

9-1-2018

Excluding a large theta graph

Guoli Ding
Louisiana State University

Emily Marshall
Arcadia University

Follow this and additional works at: https://digitalcommons.lsu.edu/mathematics_pubs

Recommended Citation

Ding, G., & Marshall, E. (2018). Excluding a large theta graph. *Journal of Combinatorial Theory. Series B*, 132, 47-79. <https://doi.org/10.1016/j.jctb.2018.03.002>

This Article is brought to you for free and open access by the Department of Mathematics at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.



ELSEVIER

Contents lists available at ScienceDirect

Journal of Combinatorial Theory,
Series B

www.elsevier.com/locate/jctb



Excluding a large theta graph

Guoli Ding^{a,1}, Emily Marshall^{b,2}^a Mathematics Department, Louisiana State University, Baton Rouge, LA, United States^b Computer Science and Mathematics Department, Arcadia University, Glenside, PA, United States

ARTICLE INFO

Article history:

Received 2 September 2016

Available online 15 March 2018

Keywords:

Theta graphs
Graph minors

ABSTRACT

A *theta graph*, denoted $\theta_{a,b,c}$, is a graph of order $a + b + c - 1$ consisting of a pair of vertices and three internally-disjoint paths between them of lengths a , b , and c . In this paper we study graphs that do not contain a large $\theta_{a,b,c}$ minor. More specifically, we describe the structure of $\theta_{1,2,t^-}$, $\theta_{2,2,t^-}$, θ_{1,t,t^-} , θ_{2,t,t^-} , and $\theta_{t,t,t}$ -free graphs where t is large. The main result is a characterization of $\theta_{t,t,t}$ -free graphs for large t . The 3-connected $\theta_{t,t,t}$ -free graphs are formed by 3-summing graphs without a long path to certain planar graphs. The 2-connected $\theta_{t,t,t}$ -free graphs are then built up in a similar fashion by 2- and 3-sums. This result implies a well-known theorem of Robertson and Chakravarti on graphs that do not have a bond containing three specified edges.

© 2018 Elsevier Inc. All rights reserved.

1. Introduction

All graphs are loopless but may have parallel edges. Undefined terminology can be found in [1]. In particular, a set of paths is called *independent* if no interior vertex of one path is contained in another path.

E-mail address: marshalle@arcadia.edu (E. Marshall).

¹ Research supported in part by NSF grant DMS-1500699.

² Work for this paper was largely done while at Louisiana State University.

In this paper, we describe the structure of graphs that do not contain certain large theta graphs as a minor. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A *theta graph*, denoted $\theta_{a,b,c}$, is a graph of order $a + b + c - 1$ consisting of a pair of vertices and three independent paths between them of lengths a , b , and c . Theta graphs have maximum degree 3 so containing a theta graph as a minor is equivalent to containing a theta graph as a topological minor. Throughout we will say G *contains* $\theta_{a,b,c}$ to mean G contains $\theta_{a,b,c}$ as a minor (or topological minor). Additionally we use the phrase G *contains a* $\theta_{a,b,c}$ *graph at* u *and* v to mean G contains as a subgraph a subdivision of $\theta_{a,b,c}$ in which u and v are the two vertices of degree 3. A graph is $\theta_{a,b,c}$ -*free* if it does not contain $\theta_{a,b,c}$.

The main goal of this paper is to describe all $\theta_{t,t,t}$ -free graphs for large integers t . In other words, we want to characterize all graphs that do not contain three long independent paths between any pair of vertices. This problem is in fact an instance of a very general problem stated below.

Problem (P): *for a given infinite set $\mathcal{H} = \{H_1, H_2, \dots\}$ of graphs where each H_i is a minor of H_{i+1} , characterize all minor-closed classes \mathcal{G} of graphs for which $\mathcal{G} \not\supseteq \mathcal{H}$.*

Note $\mathcal{G} \not\supseteq \mathcal{H}$ holds if and only if all graphs in \mathcal{G} are H_i -free for some i . Since this i can be (and can also be assumed to be) arbitrarily big, problem (P) is often loosely stated as: characterize graphs that do not contain a large H_i minor. Therefore when we say we want to characterize graphs that do not have a *large* $\theta_{t,t,t}$ minor, we are talking about solving (P) for $\mathcal{H} = \{\theta_{t,t,t} : t \geq 2\}$. Likewise, to characterize $\theta_{1,2,t}$ -free graphs for large t means we are solving (P) for $\mathcal{H} = \{\theta_{1,2,t} : t \geq 2\}$.

There are several choices of \mathcal{H} for which (P) has been solved. Along this line, the best known results are the two obtained by Robertson and Seymour which solve (P) for the class of all complete graphs [10] and for the class of all planar grids [8]. The same authors also solved (P) for the classes of all trees, all stars, and all paths [7,11]. Other classes for which (P) is solved include the class of all wheels [3] and the class of all double-paths [2].

We prove that $\theta_{t,t,t}$ -free graphs have the following structure: begin with a planar graph that contains no long paths outside of a special facial cycle and attach graphs that do not have long paths to the planar graph along edges, facial triangles, and certain facial 4-cycles. This result is stated formally in the next section. Additionally, we obtain corresponding results for all $\theta_{1,2,t^-}$, $\theta_{2,2,t^-}$, θ_{1,t,t^-} , and θ_{2,t,t^-} -free graphs.

Our result for $\theta_{t,t,t}$ -free graphs implies a result of Robertson and Chakravarti [6] concerning when three specified edges of a graph are contained in a *bond* (a minimal nonempty edge-cut of the graph). Suppose we subdivide the three specified edges sufficiently many times. Then it is easy to see that the three specified edges are contained in a bond in the original graph if and only if the subdivided graph contains $\theta_{t,t,t}$. This connection easily leads to the result of Robertson and Chakravarti, as we will see in the next section, and it also illustrates how much our result strengthens their result.

Another important goal of this paper is to develop tools for dealing with various cases of problem (P). We will prove several key lemmas that could be used in similar

situations. In particular, we will obtain a strengthened version of a result of Robertson and Seymour [9] on the embeddability of a graph on a disc.

The remainder of the paper is organized as follows. In Section 2 we formalize and state our main theorem. In Section 3 we examine graphs with a long path and look at large graphs which are necessarily present in such graphs. Section 4 describes several ways we will decompose our graphs into smaller pieces which will be useful in proofs. Section 5 includes lemmas on weighted graphs. In Section 6 we state and prove the characterizations of $\theta_{1,2,t^-}$, $\theta_{2,2,t^-}$, θ_{1,t,t^-} , and θ_{2,t,t^-} -free graphs. Section 7 extends and strengthens a result of Robertson and Seymour concerning planar drawings of graphs and crossing paths. In Section 8 we prove our main theorem for 3-connected graphs. Finally in Section 9 we complete our proof of the main theorem by considering 2-connected graphs.

2. Statement of the main theorem

Let G be a graph. For any two adjacent vertices x and y , the set of all edges between x and y is called a *parallel family* of G . A *simplification* of G , denoted $si(G)$, is a simple graph obtained from G by deleting all but one edge from each parallel family. We call G *3-connected* if $si(G)$ is 3-connected. We call G *2-connected* if either $si(G)$ is 2-connected or $si(G) = K_2$ with $\|G\| \geq 2$. Because $\theta_{a,b,c}$ is 2-connected, a graph is $\theta_{a,b,c}$ -free if and only if each of its blocks is $\theta_{a,b,c}$ -free. Therefore, we only need to determine 2-connected $\theta_{a,b,c}$ -free graphs.

For any subgraph H of G , a path P of G is an *H-path* if $E(P) \cap E(H) = \emptyset$ and the distinct ends of P are the only two vertices of P that are in H . Let C be a facial cycle of a plane graph G . If C bounds the infinite face of G then C is called the *outer cycle*; if C bounds a finite face then C is an *inner cycle*. Note that C is both inner and outer if $G = C$. For any cycle C , we always assume there is an implicit forward direction. This is for the purpose of simplifying our terminology. For any two vertices u, v of C , denote by $C[u, v]$ the forward path of C from u to v .

In our proof, it becomes convenient to consider weighted graphs. This notion also allows us to obtain a stronger result. A *weight function* of a graph G is a mapping w from $E(G)$ to the set of positive integers. A graph with a weight function is called a *weighted graph* and is denoted (G, w) . For any subgraph G' of G , the *weight* of G' , denoted $w(G')$, is the sum of $w(e)$ over all edges e of G' . We say (G, w) *contains* $\theta_{a,b,c}$ if G contains a theta graph as a subgraph for which the three independent paths have weights at least a , b , and c , respectively. Naturally, (G, w) is $\theta_{a,b,c}$ -free if it does not contain $\theta_{a,b,c}$. Our main result in fact is a characterization of $\theta_{t,t,t}$ -free weighted graphs. To describe these weighted graphs, we first define two fundamental classes of weighted graphs.

Let $r, s \geq 2$ be integers. Let \mathcal{L}_s be the class of 2-connected graphs that do not contain a path of length s . Let $\mathcal{L}_{r,s}$ be the class of weighted graphs (G, w) with $G \in \mathcal{L}_s$ and

$w(e) < r$ for all $e \in G$. It is clear that weighted graphs in $\mathcal{L}_{r,s}$ do not contain $\theta_{t,t,t}$ if $t \geq rs$.

For any integer $r \geq 2$, let \mathcal{P}_r be the class of 2-connected weighted plane graphs (G, w) such that if C is the outer cycle then G contains no C -path of weight $\geq 2r$ and $G \setminus E(C)$ contains no edge of weight $\geq r$. It is not difficult to see that weighted graphs in \mathcal{P}_r contain no $\theta_{t,t,t}$ for sufficiently large t . We do not justify this observation here since a more general statement will be proved later.

General $\theta_{t,t,t}$ -free weighted graphs will be constructed from $\mathcal{L}_{r,s}$ and \mathcal{P}_r by k -sums which are defined as follows for $k = 2, 3, 4$. For any two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of a graph G , let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. Let $H = G_1 \cap G_2$. Then the graph $(G_1 \cup G_2) \setminus E(H)$ is called a 2 -sum of G_1 and G_2 if $H \cong K_2$, a 3 -sum of G_1 and G_2 if $H \cong K_3$, and a 4 -sum of G_1 and G_2 if H is a 4-cycle. The graph H will be called the *summing edge* when $H \cong K_2$ and otherwise the *summing k -cycle*. If w_1, w_2 are weight functions of G_1, G_2 , then a k -sum ($k = 2, 3, 4$) of (G_1, w_1) and (G_2, w_2) is a new weighted graph (G, w) such that G is a k -sum of G_1, G_2 and for each $e \in G$, $w(e) = w_i(e)$ where i is such that $e \in G_i$.

Let G be a plane graph and let C be its outer cycle. An inner facial 4-cycle $R = x_1x_2x_3x_4x_1$ of G is called a *rectangle* if the four vertices of R are all on C and the two edges x_1x_2 and x_3x_4 of R are also edges of C . Note this implies there are no edges parallel to either x_1x_2 or x_3x_4 .

For any integers $r, s \geq 2$, let $\Phi(\mathcal{P}_r, \mathcal{L}_{r,s})$ denote the class of 2-connected weighted graphs obtained from weighted graphs $(G_0, w_0) \in \mathcal{P}_r$ by k -summing ($k = 2, 3, 4$) weighted graphs from $\mathcal{L}_{r,s}$ to edge-disjoint edges, inner facial triangles, and rectangles of G_0 . We call G_0 the *base graph* of G . Now we are ready to state our main theorem.

Theorem 2.1. *There exists a function $t(r, s)$ such that all weighted graphs in $\Phi(\mathcal{P}_r, \mathcal{L}_{r,s})$ are $\theta_{t,t,t}$ -free. Conversely, there also exists a function $s(t)$ such that every 2-connected $\theta_{t,t,t}$ -free weighted graph belongs to $\Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$.*

Since every graph G can be viewed as a weighted graph (G, ε) where $\varepsilon(e) = 1$ for all $e \in G$, Theorem 2.1 also characterizes unweighted graphs that are $\theta_{t,t,t}$ -free. We do not formally state this simplified characterization since its derivation is straightforward and the final formulation is almost identical to Theorem 2.1.

When t is treated as a fixed integer, Theorem 2.1 can be viewed as an approximate characterization of $\theta_{t,t,t}$ -free weighted graphs since all such graphs are contained in $\Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$ while no graph in $\Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$ contains $\theta_{t',t',t'}$ when t' is sufficiently large. On the other hand, when t is treated as a variable, it is not difficult to see that Theorem 2.1 implies the following. Suppose \mathcal{G} is a class of 2-connected weighted graphs such that every 2-connected weighted subgraph of any member of \mathcal{G} remains in \mathcal{G} , then $\mathcal{G} \not\subseteq \{\theta_{a,b,c} : a, b, c \geq 2\}$ if and only if $\mathcal{G} \subseteq \Phi(\mathcal{P}_r, \mathcal{L}_{r,s})$ for some r, s . Using our loose language we can say that Theorem 2.1 characterizes weighted graphs that do not contain a large $\theta_{t,t,t}$.

In the following we formally state the result of Robertson and Chakravarti [6] and we prove it using Theorem 2.1. Note that [6] only states the 4-connected version of the theorem.

Corollary 2.2. *Let G be a 2-connected graph with three distinct edges e, f, g . Then either G has a bond containing e, f, g or G is obtained from a 2-connected plane graph G_0 by 2- and 3-summing graphs to edges and inner facial triangles of G_0 , where e, f, g are contained in three graphs that are 2-summed to three distinct edges of the outer cycle of G_0 .*

Proof. Suppose G does not have a bond containing e, f, g . Let $t = |G|$ and let w be the weight function of G with $w(e) = w(f) = w(g) = t$ and $w(x) = 1$ for all other edges x of G . Then (G, w) is $\theta_{t,t,t}$ -free. By Theorem 2.1, (G, w) is obtained by summing weighted graphs from $\mathcal{L}_{t,s(t)}$ to $(H_0, w_0) \in \mathcal{P}_t$. Let C be the outer cycle of H_0 . Since no member of $\mathcal{L}_{t,s(t)}$ has an edge of weight $\geq t$ and since no edge of $H_0 \setminus E(C)$ has weight $\geq t$, it follows that e, f, g are all contained in C .

Let e', f', g' be three new edges parallel to e, f, g , respectively, and let J_0 be obtained from H_0 by replacing e, f, g with e', f', g' . Then H_0 can be obtained from J_0 by 2-summing 2-cycles $ee', ff',$ and gg' to three distinct edges of the outer cycle of J_0 . This observation implies G can be obtained from a plane graph J_0 by 2-, 3-, 4-summing graphs to edges, inner facial triangles, and rectangles of J_0 and such that e, f, g are contained in three graphs that are 2-summed to three distinct edges of the outer cycle of J_0 . Among all such expressions, we choose one with J_0 as small as possible. Let D be the outer cycle of this J_0 . If there is a 4-sum, then we claim that we can make J_0 smaller. To see this, let x_1, x_2, x_3, x_4 be the vertices of the sum in that order along D and with $x_1x_2, x_3x_4 \in E(D)$. Since edges x_1x_2 and x_3x_4 are deleted after the sum, without loss of generality, suppose e and f are in graphs 2-summed to edges of D between x_2 and x_3 . Let D' be the cycle formed from D by replacing the subpath of D from x_3 to x_2 with the edge x_2x_3 ; now the graph J'_0 with D' as its outer cycle is smaller than J_0 . Thus we may assume there are no 4-sums and the result follows immediately. \square

3. Unavoidable large graphs

Graphs without a sufficiently long path are necessarily $\theta_{t,t,t}$ -free. Since graphs without a long path have already been characterized by Robertson and Seymour [11], we will restrict our focus to graphs that do have a long path. The presence of a long path in a graph often implies the presence of some other large structure as well. In this section, we prove several lemmas describing these large structures.

We begin with two lemmas that describe the unavoidable large structures in connected graphs with many vertices and in trees with many leaves, respectively. These will be used in our later proofs. Denote by $\Delta(G)$ the maximum degree of a vertex in G .

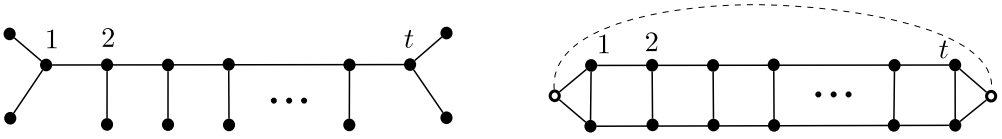


Fig. 3.1. comb_t and L_t^+ .

Lemma 3.1. *If G is simple, connected, and of order exceeding $1 + d + d(d - 1) + \dots + d(d - 1)^{p-1}$, where $d, p \geq 1$ are integers, then either $\Delta(G) > d$ or G has an induced path of length $p + 1$ starting from any specified vertex.*

Proof. Suppose $\Delta(G) \leq d$. Let $v \in V(G)$ and let n_k be the number of vertices of distance k away from v . Then $n_0 = 1$, $n_1 = d_G(v)$, and $n_k \leq n_{k-1}(d - 1)$ for all $k \geq 2$. It follows that $|G| > n_0 + n_1 + \dots + n_p$ and thus $n_{p+1} \neq 0$. Therefore, G has a vertex of distance $p + 1$ away from v , which proves the lemma. \square

Lemma 3.2. *If T is a tree with at least d^t leaves, where $d, t \geq 2$ are integers, then either $\Delta(T) > d$ or T contains a subdivision of comb_t , which is shown on the left of Fig. 3.1.*

Proof. Since contracting an edge incident with a degree 2 vertex does not change the problem, assume T has no vertex of degree 2. Since $d^t \geq 4$, T has a vertex v of degree greater than 2. If T has a path of length t starting from v (which is necessarily induced), then a comb_t subgraph can be obtained by extending this path. Assume no such path exists. Since T has at least d^t leaves, it follows that $|T| > d^t > 1 + d + d^2 + \dots + d^{t-1} > 1 + d + d(d - 1) + \dots + d(d - 1)^{t-2}$. Thus we deduce from Lemma 3.1 that $\Delta(T) > d$. \square

Denote by W_n the wheel on $n + 1$ vertices and denote by $\ell(G)$ the length of a longest path in a graph G . The next result says that a 3-connected graph with a sufficiently long path must have a big wheel minor.

Lemma 3.3 ([3], Prop. 3.8). *There exists a function $f_{3.3}(t)$ such that every 3-connected graph G with $\ell(G) \geq f_{3.3}(t)$ contains a W_t minor.*

Let L_t be the graph shown on the right of Fig. 3.1 without the dashed edge and the white vertices.

Lemma 3.4. *Let G consist of two disjoint paths $X = x_1x_2 \dots x_m$ and $Y = y_1y_2 \dots y_m$ and a matching $M = \{e_i = x_iy_{\pi(i)} : i = 1, 2, \dots, m\}$. If $m > n^2$ then G contains an L_{n+1} (topological) minor.*

Proof. Let $e_i \prec e_j$ if $i < j$ and $\pi(i) < \pi(j)$. Let F_1 be the set of maximal members of M with respect to \prec . Inductively, if F_i has been defined and $M_i = M \setminus F_1 \setminus \dots \setminus F_i \neq \emptyset$, then let F_{i+1} be the set of maximal members of M_i with respect to \prec . Note members

of each F_i can be expressed as $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ such that $i_1 < i_2 < \dots < i_k$ and $\pi(i_1) > \pi(i_2) > \dots > \pi(i_k)$. If $|F_i| > n$ for some i , then the conclusion holds since the union of paths X, Y and matching F_i contains L_{n+1} . Suppose $|F_i| \leq n$ for all i . Then $F_{n+1} \neq \emptyset$ since $m > n^2$. For each $i = 2, \dots, n+1$ and each $e \in F_i$, note there exists $f \in F_{i-1}$ with $e \prec f$. Thus there exists $e_{i_j} \in F_j$ for $j = 1, 2, \dots, n+1$ such that $e_{i_{n+1}} \prec e_{i_n} \prec \dots \prec e_{i_1}$. Now the union of X, Y and $e_{i_1}, e_{i_2}, \dots, e_{i_{n+1}}$ contains L_{n+1} . \square

Let L_t^+ be the graph shown on the right in Fig. 3.1 with the dashed edge. The next result strengthens Lemma 3.3.

Lemma 3.5. *There exists a function $f_{3.5}(t)$ such that every 3-connected graph G with $\ell(G) \geq f_{3.5}(t)$ contains W_t or L_t^+ as a topological minor.*

Proof. We will show $f_{3.5}(t) = f_{3.3}(s)$, where $s = (t-1)^r$ and $r = 1 + (t+1)^2$, satisfies the theorem. Let G be 3-connected with $\ell(G) \geq f_{3.5}(t)$. By Lemma 3.3, G has a W_s minor. This minor can be considered as a cycle C of length at least s in G , a connected subgraph G_0 of G with $V(G_0 \cap C) = \emptyset$, and a set S of s edges each incident with a vertex of G_0 and a distinct vertex of C . Let G_1 be the graph G_0 together with the edges in S . Let T be a smallest tree of G_1 containing all edges of S . Then leaves of T are precisely the s vertices on C that are incident with an edge of S . Now by Lemma 3.2, either $\Delta(T) > t-1$ or T contains a subdivision of comb_r . First suppose the former and let v be a vertex of degree at least t in T . Then T has t independent paths from v to leaves of T . Clearly, these paths together with C form a subdivision of W_t .

Next suppose T contains a subdivision T' of comb_r . Let X be the minimal path of T' that contains all the r cubic vertices of T' . Then T contains a set \mathcal{P} of r disjoint paths from X to C . Let e be an edge of C and let $Y = C \setminus e$. By viewing paths in \mathcal{P} as a matching between X and Y , we deduce from Lemma 3.4 that the union of X, Y , and paths in \mathcal{P} contains an L_{t+2} topological minor. Now this topological minor together with C contains an L_t^+ topological minor. \square

4. Decompositions

It will be helpful in later proofs to decompose graphs into smaller pieces for the purpose of better understanding their structure. In this section, we describe several ways to do this.

A *separation* of a graph G is a pair (G_1, G_2) of edge-disjoint non-spanning subgraphs of G with $G_1 \cup G_2 = G$. A set $Z \subseteq V(G)$ is a *cut* of G if $G - Z$ is disconnected. It is clear that if (G_1, G_2) is a separation then $V(G_1 \cap G_2)$ is a cut. Conversely, if Z is a cut then G has a separation (G_1, G_2) with $V(G_1 \cap G_2) = Z$. For any integer k , a *k-separation* is a separation (G_1, G_2) with $|G_1 \cap G_2| = k$ and a *k-cut* is a cut Z with $|Z| = k$. The following lemma relates k -sum with k -separation. We omit the proof since it is easy.

Lemma 4.1. (a) Let G be 2-connected and let (G_1, G_2) be a 2-separation of G with $V(G_1 \cap G_2) = \{x, y\}$. For $i = 1, 2$, let G_i^+ be obtained from G_i by adding a new edge xy . Then each G_i^+ is a 2-connected minor of G and G is a 2-sum of G_1^+ and G_2^+ .

(b) Let G be 3-connected and let (G_1, G_2) be a 3-separation of G with $V(G_1 \cap G_2) = \{x, y, z\}$. For $i = 1, 2$, let G_i^+ be obtained from G_i by adding three new edges xy, yz, xz . Then each G_i^+ is 3-connected and G is a 3-sum of G_1^+ and G_2^+ . Moreover, G_i^+ is a minor of G unless $si(G_{3-i}) = K_{1,3}$.

(c) Let G be k -connected ($k = 2, 3$) and be a k -sum of G_1, G_2 , where $|G_1|, |G_2| > k$. For $i = 1, 2$, let G'_i be obtained from G_i by deleting its summing edge (when $k = 2$) or its edges of the summing triangle (when $k = 3$). Then (G'_1, G'_2) is a k -separation of G .

For any $n \in \{2, 3\}$, a star decomposition of order n of a graph G consists of edge disjoint subgraphs H_0, \dots, H_k ($k \geq 0$) of G such that $H_0 \cup \dots \cup H_k = G$, $H_i \cap H_j$ is contained in H_0 for all $i > j > 0$, and $|H_i \cap H_0| = n$ for all $i > 0$. Equivalently, there are graphs G_0, \dots, G_k such that G is obtained by n -summing G_1, \dots, G_k to G_0 . Note that G_0 is obtained from H_0 by adding k edges (when $n = 2$) or k edge disjoint triangles (when $n = 3$), and each G_i ($i > 0$) is obtained from H_i by adding an edge (when $n = 2$) or a triangle (when $n = 3$). Under these circumstances, we will write $G = S_n(G_0; G_1, \dots, G_k)$.

Lemma 4.2. Let $e = xy$ be an edge of a 2-connected graph G of order at least three. Then G can be expressed as $S_2(G_0; G_1, \dots, G_k)$ where G_0, G_1, \dots, G_k are 2-connected minors of G such that $e \in G_0$ and $|G_i| \geq 3$ for $i > 0$. Moreover, if $\{x, y\}$ is a 2-cut of G then $si(G_0) = K_2$ and $k \geq 2$; if $\{x, y\}$ is not a 2-cut of G then either $si(G_0) = K_3$ or G_0 is 3-connected.

Proof. Suppose the result is false. Then we choose a counterexample G with $|G|$ minimum. If $si(G) = K_3$ or G is 3-connected then the lemma holds with $k = 0$; if $\{x, y\}$ is a 2-cut then the lemma also holds by Lemma 4.1(a). Thus G has a 2-separation but $\{x, y\}$ is not a 2-cut. It follows that G can be expressed as a 2-sum of two 2-connected minors G' and G'' over summing edge $e' = x'y'$. Among all possible choices, let us choose G' and G'' such that $|G'|$ is minimum with the property that $e \in G'$. Note e and e' are not parallel since $\{x', y'\}$ is a 2-cut of G , but $\{x, y\}$ is not. By the minimality of G , G' has 2-connected minors G_0, G_1, \dots, G_k of order ≥ 3 such that $e \in G_0$, $G' = S_2(G_0; G_1, \dots, G_k)$, and either $si(G_0) = K_3$ or G_0 is 3-connected. Now by the minimality of G' we also have $e' \in G_0$ since otherwise $e' \in G_i$ ($i \neq 0$) and we can find a smaller G' using the 2-cut which separates G_i from G_0 . Therefore, $G = S_2(G_0; G'', G_1, \dots, G_k)$, contradicting the choice of G , which proves the lemma. \square

We also have a 3-connected version of the last lemma. Let G be 3-connected and let $Z \subseteq V(G)$. We call (G, Z) 4-connected if for every s -separation (G_1, G_2) of G with $Z \subseteq V(G_1)$, either $s \geq 4$ or $s = 3 = |G_2| - 1$. If (G_1, G_2) is a separation of G with $Z \subseteq V(G_1)$, then we say $V(G_1 \cap G_2)$ is a Z -cut of G .

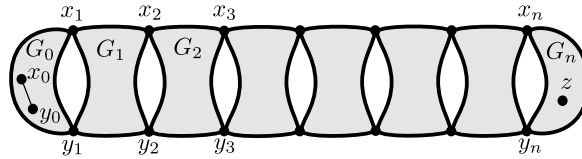


Fig. 4.1. A chain decomposition.

Lemma 4.3. *Let G be 3-connected and let $Z \subseteq V(G)$. Then G can be expressed as $S_3(G_0; G_1, \dots, G_k)$ where G_0 is a 3-connected minor of G , $Z \subseteq V(G_0)$, (G_0, Z) is 4-connected, no summing triangle is a Z -cut of G_0 , and G_1, \dots, G_k are 3-connected of order ≥ 5 . In addition, each G_i ($i > 0$) is a minor of G unless $k = 1$ and $si(G)$ has a cubic vertex z such that z is not in any triangle and $Z \subseteq \{z\} \cup N_G(z)$.*

Proof. Suppose the result is false. Then we choose a counterexample G with $|G|$ minimum. Since the result holds if (G, Z) is 4-connected, we deduce G has a 3-separation (H_1, H_2) with $Z \subseteq V(H_1)$ and $|H_2| \geq 5$. By Lemma 4.1(b), G can be expressed as a 3-sum of two 3-connected graphs G' and G'' over summing triangle T such that $Z \subseteq V(G')$ and $|G''| \geq 5$. Among all possible choices, let us choose G', G'' with $|G'|$ minimum. Note G' is a minor of G since $|G''| \geq 5$. Also note $|G'| \geq 5$ because otherwise $|G'| = 4$ and trivially (G', Z) is 4-connected so $(G_0, G_1) = (G', G'')$ would satisfy the lemma, which contradicts the choice of G . As a result, G'' is also a minor of G . By the minimality of G , G' can be expressed as $S_3(G_0; G_1, \dots, G_k)$ where G_0 is a 3-connected minor of G' , $Z \subseteq V(G_0)$, (G_0, Z) is 4-connected, no summing triangle is a Z -cut, and $|G_i| \geq 5$ for $i > 0$. By the minimality of G' , $V(T)$ must be contained in $V(G_0)$ and is not a 3-cut in G_0 ; we may assume $E(T) \subseteq E(G_0)$. It follows that G_1, \dots, G_k are all minors of G' and $G = S_3(G_0; G'', G_1, \dots, G_k)$. This contradicts the choice of G and thus the lemma is proved. \square

The previous two lemmas are about how a graph can be decomposed into a star structure with a better connected center. In the following we consider how to decompose a graph into a path structure. Let $e = x_0y_0$ be a specified edge of a 2-connected graph G . A sequence G_0, G_1, \dots, G_n ($n \geq 0$) of edge-disjoint subgraphs of G is called a *chain decomposition* of G at e with length n if

- (i) $e \in G_0$;
- (ii) for each $i = 1, \dots, n$, $(G_0 \cup \dots \cup G_{i-1}, G_i \cup \dots \cup G_n)$ is a 2-separation of G ;
- (iii) let $\{x_i, y_i\} = V((G_0 \cup \dots \cup G_{i-1}) \cap (G_i \cup \dots \cup G_n))$ for $i = 1, \dots, n$; then the pairs $\{x_0, y_0\}, \{x_1, y_1\}, \dots, \{x_n, y_n\}$ are all distinct.

We point out that $\{x_i, y_i\} \cap \{x_{i+1}, y_{i+1}\} \neq \emptyset$ is allowed. It is clear that every 2-connected G admits a chain decomposition of length 0 at any of its edges since conditions (ii–iii) are trivially satisfied. Let $a(G, e)$ denote the largest length of a chain decomposition of G at e .

Chain decompositions and star decompositions are similar, yet each allow us to focus on different aspects of a graph. A star decomposition focuses on how a graph is built

around one central piece and will be used later in the paper when we have a known subdivision in a graph and want to look at possible extensions of the subdivision. A chain decomposition looks at how a graph can be broken down into a chain of 2-connected pieces and is useful in determining long paths in a graph. The next lemma involves both decompositions.

By *operation S* we mean the operation of constructing $S_2(G_0; G_1, \dots, G_k)$ from G_0, \dots, G_k . Starting from any class of graphs we may construct more graphs by applying operation *S* repeatedly. In the following we make this more precise. Let \mathcal{G} be a class of graphs. Let \mathcal{G}_0 be the class of all pairs (G, e) such that $G \in \mathcal{G}$ and e is an edge of G . For any positive integer n , if \mathcal{G}_{n-1} has been defined, let \mathcal{G}_n consist of all pairs (G, e) for which there exist $(G_0, e) \in \mathcal{G}_0$ and $(G_i, e_i) \in \mathcal{G}_{n-1}$ ($i = 1, \dots, k$) such that G is obtained by 2-summing G_i to G_0 over e_i for all $i > 0$. We say each $(G, e) \in \mathcal{G}_n$ is constructed from graphs in \mathcal{G} by n iterations of operation *S*.

Lemma 4.4. *Let e be a specified edge of a 2-connected graph G with $a(G, e) \leq a$. Then (G, e) can be constructed from its 3-connected minors and 2-connected minors of order 2 or 3 by at most $a + 1$ iterations of operation *S*.*

Proof. Let x, y be the two ends of e . We first assume $|G| > 2$ and $\{x, y\}$ is not a 2-cut. In this case we claim (G, e) can be constructed from its 3-connected minors and 2-connected minors of order 3 within a iterations. Suppose the claim is false. Choose a counterexample with $|G|$ as small as possible. By Lemma 4.2, G has 2-connected minors G_0, G_1, \dots, G_k such that $e \in G_0$, either $si(G_0) = K_3$ or G_0 is 3-connected, $|G_i| \geq 3$ ($i > 0$), and $G = S_2(G_0; G_1, \dots, G_k)$. For each $i > 0$, let $e_i = x_i y_i$ be the summing edge of G_i . By allowing different graphs to sum over edges of G_0 from the same parallel family, we may assume $G_i - \{x_i, y_i\}$ is connected. Then $a(G_i, e_i) \leq a - 1$ because otherwise, since $G - \{x, y\}$ is connected, we would have $a(G, e) \geq a(G_0 \cup G_i, e) > a$. By the minimality of G , we deduce that each (G_i, e_i) can be constructed from its 3-connected minors and 2-connected minors of order 3 within $a - 1$ iterations. It follows that (G, e) can be constructed from its 3-connected minors and 2-connected minors of order 3 within a iterations. This conclusion contradicts the choice of G and thus proves our claim.

If $|G| = 2$ then $a(G, e) = 0$ and it is clear that (G, e) can be constructed in at most one iteration. Now suppose $G - \{x, y\}$ has $k > 1$ components. Let G_0 consist of e and k other edges parallel with e . Then G has 2-connected minors G_1, \dots, G_k of order ≥ 3 such that $G = S_2(G_0; G_1, \dots, G_k)$. For each $i = 1, \dots, k$, let G_i be summed to G_0 over e_i . Note $G_i - \{x, y\}$ is connected and $a(G_i, e_i) \leq a$ for every i . By the above claim, every (G_i, e_i) can be constructed within a iterations, which implies (G, e) can be constructed within $a + 1$ iterations. \square

5. Weighted graphs

In this section we prove a few technical lemmas on weighted graphs.

Lemma 5.1. *Let $t \geq 2$ be an integer and let (G, w) be a 2-connected weighted graph with a path of weight exceeding $(t - 2)^2$. Then G has a cycle of weight at least t and, for any two distinct vertices u, v , a uv -path of weight at least $t/2$.*

Proof. Let $P = x \dots y$ be a path of G of weight at least $(t - 2)^2 + 1$. We first show G has a cycle C of weight at least t . Let C' be a cycle containing x and y . We assume $w(C') < t$ because otherwise $C = C'$ satisfies the requirement. Then $V(P \cap C')$ divides P into at most $t - 2$ subpaths and hence at least one subpath P' must have weight at least $t - 1$. Clearly, $P' \cup C'$ contains a cycle C of weight at least t , as required. Since G is 2-connected, for every distinct pair of vertices u, v , there exist disjoint paths between $\{u, v\}$ and C (where u, v may be on C). These two paths together with C contain a uv -path of weight at least $t/2$. \square

Lemma 5.2. *Let (G, w) be a 2-connected weighted graph of order ≥ 3 and let t be a positive integer. Then one of the following holds.*

- (a) G has a 2-separation (H, J) with $V(H \cap J) = \{x, y\}$ such that neither H nor J has an xy -path of weight $\geq t$.
- (b) G has a 2-separation (H, J) with $V(H \cap J) = \{x, y\}$ such that both H and J have an xy -path of weight $\geq t$.
- (c) $G = S_2(G_0; G_1, \dots, G_k)$ such that either $si(G_0) = K_3$ or G_0 is 3-connected, and for each $i > 0$, if $e_i = x_i y_i$ is the summing edge of G_i then $G_i \setminus e_i$ has no $x_i y_i$ -path of weight $\geq t$.

Proof. Suppose the lemma is false. Let (G, w) be a counterexample on the fewest vertices. If G has no 2-separations then (c) would hold with $k = 0$. Hence G has a 2-separation (H, J) with $V(H \cap J) = \{x, y\}$ such that H has an xy -path of weight $\geq t$ but J does not. Among all such 2-separations we choose one with $|H|$ minimum. If $H - \{x, y\}$ is not connected, then H has a 2-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{x, y\}$ and since $|H|$ is minimum, H_1 and H_2 each must have an xy -path of weight $\geq t$. Thus (b) would hold so $H - \{x, y\}$ must be connected. Let H^+ be formed from H by adding a new edge $e = xy$ and let J^+ be formed similarly. By Lemma 4.2, H^+ has 2-connected minors G_0, G_1, \dots, G_k of order ≥ 3 such that $e \in G_0$, either $si(G_0) = K_3$ or G_0 is 3-connected, and $H^+ = S_2(G_0; G_1, \dots, G_k)$. For each $i > 0$, let G_i be 2-summed to G_0 over $e_i = x_i y_i$. Then the minimality of H implies $G_i \setminus e_i$ has no $x_i y_i$ -path of weight $\geq t$. It follows that $G = S_2(G_0; J^+, G_1, \dots, G_k)$ and the decomposition satisfies (c). This contradicts the choice of G and thus it proves the lemma. \square

In the next lemma we use the following terminology. Let (G, w) be a weighted graph and let (G_1, G_2) be a 2-separation of G with $V(G_1 \cap G_2) = \{x, y\}$. For $i = 1, 2$, define (G_i^+, w_i) where G_i^+ is obtained from G_i by adding a new edge $e = xy$, $w_i(e)$ is equal to the maximum weight of an xy -path in G_{3-i} , and $w_i(f) = w(f)$ for all edges f of G_i .

Lemma 5.3. (G, w) contains $\theta_{a,b,c}$ if and only if at least one of (G_1^+, w_1) and (G_2^+, w_2) contains $\theta_{a,b,c}$.

Proof. Suppose (G, w) contains $\theta_{a,b,c}$. Then G contains two vertices u, v and three independent uv -paths P_1, P_2, P_3 of weight at least a, b, c , respectively. Observe that both u, v are contained in G_i for some i because otherwise we would have $u \in G_j - \{x, y\}$ and $v \in G_{3-j} - \{x, y\}$ for some j , which is impossible. Let $T = P_1 \cup P_2 \cup P_3$ and without loss of generality, let $i = 1$. Then either $T \subseteq G_1$ or $T \cap G_2 \subseteq P_j$ for some $j \in \{1, 2, 3\}$. In the first case T is a $\theta_{a,b,c}$ contained in (G_1^+, w_1) while in the second case replacing $T \cap G_2$ by e in T results in a $\theta_{a,b,c}$ contained in (G_1^+, w_1) .

Conversely, suppose some (G_i^+, w_i) contains a $\theta_{a,b,c}$ graph T . If $e \notin T$ then T is a $\theta_{a,b,c}$ graph of (G, w) . So assume $e \in T$. Form a new theta graph T' by replacing e in T with an xy -path of G_{3-i} of weight equal to the weight of e . By the definition of $w_i(e)$ such a path exists. Then T' is a $\theta_{a,b,c}$ graph of (G, w) . \square

Lemma 5.4. Let (G, w) be $\theta_{t,t,t}$ -free, where G is 3-connected and planar. Suppose C is a facial cycle such that $|C| \geq 3t$ or C contains two edges each of weight $\geq t$. If each edge of C has the maximum weight among edges parallel to it, then G has no C -path of weight $\geq 2t$ and $G \setminus E(C)$ has no edge of weight $\geq t$.

Proof. Suppose, for the sake of contradiction, G contains a C -path P with $w(P) \geq 2t$. Let v_1 and v_2 be the two ends of P . If $C[v_1, v_2]$ and $C[v_2, v_1]$ both have weight at least t , then there is a $\theta_{t,t,t}$ in G at v_1 and v_2 . Hence one of these paths, say $C[v_1, v_2]$, has weight less than t and so $C[v_2, v_1]$ has a vertex x such that $C[v_2, x]$ and $C[x, v_1]$ each has weight at least t . Note $|P| \geq 3$ because otherwise, since C is a facial cycle and $V(P)$ is not a 2-cut, $C[v_1, v_2]$ must have only one edge and this edge is parallel to the unique edge of P . This contradicts our assumption on C since $w(P) > w(C[v_1, v_2])$. Since G is 3-connected, it has three independent paths Q_1, Q_2, Q_3 from x to distinct vertices of P , where the paths are listed in the order in which their ends appear on P . If C' is the cycle contained in $Q_1 \cup Q_3 \cup P$, then $V(Q_2 \cap C') \subseteq \{x\} \cup V(P)$; since C is a facial cycle, this implies $V(Q_2 \cap C) = \{x\}$. Therefore, $C \cup P \cup Q_2$ contains a $\theta_{t,t,t}$ at x and either v_1 or v_2 .

Suppose $G \setminus E(C)$ has an edge e with $w(e) \geq t$. Find two disjoint paths from the ends of e to C and let P be the C -path consisting of e and these two paths. Now by an argument similar to the one used above, we find a vertex x and a path Q_2 from x to y on P and then a $\theta_{t,t,t}$ in (G, w) . Previously, we required $w(P) \geq 2t$ so that at least one of the two subpaths of P divided by y would have length at least t . Now since P in this case contains an edge e of weight at least t , taking the part of P that contains e will have the same result. \square

In the next lemma, the graphs in the statement are not weighted but a weighted graph is defined and used in the proof.

Lemma 5.5. If $k \geq 1$ then $\ell(S_2(G_0; G_1, \dots, G_k)) \leq (\ell(G_0) + 2) \cdot \max\{\ell(G_1), \dots, \ell(G_k)\}$.

Proof. For each $i = 1, \dots, k$, let e_i be the edge of G_0 such that G_i is 2-summed to G_0 over e_i . Let $L = \max\{\ell(G_1), \dots, \ell(G_k)\}$. Let w be a weight function of G_0 such that $w(e_i) = L$ for $i = 1, \dots, k$ and $w(e) = 1$ for all other edges. Now we consider any longest path P of $S_2(G_0; G_1, \dots, G_k)$. Let \mathcal{Q} be the set of all maximal subpaths Q of P such that $\emptyset \neq E(Q) \subseteq E(G_i)$ for some $i \neq 0$. We modify P as follows. For each $Q \in \mathcal{Q}$, if Q is contained in G_i and the two ends of Q are the two ends of e_i then in P we replace Q by e_i . Let P' be the resulting path. Note P' is the union of a path P'' of G_0 and up to two members of \mathcal{Q} , each containing an end of P . It follows that $\|P\| \leq L + w(P'') + L \leq \ell(G_0)L + 2L = (\ell(G_0) + 2)L$. \square

6. Excluding a large restricted theta graph

In this section we prove characterizations of $\theta_{1,2,t^-}$, $\theta_{2,2,t^-}$, θ_{1,t,t^-} , and $\theta_{2,t,t}$ -free graphs for large t . For a proper subgraph H of G , a *bridge* of H or an *H-bridge* is either a subgraph of G induced by the edges of a component C of $G - V(H)$ together with the edges linking C to H , or a subgraph induced by an edge not in H but with both ends in H . We will call the second type of bridges *trivial*. The vertices of an H -bridge that are in H are the *feet* of the bridge.

We begin with $\theta_{1,2,t}$ -free graphs. The characterization is intuitive and requires only a very short proof.

Theorem 6.1. *Let $t \geq 2$ be an integer. Then every 2-connected simple graph G with $\ell(G) \geq 4t^2$ either contains $\theta_{1,2,t}$ or is a cycle.*

Proof. Let G be a 2-connected simple graph with $\ell(G) \geq 4t^2$. By Lemma 5.1, G contains a cycle C of length exceeding $2t$. If $G \neq C$, then G has a bridge B of C . Since G is 2-connected, B has at least two feet along C , say u and v . Suppose without loss of generality, $|C[u, v]| \geq |C[v, u]|$. Then $\|C[u, v]\| > t$ since $|C| > 2t$. Let Q be a uv -path of B . Then $C[u, v] \cup C[v, u] \cup Q$ is a subdivision of $\theta_{1,2,t}$ since G is simple. \square

Recall that \mathcal{L}_s is the class of 2-connected graphs without a path of length s . Let \mathcal{C} denote the class of all cycles. It is easy to see that graphs in $\mathcal{C} \cup \mathcal{L}_s$ are $\theta_{1,2,t}$ -free for sufficiently large t . Hence this theorem solves problem (P) for $\mathcal{H} = \{\theta_{1,2,t} : t \geq 2\}$: Let \mathcal{G} be a class of 2-connected simple graphs such that every 2-connected simple minor of any graph in \mathcal{G} remains in \mathcal{G} . Then $\mathcal{G} \not\supseteq \theta_{1,2,t}$ for some t if and only if $\mathcal{G} \subseteq \mathcal{C} \cup \mathcal{L}_s$ for some s . Using our loose language, we can say that a 2-connected simple graph G does not contain a large $\theta_{1,2,t}$ if and only if G is a cycle or G has no long path. Other theorems in this section provide similar characterizations concerning other theta graphs. We will not state each of these results in terms of minor-closed classes. This task is left to the reader.

A 2-connected graph is *outerplanar* if it has a plane embedding in which all vertices are on the outer cycle. Outerplanar graphs are known to be $\theta_{2,2,2}$ -free ($\theta_{2,2,2} \cong K_{2,3}$)

and thus are $\theta_{2,2,t}$ -free for all $t \geq 2$. The next result implies that a 2-connected graph G does not contain a large $\theta_{2,2,t}$ if and only if G is outerplanar or G has no long path.

Theorem 6.2. *Let $t \geq 2$ be an integer. Then every 2-connected graph G with $\ell(G) \geq 4t^2$ either contains $\theta_{2,2,t}$ or is outerplanar.*

Proof. Let G be 2-connected with $\ell(G) \geq 4t^2$ and let C be a longest cycle of G . By Lemma 5.1, $|C| > 2t$. Suppose C is not a Hamilton cycle. Then G has a nontrivial bridge B of C . Since G is 2-connected, B has at least two feet along C , say u and v . If u and v are adjacent along C , then G contains a cycle longer than C : replace the edge uv in C with a path through B of length ≥ 2 . Hence u and v are not adjacent in C and thus G contains a $\theta_{2,2,t}$ graph at u and v : one path of length ≥ 2 is through B and the other two paths are $C[u, v]$ and $C[v, u]$. Since $|C| \geq 2t$, one of these paths necessarily has length $\geq t$.

Now C is a Hamilton cycle. Suppose uv, xy are chords of C such that u, x, v, y are distinct and they appear in that forward order along C . Since $|C| = |G| \geq \ell(G) \geq 4t^2$, at least one of $C[u, x], C[x, v], C[v, y], C[y, u]$ has length $\geq t$. Without loss of generality, suppose $|C[u, x]| \geq t$. Then G contains a $\theta_{2,2,t}$ graph at u and x : the path of length $\geq t$ is $C[u, x]$ and the two paths of length ≥ 2 each use one of the edges uv and xy . Hence C has no crossing chords and G is outerplanar. \square

To describe $\theta_{1,t,t}$ -free graphs, we define a new class of graphs. For any family \mathcal{G} of 2-connected graphs, let $C(\mathcal{G})$ be the class of graphs constructed by 2-summing graphs from \mathcal{G} to a cycle. In other words, $C(\mathcal{G}) = \{S_2(C; G_1, \dots, G_k) : C \text{ is a cycle, } k \geq 0, \text{ and } G_1, \dots, G_k \in \mathcal{G}\}$.

Theorem 6.3. *There exists a function $f_{6.3}(t)$ such that every 2-connected graph G with $\ell(G) \geq f_{6.3}(t)$ either contains $\theta_{1,t,t}$ or is in $C(\mathcal{L}_{8t^2})$ where $t \geq 3$ is an integer. Additionally, all graphs in $C(\mathcal{L}_t)$ are $\theta_{1,t,t}$ -free.*

Proof. Let $w(t) = f_{3.3}(2t)$. We show that $f_{6.3}(t) = \lceil w(t) + 2 \rceil^{3t} w(t)$ satisfies the theorem. Suppose $\ell(G) \geq f_{6.3}(t)$ and further assume $G \notin C(\mathcal{L}_{8t^2})$. We need to show that G contains $\theta_{1,t,t}$.

Let $b = \max\{\ell(G') : G' \text{ is a 3-connected minor of } G\}$; we know $b < w(t)$ since otherwise, by Lemma 3.3, G contains a W_{2t} minor and hence a $\theta_{1,t,t}$. Let e be a specified edge of G and consider a chain decomposition of G given by G_0, G_1, \dots, G_n and with vertices $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n, z$, as in Fig. 4.1. If $a(G, e) < 3t$, then by Lemma 4.4, G can be constructed from 3-connected minors and graphs of order ≤ 3 by at most $3t$ iterations of operation S . By Lemma 5.5, $\ell(G) \leq (b + 2)^{3t} b < \lceil w(t) + 2 \rceil^{3t} w(t)$ which is a contradiction.

Hence assume $a(G, e) = n \geq 3t$. Since G is 2-connected, it has two independent paths from z to x_0 and y_0 . Without loss of generality, we assume one contains every x_i and the

other contains every y_i . Suppose $G_t \cup G_{t+1} \cup \dots \cup G_{n-t}$ contains a path P from some x_i to some y_j and without loss of generality, assume P does not include any other x_k or y_k . Then there is a $\theta_{1,t,t}$ in G at x_i and y_j : the path of length ≥ 1 is P , one path of length $\geq t$ includes the vertices $x_{i-1}, x_{i-2}, \dots, x_0, y_0, y_1, \dots, y_{j-1}$ and the other includes the vertices $x_{i+1}, x_{i+2}, \dots, x_n, z, y_n, y_{n-1}, \dots, y_{j+1}$. Hence no such path P exists. It follows that each G_i ($t \leq i \leq n-t$) has two components G'_i, G''_i such that G'_i contains both x_i, x_{i+1} and G''_i contains both y_i, y_{i+1} . Therefore, G can be constructed by 2-summing 2-connected graphs H_1, \dots, H_k to a cycle H_0 of length $k > t$. We choose these graphs with k maximum.

Because $G \notin C(\mathcal{L}_{8t^2})$, $\ell(H_i) \geq 8t^2$ for some i . Let xy be the summing edge of H_i . By the maximality of k , $H_i \setminus xy$ is 2-connected. Clearly, $\ell(H_i \setminus xy) \geq 4t^2$. Thus by Lemma 5.1, $H_i \setminus xy$ has a cycle C of length exceeding $2t$. Since H_i is 2-connected, it has disjoint paths from x to a vertex x' of C and from y to a vertex y' of C (where possibly $x = x'$ or $y = y'$). Now the 2-sum of H_i and H_0 contains a $\theta_{1,t,t}$ graph at x' and y' : $C[x', y']$ and $C[y', x']$ are paths of length $\geq t$ and ≥ 1 , and the other path of length $\geq t$ is the union of the xx' -path, the yy' -path, and $H_0 \setminus xy$. Consequently, G contains $\theta_{1,t,t}$.

Finally, let $G \in C(\mathcal{L}_t)$. Suppose G is formed by 2-summing graphs G_1, \dots, G_k to a cycle C . Suppose G has a $\theta_{1,t,t}$ graph at x and y . If $x \in V(G_i) \setminus V(C)$ for some i then y must also be in $V(G_i)$ because otherwise there could not be three independent paths from x to y since G_i is separated from the rest of the graph by two vertices. But now at least one of the paths of length $\geq t$ would have to remain in G_i which cannot happen since $\ell(G_i) < t$. Hence x and y must both be vertices of C . Because no G_i has a path of length $\geq t$, the two paths of length $\geq t$ in any $\theta_{1,t,t}$ must each have an interior vertex in C . But now, no matter how these vertices are oriented with respect to x and y along C , there cannot be a $\theta_{1,t,t}$. \square

The proof of the characterization of $\theta_{2,t,t}$ -free graphs requires the following lemma. Let G be a graph and let $e, f \in E(G)$. A subgraph H of G is called an ef -theta if H is a theta graph such that, if u, v are its two cubic vertices then e, f belong to different uv -paths of H and the third uv -path of H has length ≥ 2 . Suppose $e = ab, f = cd$, and $Z \subseteq V(G)$. Then we say Z separates e from f if $\{a, b\} \setminus Z \neq \emptyset, \{c, d\} \setminus Z \neq \emptyset$, and $G - Z$ has no path between $\{a, b\} \setminus Z$ and $\{c, d\} \setminus Z$.

Lemma 6.4. *Let e, f be distinct edges of a 2-connected simple graph G . Suppose no two vertices of G separate e from f . Then G contains an ef -theta unless either e, f have a common end v with $\deg_G(v) = 2$ or e, f have no common end and $G = K_4$.*

Proof. Let $e = ab$ and $f = cd$. First consider the case $a = c$. Suppose $\deg(a) \geq 3$ and let $x \in N_G(a) \setminus \{b, d\}$. Since $\{a, x\}$ does not separate e, f , there is a path P from b to d in $G - \{a, x\}$. Furthermore, since G is 2-connected, there is a path Q from x to P in $G - a$. Then the union of P, Q, e, f , and ax is an ef -theta, as required.

Now e, f is a matching. Assume G does not contain an ef -theta. We will show $G = K_4$. Because G is 2-connected, it has a cycle C containing e, f . Let P, Q be the two paths of $C \setminus \{e, f\}$. Without loss of generality, assume P is between a and c and Q is between b and d . Since $\{b, c\}$ does not separate e, f , there is a C -path with one end p in $P - c$ and one end q in $Q - b$. Note this path is a single edge pq because otherwise the union of C and this path would be a required ef -theta. Since $\{p, q\}$ does not separate e, f , there is a path R in $G - \{p, q\}$ between the two components of $C - \{p, q\}$. If the ends of R are both on P or both on Q , then the union of R, C , and pq contains an ef -theta. So one end p' of R is on P and the other end q' of R is on Q . It follows that R has only one edge $p'q'$. Furthermore, by symmetry we assume p' is between p and c along P and q' is between b and q along Q .

If $pp' \notin E(P)$, then there is an ef -theta with p and p' as the two degree 3 vertices. Hence $pp' \in E(P)$ and symmetrically $qq' \in E(Q)$. If $\{p, q'\} \neq \{a, b\}$, then since $\{p, q'\}$ does not separate e, f , there is a path S in $G - \{p, q'\}$ between the two components of $C - \{p, q'\}$. By the symmetry between $\{pq, p'q'\}$ and $\{pp', qq'\}$, we may assume that both ends of S are on P . Then $C \cup S \cup pq'$ contains an ef -theta. Thus we must have $p = a$ and $q' = b$ and similarly $p' = c$ and $q = d$ so e, f are contained in a K_4 subgraph of G . If $G \neq K_4$, then G has a vertex x not in the K_4 subgraph. G is 2-connected so G has two independent paths from x to distinct vertices of K_4 and again we can find an ef -theta. Hence the result follows. \square

To describe $\theta_{2,t,t}$ -free graphs, we use nearly outerplanar graphs. A simple graph G is *nearly outerplanar* if G has a Hamilton cycle C such that every chord crosses at most one other chord and, in addition, if two chords ab and cd do cross, then either $ac, bd \in E(C)$ or $ad, bc \in E(C)$. An edge of C is *free* if it does not belong to a 4-cycle spanned by two crossing chords. A general graph G is *nearly outerplanar* if $si(G)$ is nearly outerplanar, and *free* edges of G are those that are parallel to a free edge of $si(G)$. For any positive integer n , let \mathcal{O}_n be the class of graphs formed by 2-summing graphs from \mathcal{L}_n to free edges of nearly outerplanar graphs. Note $C(\mathcal{L}_n) \subset \mathcal{O}_n$.

Theorem 6.5. *There exist two functions $f_{6.5}(t)$ and $g_{6.5}(t)$ such that every 2-connected graph G with $\ell(G) \geq f_{6.5}(t)$ either contains $\theta_{2,t,t}$ or is in $\mathcal{O}_{g_{6.5}(t)}$, where $t \geq 3$ is an integer. Additionally, all graphs in \mathcal{O}_t are $\theta_{2,t,t}$ -free.*

This proof is very similar to that of Theorem 6.3. In both proofs we produce a long chain decomposition. The difference lies in how we analyze each part of the decomposition; in this proof we have to do more.

Proof. As in the proof of Theorem 6.3, let $w(t) = f_{3.3}(2t)$. We will show $f_{6.5}(t) = [w(t) + 2]^{3t}w(t)$ and $g_{6.5}(t) = [w(t) + 2]^t w(t)$ satisfy the theorem. Suppose $\ell(G) \geq f_{6.5}(t)$ and further assume G does not contain $\theta_{2,t,t}$. We need to show $G \in \mathcal{O}_{g_{6.5}(t)}$.

Let $b = \max\{\ell(G') : G' \text{ is a 3-connected minor of } G\}$; we know $b < w(t)$ since otherwise G contains $\theta_{2,t,t}$. Let e^* be a specified edge of G and consider a chain decompo-

sition of G given by G_0, G_1, \dots, G_n and with vertices $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n, z$ as in Fig. 4.1. If $a(G, e^*) < 3t$, then by Lemma 4.4, G can be constructed from its 3-connected minors and graphs of order ≤ 3 in at most $3t$ iterations of operation S . By Lemma 5.5, $\ell(G) \leq (b + 2)^{3t}b < [w(t) + 2]^{3t}w(t) = f_{6.5}(t)$ which is a contradiction.

Hence assume $a(G, e^*) = n \geq 3t$. Since G is 2-connected, it has a cycle C^* containing e^* and z . For each $i \in \{t, t + 1, \dots, n - t\}$, let G_i^+ be obtained from G_i by adding a new edge e_i between x_i, y_i and a new edge f_i between x_{i+1}, y_{i+1} . Then G_i^+ is 2-connected and has no 2-cut separating e_i from f_i since otherwise we could find a chain decomposition of G with $a(G, e^*) > n$. If for at least one of these i , G_i^+ contains an $e_i f_i$ -theta T at vertices u and v , then $C^* \cup (T \setminus \{e_i, f_i\})$ contains a $\theta_{2,t,t}$ also at u and v . Thus for all these i , by Lemma 6.4, either e_i, f_i have a common end and that end has only two neighbors in G_i^+ or e_i, f_i have no common end and $si(G_i^+) = K_4$. In the former case, let $e_i = ab$ and $f_i = ac$; then G_i^+ is a 2-sum of a graph F_i and a 3-vertex graph over the edge bc .

We can now conclude that there exists a nearly outerplanar graph H such that its Hamilton cycle C has length exceeding t , and G is obtained from H by 2-summing minors of G to free edges of C . The graphs which are 2-summed to C are the graphs $F_i, G_1 \cup \dots \cup G_{t-1}$, and $G_{n-t+1} \cup \dots \cup G_n$. The Hamilton cycle C runs through the graphs G_i where $i \in \{t, t+1, \dots, n-t\}$ and includes in order the vertices $x_t, x_{t+1}, \dots, x_{n-t}, y_{n-t}, y_{n-t-1}, \dots, y_t$. Because in each G_i , at most one of $x_i = x_{i+1}$ and $y_i = y_{i+1}$ holds, C has length at least t .

Choose H such that C is as long as possible. Let G_e be a graph 2-summed to a free edge e of H over an edge e' of G_e . In order to conclude $G \in \mathcal{O}_{g_{6.5}(t)}$, it suffices to show $\ell(G_e) < g_{6.5}(t)$. Suppose $a(G_e, e') = m \geq t$ and let H_0, H_1, \dots, H_m be the corresponding chain decomposition. Let u, v be the two common vertices of H_0 and H_1 . Let H_0^+ be obtained from H_0 by adding a new edge $f' = uv$. If H_0^+ contains an $e' f'$ -theta, then G contains $\theta_{2,t,t}$ where one long path goes through C and the other long path goes through $H_1 \cup \dots \cup H_m$. Thus by Lemma 6.4, either e', f' have a common end and that end has only two neighbors in H_0^+ or e', f' have no common end and $si(H_0^+) = K_4$. Each of these two cases contradicts the maximality of C . Hence $a(G_e, e') < t$. It follows from Lemmas 5.5 and 4.4 that $\ell(G_e) \leq [b + 2]^t b < [w(t) + 2]^t w(t) = g_{6.5}(t)$.

Finally, we prove every $G \in \mathcal{O}_t$ is $\theta_{2,t,t}$ -free ($t \geq 3$). Note every 2-connected minor of every graph in \mathcal{L}_t remains in \mathcal{L}_t . Similarly, every 2-connected minor of every nearly outerplanar graph remains nearly outerplanar (and free edges remain free). It follows that every 2-connected minor of every graph in \mathcal{O}_t remains in \mathcal{O}_t . Therefore, to prove every $G \in \mathcal{O}_t$ is $\theta_{2,t,t}$ -free we only need to show $\theta_{2,t,t} \notin \mathcal{O}_t$. Suppose otherwise that $\theta_{2,t,t}$ can be formed from a nearly outerplanar graph H by 2-summing $k \geq 0$ graphs $H_1, \dots, H_k \in \mathcal{L}_t$ to free edges of H . Since $\theta_{2,t,t}$ has no 4-cycle, H cannot contain crossing chords and thus H is outerplanar. Let C be the facial Hamilton cycle of H and let x, y be the two cubic vertices of $\theta_{2,t,t}$. Suppose $x \in V(H_i) \setminus V(C)$ for some $i > 0$. Then $y \in V(H_i)$ as well because H_i is separated from the rest of the graph by a 2-cut. But now H_i must contain an xy -path of length t , which contradicts the assumption $H_i \in \mathcal{L}_t$. Therefore, each H_i is a cycle and 2-summing it to H amounts to replacing an edge of C

by a path. What this means is that we may consider H_i as part of C in the first place. In other words, we may assume $k = 0$. It follows that $\theta_{2,t,t} = H$, which is impossible since $\theta_{2,t,t}$ is not outerplanar. This contradiction completes our proof. \square

7. Planar drawings versus crossing paths

An important step in proving our main result is to determine if a graph admits a planar drawing with certain vertices and edges on a facial cycle. This problem is essentially solved by Robertson and Seymour in [9]. However, their result is not strong enough for our application. In the following we first state two results from [9] and then we prove a refinement of these results.

Let C be a cycle of G . Let u, v be distinct vertices of $G - V(C)$ and let P_1, P_2, P_3 be independent uv -paths with $|P_i \cap C| \leq 1$ for every i . Then (P_1, P_2, P_3) is a *tripod* of G with respect to C if $G - \{u, v\}$ has three disjoint paths Q_1, Q_2, Q_3 between P_1, P_2, P_3 and C with Q_i from a vertex s_i on $P_i - \{u, v\}$ to a vertex t_i on C and such that if $|P_i \cap C| = 1$, then $Q_i = P_i \cap C$ so $s_i = t_i$. The paths Q_1, Q_2, Q_3 are *legs* and the vertices t_1, t_2, t_3 are the *feet* of the tripod. A pair of disjoint C -paths (P_1, P_2) is a *cross* of C if the ends of P_1 , say u and v , and the ends of P_2 , say x and y , appear in the order u, x, v, y around C . We use the following two lemmas by Robertson and Seymour which we have rephrased using our terminology.

Lemma 7.1 (Lemma (2.3) of [9]). *Let C be a cycle of a graph G and let (P_1, P_2, P_3) be a tripod with respect to C . If $|C| \geq 4$, then either G has a cross with respect to C or G has a k -separation (G_1, G_2) with $k \leq 3$, $V(C) \subseteq V(G_1)$, and $V(P_1 \cup P_2 \cup P_3) \subseteq V(G_2)$.*

Lemma 7.2 (Lemma (2.4) of [9]). *Let G be 2-connected with a cycle C of length ≥ 3 such that G has no 2-separation (G_1, G_2) with $V(C) \subseteq V(G_1)$. If G has no cross or tripod with respect to C , then G admits a planar drawing with C as a facial cycle.*

Note in these two lemmas, C has been specified. However, in our applications C will only be partially given. Our problem is to decide if the partial cycle can be completed into a cycle C so that G admits a planar drawing with C as a facial cycle. In the following we make the problem more precise.

A *circlet* Ω of a graph G consists of a cyclically ordered set of distinct vertices v_1, v_2, \dots, v_n of G , where $n \geq 4$, and a set of edges of G of the form $v_i v_{i+1}$, where $v_{n+1} = v_1$. Note not necessarily all edges of G of the given form are in Ω . Denote by $V(\Omega)$ and $E(\Omega)$ the set of vertices and edges of Ω , respectively. We call $v_i \in V(\Omega)$ *isolated* if no edge of Ω is incident with v_i . An Ω -*cycle* is a cycle C of G such that $V(\Omega) \subseteq V(C)$, $E(\Omega) \subseteq E(C)$, and the cyclic ordering of $V(\Omega)$ agrees with the ordering in C . For each i , the $v_i v_{i+1}$ -path of C that does not contain v_{i+2} is called a *segment* of C . We say (G, Ω) is *4-connected* if $(G, V(\Omega))$ is 4-connected.

Theorem 7.3. *Let Ω be a circlet of G such that G has an Ω -cycle and (G, Ω) is 4-connected. Then either G admits a planar drawing in which some facial cycle is an Ω -cycle, or G has an Ω -cycle C and a cross of C for which each segment of C contains at most two of the four ends of the two paths in the cross.*

We need the following two lemmas for proving this theorem. Several different formulations of these lemmas are known (for example, see Lemma 6.2.1 of [5]), but we were not able to find in the literature the formulation we need so we prove them here. Our proofs are similar to that of other versions of the lemmas. Let H be a subgraph of G and let J be a subgraph of H . An H -bridge B is called J -local if all feet of B are in J .

Lemma 7.4. *Let H be a subgraph of a simple graph G with $|H| \geq 3$. Let P be an H -path in G and let x, y be the two ends of P . Let B_1, \dots, B_t be all $(H \cup P)$ -bridges that are P -local. Suppose G has no k -separation (G_1, G_2) with $k < 3$ and $V(H) \subseteq V(G_1)$. Then $H_0 = P \cup B_1 \cup \dots \cup B_t$ has an xy -path Q such that no $(H \cup Q)$ -bridge is Q -local.*

Proof. For any H -path R with ends x, y , we define $\alpha(R)$ as follows. Let J_1, \dots, J_n ($n \geq 0$) be all $(H \cup R)$ -bridges that are R -local; let J_0 be the union of all other $(H \cup R)$ -bridges. Suppose $\|J_1\| \geq \|J_2\| \geq \dots \geq \|J_n\|$. Then $\alpha(R) = (\|J_0\|, \|J_1\|, \dots, \|J_n\|)$. Among all xy -paths in H_0 , let Q be a path that maximizes α lexicographically. We prove that no $(H \cup Q)$ -bridge is Q -local.

Suppose otherwise. Let J_1, \dots, J_n ($n \geq 1$) be all $(H \cup Q)$ -bridges that are Q -local, where $\|J_1\| \geq \|J_2\| \geq \dots \geq \|J_n\|$, and let J_0 be the union of all other $(H \cup Q)$ -bridges. Since G has no k -separation (G_1, G_2) with $k < 3$ and $V(H) \subseteq V(G_1)$, J_n has at least two feet. Let a, b be the two feet so that the only ab -path Q_{ab} of Q contains all feet of J_n . Let L be an ab -path in J_n that avoids all other feet of J_n and let Q' be obtained from Q by replacing Q_{ab} with L . Since J_n is a subgraph of H_0 , Q' is again an xy -path in H_0 .

Let $Z = V(Q_{ab} - \{a, b\})$. If $Z = \emptyset$, then $Q_{ab} = ab$. But $V(J_n) \neq \{a, b\}$ since G is simple so $\{a, b\}$ must be a 2-cut which contradicts the assumption that G has no k -separation (G_1, G_2) with $k < 3$ and $V(H) \subseteq V(G_1)$. Thus $Z \neq \emptyset$. Note $(H \cup Q)$ -bridges (other than J_n) that have no feet in Z are also $(H \cup Q')$ -bridges; $(H \cup Q)$ -bridges (other than J_n) that have a foot in Z are combined with $Q_{ab} - \{a, b\}$ into a single $(H \cup Q')$ -bridge J^* (which may include some subgraphs of J_n); and all other $(H \cup Q')$ -bridges are subgraphs of J_n . Since $|H| \geq 3$ and since G has no k -separation (G_1, G_2) with $k < 3$ and $V(H) \subseteq V(G_1)$, we deduce that at least one $(H \cup Q)$ -bridge $J \neq J_n$ has a foot in Z . Therefore, J^* contains J and $Q_{a,b}$, implying that at least one of the terms $\|J_0\|, \|J_1\|, \dots, \|J_{n-1}\|$ is increased (since either J is part of J_0 or J is some J_i for $i = 1, \dots, n - 1$). What this means is that $\alpha(Q')$ is lexicographically bigger than $\alpha(Q)$, contradicting the maximality of $\alpha(Q)$ and so the lemma is proved. \square

Let H be a subdivision of a graph J . Then $V(J)$ is a subset of $V(H)$ and $V(J)$ -paths of H are exactly the paths obtained by subdividing edges of J . We call these paths

branches of H . Suppose a subgraph H of G is a subdivision of another graph. Then an H -bridge B is called *unstable* if B is P -local for a branch P of H .

Lemma 7.5. *Let G and J be edge disjoint graphs with $V(J) \subseteq V(G)$. Suppose G is simple while J is loopless of order at least 3 and such that G has no k -separation (G_1, G_2) with $k < 3$ and $V(J) \subseteq V(G_1)$. If G has a subgraph H that is a subdivision of J then G has a subgraph H^* that is a subdivision J such that all H^* -bridges are stable.*

Proof. We first replace each branch of H by a single edge of G whenever it is possible. Then we repeatedly apply Lemma 7.4 to every branch of H . Note after each application of Lemma 7.4, no new unstable bridge is created. Therefore, after the final step all bridges are stable. \square

Proof of Theorem 7.3. Assume G does not have a planar drawing in which some facial cycle is an Ω -cycle. We will show G has an Ω -cycle and a cross of the cycle that satisfy the theorem. Without loss of generality we assume G is simple. Let v_1, \dots, v_n be the vertices of Ω and be ordered as in the definition of a circlet.

We need a definition. Suppose i is an index such that $v_i v_{i+1} \notin E(\Omega)$. Let $e = v_i v_{i+1}$ be a new edge and let $G' = G + e$. Let C be an Ω -cycle in G' such that e is on C . Then $C \setminus e$ is called an Ω -path of G . Suppose (P_1, P_2, P_3) is a tripod of G' with respect to C , such that v_i and v_{i+1} are both feet of this tripod. Then we call (P_1, P_2, P_3) a tripod of G with respect to the Ω -path $C \setminus e$.

Suppose G contains a tripod (P_1, P_2, P_3) with respect to an Ω -path P and without loss of generality let v_1 and v_2 be the two ends of P . We choose the tripod so that its three legs are as short as possible. Let vertices $s_1, s_2, s_3, t_1, t_2, t_3, u, v$ be as in the definition of a tripod and assume $v_1 = t_1$ and $v_2 = t_2$. Since (G, Ω) is 4-connected, $G - \{s_1, s_2, s_3\}$ has a path Q between $P_1 \cup P_2 \cup P_3$ and the union of P and the legs of the tripod. If an end of Q is on a leg of the tripod, then there is a tripod with shorter legs, a contradiction. Thus one end of Q is in $V(P) - \{t_1, t_2, t_3\}$; call this vertex q . Without loss of generality, let q be on the subpath of P from t_1 to t_3 . By symmetry we assume that the other end of Q is on the subpath of P_i from s_i to v for one of $i = 1, 2, 3$. Then the path from u to t_3 along P_3 and Qs_2 is a cross with respect to the Ω -cycle $us_1 t_1 P t_2 s_2 u$, unless $s_2 = t_2 = v_2$ and q and t_3 are on the $v_2 v_3$ segment of P . If this happens, then the subpath of P_1 from u to s_1 and the subpath of P_3 from v to s_3 is a cross with respect to the Ω -cycle that runs through the vertices $v_1 s_1 v v_2 u s_3 t_3 q v_1$ in that order. Hence G does not contain a tripod with respect to an Ω -path.

Let J be a cycle $v_1 v_2 \dots v_n v_1$ with edges not in G . Since $n \geq 4$, (G, Ω) is 4-connected, and G has an Ω -cycle; by Lemma 7.5, G has a cycle C that is a subdivision of J and all C -bridges are stable. It follows that every edge of G of the form $v_i v_{i+1}$ is on C . So C is an Ω -cycle. Since G has no planar drawing with C as a face, by Lemma 7.2 G has a cross or a tripod with respect to C . Since (G, Ω) is 4-connected, by Lemma 7.1, G has a cross; let (Q_1, Q_2) be a cross of C . Let x_1 and x_3 be the ends of Q_1 and x_2 and x_4

be the ends of Q_2 . Suppose for the sake of contradiction some segment P of C contains more than two of x_1, x_2, x_3, x_4 . Let v_1 and v_2 be the ends of P .

Suppose first that $x_1, x_2, x_3, x_4 \in P$. Let B be the C -bridge that contains Q_1 and let x be a foot of B not on P . Let Q be a path in B from x to the interior of Q_1 (or $Q_1 \cup Q_2$ if B also contains Q_2). Then $Q_1 \cup Q_2 \cup Q$ contains a cross of C so that P contains only three of the four ends. Hence without loss of generality, we can assume P contains x_1, x_2, x_3 but not x_4 .

Again let B be the C -bridge that contains Q_1 . Then B contains a path Q from an interior vertex z of Q_1 to a foot of x of B not on P . Assume Q meets Q_2 . Let C' be the path formed from $E(C) \setminus E(P)$; then C' is an Ω -path. Let y be the first vertex of intersection along Q from z to the foot. Now G contains a tripod T with respect to C' ; the feet of T are v_1, v_2, x_4 and using the vertex labels from the definition of a tripod, $\{u, v\} = \{x_2, z\}$ and $\{s_1, s_2, s_3\} = \{x_1, y, x_3\}$. Hence we can assume Q is disjoint from Q_2 . Let x be between v_1 and x_4 . Now the union of Q and the portion of Q_1 from z to x_3 together with Q_2 is a required cross with respect to C , unless x, x_4, x_3 are all contained in segment v_2v_3 . If this occurs, then Q and $C[x_1, x_2]$ is a cross with respect to the Ω -cycle that runs through the vertices $x_1zv_2x_2x_4xv_3v_1x_1$ in that order. \square

When Ω has no isolated vertices, we can further strengthen Theorem 7.3.

Theorem 7.6. *Let Ω be a circlet of G such that Ω has no isolated vertices, $|E(\Omega)| \geq 3$, G has an Ω -cycle, and (G, Ω) is 4-connected. Then either G admits a planar drawing in which some facial cycle is an Ω -cycle, or G has an Ω -cycle C and a cross of C for which among the four paths of C divided by the four ends of the two paths in the cross, at least three of them contain an edge of Ω .*

Proof. By Theorem 7.3 we assume G has an Ω -cycle C and a cross (P_1, P_2) of C for which each segment of C contains at most two of the four ends x_1, x_2, x_3, x_4 of P_1, P_2 . We need to show G has a cross of an Ω -cycle that satisfy the theorem.

Assume x_1, x_2, x_3, x_4 appear in that forward order around C . Let $Q_i = C[x_i, x_{i+1}]$ for $i = 1, 2, 3$ and $Q_4 = C[x_4, x_1]$. Suppose to the contrary that at most two of the Q_i contain edges of Ω . Then the choice of P_1, P_2 and the assumption that Ω has no isolated vertices imply that exactly two of the Q_i contain edges of Ω and these two Q_i cannot be adjacent. Without loss of generality, suppose Q_1 and Q_3 contain edges of Ω . Since $|E(\Omega)| \geq 3$, we further assume Q_1 contains at least two edges of Ω .

Re-choose (if necessary) C, P_1, P_2 so that Q_1 is as short as possible. Since G is 3-connected, $G - \{x_1, x_2\}$ has a path R from $Q_1 - \{x_1, x_2\}$ to $(C \cup P_1 \cup P_2) - V(Q_1)$. Let v be the endpoint of R on Q_1 ; then v is between two edges of Ω since otherwise the minimality of Q_1 is violated. If the other end of R is on $P_1 \cup P_2$, then $R \cup P_1 \cup P_2$ contains the desired cross. If the other end of R is on C , then R and one of P_1, P_2 form the desired cross. \square

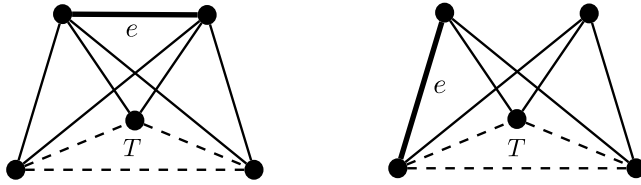


Fig. 7.1. Two minors A_1 and A_2 .

We close this section by proving the following technical lemma which we will use in the next section.

Lemma 7.7. *Suppose a 3-connected graph G has a triangle T and an edge e such that at most one end of e is in T . Then either G contains one of the two graphs in Fig. 7.1 as a minor or $G = S_3(G_0; G_1, \dots, G_k)$ where G_0 is planar with T as a facial cycle and each G_i ($i > 0$) has order ≥ 5 and is 3-summed to a facial triangle of G_0 different from T .*

Proof. We first make an observation: if $Z \subseteq V(G)$ contains at most one end of e and $|Z| \geq 3$, then G has three independent paths from a vertex outside Z to three distinct vertices of Z such that e is on one of these paths. To see this, first find two disjoint paths from the two ends of e to Z . These paths and e form a Z -path P containing e . Let z_1, z_2 be the two ends of P . Then $G - \{z_1, z_2\}$ has a path Q from $Z - \{z_1, z_2\}$ to $P - \{z_1, z_2\}$. It follows that $P \cup Q$ is the union of the three required paths.

If $V(T)$ is a 3-cut of G then we deduce from the above observation by taking $Z = V(T)$ that G contains an A_2 minor. Assume $V(T)$ is not a 3-cut. By Lemma 4.3, G has 3-connected minors G_0, G_1, \dots, G_k such that $G = S_3(G_0; G_1, \dots, G_k)$, where $T \subseteq G_0$, $(G_0, V(T))$ is 4-connected, no summing triangle is a $V(T)$ -cut, and $|G_i| \geq 5$ for all $i > 0$. If G_0 is planar, then T must be facial because $V(T)$ is not a 3-cut. Additionally, because no summing triangle is a $V(T)$ -cut, $V(T)$ is not the vertex set of any summing triangle and each summing triangle must be a facial triangle of G_0 ; hence G_0, \dots, G_k satisfies the lemma. So we assume G_0 is nonplanar.

We claim that G_0 has three independent uv -paths P_1, P_2, P_3 , for some u, v outside T , such that T meets all of these three paths. To see this, first note by Lemma 7.2, G_0 has a tripod (P_1, P_2, P_3) on T . Let Q_i, s_i, t_i be determined as in the definition of tripod. We choose the tripod with $Q_1 \cup Q_2 \cup Q_3$ as small as possible. If $s_i \neq t_i$ for some i , say for $i = 1$, then, as $(G_0, V(T))$ is 4-connected, $G_0 - \{s_1, s_2, s_3\}$ has a path P from $P_1 \cup P_2 \cup P_3$ to T . It is routine to see that the union of P and all P_i and Q_i contains a tripod with shorter legs. This contradiction shows $s_i = t_i$ for all i and thus our claim follows.

Now we consider two cases. First, suppose both ends of e are in $Z = V(P_1 \cup P_2 \cup P_3)$. Then it is straightforward to verify that either A_1 or A_2 is a minor of G . Now in the second case, we assume Z contains at most one end of e . By our earlier observation, G has three independent paths R_1, R_2, R_3 from a vertex outside Z to Z such that e is on one of these paths. If $V(R_1 \cup R_2 \cup R_3) \cap Z = V(T)$ then G contains A_2 as a minor. If $V(R_1 \cup R_2 \cup R_3) \cap Z \neq V(T)$ then $R_1 \cup R_2 \cup R_3$ contains a Z -path R such that R

contains e and at least one end of R is not in T . This situation reduces to our first case and thus G contains the required minor. \square

8. 3-connected $\theta_{t,t,t}$ -free graphs

In this section we focus on 3-connected graphs. Let (G_0, w_0) be a weighted plane graph and let $(G_1, w_1), \dots, (G_k, w_k)$ be disjoint weighted graphs with $|G_i| \geq 5$ for all $i > 0$. Denote by $S_3^p((G_0, w_0); (G_1, w_1), \dots, (G_k, w_k))$ a weighted graph (G, w) obtained by 3-summing $(G_1, w_1), \dots, (G_k, w_k)$ to inner facial triangles of (G_0, w_0) . Let $r, s \geq 2$ be integers. Let $\mathcal{L}_{r,s}^3$ be the class of 3-connected members of $\mathcal{L}_{r,s}$. Let \mathcal{P}_r^3 be the class of 3-connected members $(G, w) \in \mathcal{P}_r$ such that if C is the outer cycle of G then either $|C| \geq 3r$ or C contains at least three edges of weight at least r . Let $\Phi^3(\mathcal{P}_r^3, \mathcal{L}_{r,s}^3)$ be the class of 3-connected weighted graphs of the form $S_3^p((G_0, w_0); (G_1, w_1), \dots, (G_k, w_k))$ ($k \geq 0$) over all $(G_0, w_0) \in \mathcal{P}_r^3$ and $(G_1, w_1), \dots, (G_k, w_k) \in \mathcal{L}_{r,s}^3$ with $|G_i| \geq 5$ for all $i > 0$. In the rest of the paper we will call an edge *heavy* if its weight is at least t . The following is the main result of this section.

Theorem 8.1. *There exists a function $f_{8.1}(t)$ such that if (G, w) is 3-connected and $\theta_{t,t,t}$ -free, then one of the following holds.*

- (a) $(G, w) \in \Phi^3(\mathcal{P}_t^3, \mathcal{L}_{t, f_{8.1}(t)}^3)$,
- (b) $G \in \mathcal{L}_{f_{8.1}(t)}^3$ and either G has at most two heavy edges or G has exactly three edges and these three form a triangle.

The proof of this theorem is divided into three steps. The first two are given in two lemmas, which deal with unweighted graphs. For any integer $k \geq 2$, let W_k^+ be the graph obtained from W_{2k} with rim cycle $x_1x_2\dots x_{2k}x_1$ by first subdividing the edges x_1x_2 and $x_{k+1}x_{k+2}$ and then joining these two new vertices by an edge. Let W'_k be obtained from W_k by adding a parallel edge to each of its spokes. We define W'_k for technical purpose because now W'_k is the edge-disjoint union of k triangles and thus we can talk about 3-summing graphs to all these triangles.

Lemma 8.2. *There exists a function $f_{8.2}(t, k)$ such that every 3-connected graph with a path of length $f_{8.2}(t, k)$ either contains W_t^+ or L_t^+ as a topological minor or can be expressed as $S_3(W'_k; G_1, \dots, G_k)$, where $t \geq 2$ and $k \geq 4$ are integers and $|G_i| \geq 5$ for all i .*

Proof. Let $f_R(t)$ be the minimum integer such that every connected simple graph on at least $f_R(t)$ vertices has an induced K_{t+2} , $K_{1,3}$, or P^{2t+2} (such a function arises as an extension of Ramsey theory and its existence was proven in [4] (see Theorem 5.3)). We prove $f_{8.2}(t, k) = f_{3.5}(3k(t+1)^2 f_R(t))$ satisfies the lemma. Let G be a 3-connected graph with $\ell(G) \geq f_{8.2}(t, k)$. We assume G is simple and G does not contain L_t^+ as a topological minor. Then by Lemma 3.5, G has a subgraph H isomorphic to a subdivision of W_n where $n \geq 3k(t+1)^2 f_R(t)$. Take n to be maximal.

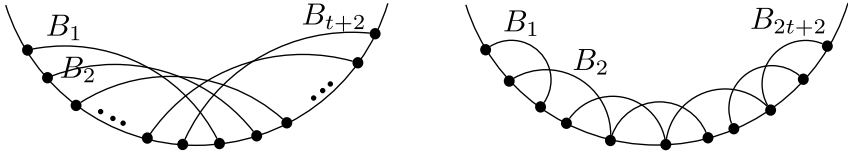


Fig. 8.1. Γ' contains an induced K_{t+2} or P^{2t+2} .

Let x_0, x_1, \dots, x_n be the non-subdividing vertices of H with x_0 corresponding to the center. For $i = 1, 2, \dots, n$, let P_i be the x_0x_i -path and Q_i be the x_ix_{i+1} -path (where $x_{n+1} = x_1$) of H . By Lemma 7.5, we may assume the feet of each H -bridge are not contained in a single P_i or Q_i . Let $E_0 = E((P_1 \cup \dots \cup P_n) - x_0)$; let $G' = (G - x_0)/E_0$ and $H' = (H - x_0)/E_0$. To simplify our notation, we consider each Q_i as a path of H' as well. Note H' is the cycle formed by the union of all paths Q_i , and because no trivial H -bridge has a foot at x_0 , there is a one-to-one correspondence between H -bridges of G and H' -bridges of G' . Moreover, since G is 3-connected and each H -bridge is stable, each H' -bridge of G' has at least two feet on H' .

For any path J of H' , define the Q -length of J to be the least number of paths Q_i whose union contains J . Suppose G' has an H' -bridge B that contains two feet u, v for which both uv -paths of H' are of Q -length $\geq t + 1$. Then $H \cup B$ contains W_t^+ as a topological minor. Hence assume any two feet of any H' -bridge are contained in a path of H' of Q -length $\leq t$. Since $n > 3t$, it follows that all feet of any H' -bridge are contained in a path of H' of Q -length $\leq t$. For each H' -bridge B , let $Q(B)$ denote the unique minimal path of H' of Q -length $\leq t$ that contains all feet of B . Generally, as n is much bigger than t , we can think of each path $Q(B)$ as a very small segment of H' ; this leads to a rough description of G' as a long cycle with bridges attached to small segments of the cycle.

To understand the structure of G' , we do not need to know all H' -bridges. Instead, knowing the “maximal” ones will be enough. Let \mathcal{B} be a minimal set of H' -bridges such that for every H' -bridge B_1 , there exists $B_2 \in \mathcal{B}$ with $Q(B_1) \subseteq Q(B_2)$. We will focus on bridges in \mathcal{B} . Let Γ be the simple graph with vertex set \mathcal{B} such that B_1 and B_2 are adjacent if $E(Q(B_1) \cap Q(B_2)) \neq \emptyset$. These graphs are proper circular arc graphs and it is clear by the definition they contain no induced claws. For any subgraph Γ' of Γ , we will say the *bridges of Γ'* to mean the bridges corresponding to the vertices of Γ' .

Suppose a component Γ' of Γ has at least $f_R(t)$ vertices. Because Γ does not contain any induced claws, Γ' contains an induced K_{t+2} or P^{2t+2} . If Γ' contains an induced K_{t+2} , then H' together with bridges of this clique contains a subdivision of the Möbius ladder as shown in Fig. 8.1, where each bridge B_i is represented by a chord joining the two ends of $Q(B_i)$. As a result, G' and hence G contains L_t^+ as a topological minor. Similarly, if Γ' contains an induced P^{2t+2} , then H' together with bridges of this path contains L_t^+ as a topological minor. Thus we conclude each component of Γ has fewer than $f_R(t)$ vertices.

For each component Γ' of Γ , let $Q(\Gamma')$ be the union of $Q(B)$ over all bridges B of Γ' . Then $Q(\Gamma')$ is a path of H' and its Q -length is less than $tf_R(t)$. Since n is much bigger

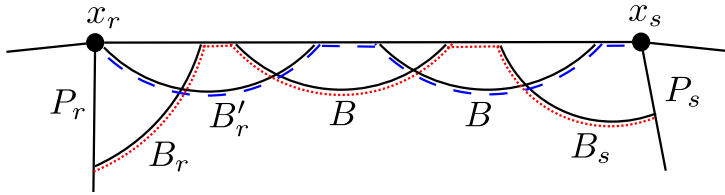


Fig. 8.2. $Q(\Gamma_i)$ and some relevant bridges.

than $tf_R(t)$, these paths again can be viewed as very short segments of H' . Let Γ_1, Γ_2 be distinct components of Γ . Observe $Q(\Gamma_1)$ and $Q(\Gamma_2)$ are edge-disjoint. We say Γ_1, Γ_2 are *linked* if $Q(\Gamma_1)$ and $Q(\Gamma_2)$ have a common end v such that v is obtained by contracting $E(P_i - x_0)$ for some i , and for each $j \in \{1, 2\}$, there is a bridge B_j for which, when viewed as an H -bridge of G , B_j has a foot in $P_i - \{x_0, x_i\}$, and when viewed as an H' -bridge of G' , B_j has a foot in $Q(\Gamma_j) - v$.

A *linkage* Λ is a maximal sequence $\Gamma_1, \dots, \Gamma_m$ of components of Γ such that $Q(\Gamma_i)$ and $Q(\Gamma_{i+1})$ are linked for $i = 1, \dots, m - 1$. Suppose there is a linkage Λ with $m \geq 4$. Let us consider each Γ_i with $2 \leq i \leq m - 1$. Let the two ends of $Q(\Gamma_i)$ be obtained by contracting $P_r - x_0$ and $P_s - x_0$; let B_r, B_s be bridges linking $P_r - \{x_0, x_r\}$ and $P_s - \{x_0, x_s\}$, respectively, to the rest of $Q(\Gamma_i)$, as shown in Fig. 8.2. Note B_r, B_s may not belong to \mathcal{B} (and B_r in the Figure is such an example). Choose two bridges of Γ_i so that the two ends of $Q(\Gamma_i)$ are feet of these two bridges, respectively. In our example B'_r and B_s are these two bridges. Since Γ_i is connected, it contains an induced path between these two bridges. Then bridges of this path together with B_r, B_s , and $Q(\Gamma_i)$ contain two disjoint paths R'_i, R''_i of G between $P_r - x_0$ and $P_s - x_0$. Now it is easy to see that the union of R'_i, R''_i ($i = 2, \dots, m - 1$) and $H - x_0$ contains L_{m-3}^+ as a topological minor.

What we have shown is that each linkage can have at most $t + 2$ terms. Let $Q(\Lambda)$ denote the union of $Q(\Gamma_i)$ over all terms Γ_i of Λ . Then $Q(\Lambda)$ is a path of H' with Q -length $< t(t + 2)f_R(t)$. Let I_Λ consist of all i such that either x_i is an interior vertex of $Q(\Lambda)$ or x_i is an end of $Q(\Lambda)$ for which G has an H -bridge with feet in both $P_i - \{x_0, x_i\}$ and $Q(\Lambda) - x_i$. Let $Q^+(\Lambda)$ be the union of $Q(\Lambda)$ (as a path of H) and P_i for all $i \in I_\Lambda$. The four shaded subgraphs in Fig. 8.3 are examples of $Q^+(\Lambda)$. For any two distinct linkages Λ_1, Λ_2 , since $Q(\Lambda_1)$ and $Q(\Lambda_2)$ are edge-disjoint, it follows that $Q^+(\Lambda_1)$ and $Q^+(\Lambda_2)$ are also edge-disjoint. Moreover, the only possible common vertices of $Q^+(\Lambda_1)$ and $Q^+(\Lambda_2)$ are x_0 and the common end of $Q(\Lambda_1)$ and $Q(\Lambda_2)$.

We claim that for every H -bridge B there exists a linkage Λ such that all feet of B are contained in $Q^+(\Lambda)$. When B is viewed as an H' -bridge, $Q(B)$ is contained in $Q(B')$ for some $B' \in \mathcal{B}$ and thus $Q(B)$ is contained in $Q(\Lambda)$ for a linkage Λ . Then the definition of I_Λ implies that, when B is viewed as an H -bridge, all feet of B are in $Q^+(\Lambda)$, which proves our claim.

For each vertex v of H' , if v is a foot of at least one H' -bridge then v is contained in $Q(B)$ for at least one $B \in \mathcal{B}$. Let Z be the set of vertices z of H' such that z is not contained in $Q(B)$ for any $B \in \mathcal{B}$. Then for each $z \in Z$ there exists i such that

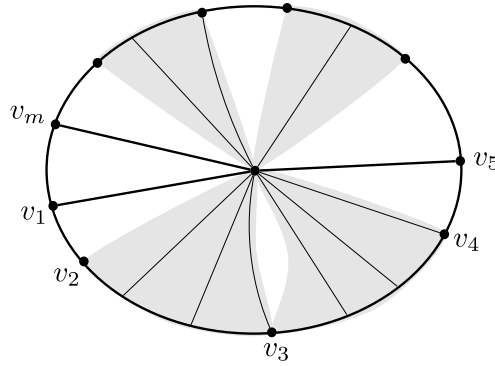


Fig. 8.3. H is divided according to H -bridges.

$z = x_i$, P_i contains only one edge x_0x_i , and x_i has degree 3 in G . In Fig. 8.3, Z consists of v_1, v_5, v_m . It follows that every vertex of H' belongs to either Z or $Q(\Lambda)$ for some linkage Λ . Let Y be the set of vertices y on the rim of H such that there is a linkage Λ for which, when $Q(\Lambda)$ is considered as a path of H , y is an end of this path. In our example, Y contains seven vertices including v_2, v_3, v_4 . Let v_1, v_2, \dots, v_m be all vertices of $Y \cup Z$, which are listed in the order they appear on the rim cycle of H .

The above discussions imply that, if $1 < |i - j| < m - 1$ then $\{x_0, v_i, v_j\}$ is a 3-cut of G separating $\{v_{i+1}, \dots, v_{j-1}\}$ from $\{v_{j+1}, \dots, v_{i-1}\}$ (indices are modulo m). It follows that G can be expressed as $S_3(W'_m; H_1, \dots, H_m)$, where W'_m has vertex set $\{x_0, v_1, \dots, v_m\}$. Note that some of the H_i may be only three vertices (for example, the graph that would be summed to $x_0v_1v_2$ in Fig. 8.3). However, by taking a smaller wheel on vertices $x_0, v_1, v_4, \dots, v_{3\lfloor m/3 \rfloor - 2}$ we have $G = S_3(W'_{\lfloor m/3 \rfloor}; G_1, \dots, G_{\lfloor m/3 \rfloor})$ and such that $|G_i| \geq 5$ for all i .

It remains to show that $|Y \cup Z| \geq 3k$. We assume $|Z| < 3k$ because otherwise we are done. We prove that there are at least $3k$ linkages, which would imply $|Y| \geq 3k$. Suppose otherwise. Since each $Q(\Lambda)$ has Q -length $< t(t + 2)f_R(t)$, at most $t(t + 2)f_R(t)$ vertices x_i are contained in each $Q(\Lambda)$. It follows that the total number of vertices x_i would be $< |Z| + 3kt(t + 2)f_R(t) < 3k(t^2 + 2t + 1)f_R(t) = n$. This contradiction completes our proof of the lemma. \square

To simplify our notation, for any class \mathcal{G} of weighted graphs, we will write $G \in \mathcal{G}$ if $(G, \varepsilon) \in \mathcal{G}$, where $\varepsilon(e) = 1$ for all edges e of G . Using this terminology, $G \in \mathcal{P}_r^3$ is equivalent to: G is a 3-connected plane graph such that if C is the outer cycle then $|C| \geq 3r$ and G has no C -path of length at least $2r$. Note wheels are examples of such graphs. Let \mathcal{L}_s^3 denote the class of 3-connected graphs in \mathcal{L}_s . Then $G \in \mathcal{L}_s^3$ if and only if $G \in \mathcal{L}_{r,s}^3$. Finally, both S_3^p and Φ^3 can be naturally restricted to unweighted graphs. That is, $S_3^p(G_0; G_1, \dots, G_k)$ is a graph obtained by 3-summing G_1, \dots, G_k , each of order ≥ 5 , to inner facial triangles of a plane graph G_0 , and $\Phi^3(\mathcal{P}_r^3, \mathcal{L}_s^3)$ is the class of 3-connected graphs of the form $S_3^p(G_0; G_1, \dots, G_k)$ ($k \geq 0$) over all $G_0 \in \mathcal{P}_r^3$ and $G_1, \dots, G_k \in \mathcal{L}_s^3$ of order ≥ 5 .

Lemma 8.3. *There exists a function $f_{8.3}(t)$ such that all 3-connected $\theta_{t,t,t}$ -free graphs belong to $\mathcal{L}_{f_{8.3}(t)}^3 \cup \Phi^3(\mathcal{P}_t^3, \mathcal{L}_{f_{8.3}(t)}^3)$.*

The main idea of the proof is to assume such a graph is not in $\mathcal{L}_{f_{8.3}(t)}^3$ and show it is in $\Phi^3(\mathcal{P}_t^3, \mathcal{L}_{f_{8.3}(t)}^3)$. From Lemma 8.2, we will start with the structure $S_3(W'_{3t}; G_1, \dots, G_{3t})$ and from here we move to the desired structure $S_3^p(G_0; H_1, \dots, H_h)$ by expanding the base graph as much as possible while still maintaining planarity. So for example if some G_i in $S_3(W'_{3t}; G_1, \dots, G_{3t})$ is planar, we want to incorporate it into G_0 . It is also possible (roughly speaking) that we may incorporate only some planar portion of a G_i into G_0 , leaving behind some H_j which is now 3-summed to G_0 .

Proof. We show $f_{8.3}(t) = f_{8.2}(2t, 3t)$ satisfies the theorem. For simplicity, let $s(t) = f_{8.3}(t)$. Suppose G is a 3-connected $\theta_{t,t,t}$ -free graph that does not belong to $\mathcal{L}_{s(t)}^3$. We will show that $G \in \Phi^3(\mathcal{P}_t^3, \mathcal{L}_{s(t)}^3)$.

We first argue that G can be expressed as $S_3^p(G_0; H_1, \dots, H_h)$, where $|H_i| \geq 5$ for all i , and G_0 has a subgraph H_0 such that H_0 is a subdivision of W_k with $k \geq 3t$ and the rim cycle of H_0 is a facial cycle of G_0 . To see this, observe that W_{2t}^+ and L_{2t}^+ both contain $\theta_{t,t,t}$ so by Lemma 8.2, G admits the required expression with $G_0 = H_0 = W'_{3t}$ and $(H_1, \dots, H_h) = (G_1, \dots, G_{3t})$. By Lemma 4.1 parts (b) and (c), the graphs G_0, H_1, \dots, H_h are all 3-connected minors of G . Among all such representations $S_3^p(G_0; H_1, \dots, H_h)$ of G , choose one with $|G_0|$ as large as possible.

Let x_0, x_1, \dots, x_k be the non-subdividing vertices of H_0 with x_0 corresponding to the center. By Lemma 5.4, $G_0 \in \mathcal{P}_t^3$. So we only need to show $H_i \in \mathcal{L}_{s(t)}^3$ for all i . To simplify notation, assume $i = 1$. We suppose H_1 has a path of length $s(t)$ and derive a contradiction. Let $y_0y_1y_2$ be the common triangle of G_0 and H_1 . Note $y_0y_1y_2$ is a face of G_0 so it is contained in some face of H_0 . Let C be the cycle bounding the region containing $y_0y_1y_2$ where C corresponds to triangle $x_0x_1x_2$ of H_0 . Since G_0 is 3-connected, there are three disjoint paths in G_0 (in fact, inside C) from $x_0x_1x_2$ to $y_0y_1y_2$. By renaming the indices of $y_0y_1y_2$, if necessary, we assume that the paths are from x_i to y_i ($i = 0, 1, 2$). Note the x_0y_0 -path is disjoint from the rim of H_0 .

Suppose at least one of y_1, y_2 , say y_2 , is not on the rim of H_0 . Since H_1 is 3-connected, $H_1 - y_2$ is 2-connected. Since H_1 has a path of length $s(t)$ (and $s(t) = f_{8.2}(2t, 3t) > 8t^2$), $H_1 - y_2$ has a path of length $4t^2$ and hence by Lemma 5.1, a y_0y_1 -path P of length at least t . Now we have a contradiction since $G_0 \cup P$ contains $\theta_{t,t,t}$ at x_0 and x_1 : one path uses P as well as the x_0y_0 -path and x_1y_1 -path, and the other two paths are in H_0 . It is important to note edges of triangle $y_0y_1y_2$ are not used in this $\theta_{t,t,t}$ since these three edges are deleted when H_1 is 3-summed to G_0 .

From the last paragraph we conclude that both y_1, y_2 are on the rim of H_0 . Since G_0 is 3-connected, y_1 and y_2 must be adjacent in H_0 . We assume that y_1y_2 is an edge of H_0 and, moreover, G_0 has no other edges parallel to y_1y_2 since all such edges can be placed in H_1 . In the following we will look, in H_1 , for a path from y_1 to y_2 together with

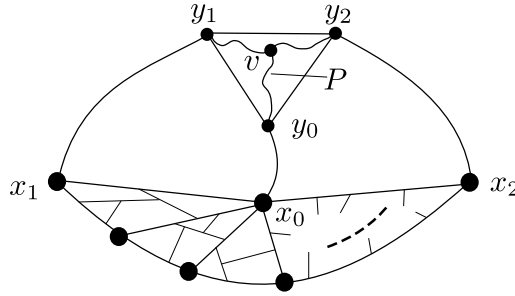


Fig. 8.4. A long spoke in G .

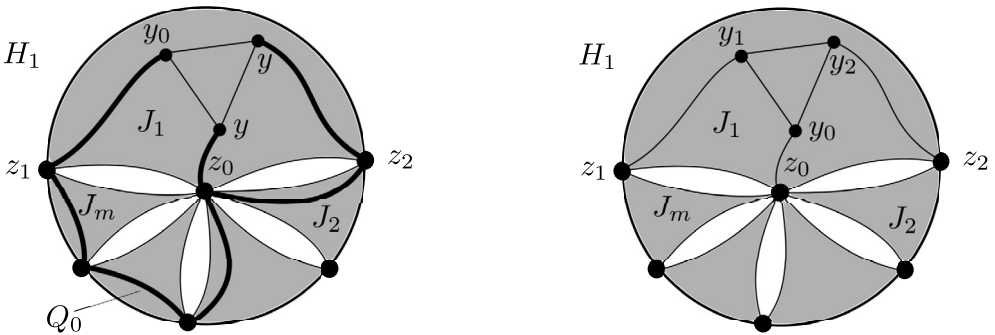


Fig. 8.5. Decomposition of H_1 into pieces.

a path P of length at least t from this path to y_0 ; call P a *long spoke*. With these two paths, there is a $\theta_{t,t,t}$ in G at x_0 and v as shown in Fig. 8.4.

Since H_1 is $\theta_{t,t,t}$ -free with $\ell(H_1) \geq s(t)$, by Lemma 8.2, $H_1 = S_3(J_0; J_1, \dots, J_{3t})$ where $J_0 = W'_{3t}$ and $|J_i| \geq 5$ for all $i > 0$. Let z_0 be the center of J_0 and $z_1 z_2 \dots z_{3t} z_1$ be its rim cycle. Without loss of generality, assume $y_0 y_1 y_2$ is contained in J_1 and $z_0 z_1 z_2$ is the common triangle of J_0 and J_1 .

Since J_1 is 3-connected, there are three disjoint paths P_i ($i = 0, 1, 2$) from z_i to the triangle $y_0 y_1 y_2$. Suppose the other end of P_0 is not y_0 . Then H_1 contains three independent paths Q_0, Q_1, Q_2 from z_0 to y_0, y_1, y_2 , respectively, as shown in the left in Fig. 8.5. Since Q_0 has length at least t , it is a long spoke and G contains a $\theta_{t,t,t}$. Hence assume P_i is from z_i to y_i ($i = 0, 1, 2$) as on the right in Fig. 8.5.

Let H'_1 be the 3-sum of J_0 and J_1 . In other words, H'_1 is obtained from H_1 by reducing each J_i ($i > 1$) to a triangle. Then H'_1 is 3-connected. Let Ω be the circllet of H'_1 with vertices $z_1, y_1, y_0, y_2, z_2, z_3, \dots, z_{3t}$, which are cyclically ordered as they are listed, and with $3t + 1$ edges from the two paths $y_1 y_0 y_2$ and $z_2 z_3 \dots z_{3t} z_1$. Note Ω is well-defined even if $y_1 = z_1$ or $y_2 = z_2$. From Lemma 4.3, we know that H'_1 can be expressed as $S_3(M_0; M_1, \dots, M_a)$ where M_i is a 3-connected minor of H'_1 with $|M_i| \geq 5$ for all $i > 0$, $V(\Omega) \subseteq V(M_0)$, (M_0, Ω) is 4-connected, and no summing triangle is a 3-cut of M_0 . Since z_0 has more than three neighbors in Ω , z_0 must belong to M_0 . It follows that

$H_1 = S_3(M_0; M_1, \dots, M_b)$ where M_{a+1}, \dots, M_b are J_2, \dots, J_{3t} , respectively. Note $H_1 \setminus y_1 y_2$ has an Ω -cycle $z_1 P_1 y_1 y_0 y_2 P_2 z_2 \dots z_{3t} z_1$, hence $M_0 \setminus y_1 y_2$ also has an Ω -cycle.

If $M_0 \setminus y_1 y_2$ admits a planar drawing so that some facial cycle F is an Ω -cycle, then we can find a larger base graph which contradicts the choice of G_0 . To see this, let G'_0 be the 3-sum of G_0 and M_0 ; G'_0 is planar. Let H'_0 be obtained from H_0 by replacing edge $y_1 y_2$ with path $F \setminus y_1 y_2$. Then H'_0 is a subdivision of W_k and the rim cycle of H'_0 is a facial cycle of G'_0 . Moreover, $G = S_3^p(G'_0; H_2, \dots, H_h, M_1, \dots, M_b)$, which contradicts the maximality of G_0 .

From Lemma 7.6, $M_0 \setminus y_1 y_2$ has an Ω -cycle F and a cross (Q_1, Q_2) of F with ends q_1, q_3 and q_2, q_4 , respectively, such that among the four paths of F divided by q_1, q_2, q_3, q_4 , at least three of them contain edges of Ω . For $i = 1, 2, 3, 4$, let $F_i = F[q_i, q_{i+1}]$, where $q_5 = q_1$. We consider two cases. Suppose path $y_1 y_0 y_2$ is contained in some F_i , say $i = 1$. Then one of q_3, q_4 , say q_3 , belongs to $\{z_3, z_4, \dots, z_{3t}\}$. It follows that Q_1 contains z_0 and thus $F_2 \cup F_3$ contains the path $z_2 z_3 \dots z_{3t} z_1$. Without loss of generality, assume $q_3 = z_{\lfloor 3t/2 \rfloor}$. Then the union of $Q_1, Q_2, F \setminus E(F_4)$, and H_0 contains $\theta_{t,t,t}$ at q_2, q_3 , which settles this first case. Now we assume $y_0 \in \{q_1, q_2, q_3, q_4\}$, and without loss of generality, $y_0 = q_1$. We claim we may further assume that $F_2 \cup F_3$ contains the path $z_2 z_3 \dots z_{3t}$. This is clear if Q_2 does not contain z_0 . If Q_2 contains z_0 then Q_1 does not contain z_0 , which implies either $F_1 \cup F_2$ or $F_3 \cup F_4$ contains path $z_2 z_3 \dots z_{3t}$. Let us assume the former, by symmetry. Then we can set $q_2 = z_2$, which proves our claim. Therefore, either $Q_1 \cup F_2$ or $Q_1 \cup F_3$ is a long spoke and hence G contains $\theta_{t,t,t}$. This completes our proof. \square

Let (G, w) be a weighted graph and suppose $G = S_d(G_0; G_1, \dots, G_k)$, where $d \in \{2, 3\}$. Then we can define weights w_0, w_1, \dots, w_k . For each $i \geq 0$, if $e \in G_i$ does not belong to any summing triangle, then $w_i(e) = w(e)$. If $e \in G_i$ belongs to a summing triangle, then $w_i(e) = 1$. We say that w_0, \dots, w_k are the *induced weights*.

Proof of Theorem 8.1. We show $f_{8.1}(t) = f_{8.3}(t)$ satisfies the theorem. Let (G, w) be 3-connected and $\theta_{t,t,t}$ -free. Assume (b) does not hold. We first claim that there exists a 3-connected plane graph G_0 such that

- if C is the outer cycle of G_0 then either $|C| \geq 3t$ or C contains at least three heavy edges, and
- $(G, w) = S_3^p((G_0, w_0); (G_1, w_1), \dots, (G_k, w_k))$, where $G_i \in \mathcal{L}_{f_{8.1}(t)}^3$ with $|G_i| \geq 5$ for $i = 1, \dots, k$.

This claim follows from Lemma 8.3 immediately if $G \notin \mathcal{L}_{f_{8.1}(t)}^3$. So we assume $G \in \mathcal{L}_{f_{8.1}(t)}^3$. Consider a cycle Q containing as many heavy edges as possible. If there is a heavy edge e not contained in Q then G has a Q -path P containing e . It is easy to see that $Q \cup P$ either contains $\theta_{t,t,t}$ or contains a cycle that contains more heavy edges. Both cases are impossible, so Q must contain all heavy edges. Let Ω be a circllet such that its edge set consists of all heavy edges, its vertex set consists of exactly vertices that are incident with at least one heavy edge, and such that Q is an Ω -cycle. Note $|E(\Omega)| \geq 3$ and $|V(\Omega)| \geq 4$ because (b) does not hold. By Lemma 4.3, G can be expressed as

$S_3(G_0; G_1, \dots, G_k)$ where G_i is a 3-connected minor of G with $|G_i| \geq 5$ for all $i > 0$, $V(\Omega) \subseteq V(G_0)$, (G_0, Ω) is 4-connected, and no summing triangle is a 3-cut of G_0 . Note G_0 contains an Ω -cycle since G has an Ω -cycle. By Theorem 7.6, G_0 admits a planar drawing with an Ω -cycle C as a facial cycle since a cross as defined in Theorem 7.6 of an Ω -cycle would result in a $\theta_{t,t,t}$. Let w_0, \dots, w_k be the induced weights. Then our claim holds with our choices of $(G_0, w_0), (G_1, w_1), \dots, (G_k, w_k)$, and C .

Let us choose G_0 satisfying the above claim with as many vertices as possible. If G_i is 3-summed to G_0 over triangle T , then we assume no edge of G_i is parallel to any edge of T since we may put all these edges in G_0 . We also assume each edge e of C has the maximum weight among all edges of G_0 that are parallel to e . By Lemma 5.4, G_0 contains no C -path of weight at least $2t$ and $w_0(e) < t$ for all edges e of $G_0 \setminus E(C)$. Hence we conclude $(G_0, w_0) \in \mathcal{P}_t^3$.

It remains to show that no G_i ($i > 0$) contains a heavy edge. Suppose to the contrary that some G_i contains a heavy edge e . Let T be the summing triangle of G_i . Then at most one end of e is in T . By the maximality of G_0 and Lemma 7.7, G_i contains a minor $A \in \{A_1, A_2\}$. Note at least one vertex of T , say v , is not on C . Thus the 3-sum of (G_0, w_0) and (G_i, w_i) contains a minor (G'_0, w'_0) obtained as follows: first we reduce $G_i \setminus E(T)$ to $A \setminus E(T)$, then we reduce $A \setminus E(T)$ to a triangle (by contracting two edges and deleting one or two edges) with vertex set $V(T)$ and such that e is on the triangle and is incident with v . Then by applying Lemma 5.4 to (G'_0, w'_0) we obtain a $\theta_{t,t,t}$. This contradiction completes our proof of the theorem. \square

9. Proving the main theorem

In this section we prove Theorem 2.1. We divide the proof into two parts.

Lemma 9.1. *There exists a function $f_{9.1}(r, s)$ such that all weighted graphs in $\Phi(\mathcal{P}_r, \mathcal{L}_{r,s})$ are $\theta_{t,t,t}$ -free, where $t = f_{9.1}(r, s)$.*

Proof. We show $f_{9.1}(r, s) = 2qr$ satisfies the theorem, where $q = (r - 1)(s - 1)$. Suppose there is a counterexample (G, w) . Then we choose one with $|G|$ minimum. Assume (G, w) is formed by k -summing ($k = 2, 3, 4$) weighted graphs $(G_1, w_1), \dots, (G_n, w_n) \in \mathcal{L}_{r,s}$ to $(G_0, w_0) \in \mathcal{P}_r$ along edges, inner facial triangles, and rectangles. Let C be the outer cycle of G_0 .

Suppose some (G_{i_0}, w_{i_0}) is 4-summed to a rectangle $x_1x_2x_3x_4x_1$ of G_0 , where x_1x_2 and x_3x_4 are edges of C . Recall that by the definition of a rectangle, this means no graph (G_{i_1}, w_{i_1}) can be 2-summed to an edge between x_1 and x_2 or x_3 and x_4 since there are no parallel edges between these vertices. We consider two cases. Assume first that G has a 2-separation (H, J) with $V(H \cap J) = \{x_j, x_{5-j}\}$ for $j = 1$ or 2 and such that $C[x_j, x_{5-j}] \subseteq H$ and $C[x_{5-j}, x_j] \subseteq J$. Define (H^+, w_H) where H^+ is obtained by adding a new edge $e = x_jx_{5-j}$ to H , $w_H(e)$ is equal to the maximum weight of an x_jx_{5-j} -path in J , and $w_H(f) = w(f)$ for all edges f of H . Also define (J^+, w_J) analogously. Then

G_0 can be expressed as a 2-sum of plane graphs G_0^H and G_0^J over e such that the outer cycles of G_0^H and G_0^J are $C[x_j, x_{5-j}] + e$ and $C[x_{5-j}, x_j] + e$, respectively, and any edges parallel to e are in G_0^H . Moreover, $(G_1, w_1), \dots, (G_n, w_n)$ can be divided into two groups such that the first group is summed to G_0^H to obtain (H^+, w_H) and the second group is summed to G_0^J to obtain (J^+, w_J) . It follows that both (H^+, w_H) and (J^+, w_J) belong to $\Phi(\mathcal{P}_r, \mathcal{L}_{r,s})$. By the minimality of G , both (H^+, w_H) and (J^+, w_J) are $\theta_{t,t,t}$ -free and thus, by Lemma 5.3, (G, w) is also $\theta_{t,t,t}$ -free. This is a contradiction and so the first case is settled.

Now in the second case, G does not have a 2-separation as described in the previous paragraph. Then the length of C must be 4 and G_{i_0} must be the only graph summed to G_0 (so $n = 1$). Therefore, G_0 consists of the 4-cycle $x_1x_2x_3x_4x_1$ and possibly more edges parallel to x_1x_4 or x_2x_3 . Consequently, G is obtained from $G_1 \setminus \{x_1x_2, x_3x_4\}$ by adding parallel edges. Since all heavy edges of G belong to C and x_1x_2 and x_3x_4 are deleted after the sum, we deduce G has at most two heavy edges. As a result, in every $\theta_{a,b,c}$ of (G, w) , at least one of its three independent paths cannot have any heavy edges. Let t^* be the largest integer so that (G, w) contains θ_{t^*,t^*,t^*} . Then $t^* \leq (r - 1)(s - 1) < f_{9.1}(r, s)$.

Now we assume that no G_i is 4-summed to G_0 . Suppose x, y are distinct vertices of G and P_1, P_2, P_3 are independent xy -paths of G . Let $p = \min\{w(P_1), w(P_2), w(P_3)\}$. We prove $p < 2qr$. If $P_j \subseteq G_i$ for some j and $i > 0$ then $p \leq w_i(P_j) \leq \max\{w_i(P) : P \text{ is a path of } G_i\} \leq (r - 1)(s - 1) < 2qr$. Henceforth we assume no G_i contains any P_j . In particular, each $G_i - V(G_0)$ contains at most one of x, y .

We modify (G_0, w_0) and P_1, P_2, P_3 as follows. Let $P = P_1 \cup P_2 \cup P_3$. For each i such that $G_i - V(G_0)$ contains neither x nor y , note $G_i \cap P$ consists of zero, one, or two $G_i \cap G_0$ -paths. If Z is such a path with ends z_1, z_2 , we change $w_0(z_1z_2)$ to $w_i(Z)$ and, in P , we replace path Z by a single edge z_1z_2 . If $G_i - V(G_0)$ contains x or y , say x , then $V(G_i \cap G_0)$ consists of three vertices z_1, z_2, z_3 , and we add a new vertex x' and three new edges $x'z_1, x'z_2, x'z_3$ to G_0 . In this case we define the weight of $x'z_j$ ($j = 1, 2, 3$) to be $w_i(Z_j)$, where Z_j is the xz_j -path contained in $G_i \cap P$. We also change $w_0(z_jz_{j'})$ to $w_i(Z_j) + w_i(Z_{j'})$. Let (G'_0, w'_0) be the modified weighted graph. Let P'_1, P'_2, P'_3 be the three modified paths and x', y' be their ends. Note $w'_0(P'_j) = w(P_j)$ for $j = 1, 2, 3$.

Note G'_0 is planar and let C' be its outer cycle. We may assume that P'_2 is inside the region bounded by cycle $P'_1 \cup P'_3$ and C' is outside this region. Let Q_1, Q_2 be two disjoint paths from C' to $P'_1 \cup P'_3$. Then $P'_1 \cup P'_2 \cup P'_3 \cup Q_1 \cup Q_2$ contains a C' -path Q' such that $P'_2 \subseteq Q'$. Since the only possible vertices in $V(G'_0) \setminus V(G_0)$ are x', y' and each of them is surrounded by a triangle of G_0 , we deduce G_0 has a C' -path Q with $w'_0(Q) = w'_0(Q')$. Therefore, $p \leq w'_0(P'_2) \leq w'_0(Q') = w'_0(Q) \leq (r - 1)(s - 1)||Q|| < 2qr$. □

Theorem 9.2. *There exists a function $s(t)$ such that every 2-connected $\theta_{t,t,t}$ -free weighted graph belongs to $\Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$.*

Proof. We prove $s(t) = 4t^2(3f_{8.1}(t) + 2)$ satisfies the theorem. Suppose there is a counterexample (G, w) . Then we choose one with $|G|$ minimum. If $|G| = 2$, since (G, w) is

$\theta_{t,t,t}$ -free, G must have at most two heavy edges and thus $(G, w) \in \mathcal{P}_t \subseteq \Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$. This contradicts the choice of (G, w) , so we assume $|G| \geq 3$. We consider three cases based on Lemma 5.2.

Case (a) holds: Let (H, J) be a 2-separation of G with $V(H \cap J) = \{x, y\}$ such that neither H nor J has an xy -path of weight at least t . It is clear that G has no heavy edges and, by Lemma 5.1, G has no path of length at least $4t^2$. Hence $(G, w) \in \mathcal{L}_{t,s(t)}$. Since G can be considered as a 2-sum of G with a 2-cycle, and any weighted 2-cycle belongs to \mathcal{P}_t , it follows that $(G, w) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$ as required.

Case (b) holds: Let (H, J) be a 2-separation of G with $V(H \cap J) = \{x, y\}$ such that H and J each have an xy -path of weight at least t . Denote by (H^+, w_H) the graph formed from H by adding an edge $e = xy$ with $w_H(e)$ equal to the weight of a heaviest xy -path in J and $w_H(f) = w(f)$ for all other edges f . Define (J^+, w_J) analogously. Now since (G, w) is a minimal counterexample and, by Lemma 5.3, both (H^+, w_H) and (J^+, w_J) are $\theta_{t,t,t}$ -free, they both belong to $\Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$.

Let $(H_0, \alpha_0) \in \mathcal{P}_t$ be the base graph for constructing (H^+, w_H) and let C_H be the outer cycle of H_0 . Let (J_0, β_0) and C_J be defined analogously. Since e is heavy, $e \in C_H$ and $e \in C_J$. Let (G_0, w_0) be the 2-sum of (H_0, α_0) and (J_0, β_0) over e , and let C be the 2-sum of C_H and C_J over e . Then G_0 is a plane graph with outer cycle C . In fact, $(G_0, w_0) \in \mathcal{P}_t$ because every C -path of G_0 is a C_H -path of H_0 or a C_J -path of J_0 , and every heavy edge of G_0 is a heavy edge of H_0 or J_0 .

Let \mathcal{G} be the set of weighted graphs that are summed to (H_0, α_0) or (J_0, β_0) in forming (H^+, w_H) and (J^+, w_J) . We claim that (G, w) is formed by summing members of \mathcal{G} to (G_0, w_0) . Since $e \in H^+$, e is not contained in any summing 3- or 4-cycle of H_0 . Moreover, every inner facial cycle of H_0 that does not contain e remains an inner facial cycle of G_0 . So summing edges and summing cycles of H_0 can still serve as a summing edge or cycle of G_0 . Similarly, summing edges and summing cycles of J_0 can still serve as a summing edge or cycle of G_0 . Therefore, the claim follows and thus $(G, w) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$.

Case (c) holds: Let $G = S_2(G_0; G_1, \dots, G_k)$ where G_0, \dots, G_k satisfy Lemma 5.2(c). Let w_0, \dots, w_k be the induced weights. By Lemma 5.1, $(G_i, w_i) \in \mathcal{L}_{t,4t^2}$ for $i = 1, \dots, k$. Moreover, heavy edges of (G, w) are exactly heavy edges of (G_0, w_0) . First suppose $si(G_0) = K_3$. If no two heavy edges of G_0 are parallel, then $(G_0, w_0) \in \mathcal{P}_t$ and thus $(G, w) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,4t^2})$. Assume G_0 has two parallel heavy edges e, f . Then they are the only two heavy edges since (G, w) is $\theta_{t,t,t}$ -free. Define (G'_0, w'_0) where G'_0 consists of e, f and a new edge g parallel to e, f , and $w'_0(e) = w(e)$, $w'_0(f) = w(f)$, $w'_0(g) = 1$. Let (G'_1, w'_1) be obtained from (G, w) by deleting e, f and adding g with weight 1. Then (G, w) is the 2-sum of (G'_0, w'_0) and (G'_1, w'_1) over g . It is clear that $(G'_0, w'_0) \in \mathcal{P}_t$ and, by Lemma 5.5, $(G'_1, w'_1) \in \mathcal{L}_{t,16t^2}$. Again we have $(G, w) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$.

Second suppose G_0 is 3-connected. By Lemma 5.3, (G_0, w_0) is $\theta_{t,t,t}$ -free. We claim that $(G_0, w_0) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,3f_{8.1}(t)})$. By Theorem 8.1, we assume $G_0 \in \mathcal{L}_{f_{8.1}(t)}$ and either (G_0, w_0) has at most two heavy edges or (G_0, w_0) has exactly three heavy edges and these three form a triangle. Our claim is clear if (G_0, w_0) has zero, one, two parallel, or three heavy edges: take the base graph to be a cycle containing all of the heavy

edges with an additional parallel edge added to each edge of the cycle. Suppose (G_0, w_0) has two adjacent heavy edges $e = xy$ and $f = xz$ with $y \neq z$. Define (G'_0, w'_0) where G'_0 is obtained from e, f by adding three new edges xy, yz, xz , and $w'_0(e) = w_0(e)$, $w'_0(f) = w_0(f)$, $w'_0(xy) = w'_0(yz) = w'_0(xz) = 1$. Let (G'_1, w'_1) be obtained from (G_0, w_0) by deleting e, f and adding xy, yz, xz of weight 1. Then (G_0, w_0) is a 3-sum of (G'_0, w'_0) and (G'_1, w'_1) . Moreover, $(G'_0, w'_0) \in \mathcal{P}_t$ and $(G'_1, w'_1) \in \mathcal{L}_{t,2f_{8.1}(t)}$ (as $G'_1 \setminus xz \cong G_0$), and thus our claim holds in this case. Finally, suppose (G_0, w_0) has two nonadjacent heavy edges $e = x_1x_4$ and $f = x_2x_3$. Define (G'_0, w'_0) where G'_0 is obtained from e, f by adding a 4-cycle $x_1x_2x_3x_4x_1$, and $w'_0(e) = w_0(e)$, $w'_0(f) = w_0(f)$, $w'_0(x_1x_2) = w'_0(x_2x_3) = w'_0(x_3x_4) = w'_0(x_4x_1) = 1$. Let (G'_1, w'_1) be obtained from (G_0, w_0) by deleting e, f and adding $x_1x_2, x_2x_3, x_3x_4, x_4x_1$ of weight 1. Then (G_0, w_0) is a 4-sum of (G'_0, w'_0) and (G'_1, w'_1) . Moreover, $(G'_0, w'_0) \in \mathcal{P}_t$ and $(G'_1, w'_1) \in \mathcal{L}_{t,3f_{8.1}(t)}$, and thus our claim is proved.

By this claim, (G_0, w_0) is formed by 2-, 3-, and 4-summing $(H_1, \alpha_1), \dots, (H_n, \alpha_n) \in \mathcal{L}_{t,3f_{8.1}(t)}$ to $(H_0, \alpha_0) \in \mathcal{P}_t$. Now weighted graphs $(G_1, w_1), \dots, (G_k, w_k)$ can be divided into groups $\mathcal{H}_0, \dots, \mathcal{H}_n$ such that (G_i, w_i) belongs to \mathcal{H}_j if G_i is 2-summed to H_j . For each $j > 0$, let (H_j^*, α_j^*) be obtained by 2-summing all weighted graphs in \mathcal{H}_j to (H_j, α_j) . Then (G, w) is obtained by 2-, 3-, 4-summing members of $\mathcal{H}_0 \cup \{(H_1^*, \alpha_1^*), \dots, (H_n^*, \alpha_n^*)\}$ to (H_0, α_0) . It remains to show $\mathcal{H}_0 \cup \{(H_1^*, \alpha_1^*), \dots, (H_n^*, \alpha_n^*)\} \subseteq \mathcal{L}_{t,s(t)}$. Since each (G_i, w_i) has no x_iy_i -path of weight $\geq t$, we must have $\mathcal{H}_0 \subseteq \mathcal{L}_{t,4t^2}$ by Lemma 5.1. Moreover, by Lemma 5.5, $\ell(H_j^*) \leq (3f_{8.1}(t) + 2)(4t^2)$ and thus $\{(H_1^*, \alpha_1^*), \dots, (H_n^*, \alpha_n^*)\} \subseteq \mathcal{L}_{t,s(t)}$. Therefore, $(G, w) \in \Phi(\mathcal{P}_t, \mathcal{L}_{t,s(t)})$, which completes our proof. \square

Proof of Theorem 2.1. The theorem is proved by Lemma 9.1 and Theorem 9.2. \square

References

- [1] R. Diestel, Graph Theory, fourth edition, Springer, Berlin, Germany, 2010.
- [2] G. Ding, Excluding a long double path minor, J. Combin. Theory Ser. B 66 (1996) 11–23.
- [3] G. Ding, S. Dziobiak, H. Wu, Large W_k - or $K_{3,t}$ -minors in 3-connected graphs, J. Graph Theory 82 (2016) 207–217.
- [4] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected binary matroids, J. Combin. Theory Ser. B 66 (1996) 334–360.
- [5] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, 2001.
- [6] N. Robertson, K. Chakravarti, Covering three edges with a bond in a nonseparable graph (Abstract), Ann. Discrete Math. 8 (1980) 247.
- [7] N. Robertson, P.D. Seymour, Graph minors. I. Excluding a forest, J. Combin. Theory Ser. B 35 (1983) 39–61.
- [8] N. Robertson, P.D. Seymour, Graph minors. V. Excluding a planar graph, J. Combin. Theory Ser. B 41 (1986) 92–114.
- [9] N. Robertson, P.D. Seymour, Graph minors IX: disjoint crossed paths, J. Combin. Theory Ser. B 49 (1990) 40–77.
- [10] N. Robertson, P.D. Seymour, Graph minors. XVI. Excluding a non-planar graph, J. Combin. Theory Ser. B 89 (2003) 43–76.
- [11] N. Robertson, P.D. Seymour, Graph minors – a survey, Surv. Combin. 103 (1985) 153–171.