u^{-\epsilon}, u^\epsilon)\,u^{-\delta}\,\lambda\,ds\,du\,\gamma(dr),$$

where B' is analogous to (3.9) satisfying

$$B'(s,u,t_k,n) \leq 1_{R_{1,k,n}} + 2\left(1 \wedge \sum_j M|\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}}\right) . 1_{R_{2,k,n}} + 4\,.\,1_{R_{3,k,n}} + \left(2\frac{t_k}{u/n^{1-\alpha/\delta}} + \frac{|s|t_k}{u^2/n^{2(1-\alpha/\delta)}}\right) 1_{R_{4,k,n}} \tag{5.4}$$

and $R_{1,k,n} \ldots, R_{4,k,n}$ are analogous to $R_{1,k}, \ldots, R_{4,k}$ with u replaced by $u/n^{1-\alpha/\delta}$. The right hand side of (5.4) is integrable with respect to $\max(u^{-\epsilon}, u^\epsilon)u^{-\delta}\,du\,ds$ when ϵ is chosen as in (3.11) as shown next. Substituting the limits of integration in regions $R_{1,k,n}, R_{3,k,n}, R_{4,k,n}$ shown by I_1, I_3, I_4, respectively, we have I_1, I_3, I_4 are finite for $1 < \delta - \epsilon < \delta < \delta + \epsilon < 2$ since

$$\begin{aligned}
\int_{-\infty}^0 \int_{-\tilde{n}s}^{\tilde{n}t_k - \tilde{n}s} u^{-\tilde{\delta}}\,du\,ds &\leq C_1\,t_k^{2-\tilde{\delta}} \\
\int_0^{t_k} \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} u^{-\tilde{\delta}}\,du\,ds &\leq C_2\,t_k^{2-\tilde{\delta}} \\
\int_{-\infty}^0 \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} \left(\frac{2}{u/\tilde{n}} + \frac{|s|}{u^2/\tilde{n}^2}\right) u^{-\tilde{\delta}}\,du\,ds &\leq C_3(t_k^{2-\tilde{\delta}} + t_k^{-\tilde{\delta}} + t_k^{1-\tilde{\delta}})
\end{aligned}$$

for $1 < \tilde{\delta} < 2$ and $\tilde{n} := n^{1-\alpha/\delta} \geq 1$, where $C_1, C_2, C_3 \in \mathbb{R}$. In $R_{2,k,n}$, we have

$$I_2 = \int_0^{t_k} \int_0^{\tilde{n}t - \tilde{n}s} \left(1 \wedge \sum_j M|\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}}\right) \max(u^{-\epsilon}, u^\epsilon)\,u^{-\delta}\,du\,ds\,.$$

As in the proof of Theorem 3.1, we consider two intervals $[0,\bar{u}]$ and $(\bar{u},t_k]$ to evaluate this integral. Over the first interval, it is finite for $1<\tilde{\delta}<1+\kappa$, and over the latter, it is proportional to $\tilde{n}^{1-\tilde{\delta}}$ which is bounded by 1. As a result, the right hand side of (5.4) is integrable if we choose $\epsilon>0$ as in (3.11).

For $u<n_0/n^{\alpha/\delta}$, we can find a dominating function for the integrand in (5.3) using Markov's inequality. As in the proof of Theorem 3.1, we have

$$\begin{aligned}\frac{1}{n^{1-\alpha/\delta}}|(\partial_u g)(s,u/n^{1-\alpha/\delta},n^{1-\alpha/\delta}r)|\,\mathbb{P}\{U>n^{\alpha/\delta}u\}\,\frac{n^{\alpha}}{h(n^{\alpha/\delta})}\\ \le 2M\frac{\mathbb{E}U}{u}\frac{n^{\alpha-\alpha/\delta}}{h(n^{\alpha/\delta})}\,|r|\sum_k|\xi_k|B'(s,u,t_k,n),\end{aligned} \tag{5.5}$$

where B' satisfies (5.4). Using (5.4) and using an inequality similar to (3.16) in view of the assumption $u<n_0/n^{\alpha/\delta}$, we can increase the right hand side of (5.5) as

$$\begin{aligned}2M\mathbb{E}U\,n_0^{\delta+\epsilon-1}\,|r|u^{-\delta-\epsilon}\sum_k|\xi_k|\;[1_{R_{1,k,n}}+2\,.\,1_{R_{2,k,n}}\sum_j M|\xi_j|^{\kappa}|r|^{\kappa}u^{\kappa}1_{\{s\le t_j\}}\\ +4\,.\,1_{R_{3,k,n}}+\left(2\frac{t_k}{u/n^{1-\alpha/\delta}}+\frac{|s|t_k}{u^2/n^{2(1-\alpha/\delta)}}\right)1_{R_{4,k,n}}]\end{aligned} \tag{5.6}$$

which is integrable over $0<u<1$ as shown for (5.4) above.

We can now use the dominated convergence theorem. Note that

$$\lim_{n\to\infty}\frac{1}{n^{1-\alpha/\delta}}\,(\partial_u g)(s,u/n^{1-\alpha/\delta},n^{1-\alpha/\delta}r)=\partial_u\lim_{n\to\infty}g(s,u/n^{1-\alpha/\delta},n^{1-\alpha/\delta}r), \tag{5.7}$$

where

$$g(s,u/n^{1-\alpha/\delta},n^{1-\alpha/\delta}r)=\Phi\left(\sum_{k=1}^{m}\xi_k ru\Big[f\Big(\frac{t_k-s}{u/n^{1-\alpha/\delta}}\Big)-f\Big(\frac{-s}{u/n^{1-\alpha/\delta}}\Big)\Big]\right) \tag{5.8}$$

and we have

$$\lim_{n\to\infty}\left[f\left(\frac{t_k-s}{u/n^{1-\alpha/\delta}}\right)-f\left(\frac{-s}{u/n^{1-\alpha/\delta}}\right)\right]=1_{\{0<s<t_k\}}(s) \tag{5.9}$$

To see (5.9), one takes the limit in regions $R_{1,k},\dots,R_{4,k}$, separately. Fig.2 illustrates the function $\tilde{f}(\cdot):=f(\frac{\cdot-s}{u/n^{1-\alpha/\delta}})$ over these regions where we consider $\tilde{f}(t_k)-\tilde{f}(0)$ as $n\to\infty$. By (3.17), (5.7), (5.8) and (5.9), we take the limit of (5.3) and then revert the integration by parts to get the limiting characteristic function of $\sum_{k=1}^m\xi_k Z(t_k)$, where

$$Z(t)=\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}r\,u\,1_{\{0<s<t\}}\,\tilde{N}'(ds,du,dr)$$

for a Poisson random measure N' with mean measure $\mu'=\lambda dsu^{-\delta-1}du\gamma(dr)$. This characterizes the limiting process by Lemma 5.1.

To complete the proof of weak convergence, it is sufficient to show that $\mathbb{E}|Z_n(t)-\mathbb{E}Z_n(t)|^{1+\kappa}\le Ct^b$ for some $b>1$ and $C>0$ in view of the proof of Theorem 3.1.

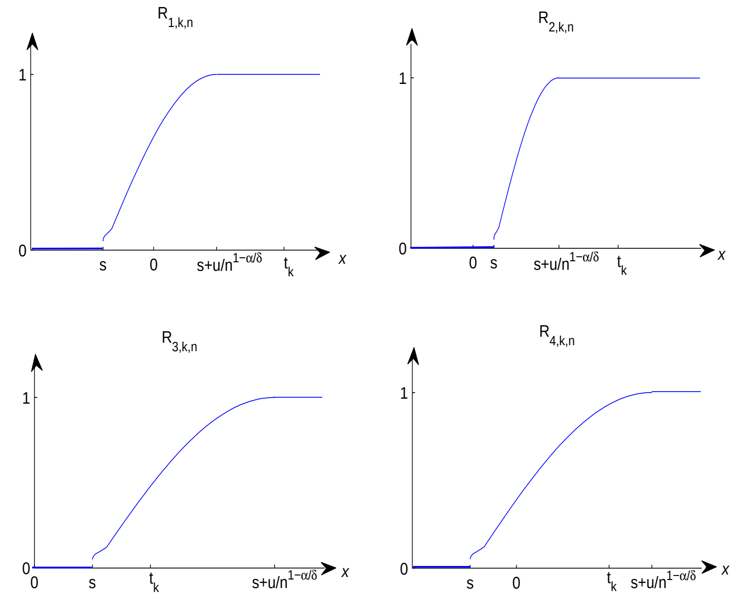

FIGURE 2. Sketch of an example $\tilde{f}(x) := f\left(\frac{x-s}{u/n^{1-\alpha/\delta}}\right)$ in different subregions for $(s, u/n^{1-\alpha/\delta})$.

In the present theorem, we need a finer estimate given in [30, Lemma 2] and used in [19, Lemma 6]. We have

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq a \int_0^\infty \left(1 - e^{-2I_n}\right) \xi^{-2-\kappa} d\xi, \tag{5.10}$$

where

$$I_n = \int\int\int \left(1 - \cos\left(\xi\, n^{1-\alpha/\delta} ru \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\right)\right) \mu_n(ds, du, dr)$$

and $a = (\int_0^\infty (1-\cos x) x^{-2-\kappa} dx)^{-1}$, which is finite with $0 < \kappa \leq 1$. Substituting μ_n and applying integration by parts, we get

$$I_n = \int\int\int \partial_u k(s, u, n^{1-\alpha/\delta} r)\, \mathbb{P}\{U > nu\} \frac{n^\alpha}{h(n^{\alpha/\delta})}\, \lambda\, ds\, du\, \gamma(dr), \tag{5.11}$$

where

$$k(s, u, r) = 1 - \cos\left(\xi\, ru \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\right). \tag{5.12}$$

For latter use, the partial derivative of k in u is found as

$$\partial_u k(s,u,r) = \sin\left(\xi ur\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\right)$$
$$\cdot\left[\xi r\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right] + \xi r\left[f'\left(\frac{t-s}{u}\right)\frac{t-s}{u} - f'\left(\frac{-s}{u}\right)\frac{-s}{u}\right]\right].$$

Making a change of variable u to $u/n^{1-\alpha/\delta}$ in (5.11), we find that I_n is equal to

$$\iiint \frac{1}{n^{1-\alpha/\delta}}\,(\partial_u k)(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r)\,\mathbb{P}\{U > n^{\alpha/\delta} u\}\,\frac{n^\alpha}{h(n^{\alpha/\delta})}\,\lambda\,ds\,du\,\gamma(dr).$$

Note the similarity of I_n to (5.3). Moreover, the inequality $\sin x \le 2^{1-\kappa}|x|^\kappa \wedge 2$ holds since $\sin x = [e^{ix} - 1 + (e^{ix}+1)]/2$ leading to estimates as in (3.6) and (3.7). It follows that

$$4M(C+\epsilon)\xi\int\int_0^\infty\int |r|\,B(s,u,t,n)\max(u^{-\epsilon}, u^{\epsilon})\,u^{-\delta}\,\lambda\,ds\,du\,\gamma(dr)$$

is an upper bound to $|I_n|$ when it is evaluated over $u \ge n_0/n^{\alpha/\delta}$ where ϵ and n_0 are as above and

$$B(s,u,t,n) \le 1_{R_{1,k,n}} + 2\,(1 \wedge M|\xi|^\kappa |r|^\kappa u^\kappa)\,1_{R_{2,k,n}}$$
$$+ 4\,.\,1_{R_{3,k,n}} + \left(2\frac{t}{u/n^{1-\alpha/\delta}} + \frac{|s|t}{u^2/n^{2(1-\alpha/\delta)}}\right)1_{R_{4,k,n}}.$$

For evaluating $|I_n|$ for smaller values of u, we have a bound similar to (5.6). Therefore, I_n is bounded by an integrable function uniformly over n by similar computations. By dominated convergence theorem, let $I = \lim_n I_n$. We find that

$$I = \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty [1 - \cos(\xi ur 1_{\{0<s<t\}})]\,u^{-\delta-1}\lambda\,ds\,du\,\gamma(dr)$$

by using the same approach for taking the limit of the characteristic function of the finite dimensional distributions above. Then we can write

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \le a\int_0^\infty \left(1 - e^{-4I}\right)\xi^{-2-\kappa}d\xi \tag{5.13}$$

for sufficiently large n, by (5.10), since $1 - e^{-x}$ is increasing in x. Simplifying I further, we have

$$I = \lambda t\int_{-\infty}^\infty\int_0^\infty [1-\cos(\xi ur)]\,u^{-\delta-1}du\,\gamma(dr) = \lambda\,t\,\xi^\delta\mathbb{E}|R|^\delta\int_0^\infty (1-\cos u)\,u^{-\delta-1}du,$$

where the second equality follows by a change of variable u to $u/(\xi|r|)$. Define the constant $\tilde{C}$ so that $I =: \tilde{C}t\xi^\delta$. Now, substituting I in (5.13) and changing ξ to $\xi/t^{1/\delta}$, we get

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \le a\,t^{\frac{1+\kappa}{\delta}}\int_0^\infty \left(1 - e^{-4\tilde{C}\xi^\delta}\right)\xi^{-2-\kappa}d\xi$$

which concludes the proof as $(1+\kappa)/\delta > 1$. □

Note that the stable process obtained in the limit is stable with a skewness parameter that depends on the distribution of R, which we have reserved as an extra random variable for applications in addition to U and S. Moreover, it has stationary and independent increments. Therefore, it is also a δ-stable Lévy motion [29, Def.7.5.1], but with scale parameter $\sigma\lambda^{1/\delta}(C_1+C_2)^{1/\delta}$ and skewness parameter $\beta = (C_1 - C_2)/(C_1 + C_2)$ by [29, pg.s 10,11], where $C_1 = \mathbb{E}R^\delta 1_{\{R>0\}}$ and $C_2 = \mathbb{E}|R|^\delta 1_{\{R<0\}}$, see also [3, pg.217].

Remark 5.3. The weak convergence result given in Theorem 5.2 is proved with Skorohod's J_1 topology. In [19], only finite dimensional distributions have been considered for a stable limit with a positive linear pulse. Its weak convergence is proved in [25] with M_1 topology on the basis that the approximating process is continuous, but the limiting process has jumps. We allow for jumps in the pulse f at 0, and hence in the shot-noise process. Therefore, weak convergence to stable Lévy process in J_1 topology is proved. The convergence is shown with M_1 topology instead of J_1 in [28] where the pulse is assumed to be monotone increasing in the context of workload input to the system. M_1 topology is considered also in [17].

Remark 5.4. Stable limits can be proved using more general theorems, e.g. as given in [16, Ch.VIII]. Conditions are formulated in [17, 21] for Poisson shot-noise processes to get a stable limit. In [21], finite dimensional distributions are considered with several examples. In [17], weak convergence is also included. Theorem 5.2 is not a special case since only finite dimensional convergence is shown in [17, Thm.5] when the centering term is the mean of the process as in the present work.

The following theorem is based on the simpler form of the mean measure.

Theorem 5.5. *Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}_+$ with $f(x) = 0$ for all $x < 0$, $f(x) = f(1) = 1$ for all $x \geq 1$ and is also differentiable with f' satisfying a Lipschitz condition a.e., and $\mathbb{E}|R|^{1+\kappa} < \infty$ for some $0 < \kappa \leq 1$ with $1 + \kappa > \delta$. Let*

$$\tilde{Z}_n(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} nr\, u \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right] \tilde{N}_n(ds, du, dr),$$

where $\tilde{N}_n = N_n - \mu_n$ and

$$\mu_n(ds, du, dr) = n^{-\delta}\lambda\, u^{-\delta-1}\, ds\, du\, \gamma(dr).$$

Then the process $\{\tilde{Z}_n(t), t \geq 0\}$, for $1 < \delta < 2$, converges in Skorohod topology on $D(0,\infty)$ to

$$(\lambda\, \mathbb{E}R^\delta 1_{\{R>0\}})^{1/\delta}\, L_1(t) + (\lambda\, \mathbb{E}|R|^\delta 1_{\{R<0\}})^{1/\delta}\, L_2(t)$$

as $n \to \infty$, where L_1 and L_2 are independent δ-stable Lévy motions with mean 0, and skewness parameter 1 and -1, respectively.

Proof. We will give only a sketch of the proof due to its similarities with the previous theorem. The characteristic function for the finite dimensional distributions of $\tilde{Z}$ can be written as

$$\exp \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} g(s, u, nr)\, \lambda\, n^{-\delta}\, u^{-\delta-1} ds\, du\, \gamma(dr)$$

with g as in (3.2). Making a change of variable u to u/n, we get

$$\exp \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} g(s, u/n, nr)\, \lambda\, u^{-\delta-1} ds\, du\, \gamma(dr) \tag{5.14}$$

Now, $g(s, u/n, nr)$ is similar to (5.8) and we take a similar limit to (5.9) with $n^{1-\gamma/\delta}$ replaced by n. This is justified by dominated convergence theorem since the integrand in (5.14) can be bounded as in the proof of Theorem 5.2. Convergence in $D(0, \infty)$ follows along the same lines, this time with $k(s, u/n, nr)$ in k of (5.12). $\square$

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