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ON APPROXIMATION RATES FOR BOUNDARY CROSSING PROBABILITIES FOR THE MULTIVARIATE BROWNIAN MOTION PROCESS

SHAUN MCKINLAY AND KONSTANTIN BOROVKOV*

Abstract. Motivated by an approximation problem from mathematical finance, we analyse the stability of the boundary crossing probability for the multivariate Brownian motion process, with respect to small changes of the boundary. Under broad assumptions on the nature of the boundary, including the Lipschitz condition (in a Hausdorff-type metric) on its time cross-sections, we obtain an analogue of the Borovkov and Novikov (2005) upper bound for the difference between boundary hitting probabilities for close boundaries in the univariate case. We also obtained upper bounds for the first boundary crossing time densities.

1. Introduction and Main Results

Let $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^{(1)}, \ldots, W_t^{(m)})\}_{t\geq 0}$ be the standard m-dimensional Brownian motion process, $\boldsymbol{W}_0 = \boldsymbol{0}$. For a fixed $T < \infty$, let $\mathcal{G}$ be the class of open sets $G \subset (0,T) \times \mathbb{R}^m$ (the first component representing time), with $(0, \boldsymbol{0}) \in \partial G$.

In a number of applied problems (one notable example being barrier options' pricing), one needs to compute the probability

$$P(G) := \mathbb{P}((t, \boldsymbol{W}_t) \in G, t \in (0,T))$$

for $\boldsymbol{W}$ to stay within a given set in the time-space G during the time interval $(0,T)$. It is well known that that can be done by solving the respective boundary value problem for the heat equation in m dimensions (see e.g. Section 4.3C in [10] for a discussion of the univariate case and [15] for an efficient numerical scheme for computing $P(G)$ for cylindric sets G). However, even in the univariate case, a closed form expression for the probability $P(G)$ is only available in a few special cases, the most famous one being when G is specified by a one-sided linear boundary. There is vast literature devoted to different approaches to computing boundary crossing probability and first hitting time densities for the univariate Brownian motion. For a recent bibliography of the published work on that topic see e.g. [9].

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Much less was done in the multivariate case. A number of studies have considered the probability

$$p_{\boldsymbol{x}}(C) := \mathbb{P}\big(\boldsymbol{x} + \boldsymbol{W}_t \in C, t \in (0,T)\big) \equiv P((0,T) \times (C - \boldsymbol{x})), \quad \boldsymbol{x} \in C,$$

when $C \subset \mathbb{R}^m$ is a cone, the simplest form of which is defined as follows. For $\boldsymbol{y} \in \mathbb{R}^m \backslash \{\mathbf{0}\}$, let $\theta(\boldsymbol{y})$ be the angle between $\boldsymbol{y}$ and the point $(1, 0, \dots, 0) \in \mathbb{R}^m$. A cone of angle $\alpha \in (0, \pi)$ is defined as $\mathcal{C}_\alpha := \{\boldsymbol{y} \in \mathbb{R}^m : 0 < \theta(\boldsymbol{y}) < \alpha\}$.

It was apparently F. Spitzer who was the first to consider the probability $p_{\boldsymbol{x}}(\mathcal{C}_\alpha)$ in the two-dimensional case. In [17], he gave an integral transform for the function $p_{\boldsymbol{x}}(\mathcal{C}_\alpha)$. This probability was later computed explicitly in [8].

More recently, the case $m \geq 3$ has been considered for "generalised cones" defined as follows. If D is a proper open connected subset of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$, the generalised cone $\mathcal{C}_D$ generated by D is the set of all rays emanating from the origin $\mathbf{0}$ and passing through D. Under some technical restrictions on D, a representation for $p_{\boldsymbol{x}}(\mathcal{C}_D)$ as an infinite series involving confluent hypergeometric functions and eigenfunctions of the Lapace-Beltrami operator on $\mathbb{S}^{m-1}$ was given in [4]. This result was later strengthened in [1], where the same analytic formula was shown to hold for a larger class of generalized cones. An alternative technique based on the reflection principle was used in [13] to compute $p_{\boldsymbol{x}}(C)$ in case of "wedges" C.

In the case of general $G \in \mathcal{G}$, a possible approach to approximate evaluation of $P(G)$ in nontrivial univariate cases is to approximate G with another set $\widetilde{G} \in \mathcal{G}$ for which the computation of $P(\widetilde{G})$ is tractable. For instance, when

$$G = \{(t, x) : t \in (0,T),\ g_-(t) < x < g_+(t)\}, \tag{1.1}$$

where $g_-(t) < g_+(t)$ are smooth enough continuous functions, one could use a $\widetilde{G}$ of the same nature but with piece-wise linear boundaries $\widetilde{g}_\pm$ approximating $g_\pm$, respectively. Recall that, for such boundaries, the problem of calculating $P(\widetilde{G})$ reduces (by conditioning on the process' values at the times where the boundaries' have their "junction points") to calculating the values of k-dimensional normal CDFs. For more detail on this technique see e.g. [16], and for a similar approach in the case of the so-called generalised Daniels' boundaries [5], see e.g. [3] and references therein. The same technique was also used in [18] to compute the boundary crossing probabilities for a class of diffusion processes which can be expressed as piecewise monotone (not necessarily one-to-one) functionals of a standard Brownian motion.

To justify the use of such approximations, however, one must provide bounds for the approximation error $|P(G) - P(\widetilde{G})|$. In the univariate case, rather tight bounds of such type were obtained for the one-dimensional Brownian motion [3] and then extended to time-homogeneous univariate diffusions process [6].

The aim of this note is to extend the outlined approximation approach to the multivariate case and provide bounds for approximation errors. Our results below are also of interest for the theory of boundary value problems for parabolic partial differential equations.

First we will introduce some notation which we will need to define a Hausdorff-type metric that proved to be the most natural one for measuring the closeness of the sets G and $\widetilde{G}$ in the context of our problem. For $H \subset [0,T] \times \mathbb{R}^m$, let

$$H_t := \{\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : (t, \boldsymbol{x}) \in H\}, \quad t \in [0,T], \tag{1.2}$$

be the time t section of H. For $A \subset \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{R}^m$, let $\rho(\boldsymbol{x}, A) := \inf_{\boldsymbol{y} \in A} \|\boldsymbol{x} - \boldsymbol{y}\|$ be the distance from $\boldsymbol{x}$ to the set A, $\|\cdot\|$ being the Euclidian norm in $\mathbb{R}^m$, A^c be the complement of A, and let

$$A^{(v)} := \begin{cases} \{\boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, A) < v\}, & v > 0, \\ \{\boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, A^c) \le -v\}^c, & v \le 0, \end{cases}$$

denote the dilation (case $v > 0$) and erosion (case $v < 0$), respectively, of the set A by the set $B_{|v|}(\mathbf{0})$, where $B_r(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^m : \|\boldsymbol{x} - \boldsymbol{y}\| < r\}$ stands for the open ball of radius $r > 0$ with centre at $\boldsymbol{x} \in \mathbb{R}^m$.

For $H \subset [0,T] \times \mathbb{R}^m$, let

$$H^{(v)} := \{(t, \boldsymbol{x}) : t \in [0,T], \boldsymbol{x} \in H_t^{(v)}\}, \quad v \in \mathbb{R}.$$

The Hausdorff distance between sets $A, \widetilde{A} \subset \mathbb{R}^m$ is defined by

$$\rho_h(A, \widetilde{A}) := \inf\{\varepsilon > 0 : A \subset \widetilde{A}^{(\varepsilon)} \text{ and } \widetilde{A} \subset A^{(\varepsilon)}\}.$$

It will be convenient for us to use the metric

$$\rho_H(A, \widetilde{A}) := \max\{\rho_h(A, \widetilde{A}), \rho_h(A^c, \widetilde{A}^c)\}.$$

For positive numbers K, β, γ, introduce the class $\mathcal{G}_{K,\beta,\gamma} \subset \mathcal{G}$ of sets G satisfying the following conditions on their cross-sections.

[**G1**] The following Lipschitz condition holds:

$$\rho_H(G_s, G_t) \le K(t - s), \quad 0 < s < t < T.$$

[**G2**] For any $t \in (0,T)$ and $\boldsymbol{g} \in \partial G_t$, there exists a ball $B_\beta(\boldsymbol{y}) \subset G_t^c$ with $\beta > 0$ and $\boldsymbol{g} \in \partial B_\beta(\boldsymbol{y})$.

[**G3**] For any $t \in (0,T)$, there exists a $v_0 > 0$ such that

$$\mathbb{E}(1 + \|\boldsymbol{W}_t\|; 0 < \rho(\boldsymbol{W}_t, G_t^c) < v) < \gamma v, \quad 0 < v < v_0.$$

Condition [**G1**] is a multivariate version of the Lipschitz condition on the boundaries $g_\pm$ of the curvilinear strip (1.1) in the univariate case. It is easy to see that [**G1**] implies that ∂G_t is a Lipschitz continuous funtion of t in Hausdorff metric, but the converse is not true. It turns out though that, in the multivariate case, Lipschitz continuity of ∂G_t will not suffice for deriving the desired bounds.

Likewise, using just the Hausdorff metric $\rho_h(G_s, G_t)$ for measuring the rate of change of the sections G_t with time, also proved to be inadequate. For example, imagine a "Swiss cheese"-type set, e.g. the unit disc with a large number of small holes in it. The Hausdorff distance between that set and the disc can be very small, but a change of the time section from that disc at time t to the "Swiss cheese set" at time $t + h$ can clearly lead to a rather high probability of the first hitting time of ∂G during $(t, t + h)$, making it impossible to obtain necessary for the derivation of our results bounds for the first hitting time density (and leading to impassable obstacles in other elements of the proof as well).

The role of [**G2**] is ensure that the section boundaries ∂G_t do not have "tight outside folds". Together with [**G1**], it means that, for any fixed $t \in (0,T)$, as the point $\boldsymbol{x}$ approaches the boundary ∂G_t, the probability that the Brownian motion starting at $\boldsymbol{x}$ at time t will not leave G during the residual time interval (t, T) decays as a linear function of $\rho(\boldsymbol{x}, \partial G_t)$, and it is exactly that rate of decay that is needed for our main result (1.3). The above-mentioned linear bound follows from the fact that the desired probability will not exceed the probability of hitting a cone with the axis parallel to the time axis and the base of radius β on the left, which is disjoint with G and which, due to conditions [**G1**] and [**G2**], one can "attach" to the boundary ∂G at a point with the time component equal to t and space component $\rho(\boldsymbol{x}, \partial G_t)$-distant from $\boldsymbol{x}$. The respective bound for the cone is obtained in Lemma 2.3 below.

Condition [**G3**] may be a bit more technical and less transparent. Roughly speaking, it means that the boundary ∂G_t "cannot have too many folds". It is only used in the proof of Proposition 2.1 establishing an upper bound for the first hitting time density. The condition enables one to "translate" the "local bounds" on the conditional probability that the hitting time of ∂G belongs to $(t, t+h)$ given the value of $\boldsymbol{W}_t$ to the required bound for the unconditional probability of the hitting time being in that interval.

The main result of the present paper is the following bound.

Theorem 1.1. *If $G \in \mathcal{G}_{K,\beta,\gamma}$, then there exists a $c = c(T, K, \beta, \gamma) < \infty$ such that*

$$P\big(G^{(\varepsilon)}\big) \leq P(G) + c\varepsilon, \quad \varepsilon > 0. \tag{1.3}$$

It turns out that, in the important special case of convex cross-sections G_t, conditions [**G2**] and [**G3**] are superfluous. While that [**G2**] holds in this case for arbitrary large $\beta > 0$ is obvious, the validity of [**G3**] is not hard to establish using the Cauchy's surface area formula that implies that the volume of the portion of the integration region $\{\boldsymbol{x} : 0 < \rho(\boldsymbol{x}, G_t^c) < v\}$ that lies in the spherical layer $\{\boldsymbol{x} : k \leq \|\boldsymbol{x}\| < k+1\}$ cannot exceed cvk^{m-1}, as demonstrated in the proof of the following assertion.

Corollary 1.2. *Assume that $G \in \mathcal{G}$ satisfies* [**G1**] *and G_t is convex for any $t \in (0, T)$. Then G also satisfies* [**G2**] *with any $\beta > 0$ and* [**G3**] *for some $\gamma < \infty$, and so the bound from Theorem 1.1 holds true.*

The next result is a trivial consequence of Theorem 1.1. We state it here because it is the natural multivariate extension of the main bound from [3].

Corollary 1.3. *Suppose $G \in \mathcal{G}_{K,\beta,\gamma}$. For any $\varepsilon > 0$, if sets $G', G'' \in \mathcal{G}$ are such that $G \subset G' \subset G^{(\varepsilon)}$, $G \subset G'' \subset G^{(\varepsilon)}$, then*

$$|P(G') - P(G'')| < c\varepsilon \tag{1.4}$$

for some constant $c = c(T, K, \beta, \gamma) < \infty$.

Remark 1.4. The form of the statement in the above assertion is somewhat different from the one in the univariate case where we basically estimated the difference $P\big(G^{(\varepsilon)}\big) - P\big(G^{(-\varepsilon)}\big)$. The multivariate situation is noticeably more complicated. In particular, in $m \geq 2$ dimensions, for a set $G \in \mathcal{G}_{K,\beta,\gamma}$ it is not necessarily true that $G^{(-\varepsilon)} \in \mathcal{G}_{K,\beta,\gamma}$, even if we allow the parameters of the class $\mathcal{G}_{K,\beta,\gamma}$ in the last instance to be different from those for the one containing G. One implication of that observation is that, without some additional restrictive assumptions, the estimation of $P\big(G^{(\varepsilon)}\big) - P\big(G^{(-\varepsilon)}\big)$ becomes then impossible. On the other hand, the framework of our Theorem 1.1 is quite simple and appears to be the most natural in the multivariate setup.

2. Proofs

Without loss of generality, we can assume in this section that $T = 1$.

The proof is based on the same idea as in [3] and uses the total probability formula representation (2.2) below for the difference between the probabilities $P\big(G^{(\varepsilon)}\big)$ and $P(G)$. This difference is just the probability that the Brownian motion leaves the set G but does not leave the slightly bigger $G^{(\varepsilon)}$. To estimate that difference one needs bounds for both the integrand and the "integrator" in the integral on the right-hand side of (2.2).

First we will deal with the latter, obtaining in Proposition 2.1 an upper bound for the density of the first hitting time of ∂G (of which the derivation relies on Lemmata 2.2–2.6). For the former, one can expect that the integrand is $O(\varepsilon)$, with the coefficient of ε in the bound increasing as t approaches 1, as less and less time will be remaining for the Brownian motion to leave $G^{(\varepsilon)}$. The desired bound for the integrand will follow from Lemma 2.7 which we use to bound the tail of the distribution of the time the Brownian motion starting at time t at a boundary point of G_t hits a cone located outside $G^{(\varepsilon)}$ and "attached" to $\partial G^{(\varepsilon)}$ at a point with the time component equal to t. As we already pointed out, conditions [**G1**] and [**G2**] make it possible to use that result to bound the probability of not hitting $\partial G^{(\varepsilon)}$ during the residual time interval $(t, 1)$.

For a measurable $H \subset [0, 1] \times \mathbb{R}^m$, let

$$\tau(H) := \inf\{t > 0 : (t, \boldsymbol{W}_t) \in \partial H\}, \tag{2.1}$$

setting $\tau(H) := 1$ when $\boldsymbol{W}_t \notin \partial H_t$, $t \in (0, 1)$. Letting $\tau := \tau(G)$, $\tau^{(\varepsilon)} := \tau(G^{(\varepsilon)})$, we have from the Markov property of the Brownian motion that, for $\varepsilon > 0$,

$$D_\varepsilon(G) := P\big(G^{(\varepsilon)}\big) - P(G) = \int_{(0,1)} \mathbb{P}\big(\tau^{(\varepsilon)} = 1 | \tau = t\big) \mathbb{P}(\tau \in dt). \tag{2.2}$$

The following proposition, establishing absolute continuity of the distribution of τ and providing upper bounds for its density, is of independent interest.

Proposition 2.1. *The random variable τ has density p on $(0, 1)$ satisfying*

$$p(t) \le 8m^2\gamma \begin{cases} \sqrt{\frac{1}{\pi t}} + \frac{m-1}{2\beta - Kt} + 2K + \frac{2}{t}, & t \in (0, \min\{\beta/K, 1\}), \\ \sqrt{\frac{K}{\pi\beta}} + \frac{\beta+2}{2t-\beta/K} + \frac{m-1}{\beta} + K, & t \in [\min\{\beta/K, 1\}, 1). \end{cases}$$

To prove the proposition, note that, for any $t \in (0, 1)$, setting $\tau_t := \inf\{s > t : (s, \boldsymbol{W}_s) \in \partial G\}$, one has, for $0 < h < 1 - t$,

$$\begin{aligned} \mathbb{P}(\tau \in (t, t+h)) &= \int_{G_t} \mathbb{P}(\tau \in (t, t+h) | \boldsymbol{W}_t = \boldsymbol{z}) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\ &= \int_{G_t} \mathbb{P}(\tau > t | \boldsymbol{W}_t = \boldsymbol{z}) \mathbb{P}(\tau_t < t + h | \boldsymbol{W}_t = \boldsymbol{z}) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}). \end{aligned} \tag{2.3}$$

Next we will bound the two factors in the integrand on the right hand side of (2.3). It will be convenient to use the notation

$$r(\boldsymbol{z}) := \rho(\boldsymbol{z}, \partial G_t) \tag{2.4}$$

(for a fixed t). The following lemma gives a bound for the first factor.

Lemma 2.2. *For $t \in (0,1)$, one has*

$$\frac{\mathbb{P}(\tau > t | \boldsymbol{W}_t = \boldsymbol{z})}{2r(\boldsymbol{z})} \le \begin{cases} \sqrt{\frac{1}{\pi t}} + \frac{2(\|\boldsymbol{z}\| + r(\boldsymbol{z}))}{t} + \frac{m-1}{2\beta - Kt} + 2K, & t < \beta/K, \\ \sqrt{\frac{K}{\pi\beta}} + \frac{\|\boldsymbol{z}\| + r(\boldsymbol{z}) + \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K, & t \ge \beta/K. \end{cases}$$

The proof of Lemma 2.2 uses our next three lemmata. The main idea is that the desired conditional boundary non-hitting probability cannot exceed the probability of not hitting a simple "more remote" set, namely, a cone "attached from the outside" to the "nearest" to $\boldsymbol{z}$ point of ∂G_t, with the axis parallel to the time axis and such that its base is on the right. For that simpler boundary, the desired bound is obtained in Lemma 2.3. This is done by reducing the problem to a univariate straight line boundary crossing probability by a diffusion process, namely, by the radial process of the m-dimensional Brownian bridge. To obtain the desired bound for the latter problem, we use the comparison Lemma 2.4 and a simple reference process (2.10), which is just an univariate arithmetic Brownian motion. For that process, the respective bound is given in Lemma 2.5.

For $u, v > 0$, introduce the (possibly truncated) cones

$$C(v,u) := \{(s, \boldsymbol{x}) \in [0,v] \times \mathbb{R}^m : \|\boldsymbol{x}\| \le u - Ks\}, \tag{2.5}$$

$$C^*(v,u) := \{(s, \boldsymbol{x}) \in [0,v] \times \mathbb{R}^m : |x_i| \le (u - Ks)/\sqrt{m},\ i = 1, \dots, m\}.$$

Clearly, $C^*(v,u) \subset C(v,u)$.

We will slightly abuse notation by denoting by $\mathbb{P}_{\boldsymbol{x}}$ the distribution on the canonical space corresponding to the Brownian motion process started at the point $\boldsymbol{W}_0 = \boldsymbol{x} \in \mathbb{R}^m$ and keeping the notation $\tau(H)$ for the stopping time (2.1) for that process.

Lemma 2.3. *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ with $\|\boldsymbol{x}\| > \beta$, we have*

$$\frac{\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,\beta)) > u | \boldsymbol{W}_t = \boldsymbol{y})}{\|\boldsymbol{x}\| - \beta} \le \begin{cases} \sqrt{\frac{2}{\pi u}} + 2\Big(\frac{2(\|\boldsymbol{y}\| - \beta)}{t} + \frac{m-1}{2\beta - Kt} + 2K\Big)^+, & u \le t/2,\ t < \beta/K, \\ \sqrt{\frac{2}{\pi u}} + 2\Big(\frac{\|\boldsymbol{y}\| - \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K\Big)^+, & u \le \beta/(2K),\ t \ge \beta/K, \end{cases}$$

where $x^+ := \max\{0, x\}$.

Note that the above upper bounds agree at $t = \beta/K$.

To prove Lemma 2.3, we will require the following two additional lemmas. For a univariate process $X = \{X_t\}_{t \ge 0}$ and $x \in \mathbb{R}$, set

$$\eta_x(X) := \inf\{t \ge 0 : X_t = x\}. \tag{2.6}$$

Lemma 2.4. *Let* $\{W_t\}_{t\geq 0}$ *be the standard univariate Brownian motion given on a filtered probability space,* $\{Y_t\}_{t\geq 0}$ *a continuous adapted process on the same space. Let* $X_t^{(1)}$ *and* $X_t^{(2)}$ *be strong unique solutions of the stochastic differential equations* (SDEs)

$$dX_t^{(i)} = a_i(t, X_t^{(i)}, Y_t)dt + dW_t, \quad X_0^{(i)} = x_0, \quad i = 1, 2,$$

where a_i *are continuous. Suppose that, for a given* $l < x_0$*, one has* $a_1(t, x, y) < a_2(t, x, y)$ *for all* $(t, x) \in [0, \infty) \times (l, \infty)$*,* $y \in \mathbb{R}$*. Then* $X_t^{(1)} < X_t^{(2)}$ *a.s. for all* $t \in (0, \eta_l(X^{(1)}))$*.*

The proof of Lemma 2.4 below follows the argument proving a somewhat weaker assertion of Lemma 4 on p.120 of [7].

Proof. Define the continuously differentiable function

$$\Delta(t) := X_t^{(2)} - X_t^{(1)} = \int_0^t (a_2(s, X_s^{(2)}, Y_s) - a_1(s, X_s^{(1)}, Y_s))\, ds, \quad t < \eta_l.$$

Then, for all points $t < \eta_l$ with $\Delta(t) = 0$, we have that $X_t^{(1)} = X_t^{(2)}$, and so at these points

$$\Delta'(t) = a_2(t, X_t^{(2)}, Y_t) - a_1(t, X_t^{(1)}, Y_t) > 0.$$

In particular, we have $\Delta(0) = 0$, $\Delta'(0+) > 0$. Therefore we can find a $\delta > 0$ such that $\Delta(t) > 0$ for all $0 < t \leq \delta$. Now suppose the set $\{t \in (0, \eta_l) : \Delta(t) = 0\}$ is not empty. Then for $t_1 := \inf\{t \in (0, \eta_l) : \Delta(t) = 0\}$ we have $\Delta(t_1) = 0$, $\Delta'(t_1) > 0$, and so there exists a $\delta_1 > 0$ such that $\Delta(t) < 0$ for $t \in [t_1 - \delta_1, t_1]$. Therefore $\Delta(t)$ changes signs on the interval $[\delta, t_1 - \delta_1]$, i.e., it takes on the value zero there, which contradicts the definition of t_1. We conclude that $\{t \in (0, \eta_l) : \Delta(t) = 0\}$ is empty a.s., and since $\Delta(t) > 0$ for sufficiently small t, $\Delta(t) > 0$ for all $t \in (0, \eta_l)$ as required. □

Recall that $\{W_t\}_{t\geq 0}$ is the standard univariate Brownian motion process.

Lemma 2.5. *For* $c \in \mathbb{R}$ *and* $\varepsilon > 0$*,*

$$\mathbb{P}(\sup_{0\leq s\leq t}(W_s - cs) < \varepsilon) \leq \varepsilon\Big(\sqrt{\frac{2}{\pi t}} + 2c^+\Big).$$

Proof. The probability on the left hand side above is known explicitly (see e.g. 1.1.4 on p.250 of [2]): denoting by Φ the standard normal distribution function,

$$\begin{aligned}
\mathbb{P}(\sup_{0\leq s\leq t}(W_s - cs) < \varepsilon) &= \Phi\big(c\sqrt{t} + \varepsilon/\sqrt{t}\big) - e^{-2c\varepsilon}\Phi\big(c\sqrt{t} - \varepsilon/\sqrt{t}\big) \\
&\leq \Phi\big(c\sqrt{t} + \varepsilon/\sqrt{t}\big) - e^{-2c^+\varepsilon}\Phi\big(c\sqrt{t} - \varepsilon/\sqrt{t}\big) \\
&\leq \Phi\big(c\sqrt{t} + \varepsilon/\sqrt{t}\big) - \Phi\big(c\sqrt{t} - \varepsilon/\sqrt{t}\big) + (1 - e^{-2c^+\varepsilon}) \\
&\leq \sup_{x\in\mathbb{R}}\Phi'(x) \times 2\varepsilon/\sqrt{t} + 2c^+\varepsilon \\
&= \varepsilon\Big(\sqrt{\frac{2}{\pi t}} + 2c^+\Big).
\end{aligned}$$

□

Proof of Lemma 2.3. Let $\boldsymbol{B} = \{\boldsymbol{B}_s = (B_s^{(1)}, \dots, B_s^{(m)})\}_{0\le s\le t}$ be an m-dimensional Brownian bridge process starting at $\boldsymbol{x} \in \mathbb{R}^m$ at time 0 and ending at $\boldsymbol{y} \in \mathbb{R}^m$ at time t.

In order to use Lemma 2.4, we will now derive an SDE for the radial process $S_s := \|\boldsymbol{B}_s\|$ of $\boldsymbol{B}$. Recall that $\boldsymbol{B}$ satisfies the SDE

$$d\boldsymbol{B}_s = \frac{\boldsymbol{y} - \boldsymbol{B}_s}{t-s} ds + d\boldsymbol{W}_s, \quad 0 < s < t, \tag{2.7}$$

(see e.g. p.64 in [2]) By Itô's formula, the squared radial process has stochastic differential

$$dS_s^2 = 2\sum_{i=1}^m B_s^{(i)} dB_s^{(i)} + m\,ds, \quad 0 < s < t. \tag{2.8}$$

Setting $\boldsymbol{\xi_s} := \boldsymbol{B}_s/S_s$, we have $\|\boldsymbol{\xi_s}\| \equiv 1$ and therefore

$$\xi_s(\boldsymbol{y}) := \boldsymbol{\xi_s}\boldsymbol{y}^T \le \|\boldsymbol{y}\|, \tag{2.9}$$

where $\boldsymbol{y}^T$ denotes the transpose of $\boldsymbol{y}$. Then, for $0 < s < t$, we have from (2.7) and (2.8) that

$$\begin{aligned} dS_s^2 &= 2\sum_{i=1}^m B_s^{(i)}\left(\frac{y_i - B_s^{(i)}}{t-s} ds + dW_s^{(i)}\right) + m\,ds \\ &= 2\left(\frac{\boldsymbol{B}_s\boldsymbol{y}^T - S_s^2}{t-s} + \frac{m}{2}\right) ds + 2\boldsymbol{B}_s d\boldsymbol{W}_s^T \\ &= 2\left(\frac{S_s\xi_s(\boldsymbol{y}) - S_s^2}{t-s} + \frac{m}{2}\right) ds + 2S_s\boldsymbol{\xi_s} d\boldsymbol{W}_s^T \\ &= 2\left(\frac{S_s\xi_s(\boldsymbol{y}) - S_s^2}{t-s} + \frac{m}{2}\right) ds + 2S_s d\widetilde{W}_s, \end{aligned}$$

where $\{\widetilde{W}_t\}_{t\ge0}$ is a standard univariate Brownian motion, and the last equality follows from Theorem 8.4.2 in [14].

Using the above SDE for $\{S_s^2\}$ and Itô's formula with $f(x) = \sqrt{x}$, we have

$$\begin{aligned} dS_s &= f'(S_s^2)dS_s^2 + \frac{1}{2}f''(S_s^2)(dS_s^2)^2 \\ &= \frac{1}{2S_s}\left[2\left(\frac{S_s\xi_s(\boldsymbol{y}) - S_s^2}{t-s} + \frac{m}{2}\right) ds + 2S_s d\widetilde{W}_s\right] - \frac{1}{8S_s^3}(2S_s)^2 ds \\ &= \left(\frac{\xi_s(\boldsymbol{y}) - S_s}{t-s} + \frac{m-1}{2S_s}\right) ds + d\widetilde{W}_s, \quad 0 < s < t. \end{aligned}$$

Now introduce, for a fixed $a < \|\boldsymbol{x}\|$ and $t_0 \in (0,t)$, the reference process

$$\overline{S}_s := \|\boldsymbol{x}\| + \left(\frac{\|\boldsymbol{y}\| - a}{t - t_0} + \frac{m-1}{2a}\right) s + \widetilde{W}_s, \quad s \ge 0, \tag{2.10}$$

Since $\|\boldsymbol{y}\| \ge \xi_s(\boldsymbol{y})$ by (2.9), Lemma 2.4 implies that, for all $s \in [0, \min\{t_0, \eta_a(S)\}]$, one has $\overline{S}_s \ge S_s$ a.s.

Consider first the case $t \geq \beta/K$ and set $t_0 := \beta/(2K)$, $a := \beta/2$. Then, for all $u \leq t_0$, one has

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,\beta)) > u|\boldsymbol{W}_t = \boldsymbol{y}) \\
&\quad = \mathbb{P}\Big(\inf_{0\leq s\leq u}(S_s - \beta + Ks) > 0\Big) \\
&\quad \leq \mathbb{P}\Big(\inf_{0\leq s\leq u}(\overline{S}_s - \beta + Ks) > 0\Big) \\
&\quad = \mathbb{P}\bigg[\inf_{0\leq s\leq u}\bigg(\|\boldsymbol{x}\| + \bigg(\frac{\|\boldsymbol{y}\| - \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta}\bigg)s + W_s - \beta + Ks\bigg) > 0\bigg] \\
&\quad = \mathbb{P}\bigg[\sup_{0\leq s\leq u}\bigg(W_s - \bigg(\frac{\|\boldsymbol{y}\| - \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K\bigg)s\bigg) < \|\boldsymbol{x}\| - \beta\bigg] \\
&\quad \leq (\|\boldsymbol{x}\| - \beta)\Bigg[\sqrt{\frac{2}{\pi u}} + 2\bigg(\frac{\|\boldsymbol{y}\| - \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K\bigg)^+\Bigg]
\end{aligned}$$

by Lemma 2.5.

Now consider the case $t < \beta/K$ and set $t_0 := t/2$, $a := \beta - Kt/2$. Then, for all $u \leq t_0$, we have

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,\beta)) > u|\boldsymbol{W}_t = \boldsymbol{y}) \\
&\quad \leq \mathbb{P}\Big(\inf_{0\leq s\leq u}(\overline{S}_s - \beta + Ks) > 0\Big) \\
&\quad = \mathbb{P}\bigg[\inf_{0\leq s\leq u}\bigg(\|\boldsymbol{x}\| + \bigg(\frac{\|\boldsymbol{y}\| - \beta + Kt/2}{t/2} + \frac{m-1}{2\beta - Kt}\bigg)s + W_s - \beta + Ks\bigg) > 0\bigg] \\
&\quad = \mathbb{P}\bigg[\sup_{0\leq s\leq u}\bigg(W_s - \bigg(\frac{2(\|\boldsymbol{y}\| - \beta)}{t} + \frac{m-1}{2\beta - Kt} + 2K\bigg)s\bigg) < \|\boldsymbol{x}\| - \beta\bigg] \\
&\quad \leq (\|\boldsymbol{x}\| - \beta)\Bigg[\sqrt{\frac{2}{\pi u}} + 2\bigg(\frac{2(\|\boldsymbol{y}\| - \beta)}{t} + \frac{m-1}{2\beta - Kt} + 2K\bigg)^+\Bigg].
\end{aligned}$$

Lemma 2.3 is proved. □

Proof of Lemma 2.2. Fix $t \in (0,1)$ and $\boldsymbol{z} \in G_t$. Reversing the time for the conditional Brownian motion process, we have for $t' \in (0,t)$,

$$\begin{aligned}
\mathbb{P}(\tau > t|\boldsymbol{W}_t = \boldsymbol{z}) &= \mathbb{P}(\boldsymbol{W}_s \in G_s, s \in (0,t)|\boldsymbol{W}_t = \boldsymbol{z}) \\
&= \mathbb{P}_{\boldsymbol{z}}(\boldsymbol{W}_s \in G_{t-s}, s \in (0,t)|\boldsymbol{W}_t = \boldsymbol{0}) \\
&\leq \mathbb{P}_{\boldsymbol{z}}(\boldsymbol{W}_s \in G_{t-s}, s \in (0,t')|\boldsymbol{W}_t = \boldsymbol{0}). \qquad (2.11)
\end{aligned}$$

One can clearly choose a $\boldsymbol{b} \in \partial G_t$ such that $\rho(\boldsymbol{z}, \boldsymbol{b}) = r(\boldsymbol{z})$. Then by condition [**G2**] there exists a ball $B_\beta(\boldsymbol{c}) \subset G_t^c$ with $\boldsymbol{b} \in \partial B_\beta(\boldsymbol{c})$. Using $B_\beta(\boldsymbol{c})$ as the base for the cone

$$C := (0, \boldsymbol{c}) + C(t,\beta),$$

it follows from Lipschitz condition [**G1**] that $G_{t-s} \subset C_s^c$, $s \in [0,t]$. Therefore

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{z}}(\boldsymbol{W}_s \in G_{t-s}, s \in (0,t')|\boldsymbol{W}_t = \boldsymbol{0}) &\leq \mathbb{P}_{\boldsymbol{z}}(\boldsymbol{W}_s \in C_s^c, s \in (0,t')|\boldsymbol{W}_t = \boldsymbol{0}) \\
&= \mathbb{P}_{\boldsymbol{z}-\boldsymbol{c}}(\tau(C(t,\beta)) > t'|\boldsymbol{W}_t = -\boldsymbol{c}). \qquad (2.12)
\end{aligned}$$

Since $\|-\boldsymbol{c}\| \le \|\boldsymbol{z}\| + r(\boldsymbol{z}) + \beta$, we immediately obtain the bounds stated in Lemma 2.2 from Lemma 2.3 with

$$t' = u := \begin{cases} t/2, & t < \beta/K, \\ \beta/(2K), & t \ge \beta/K. \end{cases}$$

□

Now we will turn to bounding the second factor on the right hand side of (2.3). Recall the definition (2.4) of $r(\boldsymbol{z})$.

Lemma 2.6. *For* $t \in (0,1)$, $\boldsymbol{z} \in G_t$, *one has*

$$\mathbb{P}(\tau_t < t+h|\boldsymbol{W}_t = \boldsymbol{z}) \le 2m \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr), \;\; h \in (0, \min\{r(\boldsymbol{z})/K, 1-t\}).$$

Proof. We will again use cones and reduction to univariate boundary crossing problems. For h from the specified interval, setting $\overline{\Phi}(x) := 1 - \Phi(x)$, we have

$$\begin{aligned}
\mathbb{P}(\tau_t < t+h|\boldsymbol{W}_t = \boldsymbol{z}) &\le \mathbb{P}(\tau(C(h, r(\boldsymbol{z}))) < h) \\
&\le \mathbb{P}(\tau(C^*(h, r(\boldsymbol{z}))) < h) \\
&\le 2m\,\mathbb{P}(\sup_{0\le s\le h}(W(s) + Ks/\sqrt{m}) \ge r(\boldsymbol{z})/\sqrt{m}) \\
&= 2m \int_0^h \frac{r(\boldsymbol{z})/\sqrt{m}}{\sqrt{2\pi}s^{3/2}} \exp\biggl(\frac{-(r(\boldsymbol{z})/\sqrt{m} - Ks/\sqrt{m})^2}{2s}\biggr) ds \qquad (2.13)\\
&= r(\boldsymbol{z})\sqrt{\frac{2m}{\pi}} e^{r(\boldsymbol{z})K/m} \int_0^h s^{-3/2} \exp\biggl(-\frac{r(\boldsymbol{z})^2}{2sm} - \frac{K^2 s}{2m}\biggr) ds \\
&\le r(\boldsymbol{z})\sqrt{\frac{2m}{\pi}} e^{r(\boldsymbol{z})K/m} \int_0^h s^{-3/2} \exp\biggl(-\frac{r(\boldsymbol{z})^2}{2sm}\biggr) ds \\
&= 4m\, e^{r(\boldsymbol{z})K/m}\overline{\Phi}(r(\boldsymbol{z})/\sqrt{hm}) \qquad (2.14)\\
&\le 2m \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr), \qquad (2.15)
\end{aligned}$$

where (2.13) follows from Kendall's formula (see e.g. relation 2.0.2 on p.295 of [2]), (2.14) follows by making the substitution $u = r(\boldsymbol{z})/\sqrt{sm}$, and (2.15) follows by using the bound $\overline{\Phi}(x) \le \frac{1}{2}e^{-x^2/2}$, $x > 0$. The lemma is proved. □

Proof of Proposition 2.1. Suppose that $t < \beta/K$. Then from (2.3) and the bounds derived in Lemmas 2.2, 2.6, we have

$$\begin{aligned}
\mathbb{P}(\tau \in (t, t+h)) &= \int_{G_t} \mathbb{P}(\tau > t|\boldsymbol{W}_t = \boldsymbol{z})\mathbb{P}(\tau_t < t+h|\boldsymbol{W}_t = \boldsymbol{z})\mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\
&\le 4m \int_{G_t} \biggl(\sqrt{\frac{1}{\pi t}} + \frac{2(\|\boldsymbol{z}\| + r(\boldsymbol{z}))}{t} + \frac{m-1}{2\beta - Kt} + 2K\biggr) \\
&\quad \times r(\boldsymbol{z}) \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr)\mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\
&= 4m\biggl(\biggl(\sqrt{\frac{1}{\pi t}} + \frac{m-1}{2\beta - Kt} + 2K\biggr)I_1 + \frac{2}{t}(I_2 + I_3)\biggr), \qquad (2.16)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_{G_t} r(\boldsymbol{z}) \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}), \\
I_2 &:= \int_{G_t} r(\boldsymbol{z}) \|\boldsymbol{z}\| \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}), \\
I_3 &:= \int_{G_t} r(\boldsymbol{z})^2 \exp\biggl(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm}\biggr) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}).
\end{aligned}$$

Set

$$Z := r(\boldsymbol{W}_t) \mathbf{1}_{\{\boldsymbol{W}_t \in G_t\}},$$

where $\mathbf{1}_E$ is the indicator of event E. Then, for $u(x) := xe^{xK/m - x^2/(2hm)}$, we have

$$I_1 = \mathbb{E} Z \exp\biggl\{\frac{ZK}{m} - \frac{Z^2}{2hm}\biggr\} = \int_0^\infty u(x) d\nu(x),$$

where $\nu(x)$ is the distribution function of Z. Integrating by parts and using the bound $\nu(x) < \gamma x$, $x \in (0, v_0)$, from [**G3**], we obtain

$$\begin{aligned}
I_1 &= [\nu(x)u(x)]_0^\infty - \int_0^\infty \nu(x) du(x) = -\int_0^\infty \nu(x) du(x) \\
&= \int_0^\infty \nu(x) \biggl(\frac{x^2}{hm} - \frac{xK}{m} - 1\biggr) \exp\biggl(\frac{xK}{m} - \frac{x^2}{2hm}\biggr) dx \\
&< \frac{1}{hm} \int_0^\infty \nu(x) x^2 \exp\biggl(\frac{xK}{m} - \frac{x^2}{2hm}\biggr) dx = \frac{1}{hm} \biggl(\int_0^{v_0} + \int_{v_0}^\infty\biggr) (\cdots)\, dx \\
&< \frac{1}{hm} \biggl(\gamma \int_0^\infty x^3 \exp\biggl(\frac{xK}{m} - \frac{x^2}{2hm}\biggr) dx + \int_{v_0}^\infty x^2 \exp\biggl(\frac{xK}{m} - \frac{x^2}{2hm}\biggr) dx\biggr) \\
&= \gamma hm \int_0^\infty s^3 e^{sK\sqrt{h/m} - s^2/2}\, ds + \sqrt{hm} \int_{v_0/\sqrt{hm}}^\infty s^2 e^{sK\sqrt{h/m} - s^2/2}\, ds \\
&= 2\gamma hm + \mathrm{o}(h),
\end{aligned}$$

where the second last relation follows by making the substitution $s = x/\sqrt{hm}$.

Using [**G3**] and following the same steps as above, we conclude that

$$I_2 < 2\gamma hm + \mathrm{o}(h).$$

Finally, it is even simpler to show that $I_3 = \mathrm{o}(h)$.

Then, from (2.16), we have

$$\mathbb{P}(\tau \in (t, t+h)) < 8m^2 \gamma h \biggl(\sqrt{\frac{1}{\pi t}} + \frac{m-1}{2\beta - Kt} + 2K + \frac{2}{t}\biggr) + \mathrm{o}(h).$$

It follows that τ has an absolutely continuous distribution specified by a density p satisfying

$$p(t) \le 8m^2 \gamma \biggl(\sqrt{\frac{1}{\pi t}} + \frac{m-1}{2\beta - Kt} + 2K + \frac{2}{t}\biggr), \quad t \in (0, \min\{\beta/K, 1\}).$$

Now consider the case when $t \geq \beta/K$. Then, from (2.3) and the bounds derived in Lemmas 2.2, 2.6, we have

$$\begin{aligned}
\mathbb{P}(\tau \in (t, t+h)) &= \int_{G_t} \mathbb{P}(\tau > t | \boldsymbol{W}_t = \boldsymbol{z}) \mathbb{P}(\tau_t < t+h | \boldsymbol{W}_t = \boldsymbol{z}) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\
&\leq 4m \int_{G_t} \bigg(\sqrt{\frac{K}{\pi\beta}} + \frac{\|\boldsymbol{z}\| + r(\boldsymbol{z}) + \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K \bigg) \\
&\quad \times r(\boldsymbol{z}) \exp\bigg(\frac{r(\boldsymbol{z})K}{m} - \frac{r(\boldsymbol{z})^2}{2hm} \bigg) \mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\
&= 4m \bigg[\bigg(\sqrt{\frac{K}{\pi\beta}} + \frac{\beta}{2t - \beta/K} + \frac{m-1}{\beta} + K \bigg) I_1 + \frac{I_2 + I_3}{t - \beta/(2K)} \bigg],
\end{aligned}$$

and therefore, for $t \in [\beta/K, 1)$,

$$\mathbb{P}(\tau \in (t, t+h)) \leq 8m^2 h\gamma \bigg(\sqrt{\frac{K}{\pi\beta}} + \frac{\beta+2}{2t - \beta/K} + \frac{m-1}{\beta} + K \bigg).$$

As above, it follows that τ has density p satisfying

$$p(t) \leq 8m^2\gamma \bigg(\sqrt{\frac{K}{\pi\beta}} + \frac{\beta+2}{2t - \beta/K} + \frac{m-1}{\beta} + K \bigg), \quad t \in [\min\{\beta/K, 1\}, 1).$$

Proposition 2.1 is proved. □

As we pointed our in the outline of the idea of the proof of Theorem 1.1, we will also use the following lemma that provides a bound for the integrand on the right hand side of (2.2).

Lemma 2.7. *For* $\boldsymbol{x} \in \mathbb{R}^m$ *and* $0 < r < \|\boldsymbol{x}\|$, *we have*

$$\frac{\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,r)) > t)}{2(\|\boldsymbol{x}\| - r)} \leq \begin{cases} \sqrt{\frac{1}{\pi t}} + \frac{m-1}{2r - Kt} + K, & t < r/K, \\ \sqrt{\frac{K}{\pi r}} + \frac{m-1}{r} + K, & t \geq r/K. \end{cases}$$

Proof. Similarly to the proof of Lemma 2.3, we will reduce the problem to a univariate straight line boundary crossing problem and then use the comparison Lemma 2.4 to obtain the desired bound for the latter problem.

Denote by $R = \{R_s\}_{s\geq 0}$ an m-dimensional Bessel process started at $\|\boldsymbol{x}\|$ at time 0. One can stipulate that

$$R_s = \sqrt{\big(\|\boldsymbol{x}\| + W_s^{(1)}\big)^2 + \big(W_s^{(2)}\big)^2 + \cdots + \big(W_s^{(m)}\big)^2}, \quad s \geq 0,$$

and so

$$\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,r)) > t) = \mathbb{P}(\inf_{0\leq s\leq t}(R_s - r + Ks) > 0). \tag{2.17}$$

As is well-known (see e.g. p.148 in [14]), R satisfies the SDE

$$dR_s = \frac{m-1}{2R_s} ds + d\widetilde{W}_s, \quad s > 0, \quad R_0 = \|\boldsymbol{x}\|,$$

$\{\widetilde{W}_s\}_{s\geq 0}$ being a standard univariate Brownian motion process.

Consider first the case $t < r/K$ and let

$$\overline{R}_s := \|\boldsymbol{x}\| + \frac{m-1}{2r-Kt}s + \widetilde{W}_s, \quad s \in [0, t/2].$$

Then, by Lemma 2.4, we have $\overline{R}_s \ge R_s$ a.s. for all $s \in [0, \min\{t/2, \eta_{r-Kt/2}(R)\}]$ (cf. (2.6)). From here, (2.17) and Lemma 2.5, one has

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,r)) > t) &\le \mathbb{P}\Big(\inf_{0\le s\le t/2}(\overline{R}_s - r + Ks) > 0\Big)\\ &= \mathbb{P}\Big(\inf_{0\le s\le t/2}\Big(\|\boldsymbol{x}\| + \frac{m-1}{2r-Kt}s + \widetilde{W}_s - r + Ks\Big) > 0\Big)\\ &= \mathbb{P}\Big[\sup_{0\le s\le t/2}\Big(\widetilde{W}_s - \Big(\frac{m-1}{2r-Kt} + K\Big)s\Big) < \|\boldsymbol{x}\| - r\Big]\\ &\le 2(\|\boldsymbol{x}\| - r)\Bigg(\sqrt{\frac{1}{\pi t}} + \frac{m-1}{2r-Kt} + K\Bigg).\end{aligned}$$

Now consider the case $t \ge r/K$ and let

$$\overline{R}_s := \|\boldsymbol{x}\| + \frac{m-1}{r}s + \widetilde{W}_s, \quad s \in [0, r/(2K)].$$

Then, by Lemma 2.4, we have $\overline{R}_s \ge R_s$ a.s. for all $s \in [0, \min\{r/(2K), \eta_{r/2}(R)\}]$, and so from (2.17) and Lemma 2.5, a similar derivation yields the bound

$$\mathbb{P}_{\boldsymbol{x}}(\tau(C(t,r)) > t) \le 2(\|\boldsymbol{x}\| - r)\Bigg(\sqrt{\frac{K}{\pi r}} + \frac{m-1}{r} + K\Bigg),$$

as required. □

Now we can complete the proof of Theorem 1.1. The integrand on the right hand side of (2.2) has the form

$$\begin{aligned}\mathbb{P}\big(\tau^{(\varepsilon)} = 1 | \tau = t\big) &= \int_{\partial G_t} \mathbb{P}\big(\tau^{(\varepsilon)} = 1, \boldsymbol{W}_t \in d\boldsymbol{z} | \tau = t\big)\\ &= \int_{\partial G_t} \mathbb{P}_{\boldsymbol{z}}\big(\boldsymbol{W}_s \in G^{(\varepsilon)}_{t+s}, s \in (0, 1-t)\big)\mathbb{P}(\boldsymbol{W}_t \in d\boldsymbol{z} | \tau = t).\end{aligned} \tag{2.18}$$

For any $\boldsymbol{z} \in \partial G_t$, by [**G2**] there is a point $\boldsymbol{y}$ such that $B_\beta(\boldsymbol{y}) \subset G_t^c$ and $\boldsymbol{z} \in \partial B_\beta(\boldsymbol{y})$. Clearly, $B_{\beta_\varepsilon}(\boldsymbol{y}) \subset \big(G_t^{(\varepsilon)}\big)^c$, where $\beta_\varepsilon := \beta - \varepsilon$ (we assume without loss of generality that $\varepsilon < \beta/2$), and so $G_t^{(\varepsilon)} \subset \big(B_{\beta_\varepsilon}(\boldsymbol{y})\big)^c$. By condition [**G1**], we then also have

$$G^{(\varepsilon)}_{t+s} \subset G_t^{(\varepsilon + Ks)} \subset \big(B_{\beta_\varepsilon - Ks}(\boldsymbol{y})\big)^c = \big((0, \boldsymbol{y}) + C(1-t, \beta_\varepsilon)\big)^c_s, \quad s \le \frac{\beta_\varepsilon}{K},$$

using notation (2.5). Therefore, since $\|\boldsymbol{z}-\boldsymbol{y}\|-\beta_\varepsilon=\varepsilon$, one has from Lemma 2.7 that

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{z}}\big(\boldsymbol{W}_s\in G^{(\varepsilon)}_{t+s}, s\in(0,1-t)\big)\\
&\quad\le \mathbb{P}_{\boldsymbol{z}-\boldsymbol{y}}(\tau(C(1-t,\beta_\varepsilon))>1-t)\\
&\quad\le 2\varepsilon\begin{cases}\sqrt{\frac{1}{\pi(1-t)}}+\frac{m-1}{2\beta_\varepsilon-K(1-t)}+K, & 1-t<\beta_\varepsilon/K,\\ \sqrt{\frac{K}{\pi\beta_\varepsilon}}+\frac{m-1}{\beta_\varepsilon}+K, & 1-t\ge\beta_\varepsilon/K.\end{cases}
\end{aligned}$$

Now it follows from (2.2) and (2.18) that

$$\begin{aligned}
D_\varepsilon(G)&\le 2\varepsilon\int_0^{1-\beta_\varepsilon/K}\left(\sqrt{\frac{K}{\pi\beta_\varepsilon}}+\frac{m-1}{\beta_\varepsilon}+K\right)p(t)\,dt\\
&\qquad+2\varepsilon\int_{1-\beta_\varepsilon/K}^1\left(\sqrt{\frac{1}{\pi(1-t)}}+\frac{m-1}{2\beta_\varepsilon-K(1-t)}+K\right)p(t)\,dt\\
&\le 2\varepsilon\left(\sqrt{\frac{K}{\pi\beta_\varepsilon}}+\frac{m-1}{\beta_\varepsilon}+K\right)+2\varepsilon\int_0^1\frac{p(t)\,dt}{\sqrt{\pi(1-t)}}.
\end{aligned}$$

Here

$$\begin{aligned}
\int_0^1\frac{p(t)\,dt}{\sqrt{\pi(1-t)}}&\le\sqrt{\frac{2}{\pi}}+\int_{1/2}^1\frac{p(t)\,dt}{\sqrt{\pi(1-t)}}\\
&\le\sqrt{\frac{2}{\pi}}+8m^2\gamma\sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{K}{\pi\beta}}+2(\beta+2)+\frac{m-1}{\beta}+K\right)=:c^*,
\end{aligned}$$

where the second inequality follows by assuming without loss of generality that $\beta/K<1/2$ and applying the second bound from Proposition 2.1. Recalling that $\beta_\varepsilon\ge\beta/2$, we conclude that

$$D_\varepsilon(G)\le 2\varepsilon\left(c^*+\sqrt{\frac{2K}{\pi\beta}}+\frac{2(m-1)}{\beta}+K\right).$$

Theorem 1.1 is proved. □

Proof of Corollary 1.2. Condition [**G2**] is clearly satisfied (with arbitrary large $\beta>0$) due to the convexity of G_t, so we only need to verify [**G3**].

We can assume without loss of generality that there exists a $\delta>0$ such that $B_\delta(\mathbf{0})\subset(\partial G)_0$ (for otherwise it is easy to see that, in view of [**G1**], one has $P(G)=0$ and the whole problem becomes trivial).

Consider first the case $t>t_0:=\delta/(2K)$ (assuming that $t_0<1$).

Introduce the sequence of spherical layers $C_k:=B_k(\mathbf{0})\backslash B_{k-1}(\mathbf{0})$, $k=1,2,\ldots$ As the cross-section G_t is convex, it follows from Cauchy's surface area formula (see e.g. Theorem 5.5.2 on p.56 in [11]) that the ($(m-1)$-dimensional) surface area of $\partial(G_t\cap B_k(\mathbf{0}))$ does not exceed the surface area of $B_k(\mathbf{0})$ which is equal to $k^{m-1}\omega_{m-1}$, where ω_{m-1} is the unit sphere area.

Therefore, again using the convexity of G_t, it follows that, for any $\varepsilon>0$, the volume of $V_k:=\big(G_t\backslash G_t^{(-\varepsilon)}\big)\cap C_k$ does not exceed $\varepsilon k^{m-1}\omega_{m-1}$. As the maximum value of the density of $\boldsymbol{W}_t$ on V_k less than or equal to the density's value on $\partial B_{k-1}(\mathbf{0})$, we conclude that

$$\begin{aligned}\mathbb{E}(1+\|\boldsymbol{W}_t\|;\boldsymbol{W}_t\in G_t\backslash G_t^{(-\varepsilon)}) &= \sum_{k=1}^{\infty}\mathbb{E}(1+\|\boldsymbol{W}_t\|;\boldsymbol{W}_t\in V_k)\\ &\le \frac{\varepsilon\omega_{m-1}}{(2\pi t)^{m/2}}\sum_{k=1}^{\infty}(1+k)k^{m-1}e^{-(k-1)^2/2t}\\ &\le \gamma\varepsilon\end{aligned}$$

for some $\gamma=\gamma(m,t_0)<\infty$, as we only consider $t\in(t_0,1)$.

Now turn to the case $t\in(0,t_0)$. By [**G1**], for such t one has $B_{\delta/2}(\mathbf{0})\subset G_t$. Choosing $v_0:=\delta/4$ (which we can assume to be less that one without loss of generality), one can employ the same argument as above but with the spherical layers $C_k:=B_{k+\delta/4}(\mathbf{0})\backslash B_{k-1+\delta/4}(\mathbf{0})$, $k=1,2,\dots$, making use of the observation that the maximum value of the density of $\boldsymbol{W}_t$ on the "innermost layer" C_1 is bounded for $t\in(0,t_0)$. The corollary is proved. □

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