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HIDDEN MARKOV CHANGE POINT ESTIMATION

ROBERT J. ELLIOTT* AND SEBASTIAN ELLIOTT

ABSTRACT. A hidden Markov model is considered where the dynamics of the hidden process change at a random ‘change point’. In principle this gives rise to a non-linear filter but closed form recursive estimates are obtained for the conditional distribution of the hidden process and of .

1. Introduction

Hidden Markov models have found many applications from speech processing to regime switching dynamics in financial models. An early description is given in the paper [4] by Rabiner and Huang and a fuller treatment can be found in the book “Hidden Markov Models: Estimation and Control” by Elliott, Aggoun and Moore. Recent results can be found in [3].

In this paper we consider the situation where a discrete time Markov chain X is observed indirectly through a second Markov chain Y . However, the dynamics of X undergo a change at a random time . Given the observed process Y , filtered recursive estimates for the conditional distribution of X and the change point are derived. This paper uses the expectation maximization, EM algorithm as discussed in [1] and [2].

2. Chain Dynamics

Suppose $X = \{X_k, k = 0, 1, \dots\}$ is a discrete time finite state Markov chain defined on a probability space (Ω, \mathcal{F}, P) . Without loss of generality, the state space can be taken to be the set of unit vectors $\{e_1, e_2, \dots, e_N\}$ in R^N , where N is the number of elements of the state space and $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)'$ is a standard basis vector in R^N .

Suppose $\tau \in \{1, 2, \dots\}$ is a random time and

$$P(\tau = k) = p_k \geq 0.$$

Write $F_k = P(\tau > k) = \sum_{l=k+1}^{\infty} p_l$.

The random time τ represents a random time at which there is a change in the dynamics of the chain X .

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In fact, if $k \leq \tau$, suppose

$$\begin{aligned} a_{ji}^1 &= P(X_{k+1} = e_j | X_k = e_i) \\ &= P(X_1 = e_j | X_0 = e_i). \end{aligned}$$

Write A^1 for the matrix $(a_{ji}^1, 1 \leq i, j \leq N)$.

If $k > \tau$, suppose

$$a_{ji}^2 = P(X_{k+1} = e_j | X_k = e_i),$$

and write A^2 for the matrix $(a_{ji}^2, 1 \leq i, j \leq N)$.

Write $\bar{Z}_k = I_{k \geq \tau}$.

The state space of \bar{Z} will be mapped onto the unit vectors $\bar{e}_1 = (1, 0)$, $\bar{e}_2 = (0, 1)$ by considering the process

$$Z_k = (1 - \bar{Z}_k)\bar{e}_1 + \bar{Z}_k\bar{e}_2 \in R^2.$$

Define the filtrations

$$\begin{aligned} \mathcal{F}_k^X &= \sigma\{X_i, i \leq k\}, \\ \mathcal{F}_k^Z &= \sigma\{Z_i, i \leq k\}, \\ \mathcal{F}_k &= \mathcal{F}_k^X \vee \mathcal{F}_k^Z. \end{aligned}$$

Lemma 2.1. *Suppose $A_k = A^1\langle Z_k, \bar{e}_1 \rangle + A^2\langle Z_k, \bar{e}_2 \rangle$, then $X_{k+1} = A_k X_k + M_{k+1} \in R^N$, where M is a sequence of Martingale increments, that is $E[M_{k+1} | \mathcal{F}_k] = 0 \in R^N$.*

Proof.

$$\begin{aligned} E[M_{k+1} | \mathcal{F}_k] &= E[X_{k+1} - A_k X_k | \mathcal{F}_k^X \vee \mathcal{F}_k^Z] \\ &= E[X_{k+1} | X_k, Z_k] - (A^1\langle Z_k, \bar{e}_1 \rangle + \\ &\quad A^2\langle Z_k, \bar{e}_2 \rangle) X_k \\ &= E[X_{k+1} (\langle Z_k, \bar{e}_1 \rangle + \langle Z_k, \bar{e}_2 \rangle) | X_k, Z_k] - \\ &\quad (A^1\langle Z_k, \bar{e}_1 \rangle + A^2\langle Z_k, \bar{e}_2 \rangle) X_k \\ &= (A^1\langle Z_k, \bar{e}_1 \rangle + A^2\langle Z_k, \bar{e}_2 \rangle) X_k - \\ &\quad (A^1\langle Z_k, \bar{e}_1 \rangle + A^2\langle Z_k, \bar{e}_2 \rangle) X_k \\ &= 0. \end{aligned}$$

□

TRANSITIONS OF Z :

Note $P(Z_{k+1} = \bar{e}_1 | \mathcal{F}_k^Z) = P(Z_{k+1} = \bar{e}_1 | Z_k)$. Now

$$\begin{aligned} P(Z_{k+1} = \bar{e}_1 | Z_k = \bar{e}_1) &= P(\tau > k + 1 | \tau > k) = \frac{F_{k+1}}{F_k}, \\ P(Z_{k+1} = \bar{e}_2 | Z_k = \bar{e}_1) &= P(\tau = k + 1 | \tau > k) = \frac{p_{k+1}}{F_k}, \\ P(Z_{k+1} = \bar{e}_1 | Z_k = \bar{e}_2) &= P(\tau > k + 1 | \tau \leq k) = 0, \\ P(Z_{k+1} = \bar{e}_2 | Z_k = \bar{e}_2) &= P(\tau \leq k + 1 | \tau \leq k) = 1. \end{aligned}$$

Lemma 2.2. Write $F_k = \begin{pmatrix} \frac{F_{k+1}}{F_k} & 0 \\ \frac{p_{k+1}}{F_k} & 1 \end{pmatrix}$. Then $Z_{k+1} = F_k Z_k + N_{k+1} \in R^2$, where $E[N_{k+1} | \mathcal{F}_k^Z] = 0 \in R^2$.

3. Observations.

The chain X is not observed directly. Rather, there is another finite state process Y which is a ‘noisy’ observation of X .

Suppose the finite state space of Y is identified with the unit vectors where $\{f_1, f_2, \dots, f_M\}$ of R^M , $f_j = (0, 0, \dots, 1, \dots, 0)' \in R^N$. We can have $M > N$, $M < N$ or $M = N$.

Suppose $c_{j,i} = P(Y_k = f_j | X_k = e_i) \geq 0$. Note $\sum_{j=1}^M c_{j,i} = 1$.

Write $C = (c_{ji}), 1 \leq i \leq N, 1 \leq j \leq M, \mathcal{F}_k^Y = \sigma\{Y_j, 0 \leq j \leq k\}$.

Lemma 3.1. $Y_k = CX_k + L_k \in R^M$, where $E[L_k | \mathcal{F}_k^X] = 0 \in R^M$.

Proof.

$$\begin{aligned}
 E[L_k | \mathcal{F}_k^X] &= E[Y_k - CX_k | \mathcal{F}_k^X] \\
 &= E[Y_k - CX_k | X_k] \\
 &= E[Y_k | X_k] - CX_k \\
 &= \sum_{j=1}^M \sum_{i=1}^N E[Y_k \langle Y_k, f_j \rangle \langle X_k, e_i \rangle | X_k] - CX_k \\
 &= \sum_{j=1}^M \sum_{i=1}^N E[\langle Y_k, f_j \rangle \langle X_k, e_i \rangle | X_k] f_j - CX_k \\
 &= \sum_{j=1}^M \sum_{i=1}^N \langle X_k, e_i \rangle c_{ji} f_j - CX_k \\
 &= 0 \in R^M.
 \end{aligned}$$

□

JOINT DISTRIBUTIONS:

Lemma 3.2.

$$\begin{aligned}
 E[X_{k+1} \otimes Z_{k+1} | X_k, Z_k] &= A^1 X_k \otimes \begin{pmatrix} \frac{F_{k+1}}{F_k} \\ \frac{p_{k+1}}{F_k} \end{pmatrix} \langle Z_k, \bar{e}_1 \rangle \\
 &\quad + A^2 X_k \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \langle Z_k, \bar{e}_2 \rangle. \tag{3.1}
 \end{aligned}$$

Proof.

$$\begin{aligned}
& E[X_{k+1} \otimes Z_{k+1} | X_k, Z_k] \\
&= E[E[X_{k+1} \otimes Z_{k+1} | X_{k+1}, X_k, Z_k] | X_k, Z_k] \\
&= E[X_{k+1} \otimes E[Z_{k+1} | Z_k] | X_k, Z_k] \\
&= E[X_{k+1} \otimes E[Z_{k+1} (\langle Z_k, \bar{e}_1 \rangle + \langle Z_k, \bar{e}_2 \rangle) | Z_k] | X_k, Z_k] \\
&= E[X_{k+1} \otimes \begin{pmatrix} F_{k+1} \\ F_k \\ \frac{F_{k+1}}{F_k} \end{pmatrix} \langle Z_k, \bar{e}_1 \rangle | X_k, Z_k] \\
&\quad + E[X_{k+1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \langle Z_k, \bar{e}_2 \rangle | X_k, Z_k] \\
&= A^1 X_k \otimes \begin{pmatrix} F_{k+1} \\ F_k \\ \frac{F_{k+1}}{F_k} \end{pmatrix} \langle Z_k, \bar{e}_1 \rangle + A^2 X_k \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \langle Z_k, \bar{e}_2 \rangle.
\end{aligned} \tag{3.2}$$

□

4. Reference Probability.

The above dynamics of X, Z, Y are under the ‘real world’ probability P . Suppose we have another probability \bar{P} under which X and Z have the same dynamics as above, but under which Y is a process independent of X and Z and under which Y is a sequence of independent, identity distributed random variables with

$$\bar{P}(Y_k = f_j | \mathcal{F}_k^X \vee \mathcal{F}_k^Z \vee \mathcal{F}_{k-1}^Y) = \bar{P}(Y_k = f_j) = 1/M.$$

Lemma 4.1. *Let*

$$\lambda_k = M \sum_{j=1}^M \langle CX_k, f_j \rangle \langle Y_k, f_j \rangle, \tag{4.1}$$

$$\Lambda_k = \prod_{i=0}^k \lambda_i. \tag{4.2}$$

The probability P can be defined by setting $\frac{dP}{d\bar{P}}|_{\mathcal{G}_k} = \Lambda_k$, where $\mathcal{G}_k = \mathcal{F}_k^X \vee \mathcal{F}_k^Y \vee \mathcal{F}_k^Z$. Then under P the dynamics of X and Z are unchanged and

$$P(Y_k = f_j | X_k = e_i) = c_{ji}.$$

That is, under P

$$Y_k = CX_k + L_k,$$

where $E[L_k | \mathcal{F}_k^X] = 0 \in R^M$.

Proof. Consider first

$$\begin{aligned}
\bar{E}[\lambda_k | \mathcal{G}_{k-1} \vee X_k] &= M \sum_{j=1}^M \bar{E}[\langle CX_k, f_j \rangle \langle Y_k, f_j \rangle | X_k] \\
&= M \sum_{j=1}^M \langle CX_k, f_j \rangle \bar{E}[\langle Y_k, f_j \rangle] \\
&= \sum_{j=1}^M \sum_{i=1}^N c_{ji} \langle X_k, e_i \rangle \\
&= 1.
\end{aligned}$$

From Bayes' theorem, (see [1]), with $\frac{dP}{d\bar{P}}|_{\mathcal{G}_k} = \Lambda_k$,

$$\begin{aligned}
&E[\langle Y_k, f_j \rangle | \mathcal{G}_{k-1} \vee X_k] \\
&= \frac{\bar{E}[\Lambda_k \langle Y_k, f_j \rangle | \mathcal{G}_{k-1} \vee X_k]}{\bar{E}[\Lambda_k | \mathcal{G}_{k-1} \vee X_k]} \\
&= \frac{\Lambda_{k-1} \bar{E}[\lambda_k \langle Y_k, f_j \rangle | \mathcal{G}_{k-1} \vee X_k]}{\Lambda_{k-1} \bar{E}[\lambda_k | \mathcal{G}_{k-1} \vee X_k]} \\
&= M \bar{E}[\langle \sum_{r=1}^M \langle CX_k, f_r \rangle \langle Y_k, f_r \rangle \rangle \langle Y_k, f_j \rangle | \mathcal{G}_{k-1} \vee X_k] \\
&= M \bar{E}[\langle CX_k, f_j \rangle \langle Y_k, f_j \rangle | \mathcal{G}_{k-1} \vee X_k] \\
&= \langle CX_k, f_j \rangle.
\end{aligned}$$

Therefore, if $X_k = e_i$,

$$P(Y_k = f_j | X_k = e_i) = E[\langle Y_k, f_j \rangle | X_k = e_i] = c_{ji}.$$

□

5. Filters.

We wish to estimate both X and Z , given the noisy observations Y . This is required under the ‘real world’ probability P . That is, we wish to determine $E[X_k \otimes Z_k | \mathcal{F}_k^Y] \in R^{N \times 2}$.

However, it is easier to work under the ‘reference probability’ \bar{P} for which the dynamics of X and Z remain unchanged but the Y sequence is i.i.d. with $P(Y_k = f_j) = 1/M$ for all k and j .

Using Bayes’ rule,

$$E[X_k \otimes Z_k | \mathcal{F}_k^Y] = \frac{\bar{E}[\Lambda_k X_k \otimes Z_k | \mathcal{F}_k^Y]}{\bar{E}[\Lambda_k | \mathcal{F}_k^Y]}.$$

Write

$$\begin{aligned}
q_k &:= \bar{E}[\Lambda_k X_k \otimes Z_k | \mathcal{F}_k^Y] \in R^{N \times 2}, \\
q_k^1 &:= \bar{E}[\Lambda_k X_k \langle Z_k, \bar{e}_1 \rangle | \mathcal{F}_k^Y], \\
q_k^2 &:= \bar{E}[\Lambda_k X_k \langle Z_k, \bar{e}_2 \rangle | \mathcal{F}_k^Y].
\end{aligned}$$

for unnormalized conditional expectations given the observations \mathcal{F}_k^Y to time k . Then

$$q_k = (q_k^1, q_k^2).$$

Note that, with $\mathbf{1}$ denoting the vector of 1's in R^N or R^2 :

$$\mathbf{1}' \cdot X_k \otimes Z_k \cdot \mathbf{1} = 1.$$

Therefore,

$$\mathbf{1}' \cdot q_k \cdot \mathbf{1} = \bar{E}[\mathbf{1}' \cdot \Lambda_k X_k \otimes Z_k \cdot \mathbf{1} | \mathcal{F}_k^Y] = \bar{E}[\Lambda_k | \mathcal{F}_k^Y].$$

Consequently, if we know q_k , the denominator $\bar{E}[\Lambda_k | \mathcal{F}_k^Y]$ is just the sum of all components of q_k .

The filter is now a recursive estimate for q_k given the observations Y .

Theorem 5.1.

$$q_{k+1}^1 = M \sum_{j=1}^M \sum_{i=1}^N \langle A^1 q_k^1, e_i \rangle \frac{F_{k+1}}{F_k} c_{ji} e_i \langle Y_{k+1}, f_j \rangle, r \quad (5.1)$$

$$q_{k+1}^2 = M \sum_{j=1}^M \sum_{i=1}^N [\langle A^1 q_k^1, e_i \rangle \frac{p_{k+1}}{F_k} + \langle A^2 q_k^2, e_i \rangle] c_{ji} e_i \langle Y_{k+1}, f_j \rangle. \quad (5.2)$$

Proof.

$$\begin{aligned} & q_{k+1}^1 \\ &= \bar{E}[\Lambda_k \lambda_{k+1} X_{k+1} \langle Z_{k+1}, \bar{e}_1 \rangle | \mathcal{F}_{k+1}^Y] \\ &= M \sum_{j=1}^M \bar{E}[\Lambda_k \langle C X_{k+1}, f_j \rangle X_{k+1} \langle Z_{k+1}, \bar{e}_1 \rangle | \mathcal{F}_{k+1}^Y] \langle Y_{k+1}, f_j \rangle \\ &= M \sum_{j=1}^M \sum_{i=1}^N \bar{E}[\Lambda_k \langle X_{k+1}, e_i \rangle \langle Z_{k+1}, \bar{e}_1 \rangle | \mathcal{F}_{k+1}^Y] c_{ji} e_i \langle Y_{k+1}, f_j \rangle \\ &= M \sum_{j=1}^M \sum_{i=1}^N \sum_{r=1}^2 \bar{E}[\Lambda_k \langle X_{k+1}, e_i \rangle \langle Z_{k+1}, \bar{e}_1 \rangle \langle Z_k, \bar{e}_r \rangle | \mathcal{F}_{k+1}^Y] \\ &\quad \times c_{ji} e_i \langle Y_{k+1}, f_j \rangle \\ &= M \sum_{j=1}^M \sum_{i=1}^N \langle A^1 q_k^1, e_i \rangle \frac{F_{k+1}}{F_k} c_{ji} e_i \langle Y_{k+1}, f_j \rangle. \end{aligned}$$

For q^2 :

$$\begin{aligned}
& q_{k+1}^2 \\
&= \bar{E}[\Lambda_k \lambda_{k+1} X_{k+1} \langle Z_{k+1}, \bar{e}_2 \rangle | \mathcal{F}_{k+1}^Y] \\
&= M \sum_{j=1}^M \bar{E}[\Lambda_k \langle C X_{k+1}, f_j \rangle X_{k+1} \langle Z_{k+1}, \bar{e}_2 \rangle | \mathcal{F}_{k+1}^Y] \langle Y_{k+1}, f_j \rangle \\
&= M \sum_{j=1}^M \sum_{i=1}^N \bar{E}[\Lambda_k \langle X_{k+1}, e_i \rangle \langle Z_{k+1}, \bar{e}_2 \rangle | \mathcal{F}_{k+1}^Y] c_{ji} e_i \langle Y_{k+1}, f_j \rangle \\
&= M \sum_{j=1}^M \sum_{i=1}^N \sum_{r=1}^2 \bar{E}[\Lambda_k \langle X_{k+1}, e_i \rangle \langle Z_{k+1}, \bar{e}_2 \rangle \langle Z_k, \bar{e}_r \rangle | \mathcal{F}_{k+1}^Y] \cdot \\
&\quad c_{ji} e_i \langle Y_{k+1}, f_j \rangle \\
&= M \sum_{j=1}^M \sum_{i=1}^N [\langle A^1 q_k^1, e_i \rangle \frac{p_{k+1}}{F_k} c_{ji} e_i \langle Y_{k+1}, f_j \rangle + \\
&\quad \langle A^2 q_k^2, e_i \rangle c_{ji} e_i \langle Y_{k+1}, f_j \rangle].
\end{aligned}$$

□

Remark 5.2. The filter is initialized by assuming the change point τ has not occurred, so $Z_0 = \bar{e}_1$, and by taking an initial distribution $p_0 = q_0$ for X_0 .

Corollary 5.3. *The normalized conditional distributions of X_k and Z_k are then given by*

$$p_k^1 = E[X_k \langle Z_k, \bar{e}_1 \rangle | \mathcal{F}_k^Y] = \frac{q_k^1}{\mathbf{1}' \cdot q_k \cdot \mathbf{1}}, \quad (5.3)$$

$$p_k^2 = E[X_k \langle Z_k, \bar{e}_2 \rangle | \mathcal{F}_k^Y] = \frac{q_k^2}{\mathbf{1}' \cdot q_k \cdot \mathbf{1}}. \quad (5.4)$$

Given the observations, the conditional probability of the change point τ is then

$$P(\tau > k) = E[\langle Z_k, \bar{e}_1 \rangle | \mathcal{F}_k^Y] = \mathbf{1}' \cdot q_k^1. \quad (5.5)$$

Remark 5.4. Recall M is the number of elements in the state space of the observation process Y . The right hand sides of the recursions (5.5) and (5.6) of Theorem 5.1 involve a factor M . However, this will cancel when we consider the normalized forms in (5.7) and (5.8). Consequently, equation (5.5) and (5.6) can be modified to give recursions for unnormalized quantities \bar{q}_k^1, \bar{q}_k^2 as:

$$\bar{q}_{k+1}^1 = \sum_{j=1}^M \sum_{i=1}^N \langle A^1 \bar{q}_k^1, e_i \rangle \frac{F_{k+1}}{F_k} c_{ji} e_i \langle Y_{k+1}, f_j \rangle, \quad (5.6)$$

$$\bar{q}_{k+1}^2 = \sum_{j=1}^M \sum_{i=1}^N [\langle A^1 \bar{q}_k^1, e_i \rangle \frac{p_{k+1}}{F_k} + \langle A^2 \bar{q}_k^2, e_i \rangle] c_{ji} e_i \langle Y_{k+1}, f_j \rangle. \quad (5.7)$$

Again the initialization is $Z_0 = \bar{e}_1$ and $\bar{q}_0 = p_0 \in R^N$.

6. A Viterbi Recursion.

As noted in our earlier papers, the Viterbi filter is given by replacing the expected value by a maximum likelihood. That is, the Viterbi estimation is given by a sequence of vectors

$$\begin{aligned} q_k^{*1} &= [q_k^{*1}(1), q_k^{*1}(2), \dots, q_k^{*1}(N)]', \\ q_k^{*2} &= [q_k^{*2}(1), q_k^{*2}(2), \dots, q_k^{*2}(N)]'. \end{aligned}$$

defined recursively by $Z_0 = \bar{e}_1$, $q_0^{*1} = p_0$, $q_0^{*2} = \mathbf{0} \in R^N$, and

$$q_{k+1}^{*1}(i) := \langle A^1 q_k^{*1}, e_i \rangle \frac{F_{k+1}}{F_k} \max_j (c_{ji} \langle Y_{k+1}, f_j \rangle), \quad (6.1)$$

$$q_{k+1}^{*2}(i) := [\langle A^1 q_k^{*1}, e_i \rangle \frac{p_{k+1}}{F_k} + \langle A^2 q_k^{*2}, e_i \rangle] \max_j (c_{ji} \langle Y_{k+1}, f_j \rangle). \quad (6.2)$$

7. Conclusion.

Hidden Markov chains, that is Markov chains observed indirectly through the observations of a second Markov chain, have been extensively studied. For a bibliography see the book [1] by Aggoun, Elliott and Moore.

In this paper we have considered a hidden Markov chain X whose dynamics undergo a change at a random time τ . Given an observed process filtered estimates for the conditional distribution of X and the change point time τ are derived.

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