Communications on Stochastic Analysis

Volume 9 | Number 3 Article 2

9-1-2015

Singular equations driven by an additive noise and applications

Nicolas Marie

Communications on Stochastic Analysis
Vol. 9, No. 3 (2015) 309-341

Serials Publications
www.serialspublications.com

SINGULAR EQUATIONS DRIVEN BY AN ADDITIVE NOISE AND APPLICATIONS

NICOLAS MARIE

Abstract. In the pathwise stochastic calculus framework, the paper deals with the general study of equations driven by an additive Gaussian noise, with a drift function having an infinite limit at point zero. An ergodic theorem and the convergence of the implicit Euler scheme are proved. The Malliavin calculus is used to study the absolute continuity of the distribution of the solution. An estimation procedure of the parameters of the random component of the model is provided. The properties are transferred on a class of singular stochastic differential equations driven by a multiplicative noise. A fractional Heston model is introduced.

1. Introduction

Let $B := (B_t)_{t\in\mathbb{R}_+}$ be a centered stochastic process with locally α-Hölder continuous paths, and consider the stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma B_t \tag{1.1}$$

where $\alpha \in]0,1[$, $x_0 \in \mathrm{I}$, $\mathrm{I} \subset \mathbb{R}$ is an interval, $\sigma \in \mathbb{R}^* := \mathbb{R} - \{0\}$ and $b : \mathrm{I} \to \mathbb{R}$ is a $[1/\alpha] + 1$ times continuously differentiable function.

Assume that $\mathrm{I} = \mathbb{R}$ and b is everywhere differentiable with bounded derivatives. Then, Equation (1.1) has a unique (pathwise) solution defined on $\mathbb{R}_+$ with locally α-Hölder continuous paths (see Friz and Victoir [9], sections 10.3 and 10.7).

If in addition B is a fractional Brownian motion (see Nualart [28], Chapter 5), the probabilistic and statistical properties of the solution of Equation (1.1) have been deeply studied by several authors (see Hairer [13], Tudor and Viens [31], Neuenkirch and Tindel [26], etc.).

Throughout the paper, $\mathrm{I} =]0,\infty[$ and

$$\lim_{x\to 0^+} b(x) = \infty.$$

The existence and the uniqueness of the solution of Equation (1.1), and the absolute continuity of its distribution for a fractional Brownian signal of Hurst parameter belonging to $]1/2,1[$ have been already studied in Hu, Nualart and Song [15].

Received 2015-5-14; Communicated by the editors.
2010 *Mathematics Subject Classification.* 60H10.
Key words and phrases. Ergodic theorem, fractional Brownian motion, Gaussian processes, Heston model, sensitivities, stochastic differential equations.

The current paper deals with a general study of Equation (1.1) in the pathwise stochastic calculus framework (see Lyons [19], Lyons and Qian [20], Gubinelli and Lejay [12], Lejay [18], Friz and Victoir [9], etc.) under the following Assumption.

Assumption 1.1. (1) The function b is $[1/\alpha]+1$ times continuously differentiable on $]0,\infty[$ and has bounded derivatives on $[\varepsilon,\infty[$ for every $\varepsilon>0$.

(2) There exists a constant $K>0$ such that :

$$\forall x>0,\, \dot{b}(x)<-K.$$

(3) There exists a constant $R>0$ such that :

$$\forall x>0,\, b(x)>-Rx.$$

(4) For every $C>0$,

$$\int_0^T b(Ct^\alpha)dt=\infty\ ;\ \forall T>0$$

or

$$\lim_{T\rightarrow 0^+}\frac{1}{T^\alpha}\int_0^T b(Ct^\alpha)dt=\infty.$$

The second section is devoted to deterministic properties of Equation (1.1) : the global existence and the uniqueness of the solution, the regularity of the Itô map, the convergence of the implicit Euler scheme and some estimates.

The third section is devoted to probabilistic and statistical properties of the solution $X(x_0)$ of Equation (1.1), obtained via its deterministic properties proved at Section 2 and various additional conditions on the signal B. In order to ensure the integrability of estimates, B is a Gaussian process in the major part of Section 3. Subsection 3.1 deals with the ergodicity of $X(x_0)$, studied in the random dynamical systems framework (see Arnold [1]). By assuming that B is a fractional Brownian motion, the existence of an attracting stationary solution of Equation (1.1) and an ergodic theorem are proved. Subsection 3.2 deals with applications of the Malliavin calculus (see Nualart [28]) to the absolute continuity of the distribution of $X_t(x_0)$ for every $t\in]0,T]$. Via Nourdin and Viens [27], a density with a suitable expression is provided. Subsection 3.3 deals with the integrability and the convergence of the implicit Euler scheme. A rate of convergence is provided. Subsection 3.4 deals with a relationship between $X(x_0)$ and an Ornstein-Uhlenbeck process. By assuming that B is a fractional Brownian motion of Hurst parameter $H\in]1/2,1[$, an estimation procedure of (H,σ) is provided by using Melichov [24], Brouste and Iacus [3], and Berzin and León [2]. On the fractional Ornstein-Uhlenbeck process, see Cheridito et al. [5] and Garrido-Atienza et al. [10].

The fourth section is devoted to the transfer of the properties established at sections 2 and 3 on a class of singular stochastic differential equations driven by a multiplicative noise. In particular, it covers and completes Marie [21] on a generalized Cox-Ingersoll-Ross model. Subsection 4.2 deals with a Heston model (see Heston [14]) in which the volatility is modeled by a fractional Cox-Ingersoll-Ross equation in order to take benefits of the long memory and of the regularity of the paths of the fractional Brownian motion as in Comte, Coutin and Renault [6].

Notations. Let $\mathrm{J} \subset \mathbb{R}$ be a compact interval.

- The space $C^0(\mathrm{J}, \mathbb{R})$ of the continuous functions from J into $\mathbb{R}$ is equipped with the uniform norm $\|.\|_{\infty,\mathrm{J}}$ defined by :

$$\|x\|_{\infty,\mathrm{J}} := \sup_{t\in\mathrm{J}} |x_t|$$

for every $x \in C^0(\mathrm{J}, \mathbb{R})$. If $\mathrm{J} = [0, T]$ with $T > 0$, the uniform norm is denoted by $\|.\|_{\infty,T}$.

- The space $C^\alpha(\mathrm{J}, \mathbb{R})$ of the α-Hölder continuous functions from J into $\mathbb{R}$ is equipped with $\|.\|_{\infty,T}$, or with the α-Hölder norm $\|.\|_{\alpha,\mathrm{J}}$ defined by :

$$\|x\|_{\alpha,\mathrm{J}} := \sup_{(s,t)\in\mathrm{J}^2\,:\,s<t} \frac{|x_t - x_s|}{|t-s|^\alpha}$$

for every $x \in C^\alpha(\mathrm{J}, \mathbb{R})$. If $\mathrm{J} = [0, T]$ with $T > 0$, the α-Hölder norm is denoted by $\|.\|_{\alpha,T}$.

- The space $C^0(\mathbb{R}_+, \mathbb{R})$ (resp. $C^\alpha(\mathbb{R}_+, \mathbb{R})$) of the continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ (resp. of the locally α-Hölder continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$) is equipped with the compact-open topology (i.e. for every sequence $(f_n)_{n\in\mathbb{N}}$ of $C^0(\mathbb{R}_+, \mathbb{R})$, $f_n \to f$ when $n \to \infty$ for the compact-open topology if and only if, for every compact subset K of $\mathbb{R}_+$,

$$\lim_{n\to\infty} \|f_n - f\|_{\infty,\mathrm{K}} = 0).$$

2. Deterministic Properties of the Solution

The section deals with the global existence and the uniqueness of the solution of Equation (1.1), the regularity of the Itô map, the convergence of the implicit Euler scheme and some estimates.

First of all, some examples of drift functions satisfying Assumption 1.1 are provided.

Example 2.1. Consider $u, v, w, \gamma, \lambda, \mu > 0$.

- Put $b_1(x) := u(vx^{-\gamma} - wx)$ for every $x > 0$. If $1 - \alpha < \alpha\gamma$, then b_1 satisfies Assumption 1.1.
- Put $b_2(x) := u/(e^{vx^\gamma} - 1) - wx$ for every $x > 0$. If $1 \leq \alpha\gamma$, then b_2 satisfies Assumption 1.1.
- Put $b_1^*(x) := \lambda \sin(\mu x)$ for every $x > 0$. If $1 - \alpha < \alpha\gamma$ (resp. $1 \leq \alpha\gamma$) and $\lambda\mu < uw$ (resp. $\lambda\mu < w$), then $b_1 + b_1^*$ (resp $b_2 + b_1^*$) satisfies Assumption 1.1.
- Put $b_2^*(x) := \lambda \log(\mu x)$ for every $x > 0$. If $1 - \alpha < \alpha\gamma$ (resp. $1 \leq \alpha\gamma$), then $b_1 + b_2^*$ (resp $b_2 + b_2^*$) satisfies Assumption 1.1.

2.1. Existence and uniqueness of the solution. The subsection deals with the global existence, the uniqueness and an estimate of the solution of Equation (1.1).

Consider the deterministic analog of Equation (1.1) :

$$x_t = x_0 + \int_0^t b(x_s)ds + \sigma w_t \tag{2.1}$$

with $w \in C^\alpha(\mathbb{R}_+, \mathbb{R})$.

By Assumption 1.1.(1), Equation (2.1) has a unique solution on $[0, T_0]$, where

$$T_0 := \inf\{t > 0 : x_t = 0\}$$

with the convention $\inf(\emptyset) = \infty$.

Proposition 2.2. *Under Assumption 1.1, Equation (2.1) has a unique* $]0,\infty[$*-valued solution on* $\mathbb{R}_+$*.*

Proof. Assume that $T_0 < \infty$ and put $y := e^{R\cdot}x$ on $[0, T_0]$. For every $t \in [0, T_0]$, by the rough change of variable formula (see Gubinelli and Lejay [12], Lemma 6) :

$$\begin{aligned} y_t &= y_0 + \int_0^t Re^{Rs}x_s ds + \int_0^t e^{Rs}dx_s \\ &= y_0 + \int_0^t b^R(s, y_s)ds + \sigma w_t^R \end{aligned} \tag{2.2}$$

where

$$b^R(t, u) = Ru + e^{Rt}b(e^{-Rt}u)$$

for every $u > 0$, and

$$w_t^R := \int_0^t e^{Rs}dw_s.$$

For $t \in [0, T_0]$ arbitrarily chosen, by Equation (2.2) :

$$y_t + \int_t^{T_0} b^R(s, y_s)ds = \sigma(w_t^R - w_{T_0}^R).$$

Then, since w^R is α-Hölder continuous on $[0, T_0]$ and $b^R(s, u) > 0$ for every $(s, u) \in \mathbb{R}_+ \times]0, \infty[$ by Assumption 1.1.(3) :

$$\begin{aligned} y_s &\leq |\sigma|\|w^R\|_{\alpha,T_0}|s - T_0|^\alpha \text{ ; } \forall s \in [0, T_0] \text{ and} \\ \int_t^{T_0} b^R(s, y_s)ds &\leq |\sigma|\|w^R\|_{\alpha,T_0}|t - T_0|^\alpha. \end{aligned}$$

Since b is strictly decreasing on $]0, \infty[$ by Assumption 1.1.(2) :

$$\begin{aligned} \int_t^{T_0} b^R(s, y_s)ds &\geq \int_t^{T_0} b(e^{-Rs}y_s)ds \\ &\geq \int_0^{T_0-t} b(\|w^R\|_{\alpha,T_0}s^\alpha)ds. \end{aligned}$$

Therefore,

$$\int_0^{T_0-t} b(\|w^R\|_{\alpha,T_0}s^\alpha)ds \leq |\sigma|\|w^R\|_{\alpha,T_0}(T_0 - t)^\alpha.$$

However,

$$\int_0^{T_0-t} b(\|w^R\|_{\alpha,T_0}s^\alpha)ds = \infty$$

or

$$\lim_{t\to T_0^-}\frac{1}{(T_0-t)^\alpha}\int_0^{T_0-t} b(\|w^R\|_{\alpha,T_0}s^\alpha)ds=\infty$$

by Assumption 1.1.(4). That contradiction finishes the proof. $\square$

Proposition 2.3. *Under Assumption 1.1, the solution x of Equation (2.1) satisfies :*

$$\|x\|_{\infty,T}\leq x_0+|b(x_0)|T+2\sigma\|w\|_{\infty,T}$$

for every $T>0$.

Proof. Let $T>0$ and $t\in[0,T]$ be arbitrarily chosen, and put

$$T_{x_0}(t):=\sup\{s\in[0,t]:x_t\leq x_0\}.$$

If $T_{x_0}(t)=t$, then $0<x_t\leq x_0$. Assume that $T_{x_0}(t)<t$. Then,

$$x_t=x_0+\int_{T_{x_0}(t)}^t b(x_s)ds+\sigma[w_t-w_{T_{x_0}(t)}].$$

By Assumption 1.1.(2) :

$$\begin{aligned}\int_{T_{x_0}(t)}^t b(x_s)ds &\leq b(x_0)[t-T_{x_0}(t)]\\ &\leq |b(x_0)|t.\end{aligned}$$

Therefore,

$$0<x_t\leq x_0+|b(x_0)|T+2\sigma\|w\|_{\infty,T}.$$

That finishes the proof. $\square$

Notation. In the sequel, the solution of Equation (2.1) with the initial condition $x_0>0$ and the driving signal $w\in C^\alpha(\mathbb{R}_+,\mathbb{R})$ is denoted by $x(x_0,w)$. For every $T>0$, the restriction of the Itô map $x(.)$ to $]0,\infty[\times C^\alpha([0,T],\mathbb{R})$ is also denoted by $x(.)$. Then,

$$x(x_0,w)|_{[0,T]}=x(x_0,w|_{[0,T]})$$

for every $x_0,T>0$ and $w\in C^\alpha(\mathbb{R}_+,\mathbb{R})$.

2.2. Regularity of the Itô map. The two following propositions deal with the regularity of the Itô map $x(.)$.

Proposition 2.4. *Under Assumption 1.1 :*

$$\|x(x_0^1,w^1)-x(x_0^2,w^2)\|_{\infty,T}\leq|x_0^1-x_0^2|+2\sigma\|w^1-w^2\|_{\infty,T}$$

for every $T>0$, $x_0^1,x_0^2>0$ and $w^1,w^2\in C^\alpha([0,T],\mathbb{R})$.

Proof. Consider $x_0^1,x_0^2>0$ and $w^1,w^2\in C^\alpha([0,T],\mathbb{R})$ for $T>0$ arbitrarily chosen. Put $x^1:=x(x^1,w^1)$, $x^2:=x(x^2,w^2)$ and

$$T_{\rm c}:=\inf\{t\in[0,T]:x_t^1=x_t^2\}.$$

Assume that $x_0^1>x_0^2$ without loss of generality. Since x^1 and x^2 are continuous on $\mathbb{R}_+$, $x_s^1>x_s^2$ for every $s\in[0,T_{\rm c}]$. Since b is strictly decreasing on $]0,\infty[$ by Assumption 1.1.(2) :

$$b(x_s^1)-b(x_s^2)\leq 0$$

for every $s \in [0, T_{\rm c}]$. Then, for every $t \in [0, T_{\rm c}]$,

$$\begin{aligned} |x_t^1 - x_t^2| &= x_t^1 - x_t^2 \\ &= x_0^1 - x_0^2 + \int_0^t [b(x_s^1) - b(x_s^2)]ds + \sigma(w_t^1 - w_t^2) \\ &\leq |x_0^1 - x_0^2| + \sigma\|w^1 - w^2\|_{\infty,T}. \end{aligned} \tag{2.3}$$

For $t \in [T_{\rm c}, T]$ arbitrarily chosen, put

$$T_{\rm c}(t) := \sup\{s \in [T_{\rm c}, t] : x_s^1 = x_s^2\}.$$

Assume that $x_t^1 > x_t^2$ without loss of generality. Since x^1 and x^2 are continuous on $\mathbb{R}_+$, $x_s^1 > x_s^2$ for every $s \in [T_{\rm c}(t), t]$. Since b is strictly decreasing on $]0, \infty[$ by Assumption 1.1.(2) :

$$b(x_s^1) - b(x_s^2) \leq 0$$

for every $s \in [T_{\rm c}(t), t]$. Then,

$$\begin{aligned} |x_t^1 - x_t^2| &= x_t^1 - x_t^2 \\ &= \int_{T_{\rm c}(t)}^t [b(x_s^1) - b(x_s^2)]ds + \sigma(w_t^1 - w_t^2) - \sigma[w_{T_{\rm c}(t)}^1 - w_{T_{\rm c}(t)}^2] \\ &\leq 2\sigma\|w^1 - w^2\|_{\infty,T}. \end{aligned} \tag{2.4}$$

In conclusion, by inequalities (2.3) and (2.4) together :

$$\|x^1 - x^2\|_{\infty,T} \leq |x_0^1 - x_0^2| + 2\sigma\|w^1 - w^2\|_{\infty,T}.$$

That finishes the proof. □

Remark 2.5. By Proposition 2.4, for every $T > 0$, the Itô map $x(.)$ is Lipschitz continuous from

$$]0, \infty[\times C^\alpha([0, T], \mathbb{R}) \text{ into } C^0([0, T],]0, \infty[),$$

where $C^\alpha([0, T], \mathbb{R})$ is equipped with $\|.\|_{\infty,T}$ or $\|.\|_{\alpha,T}$.

Proposition 2.6. *Under Assumption 1.1, the Itô map $x(.)$ is continuously differentiable from*

$$]0, \infty[\times C^\alpha([0, T], \mathbb{R}) \textit{ into } C^0([0, T],]0, \infty[)$$

for every $T > 0$.

Proof. Consider $(x_0, w) \in \mathrm{E} :=]0, \infty[\times C^\alpha([0, T], \mathbb{R})$ for $T > 0$ arbitrarily chosen,

$$m_0 \in \left]0, \min_{t\in[0,T]} x_t(x_0, w)\right[\text{ and } \varepsilon_0 := -m_0 + \min_{t\in[0,T]} x_t(x_0, w).$$

Since $x(.)$ is continuous from E into $C^0([0, T],]0, \infty[)$ by Proposition 2.4 :

$$\begin{aligned} \forall\varepsilon \in]0, \varepsilon_0], \exists\eta &> 0 : \forall(\xi, h) \in \mathrm{E}, \\ (\xi, h) &\in B_{\mathrm{E}}((x_0, w), \eta) \Longrightarrow \|x(\xi, h) - x(x_0, w)\|_{\infty,T} < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{2.5}$$

In particular, for every $(\xi, h) \in B_{\mathrm{E}}((x_0, w), \eta)$, the function $x(\xi, h)$ is $[m_0, M_0]$-valued with $[m_0, M_0] \subset]0, \infty[$ and

$$M_0 := -m_0 + \min_{t\in[0,T]} x_t(x_0, w) + \max_{t\in[0,T]} x_t(x_0, w).$$

Then, since the function b is $[1/\alpha]+1$ times continuously differentiable on $]0,\infty[$ and has bounded derivatives on $[m_0, M_0]$ by Assumption 1.1.(1) ; $x(.)$ is continuously differentiable from $B_{\mathrm{E}}((x_0, w), \eta)$ into $C^0([0,T],]0,\infty[)$ by Friz and Victoir [9], theorems 11.3 and 11.6.

That finishes the proof, because (x_0, w) has been arbitrarily chosen. □

Remark 2.7. (1) In order to derive the Itô map with respect to the driving signal at point w in the direction $h \in C^{\beta}([0,T], \mathbb{R}^d)$, $\beta \in]0,1[$ has to satisfy the condition $\alpha+\beta > 1$ to ensure the existence of the geometric $1/\alpha$-rough path over $w + \varepsilon h$ ($\varepsilon > 0$) provided at Friz and Victoir [9], Theorem 9.34 when $d > 1$. That condition can be avoided when $d = 1$, because the canonical geometric $1/\alpha$-rough path over $w + \varepsilon h$ is

$$t \in [0,T] \longmapsto \left(1, w_t + \varepsilon h_t, \dots, \frac{(w_t + \varepsilon h_t)^{[1/\alpha]}}{[1/\alpha]!}\right).$$

(2) The first order directional derivative of $x(.)$ at point $(x_0, w) \in \mathrm{E}$ in the direction $(\xi, h) \in \mathrm{E}$ is denoted by $\mathrm{D}_{(\xi,h)}x_.(x_0, w)$ and

$$\mathrm{D}_{(\xi,h)}x_t(x_0, w) = \xi + \int_0^t \dot{b}[x_s(x_0, w)]\mathrm{D}_{(\xi,h)}x_s(x_0, w)ds + \sigma h_t$$

for every $t \in [0,T]$. Then,

$$\mathrm{D}_{(\xi,h)}x_.(x_0, w) = \int_0^. (\xi + \sigma h_s) \exp\left[\int_s^. \dot{b}[x_u(x_0, w)]du\right] ds.$$

So, by Assumption 1.1.(2) :

$$|\mathrm{D}_{(\xi,h)}x_t(x_0, w)| \leq T(\xi + \sigma\|h\|_{\infty,T})$$

for every $t \in [0,T]$.

The end of the subsection is devoted to three consequences of propositions 2.4 and 2.6 on the partial Itô map $x(., w)$ for $w \in C^{\alpha}(\mathbb{R}_+, \mathbb{R})$ arbitrarily fixed.

Corollary 2.8. *Under Assumption 1.1, $x_t(., w)$ is (strictly) increasing on $]0,\infty[$ for every $t > 0$.*

Proof. By Proposition 2.6 :

$$\begin{aligned}\frac{\partial}{\partial x_0}x_t(x_0, w) &= \mathrm{D}_{(1,0)}x_t(x_0, w)\\ &= \int_0^t \exp\left[\int_s^t \dot{b}[x_u(x_0, w)]du\right] ds > 0\end{aligned}$$

for every $t > 0$. That finishes the proof. □

Corollary 2.9. *Under Assumption 1.1, there exists $x(0, w) \in C^{\alpha}(\mathbb{R}_+, \mathbb{R}_+)$ such that $x_t(0, w) > 0$ for every $t > 0$, and*

$$\lim_{x_0 \rightarrow 0} \|x(x_0, w) - x(0, w)\|_{\infty,T} = 0 \ ; \ \forall T > 0.$$

Proof. The existence of the limit $x(0,w)$ of $x(.,w)$ in $C^0(\mathbb{R}_+,\mathbb{R}_+)$ when the initial condition x_0 goes down to 0 is proved in a first step. At the second step, it is shown that $x_t(0,w)>0$ for every $t>0$.

Step 1. Consider a strictly positive real sequence $(x_0^n)_{n\in\mathbb{N}}$ such that :

$$\lim_{n\rightarrow\infty} x_0^n = 0.$$

Let $T>0$ be arbitrarily chosen. By Proposition 2.4 :

$$\|x(x_0^n,w)-x(x_0^m,w)\|_{\infty,T}\leq |x_0^n-x_0^m| \ ;\ \forall m,n\in\mathbb{N}.$$

Then, since $C^0([0,T],\mathbb{R})$ is a Banach space, we see that $x(x_0^n,w)|_{[0,T]}$ converges in $C^0([0,T],\mathbb{R}_+)$ when n goes to infinity. Since the strictly positive real sequence $(x_0^n)_{n\in\mathbb{N}}$ has been arbitrarily chosen, there exists a function $x(0,w|_{[0,T]})$ belonging to $C^0([0,T],\mathbb{R}_+)$ such that :

$$\lim_{x_0\rightarrow 0}\|x(x_0,w)-x(0,w|_{[0,T]})\|_{\infty,T}=0.$$

Consider the function $x(0,w):\mathbb{R}_+\rightarrow\mathbb{R}_+$ such that :

$$x(0,w)|_{[0,T]}:=x(0,w|_{[0,T]})$$

for every $T>0$. By construction, $x(0,w)$ is the limit of $x(.,w)$ in $C^0(\mathbb{R}_+,\mathbb{R}_+)$ when the initial condition x_0 goes down to 0.

Step 2. For $t>s\geq 0$ and $x_0>0$ arbitrarily chosen :

$$\begin{aligned}
x_t(x_0,w)-x_s(x_0,w)-\sigma(w_t-w_s) &= \int_s^t b[x_u(x_0,w)]du\\
&\geq (t-s)b\left[\sup_{u\in[s,t]} x_u(x_0,w)\right]
\end{aligned}$$

by Assumption 1.1.(2). Assume that $x_u(0,w)=0$ for every $u\in[s,t]$. Then,

$$\begin{aligned}
\lim_{x_0\rightarrow 0} x_t(x_0,w)-x_s(x_0,w)-\sigma(w_t-w_s) &\geq \lim_{x_0\rightarrow 0}(t-s)b\left[\sup_{u\in[s,t]} x_u(x_0,w)\right]\\
&= \infty
\end{aligned}$$

by Assumption 1.1.(4). However, by the first step of the proof :

$$\begin{aligned}
\lim_{x_0\rightarrow 0} x_t(x_0,w)-x_s(x_0,w)-\sigma(w_t-w_s) &= x_t(0,w)-x_s(0,w)-\sigma(w_t-w_s)\\
&< \infty.
\end{aligned}$$

Therefore, for every $s>t\geq 0$, there exists $u\in[s,t]$ such that $x_u(0,w)>0$.

In particular, there exists a strictly positive real sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n\downarrow 0$ when $n\rightarrow\infty$, and

$$x_{t_n}(0,w)>0\ ;\ \forall n\in\mathbb{N}.$$

Let $n\in\mathbb{N}$ be arbitrarily chosen. Since $x(0,w)$ is continuous on $\mathbb{R}_+$ by construction, $x_t(0,w)>0$ for every $t\in[t_n,\tau_0(t_n)[$, where

$$\tau_0(t_n):=\inf\{t>t_n : x_t(0,w)=0\}.$$

For every $t \in [0, \tau_0(t_n) - t_n[$, consider

$$\tau_{\min}(n,t) := \text{argmin}_{s \in [t_n, t_n+t]} x_s(0,w).$$

Let $t \in [0, \tau_0(t_n) - t_n[$ be arbitrarily chosen. By Assumption 1.1.(2) and Corollary 2.8 :

$$b[x_s(x_0,w)] \leq b[x_s(0,w)] \leq b[x_{\tau_{\min}(n,t)}(0,w)] < \infty$$

for every $s \in [t_n, t_n + t]$ and $x_0 > 0$. Then, by Lebesgue's theorem :

$$\begin{aligned} x_{t_n+t}(0,w) &= x_{t_n}(0,w) + \lim_{x_0 \to 0} \int_{t_n}^{t_n+t} b[x_s(x_0,w)]ds + \sigma(w_{t_n+t} - w_{t_n}) \\ &= x_{t_n}(0,w) + \int_0^t b[x_{t_n+s}(0,w)]ds + \sigma w_t^{t_n} \end{aligned}$$

with $w^{t_n} := w_{t_n+.} - w_{t_n}$ on $\mathbb{R}_+$. Therefore, $x_{t_n+.}(0,w) = x[x_{t_n}(0,w), w^{t_n}]$ on $[0, \tau_0(t_n) - t_n[$. Since $x_{t_n}(0,w) > 0$ and w^{t_n} belongs to $C^\alpha(\mathbb{R}_+, \mathbb{R})$, by Proposition 2.2 :

$$\begin{aligned} \tau_0(t_n) &= \inf\{t > 0 : x_{t_n+t}(0,w) = 0\} \\ &= \inf\{t > 0 : x_t[x_{t_n}(0,w), w^{t_n}] = 0\} \\ &= \infty. \end{aligned}$$

So, $x(0,w)$ is a $]0,\infty[$-valued function on $[t_n, \infty[$ for every $n \in \mathbb{N}$. Since $t_n \downarrow 0$ when $n \to \infty$, $x(0,w)$ is a $]0,\infty[$-valued function on $]0,\infty[$. □

Corollary 2.10. *Under Assumption 1.1 :*

$$|x_t(x_0^1,w) - x_t(x_0^2,w)| \leq |x_0^1 - x_0^2|e^{-Kt}$$

for every $x_0^1, x_0^2, t \in \mathbb{R}_+$.

Proof. Put $x^1 := x(x_0^1,w)$ and $x^2 := x(x_0^2,w)$ for $x_0^1, x_0^2 > 0$ such that $x_0^1 \neq x_0^2$. By Proposition 2.8, $x_t^1 \neq x_t^2$ for every $t \in \mathbb{R}_+$.

The function $x^1 - x^2$ satisfies :

$$x_t^1 - x_t^2 = x_0^1 - x_0^2 + \int_0^t [b(x_s^1) - b(x_s^2)]ds \; ; \forall t \in \mathbb{R}_+. \tag{2.6}$$

Let $t \in \mathbb{R}_+$ be arbitrarily chosen. By Equation (2.6) :

$$\begin{aligned} (x_t^1 - x_t^2)^2 &= (x_0^1 - x_0^2)^2 + 2\int_0^t (x_s^1 - x_s^2)d(x^1 - x^2)_s \\ &= (x_0^1 - x_0^2)^2 + 2\int_0^t (x_s^1 - x_s^2)[b(x_s^1) - b(x_s^2)]ds. \end{aligned}$$

Then,

$$\frac{\partial}{\partial t}(x_t^1 - x_t^2)^2 = 2(x_t^1 - x_t^2)^2 \frac{b(x_t^1) - b(x_t^2)}{x_t^1 - x_t^2}. \tag{2.7}$$

By Assumption 1.1.(2) :

$$\forall u > 0, \, \dot{b}(u) < -K.$$

Then, by the mean-value theorem, there exists $c_t \in]x_t^1 \wedge x_t^2, x_t^1 \vee x_t^2[$ such that :

$$\frac{b(x_t^1) - b(x_t^2)}{x_t^1 - x_t^2} = \dot{b}(c_t) < -K.$$

Therefore, by Equation (2.7) :

$$\frac{\partial}{\partial t}(x_t^1 - x_t^2)^2 \leq -2K(x_t^1 - x_t^2)^2.$$

In conclusion,

$$|x_t^1 - x_t^2| \leq |x_0^1 - x_0^2|e^{-Kt}. \tag{2.8}$$

If $x_0^1 = 0$, $x_0^2 = 0$ or $x_0^1 = x_0^2$, Inequality (2.8) holds true. □

2.3. Existence, uniqueness, and convergence of the implicit Euler scheme. Let $T > 0$ and $n \in \mathbb{N}^*$ be arbitrarily fixed, and consider a dissection $(t_0^n, t_1^n \dots, t_n^n)$ of $[0, T]$.

The subsection deals with the global existence, the uniqueness, an estimate and the convergence of the implicit Euler scheme associated to Equation (2.1) and to the dissection $(t_0^n, t_1^n, \dots, t_n^n)$:

$$x_{k+1}^n = x_k^n + b(x_{k+1}^n)(t_{k+1}^n - t_k^n) + \sigma(w_{t_{k+1}^n} - w_{t_k^n}) \tag{2.9}$$

with $x_0^n := x_0 > 0$.

Proposition 2.11. *Under Assumption 1.1, Equation (2.9) has a unique* $]0, \infty[$-*valued solution on* $\{0, \dots, n\}$.

Proof. Let $\lambda > 0$ and $\mu \in \mathbb{R}$ be arbitrarily chosen, and put $\varphi(x) := \mu + \lambda b(x) - x$ for every $x > 0$.

By Assumption 1.1.(1)-(2), the function φ is continuously differentiable on $]0, \infty[$, and

$$\dot{\varphi}(x) = \lambda \dot{b}(x) - 1 < 0$$

for every $x > 0$. So, φ is strictly decreasing on $]0, \infty[$. By Assumption 1.1.(4) :

$$\lim_{x \to 0^+} \varphi(x) = \mu + \lambda \lim_{x \to 0^+} b(x) = \infty.$$

Let $x > x_* > 0$ be arbitrarily chosen. By Assumption 1.1.(2) :

$$b(x) < -K(x - x_*) + b(x_*).$$

Then,

$$\varphi(x) < -(\lambda K + 1)x + \mu + \lambda[K x_* + b(x_*)].$$

So,

$$\lim_{x \to \infty} b(x) = -\infty.$$

Therefore, the equation $\varphi(x) = 0$ has a unique solution belonging to $]0, \infty[$.

In conclusion, by recurrence, Equation (2.9) has a unique $]0, \infty[$-valued solution on $\{0, \dots, n\}$. □

Proposition 2.12. *Under Assumption 1.1, the solution* x^n *of Equation (2.9) satisfies :*

$$\max_{k \in \{0, \dots, n\}} x_k^n \leq x_0 + |b(x_0)|T + 2\sigma\|w\|_{\infty, T}.$$

Proof. Let $k \in \{1, \dots, n\}$ be arbitrarily chosen, and put

$$n(x_0, k) := \max\{i \in \{0, \dots, k\} : x_i^n \leq x_0\}.$$

If $n(x_0, k) = k$, then $0 < x_k^n \leq x_0$. Assume that $n(x_0, k) < k$. Then,

$$\begin{aligned} x_k^n - x_{n(x_0,k)}^n &= \sum_{i=n(x_0,k)}^{k-1} x_{i+1}^n - x_i^n \\ &= \sigma[w_{t_k^n} - w_{t_{n(x_0,k)}^n}] + \sum_{i=n(x_0,k)}^{k-1} b(x_{i+1}^n)(t_{i+1}^n - t_i^n). \end{aligned}$$

By Assumption 1.1.(2) :

$$\begin{aligned} \sum_{i=n(x_0,k)}^{k-1} b(x_{i+1}^n)(t_{i+1}^n - t_i^n) &\leq b(x_0)[t_k^n - t_{n(x_0,k)}^n] \\ &\leq |b(x_0)|T. \end{aligned}$$

Therefore,

$$0 < x_k^n \leq x_0 + |b(x_0)|T + 2\sigma\|w\|_{\infty,T}.$$

That finishes the proof. $\square$

Notations. Throughout the subsection, the solution of Equation (2.1) is denoted by x instead of $x(x_0, w)$ for the sake of readability. The solution of Equation (2.9) is denoted by x^n. For every $t \in]0, T]$, put

$$x_t^n := \sum_{k=0}^{n-1} \left[x_k^n + \frac{x_{k+1}^n - x_k^n}{t_{k+1}^n - t_k^n}(t - t_k^n)\right] \mathbf{1}_{]t_k^n, t_{k+1}^n]}(t).$$

The function $t \in [0, T] \mapsto x_t^n$ is also denoted by x^n and called the step-n implicit Euler scheme associated to Equation (2.1) and to the dissection $(t_0^n, t_1^n, \dots, t_n^n)$.

In the sequel, $t_k^n := kT/n$ for every $n \in \mathbb{N}^*$ and $k \in \{0, \dots, n\}$.

Theorem 2.13. *Under Assumption 1.1 :*

$$\begin{aligned} \|x^n - x\|_{\infty,T} \leq\ & [(\|\dot{b}\|_{\infty,[x_*,x^*]}^2 + \|\dot{b}\|_{\infty,[x_*,x^*]} + 1)\|x\|_{\alpha,T} \\ & + \|b\|_{\infty,[x_*,x^*]} + \|w\|_{\alpha,T}](T^\alpha \vee T^{\alpha+2})n^{-\alpha} \end{aligned}$$

with

$$x_* := \inf_{t\in[0,T]} x_t \textit{ and } x^* := \sup_{t\in[0,T]} x_t.$$

Proof. Consider the vector $(\xi_0^n, \dots, \xi_n^n)$ defined by $\xi_k^n := x_{t_k^n}$ for $k \in \{0, \dots, n\}$. By Equation (2.1) :

$$\xi_{k+1}^n = \xi_k^n + b(\xi_{k+1}^n)(t_{k+1}^n - t_k^n) + \sigma(w_{t_{k+1}^n} - w_{t_k^n}) + \varepsilon_k^n$$

with

$$\varepsilon_k^n := -\int_{t_k^n}^{t_{k+1}^n} [b(\xi_{k+1}^n) - b(x_t)]dt$$

for every $k \in \{0, \dots, n-1\}$.

Let $k \in \{1,\dots,n\}$ and $i \in \{0,\dots,k-1\}$ be arbitrarily chosen. If $x_{i+1}^n > \xi_{i+1}^n$, since b is strictly decreasing on $]0,\infty[$ by Assumption 1.1.(2) :

$$b(x_{i+1}^n) - b(\xi_{i+1}^n) \leq 0.$$

Then,

$$\begin{aligned}
|x_{i+1}^n - \xi_{i+1}^n| &= x_{i+1}^n - \xi_{i+1}^n \\
&= x_i^n - \xi_i^n + [b(x_{i+1}^n) - b(\xi_{i+1}^n)](t_{i+1}^n - t_i^n) - \varepsilon_i^n \\
&\leq |x_i^n - \xi_i^n| + |\varepsilon_i^n|. \qquad (2.10)
\end{aligned}$$

If $x_{i+1}^n \leq \xi_{i+1}^n$, since b is strictly decreasing on $]0,\infty[$ by Assumption 1.1.(2) :

$$b(\xi_{i+1}^n) - b(x_{i+1}^n) \leq 0.$$

Then,

$$\begin{aligned}
|x_{i+1}^n - \xi_{i+1}^n| &= \xi_{i+1}^n - x_{i+1}^n \\
&= \xi_i^n - x_i^n + [b(\xi_{i+1}^n) - b(x_{i+1}^n)](t_{i+1}^n - t_i^n) + \varepsilon_i^n \\
&\leq |x_i^n - \xi_i^n| + |\varepsilon_i^n|. \qquad (2.11)
\end{aligned}$$

So, by inequalities (2.10) and (2.11) together :

$$|x_{i+1}^n - \xi_{i+1}^n| \leq |x_i^n - \xi_i^n| + |\varepsilon_i^n|.$$

By recurrence :

$$|x_k^n - \xi_k^n| \leq \sum_{i=0}^{k-1} |\varepsilon_i^n|. \qquad (2.12)$$

By Assumption 1.1.(1), b is Lipschitz continuous on $[x_*, x^*]$. Then,

$$\begin{aligned}
|\varepsilon_i^n| &\leq \|\dot{b}\|_{\infty,[x_*,x^*]} \|x\|_{\alpha,T} \int_{t_i^n}^{t_{i+1}^n} (t_{i+1}^n - t)^\alpha dt \\
&\leq \|\dot{b}\|_{\infty,[x_*,x^*]} \|x\|_{\alpha,T} \frac{T^{\alpha+1}}{n^{\alpha+1}}.
\end{aligned}$$

So, by Equation (2.12) :

$$|x_k^n - \xi_k^n| \leq \|\dot{b}\|_{\infty,[x_*,x^*]} \|x\|_{\alpha,T} \frac{T^{\alpha+1}}{n^\alpha}. \qquad (2.13)$$

Let $t \in]0,T]$ be arbitrarily chosen. There exists $k \in \{0,\dots,n-1\}$ such that $t \in]t_k^n, t_{k+1}^n]$. By Inequality (2.13) :

$$\begin{aligned}
|x_{k+1}^n - x_k^n| &\leq [|[b(x_{k+1}^n) - b(\xi_{k+1}^n)| + |b(\xi_{k+1}^n)|](t_{k+1}^n - t_k^n) \\
&\quad + \|w\|_{\alpha,T}(t_{k+1}^n - t_k^n)^\alpha \\
&\leq [[\|\dot{b}\|_{\infty,[x_*,x^*]} |x_{k+1}^n - \xi_{k+1}^n| + \|b\|_{\infty,[x_*,x^*]}]T + \|w\|_{\alpha,T} T^\alpha] n^{-\alpha} \\
&\leq [\|\dot{b}\|_{\infty,[x_*,x^*]}^2 \|x\|_{\alpha,T} + \|b\|_{\infty,[x_*,x^*]} + \|w\|_{\alpha,T}] \\
&\quad \times (T^\alpha \vee T^{\alpha+2}) n^{-\alpha}. \qquad (2.14)
\end{aligned}$$

By inequalities (2.13) and (2.14) together :

$$\begin{aligned}
|x_t^n - x_t| &\leq |x_t^n - x_k^n| + |x_k^n - \xi_k^n| + |\xi_k^n - x_t| \\
&\leq |x_{k+1}^n - x_k^n| + (\|\dot{b}\|_{\infty,[x_*,x^*]} + 1)\|x\|_{\alpha,T}(T^\alpha \vee T^{\alpha+1})n^{-\alpha} \\
&\leq [(\|\dot{b}\|_{\infty,[x_*,x^*]}^2 + \|\dot{b}\|_{\infty,[x_*,x^*]} + 1)\|x\|_{\alpha,T} + \|b\|_{\infty,[x_*,x^*]} + \|w\|_{\alpha,T}] \\
&\quad \times (T^\alpha \vee T^{\alpha+2})n^{-\alpha}.
\end{aligned}$$

That finishes the proof. □

3. Probabilistic and Statistical Properties of the Solution

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the canonical probability space associated to the stochastic process B.

The solution of Equation (1.1) is the stochastic process $X(x_0) := (X_t(x_0))_{t\in\mathbb{R}_+}$ such that :

$$X_t(x_0, \omega) := x_t[x_0, B(\omega)]$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

Notations :

- The expectation operator associated to the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$.
- For every $p > 0$, the space of random variables $U : \Omega \to \mathbb{R}$ such that $\mathbb{E}(|U|^p) < \infty$ is denoted by $L^p(\Omega, \mathbb{P})$ and equipped with its usual norm $\|.\|_p$.

Under Assumption 1.1, if B is a centered Gaussian process with locally α-Hölder continuous paths, by Proposition 2.3 together with Fernique's theorem (see Fernique [8]) :

$$\|X(x_0)\|_{\infty,T} \in L^p(\Omega, \mathbb{P})$$

for every $p, T > 0$.

The section deals with probabilistic and statistical properties of $X(x_0)$, obtained via its deterministic properties proved previously and various additional conditions on the signal B.

3.1. Ergodicity of the solution. Assume that B is a two-sided fractional Brownian motion of Hurst parameter $H \in]0,1[$ ($\alpha \in]0, H[$).

Let $\theta := (\theta_t)_{t\in\mathbb{R}}$ be the dynamical system on $(\Omega, \mathcal{A})$, called Wiener shift, such that :

$$\theta_t\omega := \omega_{t+.} - \omega_t$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}$. By Maslowski and Schmalfuss [23], $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ is an metric dynamical system (i.e.

- $(t, \omega) \in \mathbb{R} \times \Omega \longmapsto \theta_t\omega$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{A}, \mathcal{A}$-measurable.
- For every $t \in \mathbb{R}$, $\theta_t\mathbb{P} = \mathbb{P}$ where

$$(\theta_t\mathbb{P})(A) := \mathbb{P}(\{\omega \in \Omega : \theta_t\omega \in A\}) \ ; \forall A \in \mathcal{A}),$$

which is ergodic.

Lemma 3.1. *There exists a θ-invariant set $\Omega^* \in \mathcal{A}$ satisfying $\mathbb{P}(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$,*

$$\exists C(\omega) > 0 : \forall t \in \mathbb{R},\ |B_t(\omega)| \leq C(\omega)(1 + |t|^2).$$

For a proof, see Gess et al. [11], Lemma 3.3 generalizing Maslowski and Schmalfuss [23], Lemma 2.6.

Remark 3.2. (1) In the sequel, Ω^* is equipped with the trace σ-algebra

$$\mathcal{A}^* := \{A \cap \Omega^* \ ;\ A \in \mathcal{A}\}.$$

(2) $(\Omega^*, \mathcal{A}^*, \mathbb{P}, \theta)$ is also an ergodic metric dynamical system.

The map

$$X(.) : (\omega, x_0, t) \in \Omega \times \mathbb{R}_+^2 \longmapsto X_t(x_0, \omega)$$

is a continuous random dynamical system on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ over the metric dynamical systems $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ and $(\Omega^*, \mathcal{A}^*, \mathbb{P}, \theta)$.

The reader can refer to Arnold [1] on random dynamical systems.

Notation. Let $(W_t)_{t \in \mathbb{R}_+}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbb{P})$. For every $\omega \in \Omega$ and $t, T \in \mathbb{R}_+$,

$$W_{t,T}(\omega) := W_t(\theta_{-T}\omega).$$

Proposition 3.3. *Under Assumption 1.1, for every $\omega \in \Omega^*$, there exists a constant $C(\omega) > 0$ such that for every $t, T, x_0 \in \mathbb{R}_+$ and $\varepsilon \geq x_0$,*

$$|X_{t,T}(x_0, \omega) - \varepsilon| \leq \varepsilon + |b(\varepsilon)|t + C(\omega)(1 + t + T)^2.$$

Proof. Let $\omega \in \Omega^*$, $t, T \in \mathbb{R}_+$ and $\varepsilon \geq x_0 > 0$ be arbitrarily chosen, and put

$$\tau_t^-(\varepsilon, \theta_{-T}\omega) := \sup\{s \in [0, t] : X_s(x_0, \theta_{-T}\omega) \leq \varepsilon\}.$$

If $\tau_t^-(\varepsilon, \theta_{-T}\omega) = t$, then

$$|X_{t,T}(x_0, \omega) - \varepsilon| \leq \varepsilon.$$

Assume that $\tau_t^-(\varepsilon, \theta_{-T}\omega) < t$. Then,

$$\begin{aligned} X_{t,T}(x_0, \omega) &= X_{\tau_t^-(\varepsilon, \theta_{-T}\omega), T}(x_0, \omega) + \int_{\tau_t^-(\varepsilon, \theta_{-T}\omega)}^t b[X_{s,T}(x_0, \omega)]ds \\ &\quad + \sigma[B_{t,T}(\omega) - B_{\tau_t^-(\varepsilon, \theta_{-T}\omega), T}(\omega)] \\ &= \varepsilon + \int_{\tau_t^-(\varepsilon, \theta_{-T}\omega)}^t b[X_{s,T}(x_0, \omega)]ds \\ &\quad + \sigma[B_{t-T}(\omega) - B_{\tau_t^-(\varepsilon, \theta_{-T}\omega) - T}(\omega)]. \end{aligned} \tag{3.1}$$

On one hand, by Assumption 1.1.(2) :

$$\begin{aligned} \int_{\tau_t^-(\varepsilon, \theta_{-T}\omega)}^t b[X_{s,T}(x_0, \omega)]ds &\leq b(\varepsilon)[t - \tau_t^-(\varepsilon, \theta_{-T}\omega)] \\ &\leq |b(\varepsilon)|t. \end{aligned} \tag{3.2}$$

On the other hand, by Lemma 3.1, there exists a constant $C_0(\omega) > 0$, not depending on t, T, x_0 and ε, such that :

$$\begin{aligned}|B_{t-T}(\omega) - B_{\tau_t^-(\varepsilon,\theta_{-T}\omega)-T}(\omega)| &\leq C_0(\omega)[2 + |t-T|^2 + |\tau_t^-(\varepsilon,\theta_{-T}\omega) - T|^2] \\ &\leq 4C_0(\omega)(1+t+T)^2. \qquad (3.3)\end{aligned}$$

Therefore, by Equality (3.1) together with inequalities (3.2) and (3.3) :

$$0 \leq X_{t,T}(x_0,\omega) - \varepsilon \leq |b(\varepsilon)|t + 4\sigma C_0(\omega)(1+t+T)^2.$$

In conclusion, by putting $C(\omega) := 4\sigma C_0(\omega)$, for every $t, T \in \mathbb{R}_+$,

$$|X_{t,T}(x_0,\omega) - \varepsilon| \leq \varepsilon + |b(\varepsilon)|t + C(\omega)(1+t+T)^2. \qquad (3.4)$$

If $x_0 = 0$, Inequality (3.4) holds true. □

Theorem 3.4. *Under Assumption 1.1, there exists a random variable $X^* : \Omega \to \mathbb{R}_+$ belonging to $L^p(\Omega,\mathbb{P})$ for every $p > 0$, such that for every $x_0 \in \mathbb{R}_+$,*

$$|X_T(x_0) - X^* \circ \theta_T| \xrightarrow[T\to\infty]{} 0$$

almost surely and in $L^p(\Omega,\mathbb{P})$ for every $p > 0$.

Proof. Let $\omega \in \Omega^*$, $t, x_0 \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $p > 0$ be arbitrarily chosen.

Almost sure convergence. By the cocycle property of the random dynamical system $X(.)$, Corollary 2.10 and Proposition 3.3 ; there exists a constant $C(\omega) > 0$, not depending on t, n and x_0, such that for every $\varepsilon \geq x_0$,

$$\begin{aligned}|X_n(x_0,\theta_{-n}\omega) - X_{n+1}(x_0,\theta_{-(n+1)}\omega)| &= |X_n(x_0,\theta_{-n}\omega) \\ &\quad - X_n[X_1(x_0,\theta_{-(n+1)}\omega),\theta_{-n}\omega]| \\ &\leq e^{-Kn}|x_0 - X_1(x_0,\theta_{-(n+1)}\omega)| \\ &\leq e^{-Kn}[|x_0-\varepsilon| \\ &\quad + |X_1(x_0,\theta_{-(n+1)}\omega) - \varepsilon|] \qquad (3.5) \\ &\leq e^{-Kn}[|x_0-\varepsilon| + \varepsilon + |b(\varepsilon)| + C(\omega)(3+n)^2].\end{aligned}$$

Since $n^k =_{n\to\infty} \mathrm{o}(e^{Kn})$ for every $k \in \mathbb{N}$, $(X_n(x_0,\theta_{-n}\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence, and its limit $X^0(\omega)$ is not depending on x_0 because for every other initial condition $x_1 > 0$,

$$|X_n(x_0,\theta_{-n}\omega) - X_n(x_1,\theta_{-n}\omega)| \leq e^{-Kn}|x_0 - x_1| \xrightarrow[n\to\infty]{} 0.$$

For every $\varepsilon \geq x_0$,

$$\begin{aligned}|X_t(x_0,\theta_{-t}\omega) - X^0(\omega)| &\leq |X_t(x_0,\theta_{-t}\omega) - X_{[t]}(x_0,\theta_{-[t]}\omega)| \qquad (3.6) \\ &\quad + |X_{[t]}(x_0,\theta_{-[t]}\omega) - X^0(\omega)| \\ &= |X_{[t]}[X_{t-[t]}(x_0,\theta_{-t}\omega),\theta_{-[t]}\omega] - X_{[t]}(x_0,\theta_{-[t]}\omega)| \\ &\quad + |X_{[t]}(x_0,\theta_{-[t]}\omega) - X^0(\omega)| \\ &\leq e^{-K[t]}[|x_0-\varepsilon| + |X_{t-[t]}(x_0,\theta_{-t}\omega) - \varepsilon|] \qquad (3.7) \\ &\quad + |X_{[t]}(x_0,\theta_{-[t]}\omega) - X^0(\omega)| \\ &\leq e^{-K[t]}[|x_0-\varepsilon| + \varepsilon + |b(\varepsilon)| + C(\omega)(2+[t])^2] \\ &\quad + |X_{[t]}(x_0,\theta_{-[t]}\omega) - X^0(\omega)|.\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} |X_t(x_0, \theta_{-t}\omega) - X^0(\omega)| = 0 \tag{3.8}$$

because $[t]^k =_{t\to\infty} \mathrm{o}(e^{K[t]})$ for every $k \in \mathbb{N}$. By the cocycle property of the random dynamical system $X(.)$:

$$\begin{aligned} X_t[X_n(x_0, \theta_{-n}\omega), \omega] &= X_{t+n}(x_0, \theta_{-n}\omega) \\ &= X_{t+n}[x_0, \theta_{-(t+n)}(\theta_t\omega)]. \end{aligned} \tag{3.9}$$

By continuity of $X(., \omega)$ from $\mathbb{R}_+$ into $C^0(\mathbb{R}_+)$, Corollary 2.9 and (3.8) ; when n goes to infinity in Equality (3.9) :

$$X_t[X^0(\omega), \omega] = X^0(\theta_t\omega).$$

Since $(\Omega^*, \mathcal{A}^*, \mathbb{P}, \theta)$ is an ergodic metric dynamical system and X^0 is a (generalized) random fixed point of the continuous random dynamical system $X(.)$, $(X^0 \circ \theta_t)_{t\in\mathbb{R}_+}$ is a stationary solution of Equation (1.1). Therefore, for every $\omega \in \Omega^*$,

$$\lim_{t\to\infty} |X_t(x_0, \omega) - X^0(\theta_t\omega)| = 0$$

because all solutions of Equation (1.1) converge pathwise forward to each other in time by Corollary 2.10.

Convergence in $L^p(\Omega, \mathbb{P})$. Since B and $(B_{s-t} - B_{-t})_{s\in\mathbb{R}}$ have the same distribution $\mathbb{P}$, for every $U \in L^p(\Omega, \mathbb{P})$ and $s \in \mathbb{R}_+$,

$$\|X_s(x_0) \circ \theta_{-t} - U\|_p = \|X_s(x_0) - U \circ \theta_t\|_p. \tag{3.10}$$

By Inequality (3.5) and Equality (3.10), for every $\varepsilon \geq x_0$,

$$\|X_n(x_0) \circ \theta_{-n} - X_{n+1}(x_0) \circ \theta_{-(n+1)}\|_p \leq e^{-Kn}[|x_0 - \varepsilon| + \|X_1(x_0, \omega) - \varepsilon\|_p].$$

Then, since the set $L^p(\Omega, \mathbb{P})$ equipped with $\|.\|_p$ is a Banach space, there exists $X^* \in L^p(\Omega, \mathbb{P})$ such that :

$$\lim_{n\to\infty} \|X_n(x_0) \circ \theta_{-n} - X^*\|_p = 0$$

and $X^*(\omega) = X^0(\omega)$ for every $\omega \in \Omega^*$. By Inequality (3.7) and Equality (3.10), for every $\varepsilon \geq x_0$,

$$\begin{aligned} \|X_t(x_0) \circ \theta_{-t} - X^*\|_p \quad &\leq \quad e^{-K[t]} \left[|x_0 - \varepsilon| + \sup_{s\in[0,1]} \|X_s(x_0) - \varepsilon\|_p \right] \\ &\quad + \|X_{[t]}(x_0) \circ \theta_{-[t]} - X^*\|_p. \end{aligned}$$

Then,

$$\lim_{t\to\infty} \|X_t(x_0) \circ \theta_{-t} - X^*\|_p = 0.$$

Therefore, by Equality (3.10) :

$$\begin{aligned} \lim_{t\to\infty} \|X_t(x_0) - X^* \circ \theta_t\|_p &= \lim_{t\to\infty} \|X_t(x_0) \circ \theta_{-t} - X^*\|_p \\ &= 0. \end{aligned}$$

□

Corollary 3.5. *Under Assumption 1.1, for every uniformly continuous function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with polynomial growth, and every $x_0 \in \mathbb{R}_+$,*

$$\frac{1}{T}\int_0^T \varphi[X_t(x_0)]dt \xrightarrow[T\rightarrow\infty]{} \mathbb{E}[\varphi(X^*)]$$

almost surely and in $L^p(\Omega, \mathbb{P})$ for every $p > 0$.

Proof. Let $\omega \in \Omega^*$, $x_0 \in \mathbb{R}_+$ and $p > 0$ be arbitrarily chosen. Consider also

$$I_T(\varphi, x_0) := \frac{1}{T}\int_0^T \varphi[X_t(x_0)]dt \ ; \forall T > 0$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a uniformly continuous function such that :

$$\forall x \in \mathbb{R}_+, \ |\varphi(x)| \leq c(1 + x^n) \tag{3.11}$$

with $c > 0$ and $n \in \mathbb{N}^*$.

Almost sure convergence. On one hand, since φ has a polynomial growth and X^* belongs to $L^p(\Omega, \mathbb{P})$ for every $p > 0$ by Theorem 3.4, $\varphi(X^*)$ too. Moreover, $(\Omega^*, \mathcal{A}^*, \mathbb{P}, \theta)$ is an ergodic metric dynamical system, then by Birkhoff's theorem :

$$\lim_{T\rightarrow\infty} \frac{1}{T}\int_0^T \varphi[X^*(\theta_t\omega)]dt = \mathbb{E}[\varphi(X^*)]. \tag{3.12}$$

On the other hand, by Theorem 3.4 together with the uniform continuity of φ, for every $\varepsilon > 0$, there exists $T_0 > 0$ such that :

$$\forall t > T_0, \ |\varphi[X_t(x_0, \omega)] - \varphi[X^*(\theta_t\omega)]| \leq \frac{\varepsilon}{2}.$$

Then, for every $T > T_0$,

$$\frac{1}{T}\int_{T_0}^T |\varphi[X_t(x_0, \omega)] - \varphi[X^*(\theta_t\omega)]|dt \leq \frac{\varepsilon}{2}.$$

Moreover, there exists $T_1 > T_0$ such that for every $T > T_1$,

$$\frac{1}{T}\int_0^{T_0} |\varphi[X_t(x_0, \omega)] - \varphi[X^*(\theta_t\omega)]|dt \leq \frac{\varepsilon}{2}.$$

So,

$$\frac{1}{T}\left|\int_0^T [\varphi[X_t(x_0, \omega)] - \varphi[X^*(\theta_t\omega)]]dt\right| \leq \varepsilon.$$

Therefore, by definition :

$$\lim_{T\rightarrow\infty} \frac{1}{T}\int_0^T [\varphi[X_t(x_0, \omega)] - \varphi[X^*(\theta_t\omega)]]dt = 0. \tag{3.13}$$

By (3.12) and (3.13) together :

$$\lim_{T\rightarrow\infty} I_T(\varphi, x_0, \omega) = \mathbb{E}[\varphi(X^*)].$$

Convergence in $L^p(\Omega, \mathbb{P})$. For every $t \in \mathbb{R}_+$ and $q > 0$,

$$\begin{aligned}
\|X_t(x_0)\|_q &\leq \|X_t(x_0) - X^* \circ \theta_t\|_q + \|X^* \circ \theta_t\|_q \\
&= \|X_t(x_0) - X^* \circ \theta_t\|_q + \|X^*\|_q.
\end{aligned}$$

Then, since $\|X_t(x_0) - X^* \circ \theta_t\|_q \to 0$ when t goes to infinity by Theorem 3.4 :

$$\sup_{t \in \mathbb{R}_+} \|X_t(x_0)\|_q < \infty \ ; \forall q > 0.$$

Therefore, by (3.11) :

$$\begin{aligned}\sup_{T>0} \|I_T(\varphi, x_0)\|_p &\leq \sup_{t \in \mathbb{R}_+} \|\varphi[X_t(x_0)]\|_p \\ &\leq c\left[1 + \sup_{t \in \mathbb{R}_+} \mathbb{E}^{1/p}[X_t^{np}(x_0)]\right] < \infty.\end{aligned}$$

In conclusion, by Vitali's theorem :

$$\lim_{T \to \infty} \|I_T(\varphi, x_0) - \mathbb{E}[\varphi(X^*)]\|_p = 0.$$

□

Proposition 3.6. *Under Assumption 1.1, the equation $b(x) = 0$ has a unique solution $x_b > 0$ such that for every $t_* \in \mathbb{R}_+$,*

$$\tau_{t_*}(x_b) := \inf\{t > t_* : X_t(x_0) = x_b\} < \infty$$

almost surely.

Proof. By Assumption 1.1.(1)-(2), the equation $b(x) = 0$ has a unique solution $x_b > 0$ such that $b(x) > 0$ (resp. $b(x) < 0$) for every $x \in]0, x_b[$ (resp. $x > x_b$). Let $t_* \in \mathbb{R}_+$ be arbitrarily chosen, and consider $\omega \in \{\tau_{t_*}(x_b) = \infty\}$.

Without loss of generality, assume that $\sigma > 0$.

On one hand, assume that $X_{t_*}(x_0, \omega) \geq x_b$. Since $X(x_0, \omega)$ is continuous,

$$\begin{aligned}x_b &< X_t(x_0, \omega) \\ &= X_{t_*}(x_0, \omega) + \int_{t_*}^t b[X_s(x_0, \omega)]ds + \sigma[B_t(\omega) - B_{t_*}(\omega)] \\ &\leq X_{t_*}(x_0, \omega) + \sigma[B_t(\omega) - B_{t_*}(\omega)].\end{aligned}$$

However, by Molchan [25] :

$$\inf\{t > 0 : B_t(\omega) = \lambda\} < \infty$$

for every $\lambda < B_{t_*}(\omega) + 1/\sigma[x_b - X_{t_*}(x_0, \omega)]$. There is a contradiction.

On the other hand, assume that $X_{t_*}(x_0, \omega) < x_b$. Since $X(x_0, \omega)$ is continuous, $X_s(x_0, \omega) < x_b$ for every $s > t_*$. So, for every $t > t_*$,

$$\begin{aligned}x_b &> X_t(x_0, \omega) \\ &= X_{t_*}(x_0, \omega) + \int_{t_*}^t b[X_s(x_0, \omega)]ds + \sigma[B_t(\omega) - B_{t_*}(\omega)] \\ &\geq X_{t_*}(x_0, \omega) + \sigma[B_t(\omega) - B_{t_*}(\omega)].\end{aligned}$$

However, by Molchan [25] :

$$\inf\{t > 0 : B_t(\omega) = \lambda\} < \infty$$

for every $\lambda > B_{t_*}(\omega) + 1/\sigma[x_b - X_{t_*}(x_0, \omega)]$. There is a contradiction.

Therefore, $\mathbb{P}[\tau_{t_*}(x_b) = \infty] = 0$. That finishes the proof. □

Remark 3.7. By Proposition 3.6, at any time $t_* \in \mathbb{R}_+$, the stochastic process $X(x_0)$ will hit again x_b on $]t_*, \infty[$. In particular, for almost every $\omega \in \Omega$, if $X(x_0, \omega)$ has a limit when t goes to infinity, it cannot be different from x_b.

3.2. Absolute continuity of the distribution of the solution. Let $T > 0$ be arbitrarily fixed, and assume that B is a centered Gaussian process defined on $[0, T]$, with α-Hölder continuous paths.

The subsection deals with applications of the Malliavin calculus to the absolute continuity of the distribution of $X_t(x_0)$ for every $t \in]0, T]$. The reader can refer to Nualart [28] on Malliavin calculus.

Let $\mathcal{H}$ be the reproducing kernel Hilbert space of B, and consider an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$. The Wiener integral with respect to B, defined on $\mathcal{H}$, is denoted by $\mathbf{B}$. The Malliavin derivative associated to $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{H}$ and $\mathbf{B}$ is denoted by $\mathbf{D}$.

Let $\mathcal{H}^1$ be the Cameron-Martin space of B. The map $\mathbf{I} : \mathcal{H} \to \mathcal{H}^1$ defined by

$$\mathbf{I}_.(h) := \mathbb{E}[\mathbf{B}(h)B_.] \ ; \forall h \in \mathcal{H},$$

is an isometry between $\mathcal{H}$ and $\mathcal{H}^1$ (see Marie [22], Lemma 3.4).

Notations :

- The domain of the Malliavin derivative is denoted by $\mathbb{D}^{1,2}$.
- Consider a random variable $U : \Omega \to \mathbb{R}$ and a normed vector space E continuously embedded in Ω ($\mathrm{E} \hookrightarrow \Omega$). For every $\omega \in \Omega$ and $e \in \mathrm{E}$, $U^\omega(e) := U(\omega + e)$.

Until the end of the subsection, B satisfies the following assumption.

Assumption 3.8. B is a centered Gaussian process defined on $[0, T]$, with α-Hölder continuous paths, such that :

(1) The covariance function R of B satisfies $\mathrm{R}(t, t) > 0$ for every $t \in]0, T]$.
(2) $\langle \varphi_1, \psi_1 \rangle_{\mathcal{H}} \geq \langle \varphi_2, \psi_2 \rangle_{\mathcal{H}}$ for every $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{H}$ such that

$$\varphi_1(t) \geq \varphi_2(t) \geq 0 \text{ and } \psi_1(t) \geq \psi_2(t) \geq 0 \ ; \forall t \in [0, T].$$

(3) The Cameron-Martin space of B is continuously embedded in $C^\alpha([0, T], \mathbb{R})$.

Example 3.9. A fractional Brownian motion of Hurst parameter $H \in]0, 1[$ satisfies Assumption 3.8 for every $\alpha \in]0, H[$ (See Friz and Victoir [9], Section 15.2.2, and Nualart [28], Section 5.1.3).

Proposition 3.10. *Under assumptions 1.1 and 3.8, $X_t(x_0) \in \mathbb{D}^{1,2}$ and*

$$\mathbf{D}_s X_t(x_0) = \sigma \mathbf{1}_{[0,t]}(s) \exp\left[\int_s^t \dot{b}[X_u(x_0)]du\right]$$

for every $s, t \in [0, T]$. Moreover, the distribution of $X_t(x_0)$ has a density with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for every $t \in]0, T]$.

Proof. Let $\omega \in \Omega$ and $s, t \in [0, T]$ be arbitrarily chosen. Since $\mathcal{H}^1 \hookrightarrow C^\alpha([0, T], \mathbb{R})$ by Assumption 3.8.(3), by Proposition 2.6 :

$$h \in \mathcal{H}^1 \longmapsto X_t^\omega(x_0, h)$$

is continuously differentiable, namely, $X_t(x_0)$ is continuously $\mathcal{H}^1$-differentiable. Then, by Nualart [28], Proposition 4.1.3, $X_t(x_0)$ is locally differentiable in the sense of Malliavin, and

$$\langle \mathbf{D}X_t(x_0)(\omega), h\rangle_{\mathcal{H}} = \mathrm{D}_{\mathbf{I}(h)}X_t^{\omega}(x_0,0)$$

for every $h \in \mathcal{H}$. By Proposition 2.6 :

$$\begin{aligned}
\mathbf{D}_s X_t(x_0)(\omega) &= \sum_{n=0}^{\infty} h_n(s)\langle \mathbf{D}X_t(x_0)(\omega), h_n\rangle_{\mathcal{H}} \\
&= \sum_{n=0}^{\infty} h_n(s)\mathrm{D}_{\mathbf{I}(h_n)}X_t^{\omega}(x_0,0) \\
&= \sum_{n=0}^{\infty} h_n(s)\left[\int_0^t \dot{b}[X_u(x_0,\omega)]\mathrm{D}_{\mathbf{I}(h_n)}X_u^{\omega}(x_0,0)du + \sigma\mathbf{I}_t(h_n)\right] \\
&= \int_0^t \dot{b}[X_u(x_0,\omega)]\sum_{n=0}^{\infty} h_n(s)\mathrm{D}_{\mathbf{I}(h_n)}X_u^{\omega}(x_0,0)du \\
&\quad + \sigma\sum_{n=0}^{\infty} h_n(s)\mathrm{D}_{\mathbf{I}(h_n)}B_t^{\omega}(0) \\
&= \int_0^t \dot{b}[X_u(x_0,\omega)]\mathbf{D}_s X_u(x_0)(\omega)du + \sigma\mathbf{D}_s B_t(\omega).
\end{aligned}$$

Since $\mathbf{D}B_t = \mathbf{1}_{[0,t]}$, $\mathbf{D}_s X_.(x_0)(\omega)$ satisfies

$$\mathbf{D}_s X_t(x_0)(\omega) = \xi + \int_0^t \dot{b}[X_u(x_0,\omega)]\mathbf{D}_s X_u(x_0)(\omega)du$$

with $\xi = 0$ (resp. $\xi = \sigma$) for $t \in [0,s[$ (resp. $t \in [s,T]$). Then,

$$\mathbf{D}_s X_t(x_0) = \sigma\mathbf{1}_{[0,t]}(s)\exp\left[\int_s^t \dot{b}[X_u(x_0)]du\right].$$

So, by Assumption 1.1.(2) :

$$\sigma\mathbf{1}_{[0,t]}(s)\exp\left[\int_0^T \dot{b}[X_u(x_0)]du\right] \leq \mathbf{D}_s X_t(x_0) \leq \sigma\mathbf{1}_{[0,t]}(s). \tag{3.14}$$

Put $\Gamma_t := \|\mathbf{D}X_t(x_0)\|_{\mathcal{H}}^2$. By Assumption 3.8.(1)-(2) and Inequality (3.14) :

$$0 < \sigma^2\mathrm{R}(t,t)\exp\left[2\int_0^T \dot{b}[X_u(x_0)]du\right] \leq \Gamma_t \leq \sigma^2\mathrm{R}(t,t). \tag{3.15}$$

On one hand, by Inequality (3.15), $\Gamma_t \in L^p(\Omega,\mathbb{P})$ for every $p > 0$. So, $X_t(x_0) \in \mathbb{D}^{1,2}$ by Nualart [28], Lemma 4.1.2. On the other hand, by Inequality (3.15), $\Gamma_t > 0$. So, the distribution of $X_t(x_0)$ has a density with respect to the Lebesgue measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ by Bouleau-Hirsch's criterion (see Nualart [28], Theorem 2.1.3). □

Notation. The Ornstein-Uhlenbeck semigroup (resp. operator) is denoted by $\mathbf{T} := (\mathbf{T}_t)_{t\in\mathbb{R}_+}$ (resp. $\mathbf{L}$). See Nualart [28], Section 1.4.

Remark 3.11. (1) Let $t \in \mathbb{R}_+$ be arbitrarily chosen. By Nualart [28], Property (i) page 55 :

$$\forall U \in L^2(\Omega, \mathbb{P}),\, U \geq 0 \Longrightarrow T_t(U) \geq 0. \tag{3.16}$$

Since T_t is a linear map, (3.16) implies that :

$$\forall U_1, U_2 \in L^2(\Omega, \mathbb{P}),\, U_1 \geq U_2 \Longrightarrow T_t(U_1) \geq T_t(U_2).$$

(2) Let $t \in]0, T]$ be arbitrarily chosen. Bouleau-Hirsch's criterion is sufficient to prove the absolute continuity of the distribution of $X_t(x_0)$ with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, but not to provide an explicit density. Via the main result of Nourdin and Viens [27], the following proposition provides a density with a suitable expression.

Proposition 3.12. *Under assumptions 1.1 and 3.8, for every $t \in]0, T]$,*

$$\mathbb{P}_{X_t(x_0)}(dx) = \frac{\mathbb{E}[|X_t^*(x_0)|]}{2g_{X_t(x_0)}(x)} \exp\left[-\int_{\mathbb{E}[X_t(x_0)]}^x \frac{y - \mathbb{E}[X_t(x_0)]}{g_{X_t(x_0)}(y)} dy\right] dx$$

where $X_t^*(x_0) := X_t(x_0) - \mathbb{E}[X_t(x_0)]$ and

$$g_{X_t(x_0)}(x) := \mathbb{E}[\langle \mathbf{D}X_t(x_0), -\mathbf{D}\mathbf{L}^{-1}X_t(x_0)\rangle_{\mathcal{H}} | X_t(x_0) = x]$$

for every $x > 0$.

Proof. Let $t \in]0, T]$ and $s \in [0, T]$ be arbitrarily chosen. By Nourdin and Viens [27], Proposition 3.7, Inequality (3.14) and (3.16) :

$$\begin{aligned} -\mathbf{D}_s\mathbf{L}^{-1}X_t(x_0) &= \int_0^\infty e^{-u}\mathbf{T}_u[\mathbf{D}_sX_t(x_0)]du \\ &\geq \sigma\mathbf{1}_{[0,t]}(s)\int_0^\infty e^{-u}\mathbf{T}_u\left[\exp\left[\int_0^T \dot{b}[X_v(x_0)]dv\right]\right] du. \end{aligned}$$

Then, by Assumption 3.8.(1)-(2) together with Inequality (3.14) :

$$\begin{aligned} \langle \mathbf{D}X_t(x_0), -\mathbf{D}\mathbf{L}^{-1}X_t(x_0)\rangle_{\mathcal{H}} \geq\ & \sigma^2\mathrm{R}(t,t)\exp\left[\int_0^T \dot{b}[X_v(x_0)]dv\right] \times \\ & \int_0^\infty e^{-u}\mathbf{T}_u\left[\exp\left[\int_0^T \dot{b}[X_v(x_0)]dv\right]\right] du > 0. \end{aligned}$$

So,

$$\begin{aligned} g_{X_t^*(x_0)}[X_t^*(x_0)] &:= \mathbb{E}[\langle \mathbf{D}X_t^*(x_0), -\mathbf{D}\mathbf{L}^{-1}X_t^*(x_0)\rangle_{\mathcal{H}} | X_t^*(x_0)] \\ &= \mathbb{E}[\langle \mathbf{D}X_t(x_0), -\mathbf{D}\mathbf{L}^{-1}X_t(x_0)\rangle_{\mathcal{H}} | X_t^*(x_0)] > 0. \end{aligned}$$

Therefore, by Nourdin and Viens [27], Theorem 3.1 :

$$\mathbb{P}_{X_t^*(x_0)}(dx) = \frac{\mathbb{E}[|X_t^*(x_0)|]}{2g_{X_t^*(x_0)}(x)} \exp\left[-\int_0^x \frac{y}{g_{X_t^*(x_0)}(y)} dy\right] dx.$$

Together with a straightforward application of the transfer theorem, that finishes the proof. □

3.3. Integrability and convergence of the implicit Euler scheme. Let $T > 0$ be arbitrarily fixed, and assume that B is a centered Gaussian process defined on $[0,T]$, with α-Hölder continuous paths.

Let $n \in \mathbb{N}^*$ be arbitrarily chosen, and consider the dissection $(t_0^n, t_1^n, \dots, t_n^n)$ of $[0,T]$ such that $t_k^n := kT/n$ for every $k \in \{0,\dots,n\}$. Consider also the stochastic process $X^n(x_0) := (X_t^n(x_0))_{t\in[0,T]}$ such that for every $\omega \in \Omega$, $X^n(x_0,\omega)$ is the step-n implicit Euler scheme associated to Equation (2.1) driven by $B(\omega)$ and to the dissection $(t_0^n, t_1^n, \dots, t_n^n)$.

Proposition 3.13. *Under Assumption 1.1 :*

$$\|X^n(x_0) - X(x_0)\|_{\infty,T} \xrightarrow[n\rightarrow\infty]{a.s.} 0$$

with rate of convergence $\mathrm{O}(n^{-\alpha})$*. Moreover, for every* $p > 0$*,*

$$\sup_{n\in\mathbb{N}^*} \|X^n(x_0)\|_{\infty,T} \in L^p(\Omega,\mathbb{P})$$

and

$$\lim_{n\rightarrow\infty} \mathbb{E}[\|X^n(x_0) - X(x_0)\|_{\infty,T}^p] = 0.$$

Proof. By Theorem 2.13, for every $\omega \in \Omega$,

$$\|X^n(x_0,\omega) - X(x_0,\omega)\|_{\infty,T} \xrightarrow[n\rightarrow\infty]{\text{a.s.}} 0$$

with rate of convergence $\mathrm{O}(n^{-\alpha})$.

Let $p > 0$ be arbitrarily chosen. By Proposition 2.12 together with Fernique's theorem :

$$\sup_{n\in\mathbb{N}^*} \|X^n(x_0)\|_{\infty,T} \in L^p(\Omega,\mathbb{P}).$$

So, by Vitali's theorem :

$$\lim_{n\rightarrow\infty} \mathbb{E}[\|X^n(x_0) - X(x_0)\|_{\infty,T}^p] = 0.$$

□

3.4. Estimation of parameters. The subsection deals with the estimation of the Hurst parameter and of the volatility constant of Equation (1.1) by using a transformation of the observations of $X(x_0)$ and already known estimators of the Hurst parameter and of the volatility constant of the fractional Ornstein-Uhlenbeck process.

Under Assumption 1.1, for every $y_0 \in \mathbb{R}$, let $Y(y_0) := (Y_t(y_0))_{t\in\mathbb{R}_+}$ be the solution of the following Langevin equation :

$$Y_t = y_0 - R\int_0^t Y_s ds + \sigma B_t. \tag{3.17}$$

On the fractional Ornstein-Uhlenbeck process, see Cheridito et al. [5].

Proposition 3.14. *Under Assumption 1.1, for every* $x_0 > 0$*,* $y_0 \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$*,*

$$Y_t(y_0) = X_t(x_0) - (x_0 - y_0)e^{-Rt} - \int_0^t e^{-R(t-s)} b^R[X_s(x_0)]ds$$

where $b^R(x) := b(x) + Rx$ *for every* $x > 0$*.*

Proof. Let $t \in \mathbb{R}_+$, $x_0 > 0$ and $y_0 \in \mathbb{R}$ be arbitrarily chosen. The stochastic process $\Delta(x_0, y_0) := X(x_0) - Y(y_0)$ satisfies :

$$\begin{aligned}\Delta_t(x_0, y_0) &= x_0 - y_0 + \int_0^t [b[X_s(x_0)] + RY_s(y_0)]ds \\ &= x_0 - y_0 - R\int_0^t \Delta_s(x_0, y_0)ds + \int_0^t b^R[X_s(x_0)]ds.\end{aligned}$$

Then,

$$\Delta_t(x_0, y_0) = (x_0 - y_0)e^{-Rt} + \int_0^t e^{-R(t-s)} b^R[X_s(x_0)]ds.$$

That finishes the proof. □

Until the end of the subsection, B is a fractional Brownian motion of Hurst parameter $H \in]0,1[$ ($\alpha \in]0,H[$). The values of all the parameters involving in the expression of the drift function b are known.

Consider the map Θ from $C^0(\mathbb{R}_+,]0,\infty[)$ into $C^0(\mathbb{R}_+, \mathbb{R})$ such that :

$$\Theta(\varphi)(t) := \varphi(t) - \int_0^t e^{-R(t-s)} b^R[\varphi(s)]ds \ ; \forall t \in \mathbb{R}_+,$$

for every $\varphi \in C^0(\mathbb{R}_+,]0,\infty[)$. By Proposition 3.14 :

$$Y(x_0) = \Theta[X(x_0)].$$

Since the parameter (H, σ) doesn't involve in the expression of the map Θ, an observation $x(x_0)$ of $X(x_0)$ provides an observation of $Y(x_0)$ by applying the transformation Θ to $x(x_0)$. So, since Equation (1.1) has the same additive noise σB than Equation (3.17), a consistent estimator of (H, σ) for the fractional Ornstein-Uhlenbeck process $Y(x_0)$ provides an estimation of the real value of (H, σ) from the observation $y(x_0) := \Theta[x(x_0)]$.

For $H \in]1/2, 1[$, there are several papers dealing with the estimation of the Hurst parameter and of the volatility constant of the fractional Ornstein-Uhlenbeck process. Some estimators use the whole path of $Y(x_0)$ (see Berzin and León [2]), and some estimators use discrete observations of $Y(x_0)$ (see Melichov [24] or Brouste and Iacus [3]).

In order to get an observation of $Y(x_0)$ at the time $t \in \mathbb{R}_+$ from $y(x_0)$, the whole path $x(x_0)$ has to be known until the time t by construction of the map Θ. In practice, only discrete observations of $X(x_0)$ are available. So, with the same arguments, the end of the subsection deals with a consistent estimator of H for $X^n(x_0)$ instead of $X(x_0)$.

Under Assumption 1.1, let $Y^n(x_0)$ be the step-n implicit Euler scheme associated to Equation (3.17) and to the dissection $(t_0^n, t_1^n, \dots, t_n^n)$ of $[0, T]$:

$$Y_{k+1}^n = Y_k^n - RY_{k+1}^n(t_{k+1}^n - t_k^n) + \sigma(B_{t_{k+1}^n} - B_{t_k^n}) \tag{3.18}$$

with $Y_0^n(x_0) := x_0$. On the implicit Euler schemes associated to the fractional Langevin equation, see Garrido-Atienza et al. [10].

Proposition 3.15. *Under Assumption 1.1, for every $k \in \{1, \dots, n\}$,*

$$Y_k^n(x_0) = X_k^n(x_0) - Tn^{-1}\sum_{i=1}^{k}(1 + RTn^{-1})^{i-1-k} b_R[X_i^n(x_0)].$$

Proof. For any $k \in \{1, \dots, n\}$, the stochastic process $\Delta^n(x_0) := X^n(x_0) - Y^n(x_0)$ satisfies :

$$\begin{aligned}\Delta_{k+1}^n(x_0) &= X_k^n(x_0) - Y_k^n(x_0) + Tn^{-1}[b[X_{k+1}^n(x_0)] + RY_{k+1}^n(x_0)] \\ &= \Delta_k^n(x_0) + Tn^{-1}[b_R[X_{k+1}^n(x_0)] - R\Delta_{k+1}^n(x_0)].\end{aligned}$$

Then,

$$\Delta_{k+1}^n(x_0) = (1 + RTn^{-1})^{-1}\Delta_k^n(x_0) + (1 + RTn^{-1})^{-1} b_R[X_k^n(x_0)].$$

So,

$$\Delta_k^n(x_0) = Tn^{-1}\sum_{i=1}^{k}(1 + RTn^{-1})^{i-1-k} b_R[X_i^n(x_0)].$$

That finishes the proof. □

Consider the map Θ^n from $]0, \infty[^{\mathbb{N}}$ into $\mathbb{R}^{\mathbb{N}}$ such that :

$$\Theta_k^n(u) := \begin{cases} u_0, & \text{if} \quad k = 0 \\ u_k - Tn^{-1}\displaystyle\sum_{i=1}^{k}(1 + RTn^{-1})^{i-1-k} b_R(u_i), & \text{if} \quad k \in \{1, \dots, n\}\end{cases}$$

for every $u \in]0, \infty[^{\mathbb{N}}$. By Proposition 3.15 :

$$Y^n(x_0) = \Theta^n[X^n(x_0)].$$

As for the continuous time models, the transformation Θ^n provides an observation of $Y^n(x_0)$ from an observation of $X^n(x_0)$.

Definition 3.16. Consider a stochastic process $W := (W_t)_{t \in \mathbb{R}_+}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathrm{I}_n \subset \{0, \dots, n\}$, and $k \in \mathbb{N}$ such that $k + \max \mathrm{I}_n \leq n$. The *$k$-quadratic variation* of W with respect to the dissection $(t_0^n, t_1^n, \dots, t_n^n)$ and to the index set I_n is

$$\mathrm{V}_{\mathrm{I}_n, k}(W) := \sum_{i \in \mathrm{I}_n}(\Delta_k W)_i^2$$

with $(\Delta_k W)_i := W_{t_{i+k}^n} - W_{t_i^n}$ for every $i \in \mathrm{I}_n$.

Proposition 3.17. *Consider $\mathrm{I}_n \subset \{0, \dots, n\}$, and $k \in \mathbb{N}$ such that $k + \max \mathrm{I}_n \leq n$.*

$$|\mathrm{V}_{\mathrm{I}_n, k}[Y^n(x_0)] - \mathrm{V}_{\mathrm{I}_n, k}[Y(x_0)]| \xrightarrow[n \to \infty]{} 0$$

almost surely and in $L^p(\Omega, \mathbb{P})$ for every $p > 0$.

Proof. Let $i \in \mathrm{I}_n$ and $p > 0$ be arbitrarily chosen.

$$\begin{aligned}
|[\Delta_k Y^n(x_0)]_i^2 - [\Delta_k Y(x_0)]_i^2| &\leq 2Y^* \\
&\quad \times \sum_{j=1}^{k} |Y_{i+j}^n(x_0) - Y_{i+j-1}^n(x_0) - [Y_{i+j}(x_0) - Y_{i+j-1}(x_0)]| \\
&\leq 2RY^* \sum_{j=1}^{k} \left| Y_{i+j}^n(x_0) T n^{-1} - \int_{t_{i+j-1}^n}^{t_{i+j}^n} Y_t(x_0) dt \right| \\
&\leq 2RY^* \sum_{j=1}^{k} \int_{t_{i+j-1}^n}^{t_{i+j}^n} |Y_{i+j}^n(x_0) - Y_t(x_0)| dt \\
&\leq 2RY^* \left[T n^{-1} \sum_{j=1}^{k} |Y_{i+j}^n(x_0) - Y_{t_{i+j}^n}(x_0)| \right. \qquad (3.19) \\
&\quad \left. + \sum_{j=1}^{k} \int_{t_{i+j-1}^n}^{t_{i+j}^n} |Y_{t_{i+j}^n}(x_0) - Y_t(x_0)| dt \right]
\end{aligned}$$

with

$$\begin{aligned}
Y^* &:= \|Y(x_0)\|_{\infty,T} + \sup_{n \in \mathbb{N}^*} \|Y^n(x_0)\|_{\infty,T} \\
&\in L^p(\Omega, \mathbb{P}).
\end{aligned}$$

On one hand, by Garrido-Atienza et al. [10], Theorem 1 ; there exists $C(\alpha, H, T) \in L^p(\Omega, \mathbb{P})$ such that :

$$\|Y^n(x_0) - Y(x_0)\|_{\infty,T} \leq C(\alpha, H, T) T^\alpha n^{-\alpha}.$$

So,

$$T n^{-1} \sum_{j=1}^{k} |Y_{i+j}^n(x_0) - Y_{t_{i+j}^n}(x_0)| \leq C(\alpha, H, T) T^{\alpha+1} n^{-\alpha}. \qquad (3.20)$$

On the other hand, the paths of $Y(x_0)$ are α-Hölder continuous on $[0, T]$ with $\|Y(x_0)\|_{\alpha,T} \in L^p(\Omega, \mathbb{P})$. So,

$$\begin{aligned}
\sum_{j=1}^{k} \int_{t_{i+j-1}^n}^{t_{i+j}^n} |Y_{t_{i+j}^n}(x_0) - Y_t(x_0)| dt &\leq \|Y(x_0)\|_{\alpha,T} \sum_{j=1}^{k} \int_{t_{i+j-1}^n}^{t_{i+j}^n} (t_{i+j}^n - t)^\alpha dt \\
&\leq \|Y(x_0)\|_{\alpha,T} T^{\alpha+1} n^{-\alpha}. \qquad (3.21)
\end{aligned}$$

By Inequality (3.19) together with inequalities (3.20) and (3.21) :

$$|\mathrm{V}_{\mathrm{I}_n,k}[Y^n(x_0)] - \mathrm{V}_{\mathrm{I}_n,k}[Y(x_0)]| \leq 2RY^*[C(\alpha, H, T) + \|Y(x_0)\|_{\alpha,T}] T^{\alpha+1} n^{-\alpha}.$$

That finishes the proof. □

Consider $\mathrm{I}_n^1 := \{0, \dots, n\}$, $\mathrm{I}_n^2 := \{2i \ ; \ i \in \{0, \dots, [n/2]\}$,

$$\widehat{H}_n := \frac{1}{2} - \frac{1}{2\log(2)} \log\left[\frac{\mathrm{V}_{\mathrm{I}_n^1,1}[Y(x_0)]}{\mathrm{V}_{\mathrm{I}_n^2,2}[Y(x_0)]}\right] \text{ and } \widehat{h}_n := \frac{1}{2} - \frac{1}{2\log(2)} \log\left[\frac{\mathrm{V}_{\mathrm{I}_n^1,1}[Y^n(x_0)]}{\mathrm{V}_{\mathrm{I}_n^2,2}[Y^n(x_0)]}\right].$$

Proposition 3.18. *$\widehat{h}_n$ is a strongly consistent estimator of H.*

Proof. By Proposition 3.17 together with the uniform continuity of log on $]0,\infty[$:

$$\begin{aligned}|\widehat{h}_n - \widehat{H}_n| &\leq \frac{1}{2\log(2)}[|\log[\mathrm{V}_{\mathrm{I}_n^1,1}[Y(x_0)]] - \log[\mathrm{V}_{\mathrm{I}_n^1,1}[Y^n(x_0)]]| \\ &\quad + |\log[\mathrm{V}_{\mathrm{I}_n^2,2}[Y(x_0)]] - \log[\mathrm{V}_{\mathrm{I}_n^2,2}[Y^n(x_0)]]|] \xrightarrow[n\to\infty]{\text{a.s.}} 0.\end{aligned}$$

Moreover, by Melichov [24], Section 3.2.1 ; $\widehat{H}_n$ is a strongly consistent estimator of H. So,

$$\begin{aligned}|\widehat{h}_n - H| &\leq |\widehat{h}_n - \widehat{H}_n| + |\widehat{H}_n - H| \\ &\xrightarrow[n\to\infty]{\text{a.s.}} 0.\end{aligned}$$

□

4. Application to Singular Equations Driven by a Multiplicative Noise

Let $F :]0,\infty[\to\mathbb{R}$ be a function satisfying the following assumption.

Assumption 4.1. (1) The function F is $[1/\alpha]+2$ times continuously differentiable on $]0,\infty[$.
(2) The function F is strictly monotonic on $]0,\infty[$.

Under Assumption 1.1, Equation (1.1) with the initial condition $x_0 > 0$ has a unique $]0,\infty[$-valued solution $X(x_0)$ on $\mathbb{R}_+$ by Proposition 2.2. Then, under Assumption 4.1, by the rough change of variable formula :

$$\begin{aligned}F[X_t(x_0)] &= F(x_0) + \int_0^t \dot{F}[X_s(x_0)]dX_s(x_0) \\ &= F(x_0) + \int_0^t \dot{F}[X_s(x_0)]b[X_s(x_0)]ds + \sigma\int_0^t \dot{F}[X_s(x_0)]dB_s\end{aligned}$$

for every $t\in\mathbb{R}_+$. Therefore, by putting $\mathrm{I} := F(]0,\infty[)$, with the initial condition $z_0\in\mathrm{I}$, the following equation has a unique I-valued solution $Z(z_0) := (Z_t(z_0))_{t\in\mathbb{R}_+}$ on $\mathbb{R}_+$:

$$Z_t = z_0 + \int_0^t G(Z_s)H(Z_s)ds + \sigma\int_0^t H(Z_s)dB_s \tag{4.1}$$

with $G := b\circ F^{-1}$ and $H := \dot{F}\circ F^{-1}$.

Example 4.2. Consider $\kappa\in\mathbb{R}^*$ and $u,v,w,\gamma > 0$ such that $1-\alpha < \alpha\gamma$. Put $b(x) := u(vx^{-\gamma} - wx)$ and $F_\kappa(x) := x^\kappa$ for every $x > 0$. The function b (resp. F_κ) satisfies Assumption 1.1 (resp. Assumption 4.1). Then, Equation (4.1) becomes :

$$Z_t = z_0 + \kappa u\int_0^t [vZ_s^{1-(\gamma+1)/\kappa} - wZ_s]ds + \kappa\sigma\int_0^t Z_s^{1-1/\kappa}dB_s.$$

On one hand, assume that $\kappa = \gamma+1$, $u = 1/(\gamma+1)$ and $\sigma = \zeta/(\gamma+1)$ with $\zeta\in\mathbb{R}^*$. Then, by putting $\beta := 1 - 1/(\gamma+1)$, Equation (4.1) becomes

$$Z_t = y_0 + \int_0^t (v - wZ_s)ds + \zeta\int_0^t Z_s^\beta dB_s$$

and $\beta\in]1-\alpha,1[$. So, in that case, Equation (4.1) is the generalized Cox-Ingersoll-Ross model partially studied in Marie [21].

On the other hand, assume that $\kappa = -(\gamma+1)$, $u = 1/(\gamma+1)$ and $\sigma = -\zeta^*/(\gamma+1)$ with $\zeta^* \in \mathbb{R}^*$. Then, by putting $\beta^* := 1/(\gamma+1)$, Equation (4.1) becomes

$$Z_t = z_0 + \int_0^t Z_s(w - vZ_s)ds + \zeta^* \int_0^t Z_s^{1+\beta^*} dB_s$$

and $\beta^* \in]0, \alpha[$. So, in that case, Equation (4.1) is a generalized Verhulst's model, studied for $\beta^* = 0$ and a fractional Brownian signal in Huy and Nguyen [16].

The section deals with how to transfer the probabilistic and statistical properties established at Section 3 on the solution of Equation (1.1) to the stochastic process $Z(z_0)$. A fractional Heston model is also introduced.

Proposition 4.3. *Under assumptions 1.1 and 4.1, the Itô map associated to the deterministic analog of Equation (4.1) is continuously differentiable from*

$$\mathrm{I} \times C^{\alpha}([0,T], \mathbb{R}) \text{ into } C^0([0,T], \mathrm{I})$$

for every $T > 0$.

Proof. Let $T > 0$ be arbitrarily chosen. By Proposition 2.6, $x(.)$ is continuously differentiable from

$$]0, \infty[\times C^{\alpha}([0,T], \mathbb{R}) \text{ into } C^0([0,T],]0, \infty[).$$

Moreover, by Assumption 4.1, F and F^{-1} are $[1/\alpha] + 2$ times continuously differentiable on $]0, \infty[$ and I respectively. So, the map

$$(z_0, w) \longmapsto F \circ x[F^{-1}(z_0), w]$$

is continuously differentiable from

$$\mathrm{I} \times C^{\alpha}([0,T], \mathbb{R}) \text{ into } C^0([0,T], \mathrm{I}).$$

□

4.1. Probabilistic and statistical properties of the solution.

Assume that B is a centered Gaussian process with locally α-Hölder continuous paths.

The subsection deals with how to transfer probabilistic properties established at Section 3 on the solution of Equation (1.1) to the stochastic process $Z(z_0)$.

In the sequel, the function F satisfies the following assumption.

Assumption 4.4. The function F is defined and uniformly continuous on $\mathbb{R}_+$, satisfies Assumption 4.1, and

$$\forall x \in \mathbb{R}_+,\ |F(x)| \leq C(1 + x^k)$$

with $C > 0$ and $k \in \mathbb{N}^*$.

Example 4.5. The function F_κ with $\kappa > 0$ satisfies Assumption 4.4.

Proposition 4.6. *For every $T > 0$, under assumptions 1.1 and 4.4 :*

$$\|Z(z_0)\|_{\infty,T} \in L^p(\Omega, \mathbb{P})$$

for every $p > 0$.

Proof. Let $T > 0$ and $t \in [0,T]$ be arbitrarily chosen. By Proposition 2.3 :

$$0 < X_t[F^{-1}(z_0)] \leq F^{-1}(z_0) + |b[F^{-1}(z_0)]|T + 2\sigma\|B\|_{\infty,T}.$$

So, by Jensen's inequality :

$$\begin{array}{rcl}|Z_t(z_0)| & = & |F[X_t[F^{-1}(z_0)]]| \\ & \leq & C[1 + X_t^k[F^{-1}(z_0)]] \\ & \leq & C(T)(1 + \|B\|_{\infty,T}^k)\end{array}$$

with $C(T) > 0$ (deterministic). Therefore, by Fernique's theorem :

$$\|Z(z_0)\|_{\infty,T} \in L^p(\Omega, \mathbb{P})$$

for every $p > 0$. □

Proposition 4.7. *Assume that B is a two-sided fractional Brownian motion of Hurst parameter $H \in]0,1[$. Under assumptions 1.1 and 4.4, for every uniformly continuous function $\psi : \mathrm{I} \rightarrow \mathbb{R}$ with polynomial growth, and every $z_0 \in \mathrm{I}$,*

$$\frac{1}{T}\int_0^T \psi[Z_t(z_0)]dt \xrightarrow[T\rightarrow\infty]{} \mathbb{E}[\psi[F(X^*)]]$$

almost surely and in $L^p(\Omega, \mathbb{P})$ for every $p > 0$.

Proof. Let an uniformly continuous function $\psi : \mathrm{I} \rightarrow \mathbb{R}$ with polynomial growth be arbitrarily chosen. Since $F : \mathbb{R}_+ \rightarrow \mathrm{I}$ is also uniformly continuous with polynomial growth, the function $\varphi := \psi \circ F$ satisfies the conditions of Corollary 3.5. Then, for every $z_0 \in \mathrm{I}$,

$$\begin{array}{rcl}\displaystyle\frac{1}{T}\int_0^T \psi[Z_t(z_0)]dt & = & \displaystyle\frac{1}{T}\int_0^T \varphi[X_t[F^{-1}(z_0)]]dt \\ & & \xrightarrow[T\rightarrow\infty]{} \mathbb{E}[\varphi(X^*)]\end{array}$$

almost surely and in $L^p(\Omega, \mathbb{P})$ for every $p > 0$. That finishes the proof. □

Proposition 4.8. *Let $T > 0$ be arbitrarily fixed. Under assumptions 1.1, 3.8 and 4.4 with $F^{-1} \in C^1(\bar{\mathrm{I}}, \mathbb{R}_+)$, the distribution of $Z_t(z_0)$ has a density with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for every $t \in]0,T]$.*

Proof. Let $t \in]0,T]$ be arbitrarily chosen. By Proposition 3.10 ; the distribution of $X_t[F^{-1}(z_0)]$ has a density $\mathbf{f}_t$ with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. So, by a straightforward application of the transfer theorem :

$$\mathbb{P}_{Z_t(z_0)}(dz) = \frac{\mathbf{f}_t[F^{-1}(z)]}{\dot{F}[F^{-1}(z)]}dz. \tag{4.2}$$

Therefore, the distribution of $Z_t(z_0)$ has a density with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. □

Example 4.9. The function F_κ with $\kappa \in]0,1]$ satisfies Assumption 4.4 with $F_\kappa^{-1} \in C^1(\bar{\mathrm{I}}, \mathbb{R}_+)$.

Remark 4.10. Let $t \in]0,T]$ be arbitrarily chosen. The density $\mathbf{f}_t$ can be the one provided at Proposition 3.12. So, Equality (4.2) together with Proposition 3.12 provide a density, with a suitable expression, of the distribution of $Z_t(z_0)$.

Proposition 4.11. *Consider $T > 0$ and assume that B is a centered Gaussian process defined on $[0, T]$, with α-Hölder continuous paths. Put*

$$Z_t^n(z_0) := F[X_t^n[F^{-1}(z_0)]]$$

for every $t \in [0, T]$ and $n \in \mathbb{N}^*$.

(1) *Under assumptions 1.1 and 4.1 :*

$$\|Z^n(z_0) - Z(z_0)\|_{\infty,T} \xrightarrow[n\rightarrow\infty]{a.s.} 0$$

with rate of convergence $\mathrm{O}(n^{-\alpha})$.

(2) *Under assumptions 1.1 and 4.4, for every $p > 0$,*

$$\sup_{n\in\mathbb{N}^*} \|Z^n(x_0)\|_{\infty,T} \in L^p(\Omega, \mathbb{P})$$

and

$$\lim_{n\rightarrow\infty} \mathbb{E}[\|Z^n(z_0) - Z(z_0)\|_{\infty,T}^p] = 0.$$

Proof. Let $p > 0$ be arbitrarily chosen. Put

$$X_* := \inf_{n\in\mathbb{N}^*} \inf_{t\in[0,T]} X_t[F^{-1}(z_0)] \wedge X_t^n[F^{-1}(z_0)]$$

and

$$X^* := \sup_{n\in\mathbb{N}^*} \sup_{t\in[0,T]} X_t[F^{-1}(z_0)] \vee X_t^n[F^{-1}(z_0)].$$

Under Assumption 4.1, the function F is Lipschitz continuous on $[X_*, X^*]$. So,

$$\begin{aligned} \|Z^n(z_0) - Z(z_0)\|_{\infty,T} &\leq \|\dot{F}\|_{\infty,[X_*,X^*]}\|X^n[F^{-1}(z_0)] - X[F^{-1}(z_0)]\|_{\infty,T} \\ &\xrightarrow[n\rightarrow\infty]{\text{a.s.}} 0 \end{aligned}$$

with rate of convergence $\mathrm{O}(n^{-\alpha})$, by Proposition 3.13. Under Assumption 4.4, for every $t \in [0, T]$,

$$0 < Z_t^n(z_0) \leq C[1 + |X_t^n[F^{-1}(z_0)]|^k].$$

Then, by Proposition 3.13 :

$$\sup_{n\in\mathbb{N}^*} \|Z^n(z_0)\|_{\infty,T} \in L^p(\Omega, \mathbb{P})$$

for every $p > 0$. So, by Vitali's theorem :

$$\lim_{n\rightarrow\infty} \mathbb{E}[\|Z^n(z_0) - Z(z_0)\|_{\infty,T}^p] = 0.$$

□

Assume that B is a fractional Brownian motion of Hurst parameter $H \in]0, 1[$. The values of all the parameters involving in the expressions of b and F are supposed to be known.

As established at Subsection 3.3, $Y(x_0) = \Theta[X(x_0)]$ for every $x_0 > 0$. So,

$$Y[F^{-1}(z_0)] = \Xi[Z(z_0)]$$

with $\Xi := \Theta \circ F^{-1}$. Since the parameter (H, σ) doesn't involve in the expression of the map Ξ, an observation $z(z_0)$ of $Z(z_0)$ provides an observation of $Y[F^{-1}(z_0)]$ by applying the transformation Ξ to $z(z_0)$. Therefore, a consistent estimator of

(H,σ) for the Ornstein-Uhlenbeck process $Y[F^{-1}(z_0)]$ provides an estimation of the real value of (H,σ) from the observation $y[F^{-1}(z_0)] := \Xi[z(z_0)]$.

The same arguments work on $\Xi^n := \Theta^n \circ F^{-1}$ applied to $Z^n(z_0)$ instead of Ξ applied to $Z(z_0)$.

4.2. A fractional Heston model. Financial market models with a fractional stochastic volatility have been already studied in several papers. For instance, the volatility in the Heston model (see Heston [14]) has been replaced by a fractional process in Comte, Coutin and Renault [6], taking benefits of its long memory. The subsection deals with another fractional Heston model.

On fractional financial market models, see also Rogers [30] and Cheridito [4].

Let $T > 0$ be arbitrarily fixed, and assume that B is a fractional Brownian motion of Hurst parameter $H \in]0,1[$, defined on $[0,T]$. The filtration generated by B is denoted by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$. By Decreusefond and Ustünel [7] or Nualart [28], Section 5.1.3 ; there exists a unique Brownian motion B^*, generating the same filtration $\mathbb{F}$ than B, such that :

$$B_t := \int_0^t K_H(t,s)dB_s^* \text{ ; } \forall t \in [0,T]$$

where

$$K_H(t,s) := \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\mathrm{F}(1/2-H, H-1/2, H+1/2, 1-t/s)\mathbf{1}_{[0,t[}(s)$$

for every $(s,t) \in \mathbb{R}_+^2$, and F is the Gauss hyper-geometric function (see Lebedev [17]).

Let $\mathbb{H}^2$ be the space consisting of $\mathbb{F}$-progressively measurable stochastic processes $(H_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left(\int_0^T H_t^2 dt\right) < \infty.$$

Since $\mathbb{F}$ is the filtration generated by both B and B^*, stochastic processes of $\mathbb{H}^2$ are integrable with respect to B^* in the sense of Itô.

Notation. For every $H \in \mathbb{H}^2$, the Itô stochastic integral of H with respect to B^* is denoted by

$$\left(\int_0^t H_s \mathrm{d}B_s^*\right)_{t\in[0,T]}.$$

Consider the following generalization of the Heston model :

$$S_t = S_0 + \int_0^t \mu_u S_u du + \int_0^t \varphi(Z_u)S_u \mathrm{d}B_u^* \text{ ; } S_0 > 0 \tag{4.3}$$

$$Z_t = z_0 + \int_0^t (v - wZ_u)du + \zeta \int_0^t Z_u^\beta dB_u \text{ ; } z_0 > 0 \tag{4.4}$$

where $\mu \in C^0([0,T],\mathbb{R})$, $v,w > 0$, $\zeta \in \mathbb{R}^*$, $\beta \in]1-H,1[$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that :

$$\forall x \in \mathbb{R}_+, |\varphi(x)| \leq c(1+x^n) \tag{4.5}$$

with $c > 0$ and $n \in \mathbb{N}^*$.

Proposition 4.12. *Equation (4.4) has a unique pathwise solution $Z(z_0)$ such that $Z^{\varphi}(z_0) := \varphi \circ Z(z_0) \in \mathbb{H}^2$, and Equation (4.3) has a unique solution $S(z_0)$ in the sense of Itô such that :*

$$S_t(z_0) := S_0 \exp\left[\int_0^t \left[\mu_s - \frac{1}{2}\varphi^2[Z_s(z_0)]\right] ds + \int_0^t \varphi[Z_s(z_0)]\mathrm{d}B_s^*\right]$$

for every $t \in [0, T]$.

Proof. Put $x_0 := z_0^{1-\beta}$, $\sigma := \zeta(1-\beta)$ and

$$b(x) := (1-\beta)(vx^{-\gamma} - wx) \ ; \ \forall x > 0$$

where $\gamma := \beta/(1-\beta)$. By Proposition 2.2 together with the rough change of variable formula, the stochastic process $Z(z_0)$ defined by

$$Z_t(z_0) := X_t^{\gamma+1}(x_0) \ ; \ \forall t \in [0, T],$$

is the unique pathwise solution of Equation (4.4).

Let $\alpha \in]0, H[$ be arbitrarily chosen. For every $\omega \in \Omega$ and $t \in [0, T]$, $Z_t^{\varphi}(z_0, \omega)$ is the image of $(B_s(\omega))_{s\in[0,t]}$ by the map

$$\varphi \circ x_t^{\gamma+1}(x_0, .),$$

which is continuous from $C^{\alpha}([0, t], \mathbb{R})$ into $\mathbb{R}$ by Proposition 2.4. So, $Z_t^{\varphi}(z_0)$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$. In other words, the stochastic process $Z^{\varphi}(z_0)$ is $\mathbb{F}$-adapted, and even $\mathbb{F}$-progressively measurable because the paths of $X(x_0)$ are continuous. By Proposition 4.6 together with (4.5), $Z^{\varphi}(z_0)$ belongs to $\mathbb{H}^2$.

Therefore, $Z^{\varphi}(z_0)$ is integrable with respect to B^* in the sense of Itô, and by Itô's formula (see Revuz and Yor [29], Theorem IV.3.3), the stochastic process $S(z_0)$ defined above is the unique solution, in the sense of Itô, of Equation (4.3). □

According to the usual definition of the Heston model, put $\varphi(x) := \sqrt{x}$ for every $x \in \mathbb{R}_+$.

Consider a financial market consisting of one risky asset of prices process $S(z_0)$ and one risk-free asset of prices function S^0, which is the solution of the following ordinary differential equation :

$$S_t^0 = S_0^0 + \int_0^t r_u S_u^0 du \tag{4.6}$$

where $r \in C^0([0, T], \mathbb{R})$. Since $S(z_0)$ is the solution of Equation (4.3) in the sense of Itô and S^0 is the solution of Equation (4.6), by the integration by part formula (see Revuz and Yor [29], Proposition IV.3.1), the actualized prices process $\widetilde{S}(z_0) := S(z_0)/S^0$ is the solution, in the sense of Itô, of the following stochastic differential equation :

$$\widetilde{S}_t = \widetilde{S}_0 + \int_0^t \sqrt{Z_s(z_0)}\widetilde{S}_s \mathrm{d}B_s^*(z_0)$$

with

$$B_t^*(z_0) := \int_0^t \frac{\mu_s - r_s}{\sqrt{Z_s(z_0)}} ds + B_t^*, \quad t \in [0, T]$$

References

1. Arnold, L.: *Random Dynamical Systems.* Springer Monographs in Mathematics SMM, Springer, 1998.
2. Berzin, C. and León, J. R.: *Estimation in Models Driven by Fractional Brownian Motion.* Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 44, 2, 191–213, 2008.
3. Brouste, A. and Iacus, S.: *Parameter Estimation for the Discretely Observed Fractional Ornstein-Uhlenbeck Process and the Yuima R Package.* Comput. Stat. 28, 1529–1547, 2013.
4. Cheridito, P.: *Regularizing Fractional Brownian Motion with a View Towards Stock Price Modeling.* Thèse de doctorat de l'université de Zürich, 2001.
5. Cheridito, P., Kawaguchi, H., and Maejima, M.: *Fractional Ornstein-Uhlenbeck Processes.* Electronic Journal of Probability 8, 3, 1–14, 2003.
6. Comte, F., Coutin, L., and Renault, E.: *Affine Fractional Stochastic Volatility Models.* Annals of Finance 8, 2–3, 337–378, 2012.
7. Decreusefond, L. and Ustünel, A. S.: *Stochastic Analysis of the Fractional Brownian Motion.* Potential Analysis 10, 177–214, 1999.
8. Fernique, X.: *Intégrabilité des vecteurs gaussiens.* C.R. Acad. Sci. Paris Sér. A-B 270:A1698-A1699, 1970.
9. Friz, P. and Victoir, N.: *Multidimensional Stochastic Processes as Rough Paths : Theory and Applications.* Cambridge Studies in Applied Mathematics 120, Cambridge University Press, 2010.
10. Garrido-Atienza, M. J., Kloeden, P. E., and Neuenkirch, A.: *Discretization of Stationary Solutions of Stochastic Systems Driven by Fractional Brownian Motion.* Appl. Math. Optim. 60, 151–172, 2009.
11. Gess, B., Liu, W., and Röckner, M.: *Random Attractors for a Class of Stochastic Partial Differential Equations driven by General Additive Noise.* Journal of Differential Equations 251, 1225–1253, 2011.
12. Gubinelli, M. and Lejay, A.: *Global Existence for Rough Differential Equations under Linear Growth Condition.* hal-00384327.
13. Hairer, M.: *Ergodicity of Stochastic Differential Equations Driven by Fractional Brownian Motion.* Ann. Probab. 33, 2, 703–758, 2005.
14. Heston, S. L.: *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.* The Review of Financial Studies 6, 327–343, 1993.
15. Hu, Y., Nualart, D., and Song, X.: *A Singular Stochastic Differential Equation Driven by Fractional Brownian Motion.* Statistics and Probability Letters 78, 14, 2075–2085, 2008.
16. Huy, D. P. and Nguyen, T. T.: *A Note on Fractional Stochastic Verhulst Equation with Small Perturbation.* Soochow Journal of Mathematics 28, 1, 57–64, 2002.
17. Lebedev, N. N.: *Special Functions and their Applications.* Revised English edition. Translated and edited by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, NJ, 1965.
18. Lejay, A.: *Controlled Differential Equations as Young Integrals : A Simple Approach.* Journal of Differential Equations 248, 1777–1798, 2010.
19. Lyons, T.: *Differential Equations Driven by Rough Signals.* Rev. Mat. Iberoamericana 14, 2, 215–310, 1998.
20. Lyons, T. and Qian, Z.: *System Control and Rough Paths.* Oxford Mathematical Monographs, Oxford University Press, 2002.
21. Marie, N.: *A Generalized Mean-Reverting Equation and Applications.* ESAIM:PS, DOI:10.1051/ps/2014002, 2014.
22. Marie, N.: *Sensitivities via Rough Paths.* ESAIM:PS, DOI:10.1051/ps/2015001, 2015.
23. Maslowski, B. and Schmalfuss, B.: *Random Dynamical Systems and Stationary Solutions of Differential Equations Driven by the Fractional Brownian Motion.* Stoch. Anal. Appl. 22, 1577–1607, 2004.
24. Melichov, D.: *On Estimation of the Hurst Index of Solutions of Stochastic Differential Equations.* Doctoral dissertation, Vilnius Gediminas Technical University, 2011.
25. Molchan, G. M.: *On the Maximum of Fractional Brownian Motion.* Theory. Probab. Appl. 44, 97–102, 2000.

26. Neuenkirch, A. and Tindel, S.: *A Least Square-Type Procedure for Parameter Estimation in Stochastic Differential Equations with Additive Fractional Noises.* arXiv:1111.1816v1.
27. Nourdin, I. and Viens, F.: *Density Estimates and Concentration Inequalities with Malliavin Calculus.* Electronic Journal of Probability 14, 78, 2287–2309, 2009.
28. Nualart, D.: *The Malliavin Calculus and Related Topics. Second Edition.* Probability and Its Applications, Springer, 2006.
29. Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion. Third Edition.* A Series of Comprehensive Studies in Mathematics 293, Springer, 1999.
30. Rogers, L. C. G.: *Arbitrage with Fractional Brownian Motion.* Mathematical Finance 7, 1, 95–105, 1997.
31. Tudor, C. A. and Viens, F. G.: *Statistical Aspect of the Fractional Stochastic Calculus.* Ann. Statist. 35, 3, 1183–1212, 2007.

Nicolas MARIE: Laboratoire Modal'X, Université Paris 10, Nanterre, 92000, France
E-mail address: nmarie@u-paris10.fr

and

Laboratoire ISTI, ESME Sudria, Paris, 75015, France
E-mail address: marie@esme.fr
URL: http://nicolas.marie.math.free.fr