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## PERIODIC SOLUTIONS OF DISCRETE GENERALIZED SCHRÖDINGER EQUATIONS ON CAYLEY TREES

FUMIO HIROSHIMA, JÓZSEF LÖRINCZI, AND UTKIR ROZIKOV

ABSTRACT. In this paper we define a discrete generalized Laplacian with arbitrary real power on a Cayley tree. This Laplacian is used to define a discrete generalized Schrödinger operator on the tree. The case discrete fractional Schrödinger operators with index  $0 < \alpha < 2$  is considered in detail, and periodic solutions of the corresponding fractional Schrödinger equations are described. This periodicity depends on a subgroup of a group representation of the Cayley tree. For any subgroup of finite index we give a criterion for eigenvalues of the Schrödinger operator under which periodic solutions exist. For a normal subgroup of infinite index we describe a wide class of periodic solutions.

### 1. Introduction

The spectral properties of Schrödinger operators on graphs have many applications in physics and they have been intensively studied since the late 1990s. In [14] Klein proved the existence of a purely absolutely continuous spectrum on the Cayley tree, under weak disorder.

The paper [18] reviews the main aspects and problems in the Anderson model on the Cayley tree. It shows whether wave functions are extended or localized is related to the existence of complex solutions of a certain non-linear equation. In [15] sufficient conditions for the presence of the absolutely continuous spectrum of a Schrödinger operator on a rooted Cayley tree are given; see [2] for more results on the model. In [8] the spectrum of a Schrödinger operator on the  $d$ -dimensional lattice  $\mathbb{Z}^d$  is studied.

In recent years, fractional calculus has received a great deal of attention. Equations involving fractional derivatives and fractional Laplacians have been studied by various authors, see e.g. [1, 6, 20] and the references therein. In probability theory fractional Laplacians appear as generators of stable processes [11]. Non-local (such as fractional) Schrödinger operators are currently increasingly studied [7, 9, 12, 13, 16, 17]. Their counterparts on lattices are currently much less understood than local discrete Schrödinger operators. In [10] a new phenomenon on the spectral edge behaviour of non-local lattice Schrödinger operators with a  $\delta$ -interaction has been reported. In this paper we study solutions of generalized Schrödinger equations on Cayley trees.

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The paper is organized as follows. In Section 2 we define a discrete generalized Laplacian (with power  $\alpha \in \mathbb{R}$ ) on a Cayley tree. This Laplacian is then used to define a discrete generalized Schrödinger operator on the Cayley tree. In Section 3 we consider fractional Schrödinger operators with exponents  $0 \leq \alpha \leq 2$ , and describe periodic solutions of the corresponding fractional Schrödinger equations. This periodicity depends on a subgroup of a group representation  $G_k$  of the Cayley tree, i.e., for a given subgroup  $\widehat{G}$  of  $G_k$  we define a notion of a  $\widehat{G}$ -periodic solution. Related ideas of  $\widehat{G}$ -periodic Gibbs measures and periodic  $p$ -harmonic functions have been considered in [21, 22, 24] before. The notion of  $\widehat{G}$ -periodic solutions can, however, be considered for arbitrary graphs and groups. For any subgroup of finite index we give a criterion for eigenvalues of the fractional Schrödinger operator under which there are periodic solutions. Also, we describe a wide class of periodic solutions for a normal subgroup of infinite index.

## 2. A Discrete Fractional Laplacian on Cayley Trees

Recall that a *Cayley tree*  $\Gamma^k$  is an infinite  $k$ -regular tree ( $k \geq 1$ ), i.e., a connected graph on a countably infinite set of vertices, with no cycles and in which every vertex has degree  $k + 1$ . We denote  $\Gamma^k = (V, L)$ , where  $V$  is the vertex set and  $L$  the edge set of  $\Gamma^k$ . Two vertices  $x$  and  $y$  are called *adjacent* if there exists an edge  $l \in L$  connecting them. In such a case we will use the notation  $l = \langle x, y \rangle$ . Also, we denote the set of adjacent vertices of  $x$  by  $S(x) = \{y \in V : \langle x, y \rangle \in L\}$  and call it the *neighbourhood* of  $x$ .

An alternating sequence  $(x_1, \langle x_1, x_2 \rangle, x_2, \dots, x_{n-1}, \langle x_{n-1}, x_n \rangle, x_n)$  of vertices and edges is a *path*. The *length* of a path is the number of edges connecting the two end-vertices of the path. On  $\Gamma^k$  one can define a natural *distance*  $d(x, y)$  as the length of the shortest path connecting  $x$  and  $y$ .

For any  $x \in V$  denote

$$W_m(x) = \{y \in V : d(x, y) = m\}, \quad m \geq 1.$$

Let  $u : V \rightarrow \mathbb{C}$  be a function. The *Laplacian*  $\Delta$  on a Cayley tree is defined by

$$(\Delta u)(x) = \sum_{y \in S(x)} (u(y) - u(x)). \quad (2.1)$$

For  $n \in \mathbb{N}$  we denote by  $\Delta^n$  the  $n$ -fold iteration of  $\Delta$ , i.e.,  $\Delta^n = \Delta(\Delta^{n-1})$ . We have the following expression.

**Lemma 2.1.** *For any  $n \in \mathbb{N}$  and  $x \in V$  the following holds*

$$\begin{aligned} (\Delta^n u)(x) = & \sum_{j=0}^{n-1} \left[ (-k-1)^j \binom{n}{j} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j} \in S(y_{n-j-1})} u(y_{n-j}) \right] \\ & + (-k-1)^n u(x), \end{aligned} \quad (2.2)$$

with  $y_0 = x$ .

*Proof.* We proceed by induction. For  $n = 1$  this is true by (2.1). For  $n = 2$  using (2.1) we obtain

$$\begin{aligned} (\Delta^2 u)(x) &= (\Delta \Delta u)(x) = \sum_{y \in S(x)} ((\Delta u)(y) - (\Delta u)(x)) \\ &= \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} u(y_2) - 2(k+1) \sum_{y_1 \in S(x)} u(y_1) + (k+1)^2 u(x). \end{aligned}$$

Now we assume that the formula (2.2) is true for  $n$  and shall prove it for  $n+1$ :

$$\begin{aligned} (\Delta^{n+1} u)(x) &= (\Delta \Delta^n u)(x) = \sum_{y \in S(x)} ((\Delta^n u)(y) - (\Delta^n u)(x)) \\ &= \sum_{y \in S(x)} \left( \sum_{j=0}^{n-1} \left[ (-k+1)^j \binom{n}{j} \sum_{y_1 \in S(y)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j} \in S(y_{n-j-1})} u(y_{n-j}) \right] \right. \\ &\quad \left. + (-k+1)^n u(y) \right) \\ &\quad - \sum_{j=0}^{n-1} \left[ (-k+1)^j \binom{n}{j} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j} \in S(y_{n-j-1})} u(y_{n-j}) \right] \\ &\quad \left. + (-k+1)^n u(x) \right) \\ &= \sum_{y \in S(x)} \sum_{y_1 \in S(y)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_n \in S(y_{n-1})} u(y_n) \\ &\quad + \sum_{j=1}^{n-1} \left[ (-k+1)^j \binom{n}{j} \sum_{y \in S(x)} \sum_{y_1 \in S(y)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j} \in S(y_{n-j-1})} u(y_{n-j}) \right] \\ &\quad \left. + (-k+1)^n \sum_{y \in S(x)} u(y) \right) \\ &\quad - (k+1) \sum_{j=0}^{n-1} \left[ (-k+1)^j \binom{n}{j} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j} \in S(y_{n-j-1})} u(y_{n-j}) \right] \\ &\quad \left. - (k+1)(-k+1)^n \binom{n}{n} u(x) \right). \tag{2.3} \end{aligned}$$

Now denote variable  $y$  by  $y_1$  and  $y_i$  by  $y_{i+1}$  for  $i = 1, \dots, n$ , then using  $\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}$  we get

$$\begin{aligned} \text{RHS of (2.3)} &= \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n+1} \in S(y_n)} u(y_{n+1}) \\ &\quad + \sum_{j=1}^n \left[ (-k+1)^j \binom{n}{j} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j+1} \in S(y_{n-j})} u(y_{n-j+1}) \right] \\ &\quad + \sum_{j'=1}^n \left[ (-k+1)^{j'} \binom{n}{j'-1} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j'+1} \in S(y_{n-j'})} u(y_{n-j'+1}) \right] \\ &\quad + (-k+1)^{n+1} u(x) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n+1} \in S(y_n)} u(y_{n+1}) \\
 &+ \sum_{j=1}^n \left[ (-k+1)^j \left( \binom{n}{j} + \binom{n}{j-1} \right) \sum_{y_1 \in S(x)} \cdots \sum_{y_{n-j+1} \in S(y_{n-j})} u(y_{n-j+1}) \right] \\
 &\quad + (-k+1)^{n+1} u(x) \\
 &= \sum_{j=0}^n \left[ (-k+1)^j \binom{n+1}{j} \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{n-j+1} \in S(y_{n-j})} u(y_{n-j+1}) \right] \\
 &\quad + (-k+1)^{n+1} u(x).
 \end{aligned}$$

□

It is easily seen that

$$\sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} u(y_2) = \sum_{y \in W_2(x)} u(y) + (k+1)u(x). \tag{2.4}$$

Next we define real powers  $\Delta^\alpha$  of the Laplacian in such a way that when  $\alpha$  takes a positive integer value  $n$ , then  $\Delta^n$  is recovered. To define a fractional Laplacian on the Cayley tree we make the change

$$\binom{\alpha}{j} \mapsto \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha+1-j)}.$$

Note that this is the multi-dimensional version of the discrete fractional difference operator introduced by Andersen for a sequence  $(u_n)_{n \in \mathbb{N}}$  of real numbers (see [6])

$$\Delta^\alpha u_n = \sum_{j=0}^\infty (-1)^j \binom{\alpha}{j} u_{n+j}.$$

**Definition 2.2 (Generalized Laplacian on a Cayley tree).** We define the *generalized Laplacian* of order  $\alpha \geq 0$  on a Cayley tree by

$$\begin{aligned}
 (\Delta^\alpha u)(x) &= \sum_{j=0}^{[\alpha]-1} \left[ A(\alpha, j) \sum_{y_1 \in S(x)} \sum_{y_2 \in S(y_1)} \cdots \sum_{y_{[\alpha]-j} \in S(y_{[\alpha]-j-1})} u(y_{[\alpha]-j}) \right] \\
 &\quad + (-k+1)^{[\alpha]} u(x),
 \end{aligned} \tag{2.5}$$

where  $A(\alpha, j) = \frac{(-k+1)^j \Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha+1-j)}$  and  $[\alpha]$  denotes the integer part of  $\alpha$ . When  $0 < \alpha < 2$ , we call  $\Delta^\alpha$  *fractional Laplacian* of order  $\alpha$ .

A Schrödinger equation of a fractional power  $\alpha$  can be written as

$$(\Delta^\alpha u)(x) = Eu(x) - v(x)u(x). \tag{2.6}$$

Here  $u(x) \in \mathbb{C}$  is the value of the wave function at vertex  $x$ ,  $E \in \mathbb{R}$  is the energy of the particle, and  $v = \{v(x)\}_{x \in V}$  is a given real-valued function.

### 3. Periodic Wave Functions

In what follows we consider the case  $0 \leq \alpha \leq 2$ .

**3.1. Case:**  $0 \leq \alpha < 1$ . We have by (2.5)

$$(\Delta^\alpha u)(x) = -(k + 1)^{[\alpha]} u(x) = u(x), \tag{3.1}$$

i.e., the operator  $\Delta^\alpha$  is an identity operator. Consequently, from (2.6) we obtain

$$u(x) = Eu(x) - v(x)u(x). \tag{3.2}$$

From this equation we get  $(E - 1 - v(x))u(x) = 0$ . A straightforward analysis of this equation gives the following.

**Proposition 3.1.** *For given  $E$  and function  $v$  the equation (3.2) has the solution*

$$u(x) = \begin{cases} 0, & \text{if } x \in \{y : v(y) + 1 \neq E\} \\ u_*(x), & \text{if } x \in \{y : v(y) + 1 = E\}, \end{cases}$$

where  $u_*$  is an arbitrary function.

*Remark 3.2.* If one wants to have a fractional Laplacian  $\Delta_1^\alpha$  with the property that it is not identical for any  $\alpha > 0$ , then this operator can be defined using operator (2.5) as

$$(\Delta_1^\alpha u)(x) = \begin{cases} (\Delta^{\alpha+1}u)(x), & \text{if } \alpha \in [0, +\infty) \setminus \{0, 1, 2, \dots\} \\ (\Delta^\alpha u)(x), & \text{if } \alpha \in \{0, 1, 2, \dots\}. \end{cases} \tag{3.3}$$

The Schrödinger equation (2.6) corresponding to the operator  $\Delta_1^\alpha$  is

$$(\Delta_1^\alpha u)(x) = Eu(x) - v(x)u(x). \tag{3.4}$$

We note that if we know the solutions of the equation (2.6), then using (3.3) we can find solutions of (3.4). Thus it is sufficient to consider the equation (2.6) for the operator  $\Delta^\alpha$ .

**3.2. Case:**  $\alpha = 1$ . In this case the Schrödinger equation reads

$$\sum_{y \in S(x)} u(y) = Eu(x) - v(x)u(x). \tag{3.5}$$

**3.2.1. Properties of group representations of the Cayley tree.** To study solutions of equation (3.5) and equations considered below first we give some properties of a group representation of the Cayley tree. Let  $G_k$  be a free product of  $k + 1$  cyclic groups of the second order with generators  $a_1, a_2, \dots, a_{k+1}$ , respectively. Any element  $x \in G_k$  has the following form:

$$x = a_{i_1} a_{i_2} \dots a_{i_n}, \quad \text{where } 1 \leq i_m \leq k + 1, \quad m = 1, \dots, n.$$

It is known [3] that there exists a one-to-one correspondence between the set of vertices  $V$  of the Cayley tree  $\Gamma^k$  and the group  $G_k$ . In Fig. 1 some elements of  $G_k$  are shown (see [23, Chapter 1] for detailed properties of this group representation).

Let  $K \subset G_k$  be an arbitrary normal subgroup of index  $r$  of the group  $G_k$  (see [4, 23, 22] for examples of subgroups of  $G_k$ ). We introduce the following equivalence relation on the set  $G_k$ :  $x \sim y$  if  $xy^{-1} \in K$ . Let  $G_k/K = \{K_1, K_2, \dots, K_r\}$  be the factor-group with respect to  $K$ . Denote

$$q_i(x) = |W_1(x) \cap K_i|, \quad s_i(x) = |W_2(x) \cap K_i|, \quad i = 1, \dots, r, \quad x \in G_k,$$

where  $|\cdot|$  is the counting measure of a set. The following lemma is known [19, 21].

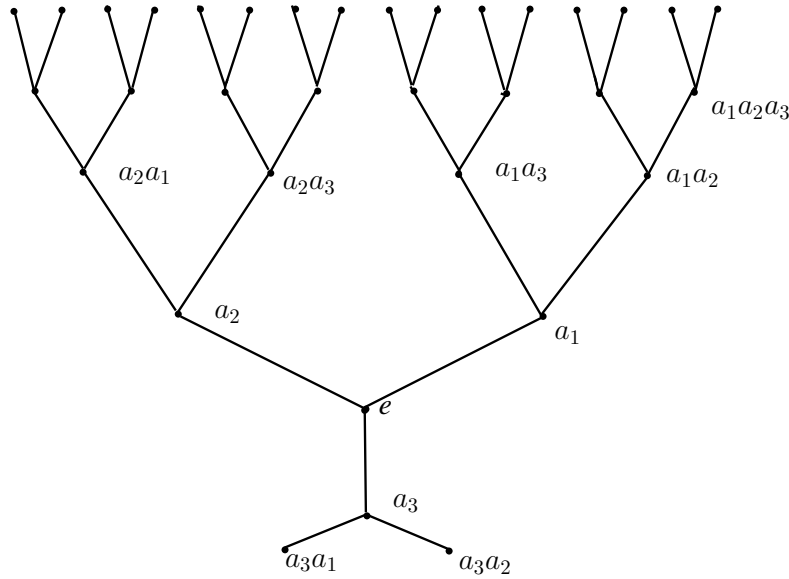


FIGURE 1. Cayley tree of order two, i.e.,  $k = 2$ .

**Lemma 3.3.** *If  $x \sim y$ , then  $q_i(x) = q_i(y)$  and  $s_i(x) = s_i(y)$  for any  $i = 1, \dots, r$ .*

By Lemma 3.3 it follows that  $q_j(x)$  and  $s_j(x)$  corresponding to  $K$  have the following form

$$q_j(x) = q_{ij}, \quad s_j(x) = s_{ij} \quad \text{for all } x \in K_i.$$

Since the set  $V$  of vertices has the group representation  $G_k$ , without loss of generality we identify  $V$  with  $G_k$ , i.e., we may replace  $V$  with  $G_k$ .

**Definition 3.4.** Let  $K$  be a subgroup of  $G_k, k \geq 1$ . A function  $u : x \in G_k \rightarrow u(x) \in \mathbb{C}$  is called  $K$ -periodic if  $u(yx) = u(x)$  for all  $x \in G_k$  and  $y \in K$ . A  $G_k$ -periodic function  $u$  is called translation-invariant.

**3.2.2. The case of finite index.** Let  $G_k/K = \{K_1, \dots, K_r\}$  be a factor group, where  $K$  is a normal subgroup of index  $r \geq 1$ . Note that any  $K$ -periodic function  $u$  has the form

$$u(x) = u_j, \quad \forall x \in K_j, \quad j = 1, \dots, r. \tag{3.6}$$

Assume that  $v = \{v(x)\}_{x \in V}$  is  $K$ -periodic, i.e.  $v(x) = v_j$  for any  $x \in K_j$  then from (3.5) we get

$$\sum_{j=1}^r q_{ij} u_j = (E - v_i) u_i, \quad i = 1, 2, \dots, r. \tag{3.7}$$

This system can be written as

$$\sum_{j=1: j \neq i}^r q_{ij} u_j - (E - q_{ii} - v_i) u_i = 0, \quad i = 1, \dots, r. \tag{3.8}$$

For a given normal subgroup  $K$  of index  $r \geq 1$  we denote by  $D_K(E, v)$  the determinant of the system (3.8). Thus the following holds.

**Theorem 3.5.** *For given  $K$  and a  $K$ -periodic  $v$  there exists a  $K$ -periodic wave function  $u \neq 0$  if and only if  $E$  is a solution to  $D_K(E, v) = 0$ .*

**Example 3.6.** (1) In case  $K = G_k$ , we have translation invariant wave functions, i.e.,  $u(x) = u_1$  for all  $x \in V$ . In this case  $D_K(E, v) = E - v_1 - k - 1 = 0$ , thus a translation invariant wave function exists if and only if  $E = v_1 + k + 1$ .

(2) Let  $K = G_k^{(2)}$  be the subgroup in  $G_k$  consisting of all words of even length. Clearly, it is a subgroup of index 2. In this case we have  $D_K(E, v) = (E - v_1)(E - v_2) - (k + 1)^2 = 0$ . For any  $E$  satisfying this equation there are  $G_k^{(2)}$ -periodic wave functions having the form

$$u(x) = \begin{cases} u_1, & \text{if } x \in G_k^{(2)} \\ u_2, & \text{if } x \in G_k \setminus G_k^{(2)}. \end{cases}$$

(3) Let  $k = 2$ ,  $K = H_0$ , with  $H_0 = \{x \in G_2 : \omega_x(a_1) - \text{even}, \omega_x(a_2) - \text{even}\}$ , where the number of letters  $a_i$  of the word  $x$  is denoted by  $\omega_x(a_i)$ . Note that  $H_0$  is a normal subgroup of index 4. The group  $G_2/H_0 = \{H_0, H_1, H_2, H_3\}$  has the following elements:

$$\begin{aligned} H_1 &= \{x \in G_2 : \omega_x(a_1) - \text{even}, \omega_x(a_2) - \text{odd}\} \\ H_2 &= \{x \in G_2 : \omega_x(a_1) - \text{odd}, \omega_x(a_2) - \text{even}\} \\ H_3 &= \{x \in G_2 : \omega_x(a_1) - \text{odd}, \omega_x(a_2) - \text{odd}\}. \end{aligned}$$

In Fig. 2 the partitions of  $\Gamma^2$  with respect to  $H_0$  are shown. The elements of the class  $H_i$ ,  $i = 0, 1, 2, 3$  are denoted by  $i$ . In this case an  $H_0$ -periodic wave function  $u$  has the form  $u(x) = u_i$  if  $x \in H_i$ . Such a non-zero function exists if and only if  $E$  satisfies the equation

$$D_{H_0}(E, v) = \begin{vmatrix} v_0 + 1 - E & 1 & 1 & 0 \\ 1 & v_1 + 1 - E & 0 & 1 \\ 1 & 0 & v_2 + 1 - E & 1 \\ 0 & 1 & 1 & v_3 + 1 - E \end{vmatrix} = 0.$$

**3.2.3. The case of infinite index.** We consider a normal subgroup  $\mathcal{H}_0$  of infinite index constructed for  $M = \{1, 2\}$  as follows. Let the mapping  $\pi_M : \{a_1, \dots, a_{k+1}\} \rightarrow \{a_1, a_2\} \cup \{e\}$  be defined by

$$\pi_M(a_i) = \begin{cases} a_i, & \text{if } i \in M \\ e, & \text{if } i \notin M. \end{cases}$$

Denote by  $G_M$  the free product of cyclic groups  $\{e, a_i\}$ ,  $i \in M$ . Consider

$$f_M(x) = f_M(a_{i_1} a_{i_2} \dots a_{i_m}) = \pi_M(a_{i_1}) \pi_M(a_{i_2}) \dots \pi_M(a_{i_m}).$$



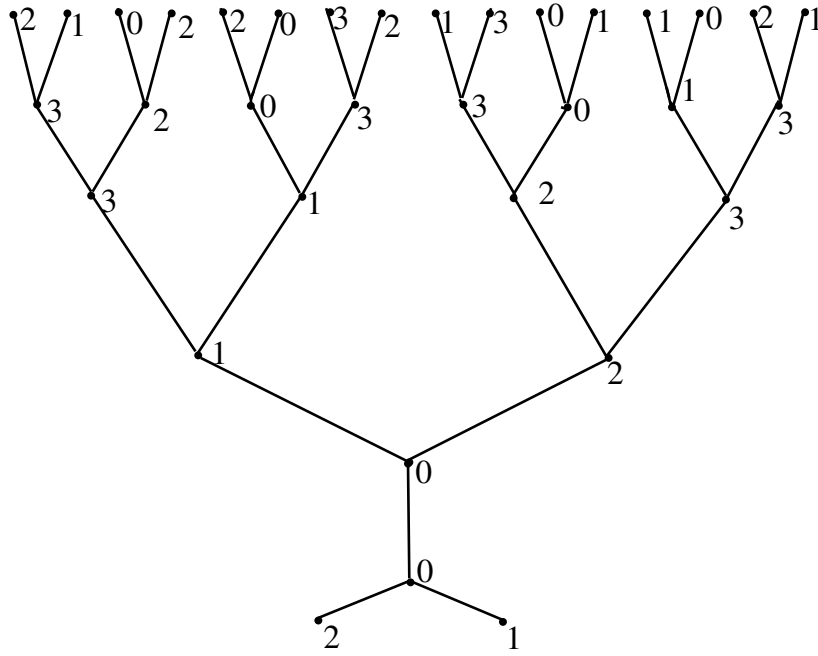


FIGURE 2. Partition of  $\Gamma^2$  by subgroup  $H_0$  of index 4.

Then it is seen that  $f_M$  is a homomorphism and hence  $\mathcal{H}_0 = \{x \in G_k : f_M(x) = e\}$  is a normal subgroup of infinite index. The factor group has the form

$$G_k/\mathcal{H}_0 = \{\mathcal{H}_0, \mathcal{H}_0(a_1), \mathcal{H}_0(a_2), \mathcal{H}_0(a_1a_2), \dots\},$$

where  $\mathcal{H}_0(y) = \{x \in G_k : f_M(x) = y\}$ . With the notations

$$\mathcal{H}_m = \mathcal{H}_0(\underbrace{a_1a_2 \dots}_m), \quad \mathcal{H}_{-m} = \mathcal{H}_0(\underbrace{a_2a_1 \dots}_m)$$

the factor group can be represented as

$$G_k/\mathcal{H}_0 = \{\dots, \mathcal{H}_{-2}, \mathcal{H}_{-1}, \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots\}.$$

The partition of the Cayley tree  $\Gamma^2$  with respect to  $\mathcal{H}_0$  is shown in Fig. 3 (the elements of the class  $\mathcal{H}_i$ ,  $i \in \mathbb{Z}$ , are denoted by  $i$ ).

Any  $\mathcal{H}_0$ -periodic wave function has the form

$$u(x) = u_m \text{ if } x \in \mathcal{H}_m, \quad m \in \mathbb{Z}.$$

We note that (see [24]) if  $x \in \mathcal{H}_m$ , then

$$q_{m-1}(x) = |W_1(x) \cap \mathcal{H}_{m-1}| = 1, \quad q_m(x) = k - 1, \quad q_{m+1}(x) = 1.$$

Using these equalities we get (see Fig. 3)

$$\begin{aligned} s_{m-2}(x) &= |W_2(x) \cap \mathcal{H}_{m-2}| = 1, \quad s_{m-1}(x) = 2(k - 1) \\ s_m(x) &= (k - 1)(k - 2), \quad s_{m+1}(x) = 2(k - 1), \quad s_{m+2}(x) = 1. \end{aligned}$$

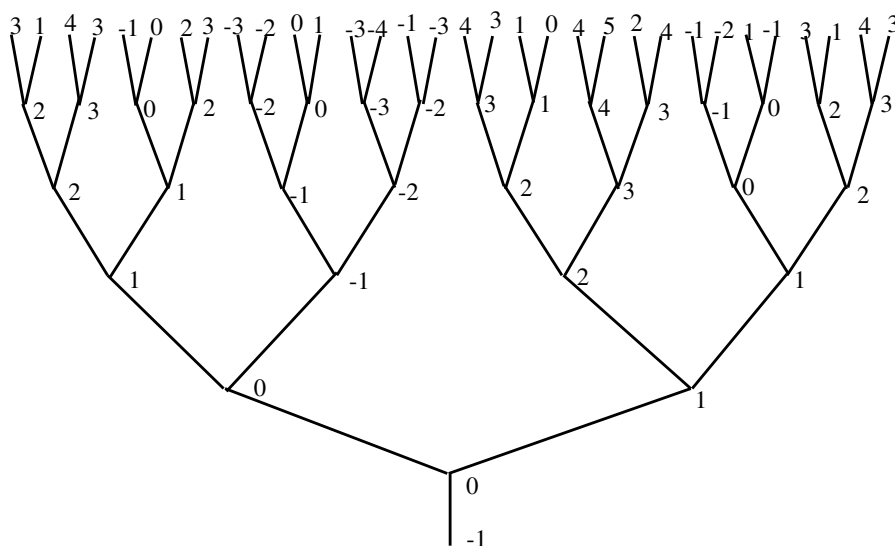


FIGURE 3. The partition of the Cayley tree  $\Gamma^2$  w.r.t.  $\mathcal{H}_0$ , the elements of the class  $\mathcal{H}_i$ ,  $i \in \mathbb{Z}$ , are denoted by  $i$ .

Assuming that  $v$  is  $\mathcal{H}_0$ -periodic, i.e.,  $v(x) = v_m$  for any  $x \in \mathcal{H}_m$ , equation (3.5) gives (see Fig. 3)

$$(E - v_n)u_n = u_{n-1} + (k - 1)u_n + u_{n+1}, \tag{3.9}$$

where  $n \in \mathbb{Z}$ . For simplicity, assume that  $v_n \equiv v_0$ , i.e.,  $v$  is translation invariant. Then equation (3.9) becomes a linear homogeneous recurrence equation of order 2, i.e.

$$u_{n+1} + (k + v_0 - E - 1)u_n + u_{n-1} = 0. \tag{3.10}$$

Recall that the solution procedure of a general linear homogeneous recurrence equation with constant coefficients of order  $d$  is given by

$$u_n + c_1u_{n-1} + c_2u_{n-2} + \dots + c_du_{n-d} = 0. \tag{3.11}$$

Let  $p(t)$  be the characteristic polynomial, i.e.,

$$p(t) = t^d + c_1t^{d-1} + c_2t^{d-2} + \dots + c_d.$$

Suppose  $\lambda_i$  is a root of  $p(t)$  having multiplicity  $r_i$ . Write  $u_n$  as a linear combination of *all* the roots counting multiplicity with arbitrary coefficients  $b_{ij}$ :

$$u_n = \sum_i (b_{i1} + b_{i2}n + b_{i3}n^2 + \dots + b_{ir_i}n^{r_i-1}) \lambda_i^n. \tag{3.12}$$

This gives the general solution to equation (3.11). Applying this procedure to equation (3.10), we have

$$p(\lambda) = \lambda^2 + (k - 1 + v_0 - E)\lambda + 1 = 0. \tag{3.13}$$

An analysis of this equation by using formula (3.12) yields the following result.

**Theorem 3.7.** *For  $\mathcal{H}_0$  and a translation invariant  $v$  the following hold:*

- (1) *If  $E \in (k + v_0 - 3, k + v_0 + 1)$ , then an  $\mathcal{H}_0$ -periodic wave function is  $u(x) = C_1\lambda_1^n + C_2\lambda_2^n$  for any  $x \in \mathcal{H}_n$ ,  $n \in \mathbb{Z}$ , where  $\lambda_1, \lambda_2$  are (complex) solutions to equation (3.13).*
- (2) *If  $E = k + v_0 - 3$ , then there is an  $\mathcal{H}_0$ -periodic function  $u(x) = (-1)^n(C_1 + nC_2)$ ,  $x \in \mathcal{H}_n$ ,  $n \in \mathbb{Z}$ .*
- (3) *If  $E = k + v_0 + 1$ , then there is an  $\mathcal{H}_0$ -periodic function  $u(x) = C_1 + nC_2$ ,  $x \in \mathcal{H}_n$ .*
- (4) *If  $E \in (-\infty, k + v_0 - 3) \cup (k + v_0 + 1, +\infty)$ , then an  $\mathcal{H}_0$ -periodic wave function is  $u(x) = C_1\lambda_1^n + C_2\lambda_2^n$  for any  $x \in \mathcal{H}_n$ ,  $n \in \mathbb{Z}$ , where  $\lambda_1, \lambda_2$  are (real) solutions to equation (3.13).*

*These solutions are in each case unique up to the choice of the constant prefactors.*

**3.3. Case:**  $1 < \alpha < 2$ . In this case we have from (2.5)

$$(\Delta^\alpha u)(x) = A(\alpha, 0) \sum_{y \in S(x)} u(y) + A(\alpha, 1)u(x) = \sum_{y \in S(x)} u(y) - \frac{(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}u(x).$$

Consequently, from (2.6) we get an equation similar to equation (3.5), with  $E$  replaced by  $E' = E + \frac{(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}$ . Thus the analysis of this equation is very similar to the analysis of equation (3.5).

**3.4. Case:**  $\alpha = 2$ . Using formula (2.4) for  $\alpha = n = 2$ , we get

$$(\Delta^2 u)(x) = \sum_{z \in W_2(x)} u(z) - 2(k+1) \sum_{y \in W_1(x)} u(y) + (k+1)(k+2)u(x). \quad (3.14)$$

The Schrödinger equation corresponding to the bi-Laplacian is

$$(\Delta^2 u)(x) = Eu(x) - v(x)u(x). \quad (3.15)$$

Now we discuss the periodic solutions of (3.15).

**3.4.1. The case of finite index.** Let  $G_k/K = \{K_1, \dots, K_r\}$  be a factor group, where  $K$  is a normal subgroup of index  $r \geq 1$ . Assume that  $v = \{v(x)\}_{x \in G_k}$  is  $K$ -periodic, i.e.,  $v(x) = v_j$  for any  $x \in K_j$ . Then by (3.15) and (3.14) for any  $i = 1, 2, \dots, r$  we get

$$\sum_{j=1}^r s_{ij}u_j - 2(k+1) \sum_{j=1}^r q_{ij}u_j + (k+1)(k+2)u_i = (E - v_i)u_i. \quad (3.16)$$

This system can be written as

$$\sum_{j=1: j \neq i}^r (s_{ij} - 2(k+1)q_{ij})u_j - (E - (k+1)(k+2) + 2(k+1)q_{ii} - s_{ii} - v_i)u_i = 0, \quad (3.17)$$

where  $i = 1, 2, \dots, r$ .

For a given normal subgroup  $K$  of index  $r \geq 1$  we denote by  $\mathcal{D}_K(E, v)$  the determinant of the system (3.17). Thus we have the following result.

**Theorem 3.8.** *For a given  $K$  and a  $K$ -periodic  $v$  there exists a  $K$ -periodic solution  $u \neq 0$  to the equation (3.15) if and only if  $E$  is a solution of  $\mathcal{D}_K(E, v) = 0$ .*

**Example 3.9.** We consider some examples of subgroups introduced in Subsection 3.2.2.

(1) In case  $K = G_k$  we have translation invariant solutions, i.e.,  $u(x) = u_1$  for all  $x \in V$ . In this case  $\mathcal{D}_K(E, v) = E - v_1 = 0$ . Thus a non-zero, translation invariant wave function exists if and only if  $E = v_1$ .

(2) For  $K = G_k^{(2)}$  we have

$$\mathcal{D}_K(E, v) = (E - v_1)(E - v_2) - 2(k + 1)^2(2E - (v_1 + v_2)) = 0.$$

For any  $E$  satisfying this equation there exist  $G_k^{(2)}$ -periodic solutions of equation (3.15).

(3) Let  $K = H_0$ . By Fig. 2 it is clear that

$$\begin{aligned} q_{00} &= q_{01} = q_{02} = q_{10} = q_{11} = q_{13} = q_{20} = q_{22} = q_{23} = q_{31} = q_{32} = q_{33} = 1 \\ q_{03} &= q_{12} = q_{21} = q_{30} = 0 \\ s_{ij} &= \begin{cases} 0, & \text{if } i = j \\ 2, & \text{if } i \neq j. \end{cases} \end{aligned}$$

Hence an  $H_0$ -periodic function exists if and only if  $E$  satisfies the equation

$$\mathcal{D}_{H_0}(E, v) = \begin{vmatrix} v_0 + 6 - E & -4 & -4 & 2 \\ -4 & v_1 + 6 - E & 2 & -4 \\ -4 & 2 & v_2 + 6 - E & -4 \\ 2 & -4 & -4 & v_3 + 6 - E \end{vmatrix} = 0.$$

**3.4.2. The case of infinite index.** Consider  $\mathcal{H}_0$  constructed in Subsection 3.2.3. Then from equation (3.15) using (3.14) and the above expressions of  $q_m(x)$  and  $s_m(x)$  we obtain

$$(E - v_m)u_m = u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}, \tag{3.18}$$

where  $m \in \mathbb{Z}$ . For simplicity, we assume that  $v_n \equiv v_0$ , i.e.,  $v$  is translation invariant. Then equation (3.18) yields the characteristic equation

$$\lambda^4 - 4\lambda^3 + (6 + v_0 - E)\lambda^2 - 4\lambda + 1 = 0. \tag{3.19}$$

We make use again of the general argument (3.11)-(3.12). Denoting  $\xi = \lambda + 1/\lambda$ , from (3.19) we get

$$(\xi - 2)^2 = E - v_0. \tag{3.20}$$

By a simple analysis of (3.20) and then of  $\lambda + 1/\lambda = \xi$ , we obtain the following results.

(1) If  $E < v_0$ , then equation (3.19) has the four distinct complex solutions

$$\hat{\lambda}_{1,2} = \frac{1}{2} \left( z \pm \sqrt{z^2 - 4} \right), \quad \hat{\lambda}_{3,4} = \frac{1}{2} \left( \bar{z} \pm \sqrt{\bar{z}^2 - 4} \right),$$

$$\text{where } z = 2 + i\sqrt{v_0 - E}, \bar{z} = 2 - i\sqrt{v_0 - E}.$$

(2) If  $E = v_0$ , then equation (3.19) has the unique solution  $\lambda = 1$  with multiplicity 4.

- (3) If  $v_0 < E < 16 + v_0$ , then equation (3.19) has the two real solutions

$$\lambda_{1,2} = \frac{2 + \sqrt{E - v_0} \pm \sqrt{(2 + \sqrt{E - v_0})^2 - 4}}{2} \quad (3.21)$$

and the two complex non-real solutions

$$\tilde{\lambda}_{3,4} = \frac{2 - \sqrt{E - v_0} \pm i\sqrt{4 - (2 - \sqrt{E - v_0})^2}}{2}. \quad (3.22)$$

- (4) If  $E = 16 + v_0$ , then equation (3.19) has three real solutions

$$\lambda_{1,2} = 3 \pm 2\sqrt{2}, \quad \lambda_3 = -1, \quad (3.23)$$

where  $-1$  has multiplicity 2.

- (5) If  $E > 16 + v_0$ , then equation (3.19) has the four real solutions  $\lambda_1, \lambda_2$  given by (3.21) and

$$\lambda_{3,4} = \frac{2 - \sqrt{E - v_0} \pm \sqrt{(2 - \sqrt{E - v_0})^2 - 4}}{2}. \quad (3.24)$$

Thus we have proved the following result.

**Theorem 3.10.** *For  $\mathcal{H}_0$  and a translation invariant  $v$  (i.e.,  $v \equiv v_0$ ) the following hold:*

- (1) *If  $E < v_0$ , then there exists an  $\mathcal{H}_0$ -periodic function of the form*

$$u(x) = u_m = C_1 \hat{\lambda}_1^m + C_2 \hat{\lambda}_2^m + C_3 \hat{\lambda}_3^m + C_4 \hat{\lambda}_4^m, \quad x \in \mathcal{H}_m, \quad m \in \mathbb{Z},$$

*where  $\hat{\lambda}_i$ ,  $i = 1, 2, 3, 4$  are the complex numbers above.*

- (2) *If  $E = v_0$ , then equation (3.15) has an  $\mathcal{H}_0$ -periodic solution*

$$u(x) = u_m = C_0 + C_1 m + C_2 m^2 + C_3 m^3, \quad x \in \mathcal{H}_m, \quad m \in \mathbb{Z}.$$

- (3) *If  $v_0 < E < 16 + v_0$ , then equation (3.15) has an  $\mathcal{H}_0$ -periodic solution*

$$u(x) = u_m = C_1 \lambda_1^m + C_2 \lambda_2^m + C_3 \tilde{\lambda}_3^m + C_4 \tilde{\lambda}_4^m, \quad x \in \mathcal{H}_m, \quad m \in \mathbb{Z},$$

*where  $\lambda_1, \lambda_2$  are given by (3.21), and  $\tilde{\lambda}_3, \tilde{\lambda}_4$  are defined in (3.22) above.*

- (4) *If  $E = 16 + v_0$ , then equation (3.15) has an  $\mathcal{H}_0$ -periodic solution*

$$u(x) = u_m = C_1 (3 - 2\sqrt{2})^m + C_2 (3 + 2\sqrt{2})^m + (C_3 + mC_4)(-1)^m,$$

*for all  $x \in \mathcal{H}_m$  and  $m \in \mathbb{Z}$ .*

- (5) *If  $E > 16 + v_0$ , then equation (3.15) has an  $\mathcal{H}_0$ -periodic solution*

$$u(x) = u_m = C_1 \lambda_1^m + C_2 \lambda_2^m + C_3 \lambda_3^m + C_4 \lambda_4^m, \quad x \in \mathcal{H}_m, \quad m \in \mathbb{Z},$$

*where  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  are given by (3.21) and (3.24).*

*These solutions are in each case unique up to the choice of the constant prefactors.*

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