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Maximilian Bock

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ON THE REGULARITY OF ONE PARAMETER TRANSFORMATION GROUPS IN BARRELED LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

MAXIMILIAN BOCK

to Christine and Sina

Abstract. In a barreled locally convex topological vector space E the pointwise convergence of a family $(T_\theta)_{\theta\in\mathbb{R}}$ of operators at a point $\theta_0 \in \mathbb{R}$ implies the uniform convergence on compact subsets of E at θ_0. For differentiable one-parameter transformation groups in $GL(E)$ we prove the stronger convergence property of regularity, namely, for each seminorm p of E there exists a seminorm q of E such that

$$\lim_{\theta\to 0} \sup_{q(x)\leq 1} p\left(\frac{T_\theta x - x}{\theta} - T'x\right) = 0,$$

where T' is the infinitesimal generator of $(T_\theta)_{\theta\in\mathbb{R}}$. In particular the convergence is uniform on all bounded subsets of E.

1. Introduction

Our starting point is a remark in [5, p. 120] mentioning that regularity is even more than differentiability, stated in the context of the white noise test functions, which form a nuclear (F)-space. In [1] the authors showed the heredity of differentiable one-parameter transformation groups under second quantization in white noise context. They emphasized that in contrast to [3, Theorem 4.1, p.82], where the heredity of regularity of differentiable one-parameter transformation groups under second quantization is proved, regularity is not needed. In this article we intend to clarify the relationships.

Furthermore we prove the heredity of regularity of differentiable one-parameter transformation groups under duality in the case of a reflexive Fréchet space.

2. Preliminaries on Locally Convex Topological Vector Spaces

We recall some well known statements about locally convex topological vector spaces, which we need for our investigation.

Let E, F be locally convex topological vector spaces topologized by the systems of semi-norms P respective Q. We define on $L(E, F)$ the following systems of seminorms:

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$$(1)\ \ P_\sigma := \left\{ \sup_{x\in A} q(Tx) \,|\, q \in Q, |A| < \infty \right\}, \quad (T \in L(E,F))$$
$$(2)\ \ P_b := \left\{ \sup_{x\in A} q(Tx) \,|\, q \in Q, A \text{ bounded } \right\}, \quad (T \in L(E,F)).$$

We use the notations $L_\sigma(E,F)$and $L_b(E,F)$ to clarify the used topology. The topology defined by P_σ is known as weak topology on $L(E,F)$, the topology defined by P_b is known as strong topology on $L(E,F)$.

For the dual space E^* of E endowed with the strong dual topology we use the notation E^*_b. For the dual space E^* of E endowed with the weak dual topology we use the notation E^*_σ. The following statement can be found in [2, 1.2 Satz, p. 72].

Proposition 2.1. *If E is semi-reflexive then E^*_b is barreled.*

Definition 2.2. Let $T \in L(E,F)$. The mapping $T^* : F^*_b \to E^*_b$ defined by

$$T^*(y^*) = y^* \circ T, \quad y^* \in F^*,$$

is called the *dual mapping* of T. Note that $T^* \in L(F^*_b, E^*_b)$.

The following statement is useful in order to transfer convergence processes to dual mappings, for the proof, see [2, 2.7 Satz, p. 84].

Theorem 2.3. *Let F be barreled or a metric locally convex vector space. Then the mapping*

$$^* : L_b(E,F) \longrightarrow L_b(F^*_b, E^*_b), \ \ T \longmapsto T^*$$

is continuous.

3. Regular One-parameter Groups

Let (X,P) be a barreled locally convex topological vector space over $\mathbb{C}$, where P is a family of semi-norms topologizing X. For each $p \in P$ the set $N_p := \{x \in X \,|\, p(x) = 0\}$ is a linear space.

With B_p we denote the completion of ${}^{X}\!/_{N_p}$. Note that X is a dense subspace of $\operatorname{proj}\lim_{p\in P} B_p$.

By [2, 2.1 Satz, p. 58] we have the following representation of the dual space:

$$X^* = \bigcup_{p\in P} B^*_p.$$

For $x^* \in B^*_p$ we define

$$p^*(x^*) := \sup_{p(x)\le 1} \{|\langle x^*, x\rangle| \ | \ x \in X\}.$$

By a Hahn-Banach argument, we have for all $x \in X$:

$$p(x) = \sup \left\{|\langle x^*, x\rangle| \ : x^* \in B^*_p \text{ and } p^*(x^*) \le 1\right\}$$

for all semi-norms $p \in P$.

Definition 3.1. Let $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ be a family of operators in $L(X)$ such that for all $\theta_1,\ \theta_2 \in \mathbb{R}$, we have

$$\Omega_{\theta_1+\theta_2} = \Omega_{\theta_1} \circ \Omega_{\theta_2}, \quad \Omega_0 = \mathrm{Id},$$

where Id denotes the identity operator. Obviously, $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ is a subgroup of $GL(X)$ and is called a *one-parameter subgroup* of $GL(X)$.

Definition 3.2. A one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of $GL(X)$ is called *differentiable* if

$$\lim_{\theta\to 0} \frac{\Omega_\theta\phi - \phi}{\theta},$$

converges in X for any $\phi \in X$. In that case a linear operator Ω' from X into itself is defined by

$$\Omega'\phi := \lim_{\theta\to 0} \frac{\Omega_\theta\phi - \phi}{\theta}.$$

Ω' is called the *infinitesimal generator* of the differentiable one-parameter subgroup.

Proposition 3.3. *Let $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset GL(X)$ be a differentiable one-parameter subgroup. Then Ω' is continuous, i.e. $\Omega' \in L(X)$. Moreover we have uniform convergence on every compact subset of X, i.e.:*

$$\lim_{\theta\to 0} \sup_{\phi\in K} p\left(\frac{\Omega_\theta\phi - \phi}{\theta} - \Omega'\phi\right) = 0$$

for any $p \in P$ and any compact subset $K \subset X$.

Proof. The assertion follows by an application of the Banach-Steinhaus theorem.

(For the uniform convergence on compact subsets see e.g. [6, 4.6 Theorem, p. 86]. Apply the theorem to any sequence $\theta_k \to 0$. Remember that the assertion is valid, if and only if it is valid for each sequence $\theta_k \to 0$, because $\mathbb{R}$ is a metric space.) □

Definition 3.4. A one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of $GL(X)$ is called *regular* if it is differentiable and there exists $\Omega' \in L(X)$ such that for any $p \in P$ there is a $q \in P$ satisfying

$$\lim_{\theta\to 0} \sup_{q(\phi)\le 1} p\left(\frac{\Omega_\theta\phi - \phi}{\theta} - \Omega'\phi\right) = 0.$$

In this case the infinitesimal generator Ω' is called a *regular generator*.

In the following we collect facts about one-parameter subgroups, for details see e.g. [5, Section 5.2].

Lemma 3.5. *Let $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ be a differentiable one-parameter subgroup of $GL(X)$. Then*

(i) $\mathbb{R} \to L_\sigma(X),\ \theta \mapsto \Omega_\theta$ *is continuous. Here the space $L_\sigma(X)$ is equipped with the topology of pointwise convergence.*

(ii) $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ *is infinitely many differentiable at each $\theta \in \mathbb{R}$ and*

$$\forall n \in \mathbb{N}: \ \frac{d^n}{d\theta^n}\Omega_\theta = (\Omega')^n \circ \Omega_\theta = \Omega_\theta \circ (\Omega')^n$$

(iii) $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ *is uniquely defined by it's infinitesimal generator, see* [5, Proposition 5.2.2, p. 119].

Lemma 3.6. *Let* $T:\mathbb{R}\to L_\sigma(X)$ *be a continuous mapping. If* K *is a compact subset of* $\mathbb{R}$*, then* $T(K)$ *is equicontinuous.*

Proof. $T(K)$ is a compact subset of $L_\sigma(X)$, hence pointwisely bounded, by the definition of the topology of pointwise convergence. The theorem of Banach and Steinhaus then completes the proof. □

Theorem 3.7 (Regularity)**.** *Let* $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ *be a differentiable one-parameter subgroup of* $GL(X)$*. Then* $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ *is regular.*

Proof. Let $p\in P$ and $\xi\in X$, $\eta\in B_p^*$. We define for all $t\in\mathbb{R}$:

$$f(t) := \langle \eta, \Omega_t \xi\rangle\,.$$

Then we have by Lemma 3.5

$$f'(t) = \langle \eta, \Omega' \Omega_t \xi\rangle\,,$$
$$f''(t) = \left\langle \eta, (\Omega')^2 \Omega_t \xi\right\rangle.$$

Now let $\theta_0>0$ be fixed. Then by (i) in Lemma 3.5 and Lemma 3.6, it follows that $\left\{(\Omega')^2\Omega_t\right\}_{|t|\le\theta_0}$ is equicontinuous. Thus there exists $q\in P$, $q\ge p$ and $K=K(\theta_0,p,q,\Omega')>0$ such that

$$\max_{|t|\le\theta_0}|f''(t)| \le K q\,(\xi)\,p^*\,(\eta)\,.$$

Let $\theta\in\mathbb{R}$ with $|\theta|\le\theta_0$. By Taylor expansion we have

$$\begin{aligned}|f(\theta)-f(0)-\theta\cdot f'(0)| &\le \frac{|\theta|^2}{2}\max_{|t|\le\theta_0}|f''(t)|\\ &\le \frac{|\theta|^2}{2} K q\,(\xi)\,p^*\,(\eta)\end{aligned}$$

and for $\theta\ne 0$

$$\sup_{q(\xi)\le 1}\ \sup_{p^*(\eta)\le 1}\left|\frac{\langle\eta,\Omega_\theta\xi\rangle-\langle\eta,\xi\rangle}{\theta}-\langle\eta,\Omega'\xi\rangle\right| \le \frac{|\theta|}{2}K.$$

Hence

$$\sup_{q(\xi)\le 1} p\left(\frac{\Omega_\theta\xi-\xi}{\theta}-\Omega'\xi\right)\le\frac{|\theta|}{2}K$$

such that

$$\lim_{\theta\to 0}\sup_{q(\xi)\le 1} p\left(\frac{\Omega_\theta\xi-\xi}{\theta}-\Omega'\xi\right)=0.$$

□

4. Heredity of Regularity of Differentiable One-parameter Transformation Groups

Proposition 4.1. *Let X be a reflexive Fréchet space. If $(T_\theta)_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(X)$, then $(T_\theta^*)_{\theta\in\mathbb{R}}$ is a regular one-parameter subgroup of $GL(X^*)$.*

Proof. By Theorem 3.7 it follows:

$$\lim_{\theta\to 0}\sup_{\phi\in B} p\left(\frac{T_\theta\phi-\phi}{\theta}-T'\phi\right)=0$$

for any $p\in P$ and any bounded subset $B\subset X$. By Proposition 2.1 and Theorem 2.3 it then follows

$$\lim_{\theta\to 0}\sup_{\phi\in B^*} p^*\left(\frac{T_\theta^*\phi-\phi}{\theta}-(T')^*\phi\right)=0$$

for any $p\in P$ and any bounded subset $B^*\subset X^*$. Because X_b^* is barreled, the statement follows immediately by Theorem 3.7 □

In the case of a barreled topological vector space X we can at least state that the differential quotient $\frac{T_\theta^*\phi-\phi}{\theta}$ converges in $L_b(X_b^*)$. This is an immediate consequence of Theorem 2.3:

Corollary 4.2. *Let X be a barreled topological vector space. If $(T_\theta)_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(X)$, then*

$$\lim_{\theta\to 0}\sup_{\phi\in B^*} p^*\left(\frac{T_\theta^*\phi-\phi}{\theta}-(T')^*\phi\right)=0$$

for any $p\in P$ and any bounded subset $B^*\subset X^*$.

5. Summary and Bibliographical Notes

In [5, p. 120] an [1] there was explicitly mentioned the formal difference between differentiability and regularity of one-parameter transformation groups even in the case of white noise test functions, which constitute a nuclear Fréchet space. By this paper we intended to clarify the relationships. Especially considering differentiable one-parameter groups on the nuclear (F)-space of white noise test functions we have seen that these are already regular.

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Maximilian Bock: Saarstahl AG, Informatik, Bismarckstr. 57-59, 66333 Völklingen, Germany
E-mail address: maximilianbock61@gmail.com