Communications on Stochastic Analysis

Volume 9 | Number 4 Article 3

12-1-2015

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Communications on Stochastic Analysis
Vol. 9, No. 4 (2015) 467-476

Serials Publications
www.serialspublications.com

DOOB'S DECOMPOSITION THEOREM FOR NEAR-SUBMARTINGALES

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Abstract. We study the discrete parameter case of near-martingales, near-submartingales, and near-supermartingales. In particular, we prove Doob's decomposition theorem for near-submartingales. This generalizes the classical case for submartingales.

1. Motivation From Non-adapted Stochastic Integral

Let $B(t)$, $t \geq 0$, be a Brownian motion starting at 0 and $\{\mathcal{F}_t\}$ the filtration given by $B(t)$, namely, $\mathcal{F}_t = \sigma\{B(s);\, 0 \leq s \leq t\}$, $t \geq 0$. The Itô integral $\int_a^b f(t)\, dB(t)$ (see, e.g., the book [8]) is defined for $\{\mathcal{F}_t\}$-adapted stochastic processes $f(t)$ with almost all sample paths being in $L^2[a, b]$. Several extensions of the Itô theory of stochastic integration to cover non-adapted integrands have been introduced and extensively studied by, just to mention a few names, Buckdahn [3], Dorogovtsev [4], Hitsuda [5], Itô [6], Kuo–Potthoff [10], León–Protter [12], Nualart–Pardoux [13], Pardoux–Protter [14], Russo–Vallois [15], and Skorokhod [16].

In particular, in his lecture for the 1976 Kyoto Symposium, Itô [6] gave rather elegant ideas to define the following non-adaptive stochastic integral

$$(\mathrm{I}) \int_0^t B(1)\, dB(s), \quad 0 \leq t \leq 1, \tag{1.1}$$

namely, enlarging the σ-field $\mathcal{F}_t$ to $\mathcal{G}_t = \sigma\{B(1), B(s);\, 0 \leq s \leq t\}$, $0 \leq t \leq 1$, so that the integrand $B(1)$ is adaptive and $B(t)$ is a quasimartingale with respect to the filtration $\{\mathcal{G}_t\}$. Then the stochastic integral in equation (1.1) is defined as a stochastic integral with respect to a quasimartingale and has the value

$$(\mathrm{I}) \int_0^t B(1)\, dB(s) = B(1)B(t), \quad 0 \leq t \leq 1. \tag{1.2}$$

On the other hand, the Hitsuda–Skorokhod integral (see [5] [16]) can be expressed in terms of a white noise integral (see the book [7]) and has the value

$$(\mathrm{HS}) \int_0^t B(1)\, dB(s) = \int_0^t \partial_s^* B(1)\, ds = B(1)B(t) - t, \quad 0 \leq t \leq 1. \tag{1.3}$$

Received 2015-9-17; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60G42, 60G48; Secondary 60G50, 60H05.
Key words and phrases. Brownian motion, stochastic integral, Hitsuda–Skorokhod integral, conditional expectation, martingale, near-martingale, near-submartingale, near-supermartingale, Doob's decomposition theorem, instantly independent sequence.
*This work was supported by JSPS Grant-in-Aid Scientific Research 15K04940.

Being motivated by Itô's ideas and observing the different values of equations (1.2) and (1.3), we have defined in [1] [2] the stochastic integral $\int_0^t B(1)\,dB(s)$ in the following way. Decompose the integrand $B(1)$ as

$$B(1) = B(t) + \big(B(1) - B(t)\big),$$

where the first term $B(t)$ is the Itô part of $B(1)$ and the second term $B(1) - B(t)$ is the counterpart of $B(1)$. For the Itô part, the evaluation points are the left endpoints of subintervals, while the evaluation points for the counterpart are the right endpoints of subintervals. Thus for $0 \le t \le 1$, we have

$$\begin{aligned}
\int_0^t B(1)\,dB(s) &= \int_0^t \big[B(s) + \big(B(1) - B(s)\big)\big]\,dB(s) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \big[B(s_{i-1}) + \big(B(1) - B(s_i)\big)\big]\big(B(s_i) - B(s_{i-1})\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \big[B(1) - \big(B(s_i) - B(s_{i-1})\big)\big]\big(B(s_i) - B(s_{i-1})\big) \\
&= \lim_{n\to\infty} \Big(B(1) \sum_{i=1}^n \big(B(s_i) - B(s_{i-1})\big) - \sum_{i=1}^n \big(B(s_i) - B(s_{i-1})\big)^2\Big) \\
&= B(1)B(t) - t, \qquad (1.4)
\end{aligned}$$

where the limit is convergence in probability. Note that this value is the same as the Hitsuda–Skorokhod integral in equation (1.3).

There is an intrinsic difference between the stochastic processes

$$X_t = B(1)B(t) - t, \quad Y_t = B(1)B(t), \quad 0 \le t \le 1, \qquad (1.5)$$

given by equations (1.4) and (1.2), respectively. For any $s \le t$, we see that

$$E[X_t \,|\, \mathcal{F}_s] = B(s)^2 - s. \qquad (1.6)$$

In particular, put $t = s$ to get

$$E[X_s \,|\, \mathcal{F}_s] = B(s)^2 - s. \qquad (1.7)$$

It follows from equations (1.6) and (1.7) that

$$E[X_t \,|\, \mathcal{F}_s] = E[X_s \,|\, \mathcal{F}_s], \quad \forall\, s \le t. \qquad (1.8)$$

On the other hand, it is easy to check that the stochastic process $Y_t = B(1)B(t)$ in equation (1.5) does not satisfy equation (1.8). This leads to the following concept introduced in [11].

Definition 1.1. A stochastic process X_t with $E|X_t| < \infty$ for $a \le t \le b$ is called a *near-martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if it satisfies the condition in equation (1.8).

We can define *near-submartingale* and *near-supermartingale* with respect to a filtration $\{\mathcal{F}_t\}$ by the following respective conditions:

$$E[X_t \,|\, \mathcal{F}_s] \ge E[X_s \,|\, \mathcal{F}_s], \quad \forall\, s \le t, \qquad (1.9)$$

and

$$E[X_t \,|\, \mathcal{F}_s] \le E[X_s \,|\, \mathcal{F}_s], \quad \forall\, s \le t.$$

Observe that if a stochastic process X_t is adapted to a filtration $\{\mathcal{F}_t\}$, then near-martingale, near-submartingale, and near-supermartingale reduce to martingale, submartingale, and supermartingale, respectively.

In this paper we will study the discrete parameter case of near-martingales and near-submartingales. In particular, we will prove Doob's decomposition theorem for near-submartingales.

2. Near-martingales and Near-submartingales

Let $\{\mathcal{F}_n\,; 1 \le n \le N\}$ be a fixed filtration, i.e., an increasing sequence of σ-fields.

Definition 2.1. A sequence $X_n, 1 \le n \le N$, of integrable random variables is called a *near-martingale* with respect to $\{\mathcal{F}_n\,; 1 \le n \le N\}$ if

$$E[X_{n+1} \,|\, \mathcal{F}_n] = E[X_n \,|\, \mathcal{F}_n], \quad \forall\, 1 \le n \le N-1. \tag{2.1}$$

Remark 2.2. It is easy to see that the equality in equation (2.1) is equivalent to the equality:

$$E[X_m \,|\, \mathcal{F}_n] = E[X_n \,|\, \mathcal{F}_n], \quad \forall\, 1 \le n \le m \le N. \tag{2.2}$$

Similarly, we can define *near-submartingale* and *near-supermartingale* just by replacing the equality sign in equation (2.1) with $\ge$ and $\le$, respectively. They also have the corresponding equivalent conditions as in equation (2.2).

Obviously, if a sequence $X_n, 1 \le n \le N$, is adapted to $\{\mathcal{F}_n\,; 1 \le n \le N\}$, then near-martingale, near-submartingale, and near-supermartingale are martingale, submartingale, and supermartingale, respectively.

Example 2.3. Take a sequence $\xi_1, \xi_2, \dots, \xi_N$ of independent random variables with mean 0. Let $\{\mathcal{F}_n\}$ be the filtration given by $\mathcal{F}_n = \sigma\{\xi_k; 1 \le k \le n\}$. Put

$$S_n = \xi_1 + \cdots + \xi_n, \quad X_n = S_N - S_n, \quad 1 \le n \le N. \tag{2.3}$$

The sequence $S_n, 1 \le n \le N$, is a martingale. On the other hand,

$$\begin{aligned} E[X_{n+1}|\mathcal{F}_n] &= E[\xi_{n+2} + \cdots + \xi_N|\mathcal{F}_n] \\ &= E(\xi_{n+2} + \cdots + \xi_N) \\ &= 0. \end{aligned}$$

Similarly, we have $E[X_n|\mathcal{F}_n] = 0$. Thus $E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n]$, which shows that $X_n, 1 \le n \le N$, is a near-martingale.

Furthermore, suppose $\xi_n, n \ge 1$, is a sequence of independent random variables with mean 0. For fixed N, $X_n = S_N - S_n, 1 \le n \le N$, is a near-martingale as shown above. However, $X_n = S_N - S_n, n \ge N$, is a martingale.

Example 2.4. Let $\xi_1, \xi_2, \dots, \xi_N$ be a sequence of independent random variables with mean 0 and $\mathrm{var}(\xi_n) = \sigma_n^2$. Let $\mathcal{F}_n = \sigma\{\xi_k; 1 \le k \le n\}$. Put

$$S_n = \xi_1 + \cdots + \xi_n, \quad X_n = S_n S_N - \sum_{k=1}^{n} \sigma_k^2, \quad 1 \le n \le N. \tag{2.4}$$

It is easy to check that

$$
\begin{aligned}
E[X_{n+1}\,\big|\,\mathcal{F}_n] &= E\Big[S_{n+1}S_N - \sum_{k=1}^{n+1}\sigma_k^2\,\Big|\,\mathcal{F}_n\Big] \\
&= E[(S_n+\xi_{n+1})(S_n+\xi_{n+1}+\cdots\xi_N)\,\big|\,\mathcal{F}_n] - \sum_{k=1}^{n+1}\sigma_k^2 \\
&= S_n^2 + \sigma_{n+1}^2 - \sum_{k=1}^{n+1}\sigma_k^2 \\
&= S_n^2 - \sum_{k=1}^{n}\sigma_k^2. \qquad (2.5)
\end{aligned}
$$

Similarly, we can easily derive

$$
E[X_n|\mathcal{F}_n] = S_n^2 - \sum_{k=1}^{n}\sigma_k^2. \qquad (2.6)
$$

It follows from equations (2.5) and (2.6) that $E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n]$. Hence the sequence $X_n = S_nS_N - \sum_{k=1}^n \sigma_k^2, 1 \le n \le N$, is a near-martingale.

Moreover, let $\xi_n, n \ge 1$, be a sequence of independent random variables with mean 0 and $\mathrm{var}(\xi_n) = \sigma_n^2$. Take $\mathcal{F}_n = \sigma\{\xi_k; 1 \le k \le n\}$. Define S_n and X_n as in equation (2.4). For fixed N, the sequence $X_n, 1 \le n \le N$, is a near-martingale as shown above. On the other hand, the sequence $X_n, n \ge N$, is a martingale.

Theorem 2.5. *Let $S_n, 1 \le n \le N$, be a square integrable martingale with respect to a filtration $\{\mathcal{F}_n; 1 \le n \le N\}$. Then*

$$
V_n = S_n(S_N - S_n), \quad 1 \le n \le N,
$$

is a near-martingale.

Proof. Note that

$$
V_{n+1} - V_n = (S_{n+1} - S_n)S_N - S_{n+1}^2 + S_n^2. \qquad (2.7)
$$

Hence we have

$$
\begin{aligned}
E[V_{n+1} - V_n|\mathcal{F}_n] &= E[(S_{n+1}-S_n)S_N|\mathcal{F}_n] - E[S_{n+1}^2|\mathcal{F}_n] + E[S_n^2|\mathcal{F}_n] \\
&= E\big\{E[(S_{n+1}-S_n)S_N|\mathcal{F}_{n+1}]\,|\mathcal{F}_n\big\} - E[S_{n+1}^2|\mathcal{F}_n] + S_n^2 \\
&= E\big\{(S_{n+1}-S_n)E[S_N|\mathcal{F}_{n+1}]\,|\mathcal{F}_n\big\} - E[S_{n+1}^2|\mathcal{F}_n] + S_n^2 \\
&= E\big\{(S_{n+1}-S_n)S_{n+1}\,|\mathcal{F}_n\big\} - E[S_{n+1}^2|\mathcal{F}_n] + S_n^2 \\
&= -S_nE[S_{n+1}|\mathcal{F}_n] + S_n^2 \\
&= -S_n^2 + S_n^2 \\
&= 0.
\end{aligned}
$$

Hence $E[V_{n+1}|\mathcal{F}_n] = E[V_n|\mathcal{F}_n]$ and so $V_n, 1 \le n \le N$, is a near-martingale. □

Theorem 2.6. *Suppose $S_n, n = 1, 2, \ldots,$ is a square integrable martingale with respect to a filtration $\{\mathcal{F}_n; 1 \leq n \leq N\}$. For a fixed natural number N, let*

$$V_n = S_n(S_N - S_n), \quad n = 1, 2, \ldots.$$

Then

(1) $V_n, 1 \leq n \leq N$, *is a near-martingale,*
(2) $V_n, n \geq N$, *is a supermartingale.*

Proof. The first assertion follows from Theorem 2.5. To prove the second assertion, we use equation (2.7) to show that for $n \geq N$,

$$\begin{aligned} E[V_{n+1} - V_n|\mathcal{F}_n] &= S_N E[(S_{n+1} - S_n)|\mathcal{F}_n] - E[S_{n+1}^2|\mathcal{F}_n] + S_n^2 \\ &= -E[S_{n+1}^2|\mathcal{F}_n] + S_n^2 \\ &\leq 0, \end{aligned}$$

since S_n^2 is a submartingale. Thus $E[V_{n+1}|\mathcal{F}_n] \leq E[V_n|\mathcal{F}_n]$ for $n \geq N$. But the sequence V_n, $n \geq N$, is adapted to the filtration $\{\mathcal{F}_n\}$. Therefore, we have

$$E[V_{n+1}|\mathcal{F}_n] \leq V_n, \quad n \geq N.$$

This shows that $V_n, n \geq N$, is a supermartingale. □

3. Doob's Decomposition Theorem

In this section we prove Doob's decomposition theorem for near-submartingales.

Theorem 3.1. *Let $X_n, n \geq 1$, be a near-submartingale with respect to a filtration $\{\mathcal{F}_n\}$. Then there exists a unique decomposition*

$$X_n = M_n + A_n, \quad n \geq 1, \tag{3.1}$$

with M_n and A_n satisfying the following conditions:

(1) $M_n, n \geq 1$, *is a near-martingale.*
(2) $A_1 = 0$.
(3) A_n *is $\mathcal{F}_{n-1}$-measurable for $n \geq 2$.*
(4) A_n *is inceasing almost surely.*

Proof. • **Existence of a decomposition**

Define $A_1 = 0$ and $M_1 = X_1$. Then we have equation (3.1) for $n = 1$. To find A_2 and M_2 such that

$$X_2 = M_2 + A_2$$

with desired properties, we take conditional expectation with respect to $\mathcal{F}_1$:

$$\begin{aligned} E[X_2|\mathcal{F}_1] &= E[M_2|\mathcal{F}_1] + E[A_2|\mathcal{F}_1] \\ &= E[M_1|\mathcal{F}_1] + A_2 \\ &= E[X_1|\mathcal{F}_1] + A_2. \end{aligned}$$

Therefore, we define

$$A_2 = E[X_2|\mathcal{F}_1] - E[X_1|\mathcal{F}_1], \quad M_2 = X_2 - A_2.$$

Then we have equation (3.1) for $n = 2$. Observe that A_2 is $\mathcal{F}_1$-measurable and $A_1 \leq A_2$ almost surely since $\{X_n\}$ is a near-submartingale.

Inductively, we repeat the above arguments to define A_n and M_n for $n \geq 3$ by

$$\begin{aligned} A_n &= \sum_{k=2}^{n} \Big(E[X_k|\mathcal{F}_{k-1}] - E[X_{k-1}|\mathcal{F}_{k-1}] \Big), \\ M_n &= X_n - A_n. \end{aligned}$$

Then we have equation (3.1) for $n \geq 3$. Notice that A_n is $\mathcal{F}_{n-1}$-measurable and $A_{n-1} \leq A_n$ almost surely since $\{X_n\}$ is a near-submartingale.

Now, we need to show that $M_n, n \geq 1$, is a near-martingale with respect to $\{\mathcal{F}_n\}$. Note that for $n \geq 2$, we have

$$M_n = X_n - \sum_{k=2}^{n} \Big(E[X_k|\mathcal{F}_{k-1}] - E[X_{k-1}|\mathcal{F}_{k-1}] \Big),$$

which yields the equality

$$M_n - M_{n-1} = X_n - X_{n-1} - E[X_n|\mathcal{F}_{n-1}] + E[X_{n-1}|\mathcal{F}_{n-1}].$$

Then we take conditional expectation with respect to $\mathcal{F}_{n-1}$ to show that

$$E[M_n - M_{n-1}|\mathcal{F}_{n-1}] = 0,$$

namely, $E[M_n|\mathcal{F}_{n-1}] = E[M_{n-1}|\mathcal{F}_{n-1}]$. Hence $M_n, n \geq 1$, is a near-martingale with respect to $\{\mathcal{F}_n\}$.

• **Uniqueness of a decomposition**

Suppose we have two such decompositions

$$X_n = M_n + A_n = N_n + B_n, \quad n \geq 1. \tag{3.2}$$

Then we have

$$M_n - N_n = B_n - A_n, \quad n \geq 1. \tag{3.3}$$

For $n = 1$, we have $B_1 = A_1 = 0$. Hence $M_1 = N_1$. For $n \geq 2$, take the conditional expectation of equation (3.3) with respect to $\mathcal{F}_{n-1}$ to get

$$E[M_n - N_n|\mathcal{F}_{n-1}] = E[B_n - A_n|\mathcal{F}_{n-1}] = B_n - A_n, \tag{3.4}$$

where in the last equality we have used the fact that A_n and B_n are $\mathcal{F}_{n-1}$-measurable. On the other hand, use equation (3.3) for $n-1$ and the fact that M_n and N_n are near-martingales to get

$$\begin{aligned} E[M_n - N_n|\mathcal{F}_{n-1}] &= E[M_{n-1} - N_{n-1}|\mathcal{F}_{n-1}] \\ &= E[B_{n-1} - A_{n-1}|\mathcal{F}_{n-1}] \\ &= B_{n-1} - A_{n-1}, \end{aligned} \tag{3.5}$$

where the last equality holds since B_{n-1} and A_{n-1} are $\mathcal{F}_{n-2}$-measurable and so are $\mathcal{F}_{n-1}$-measurable. Thus by equations (3.4) and (3.5),

$$B_n - A_n = B_{n-1} - A_{n-1}, \quad n \geq 2.$$

This equation together with $A_1 = B_1$ implies that $A_n = B_n$ almost surely for all $n \geq 1$. Then by equation (3.2) we have $M_n = N_n$ almost surely for all $n \geq 1$. Hence the decomposition is unique. □

Example 3.2. Let $\xi_n, n \geq 1$, be a sequence of independent random variables with mean 0 and $\mathrm{var}(\xi_n) = \sigma_n^2$. Take $\mathcal{F}_n = \sigma\{\xi_k; 1 \leq k \leq n\}$. Define $S_n = \xi_1 + \cdots + \xi_n$. For fixed N, consider the sequence

$$X_n = S_n S_N, \quad 1 \leq n \leq N. \tag{3.6}$$

First we show that the sequence $X_n, 1 \leq n \leq N$, is a near-submartingale. It is easy to see that

$$\begin{aligned} E[X_{n+1}|\mathcal{F}_n] &= E[S_{n+1}S_N|\mathcal{F}_n] \\ &= E[(S_n + \xi_{n+1})(\xi_1 + \cdots + \xi_N)|\mathcal{F}_n] \\ &= E[(S_n + \xi_{n+1})^2|\mathcal{F}_n] \\ &= E[S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2|\mathcal{F}_n] \\ &= S_n^2 + \sigma_{n+1}^2. \end{aligned} \tag{3.7}$$

On the other hand, we have

$$E[X_n|\mathcal{F}_n] = E[S_n S_N|\mathcal{F}_n] = S_n E[S_N|\mathcal{F}_n] = S_n^2. \tag{3.8}$$

By equations (3.7) and (3.8), we have $E[X_{n+1}|\mathcal{F}_n] \geq E[X_n|\mathcal{F}_n]$ almost surely. Hence $X_n, 1 \leq n \leq N$, is a near-submartingale.

To find the Doob decomposition of $X_n, 1 \leq n \leq N$, recall from Example 2.4 that the sequence

$$Z_n \equiv S_n S_N - \sum_{k=1}^{n} \sigma_k^2, \quad 1 \leq n \leq N,$$

is a near-martingale. This motivates us to define M_n and A_n by

$$M_n = \begin{cases} S_1 S_N, & \text{if } n = 1, \\ S_n S_N - \sum_{k=2}^n \sigma_k^2, & \text{if } n \geq 2. \end{cases}$$

and

$$A_n = \begin{cases} 0, & \text{if } n = 1, \\ \sum_{k=2}^n \sigma_k^2, & \text{if } n \geq 2. \end{cases}$$

Note that $M_n = Z_n + \sigma_1^2$. Hence M_n is a near-martingale. Then we can easily see that the Doob decomposition of $S_n S_N$ is given by

$$S_n S_N = M_n + A_n, \quad 1 \leq n \leq N.$$

We need to point out a difference between martingale case and near-martingale case. Suppose X_n is a square integrable martingale. It is well known that X_n^2 is a submartingale. However, for a square integrable near-martingale X_n, it is not true in general that X_n^2 is a near-submartingale. For instance, the sequence $X_n = S_N - S_n, 1 \leq n \leq N$, in Example 2.3 is a near-martingale. However, it is easy to check that $X_n^2, 1 \leq n \leq N$, is not a near-submartingale. In fact, it is a near-supermartingale.

4. Instantly Independent Sequences

Note that martingales must be adapted with respect to an associated filtration. In [11], we introduced the concept of instantly independent stochastic processes, which play the counterpart role of adapted stochastic processes. Thus for the discrete case, we have instantly independent sequences of random variables.

Definition 4.1. A sequence $\{\Phi_n\}$ of random variables is said to be *instantly independent* with respect to a filtration $\{\mathcal{F}_n\}$ if Φ_n and $\mathcal{F}_n$ are independent for each n.

We have the following two basic properties of instantly independent sequences of random variables.

Theorem 4.2. *If X_n is a near-martingale, then EX_n is a constant (independent of n). Conversely, if EX_n is a constant and X_n is instantly independent, then X_n is a near-martingale.*

Proof. Suppose X_n is a near-martingale. Then we have

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n], \quad \forall\, n \geq 1.$$

Upon taking expectation, we immediately get $EX_{n+1} = EX_n$ for all $n \geq 1$. Hence EX_n is a constant. Conversely, suppose EX_n is a constant and X_n is instantly independent with respect to a filtration $\{\mathcal{F}_n\}$. Then

$$\begin{aligned} E[X_{n+1}|\mathcal{F}_n] &= E\big\{E[X_{n+1}|\mathcal{F}_{n+1}]\big|\mathcal{F}_n\big\} \\ &= E\big\{EX_{n+1}\big|\mathcal{F}_n\big\} \\ &= EX_{n+1} \\ &= c, \end{aligned}$$

where c is a constant. On the other hand, since X_n and $\mathcal{F}_n$ are independent, we have

$$E[X_n|\mathcal{F}_n] = EX_n = c.$$

Hence $E[X_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n]$ and so $X_n, n \geq 1$, is a near-martingale. □

Theorem 4.3. *Suppose X_n is a square integrable martingale and Φ_n is a square integrable sequence of instantly independent random variables with $E\Phi_n$ being a constant. Then the product $X_n\Phi_n$ is a near-martingale.*

Proof. Using the assumptions we can easily derive

$$\begin{aligned} E[X_{n+1}\Phi_{n+1}|\mathcal{F}_n] &= E\big\{E[X_{n+1}\Phi_{n+1}|\mathcal{F}_{n+1}]\,\big|\,\mathcal{F}_n\big\} \\ &= E\big\{X_{n+1}E[\Phi_{n+1}|\mathcal{F}_{n+1}]\,\big|\,\mathcal{F}_n\big\} \\ &= E\big\{X_{n+1}E\Phi_{n+1}\,\big|\,\mathcal{F}_n\big\} \\ &= E\Phi_{n+1}\cdot E[X_{n+1}\,|\,\mathcal{F}_n] \\ &= cX_n, \end{aligned} \tag{4.1}$$

where $c = E\Phi_n$ is a constant. On the other hand, we have

$$E[X_n\Phi_n|\mathcal{F}_n] = X_nE[\Phi_n|\mathcal{F}_n] = X_nE\Phi_n = cX_n. \tag{4.2}$$

It follows from equations (4.1) and (4.2) that $E[X_{n+1}\Phi_{n+1}|\mathcal{F}_n] = E[X_n\Phi_n|\mathcal{F}_n]$ almost surely. Hence X_n is a near-martingale. □

Example 4.4. Let $\xi_1, \xi_2, \ldots, \xi_N$ be a sequence of independent random variables with mean 0 and finite variances. Let $\mathcal{F}_n = \sigma\{\xi_k; 1 \leq k \leq n\}$. Put

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n.$$

Then S_n is a martingale with respect to the filtration $\{\mathcal{F}_n\}$. Let θ be a real-vlaued function on $\mathbb{R}$. For fixed N, assume that the random variables

$$\theta(S_N - S_n), \quad 1 \leq n \leq N,$$

are square integrable. Then the following sequence

$$\Phi_n = \theta(S_N - S_n) - E\theta(S_N - S_n), \quad 1 \leq n \leq N,$$

is instantly independent with respect to the filtration $\{\mathcal{F}_n\}$ with mean 0. Hence by Theorem 4.3 the sequence

$$Y_n = S_n\Big(\theta(S_N - S_n) - E\theta(S_N - S_n)\Big), \quad 1 \leq n \leq N,$$

is a near-martingale.

Acknowledgment. The mathematical concepts and the results in this paper were obtained in many discussions with K. Saitô during Kuo's visits to Meijo University since 2011. Kuo would like to give his deepest appreciation to Professor Saitô for the invitations and for the warm hospitality.

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