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A chain theorem for 4-connected graphs

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ABSTRACT

A sequence of 4-connected graphs $G_0, G_1, \dots, G_n$ is called a (G_0, G_n) -chain if each G_i ($i < n$) has an edge e_i such that $G_i/e_i = G_{i+1}$. A classical result of Martinov states that for every 4-connected graph G there exists a (G, H) -chain such that $H \in \mathcal{C} \cup \mathcal{L}$, where $\mathcal{C} = \{C_n^2 : n \geq 5\}$ and $\mathcal{L} = \{L : L \text{ be the line graph of a cyclically 4-edge-connected cubic graph}\}$. This result is strengthened in this paper as follows. Suppose G is 4-connected and $G \notin \mathcal{C} \cup \mathcal{L}$. Then there exists a (G, C_6^2) -chain if G is planar and a (G, C_5^2) -chain if G is nonplanar.

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1. Introduction

The purpose of this paper is to improve a well-known classical chain theorem of Martinov [10] for 4-connected graphs. We begin by formally stating this result.

All graphs considered in this paper are simple. In particular, G/e denotes the graph obtained from G by contracting an edge e and then deleting parallel edges. For each integer $n \geq 5$, let C_n^2 be the graph obtained from an n -cycle C_n by joining vertices of distance two in the cycle. Notice that C_5^2 is K_5 and C_6^2 is the octahedron. In general, C_n^2 is 4-connected, and it is planar if and only if n is even. Let $\mathcal{C} = \{C_n^2 : n \geq 5\}$.

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A graph G with at least six vertices is called *cyclically k -edge-connected* if the deletion of fewer than k edges from G does not create two components which both contain at least one cycle. We remark that all cyclically 4-edge-connected cubic graphs can be constructed from either $K_{3,3}$ or the cube by repeatedly applying an operation known as “adding a handle” [3–6,12]. Let $\mathcal{L} = \{L : L \text{ be the line graph of an cyclically 4-edge-connected cubic graph}\}$. The following is the chain theorem of Martinov [10] (a similar result using both edge contractions and deletions was given by Fontet [3]).

Theorem 1.1. *For every 4-connected graph G there exists a sequence of 4-connected graphs $G_0, G_1, \dots, G_n$ such that $G_0 = G$, $G_n \in \mathcal{C} \cup \mathcal{L}$, and every G_i ($i < n$) has an edge e_i for which $G_i/e_i = G_{i+1}$.*

This result provides a very useful tool for analyzing 4-connected graphs. Under the current setting, it says that every 4-connected graph G can be reduced, within the class of 4-connected graphs, to a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeatedly contracting edges. If we reverse this process then the theorem tells us that every desired (usually unknown) 4-connected graph G can be constructed from a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeated “uncontractions”. This approach is used successfully in characterizing 4-connected graphs that do not contain a minor isomorphic to the cube [8], to the octahedron [7], to the octahedron plus an edge [9], or to the complement of P_7 [2]. However, this theorem has two major defects which limit its further applications.

First, for a general 4-connected graph G , the starting graph G_n in the construction sequence could be any graph in $\mathcal{C} \cup \mathcal{L}$. What this means is that, in order to obtain G we have to consider infinitely many possible choices for G_n , and this increases the complexity of our analysis. It would be nice if we can narrow down the choices for G_n . The second defect of the theorem, which causes an even bigger problem, is that G_n could be planar even if G is nonplanar. As a consequence, in order to construct G , we have to examine many planar graphs, which often are useless for constructing G . The following is the main result of this paper, which addresses both concerns. Let us call a sequence as described in Theorem 1.1 a (G, G_n) -chain.

Theorem 1.2. *Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. If G is planar then there exists a (G, C_6^2) -chain; if G is nonplanar then there exists a (G, K_5) -chain.*

As an application, we prove the following main result of [9]. Let Oct^+ denote the unique graph obtained from the octahedron by adding an edge.

Theorem 1.3. *If a 4-connected nonplanar graph G has no Oct^+ -minor then $G = C_{2n+1}^2$ for some $n \geq 2$.*

Proof. Suppose the result is false. By Theorem 1.2, either there exists a (G, K_5) -chain or $G = L(H)$ for a nonplanar cubic graph H . The second case is impossible since $L(H)$ contains $L(K_{3,3})$, which contains Oct^+ . The first case is impossible either because G has

to contain one of the three 4-connected uncontractions of K_5 , which are K_6 , $K_6 \setminus e$, Oct^+ , yet all of them contain Oct^+ . ■

We close this section by introducing a few definitions. For any graph G , let $V(G)$ and $E(G)$ denote the vertex set and edge set of G , respectively. If $X \subseteq V(G)$, let $N_G(X) = \{y \in V(G) - X : xy \in E(G) \text{ for some } x \in X\}$. Members of $N_G(X)$ are *neighbors* of X and the set $N_G(X)$ is the *neighborhood* of X . For any $x \in V(G)$, we will write $N_G(x)$ for $N_G(\{x\})$. As usual, $|N_G(x)|$ is the *degree* of x , which is denoted by $d_G(x)$. Let $E_G(x)$ stand for the set of edges of G that are incident with x . We will drop the subscript G if there is no need to emphasize G .

Let $G = (V, E)$ be a graph. A set $T \subseteq V$ is called a *k-separator* of G if $|T| = k$ and $G - T$ is disconnected. As usual, G is *k-connected* if $|V| > k$ and G has no k' -separator with $k' < k$. Let G be a *k-connected* graph. An edge e of G is said to be *k-contractible* if G/e is again *k-connected*. We may simply call e *contractible* if k is clear from the context. The new vertex of G/e will be denoted by $\bar{e}$. Observe that an edge xy of a non-complete graph G is not *k-contractible* if and only if G has a *k-separator* containing both x and y .

Let T be a *k-separator* of a *k-connected* graph G . A *T-fragment* of G is the vertex set of a union of at least one but not all components of $G - T$. We often leave out the prefix T when we do not need to emphasize it. If A is a fragment of G then it is clear that $N(A)$ is a *k-separator*. Let us define $\bar{A} = V(G) - A - N(A)$. Then $\bar{A}$ is also a fragment of G with $N(A) = N(\bar{A})$. Notice that, for any $x \in A$, x has no neighbors in $\bar{A}$.

The organization of this paper is as follows. In Section 2, we establish a few lemmas on contractible edges. Then, in Section 3, we prove our key lemma. Finally, we prove Theorem 1.2 in Section 4.

2. Contractible edges

In this section we present a few lemmas on contractible edges. We first establish that every 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$ can be reduced to K_5 or the octahedron. Our proof is divided into two steps.

Lemma 2.1. *Suppose a k -connected ($k \geq 2$) graph G has a contractible edge $e = xy$ such that $d_{G/e}(\bar{e}) = k$ and the neighborhood of $\bar{e}$ does not contain $K_{2,k-2}$ as a subgraph. Then G has an edge e' such that G/e' is isomorphic to a graph obtained from G/e by adding at least one extra edge.*

Proof. Let $N_{G/e}(\bar{e}) = \{z_1, z_2, \dots, z_k\} = Z$. Then $N_G(x) \subseteq Z \cup \{y\}$ and $N_G(y) \subseteq Z \cup \{x\}$. Since $d_G(x) \geq k$ and $d_G(y) \geq k$, we may assume, by adjusting the indices if necessary, that $N_G(x) \supseteq Z - \{z_1\}$ and $N_G(y) \supseteq Z - \{z_2\}$. By considering G/e and G/xz_2 as graphs obtained by adding edges to $G - x$, we observe that G/xz_2 is isomorphic to the graph obtained from G/e by adding edges $z_2z_3, z_2z_4, \dots, z_2z_k$ (and also possibly z_2z_1). If $e' = xz_2$ does not satisfy the lemma, then $z_2z_3, z_2z_4, \dots, z_2z_k$ are all edges of G/e . This

implies that $z_1 z_3, z_1 z_4, \dots, z_1 z_k$ are not all edges of G/e and thus $e' = yz_1$ satisfies the lemma. ■

Corollary 2.2. *Suppose e is an edge of a 4-connected graph G such that $G/e \in \mathcal{C} \cup \mathcal{L}$. Then, unless $G/e = C_5^2$ or C_6^2 , G has an edge e' such that G/e' is isomorphic to a graph obtained from G/e by adding at least one extra edge.*

Proof. If $G/e = L(H)$, where H is an cyclically 4-edge-connected cubic graph, then the neighborhood of $\bar{e}$ induces a matching since H is triangle free. Thus the result holds by Lemma 2.1. If $G/e = C_n^2$ for some $n \geq 7$, then the neighborhood of $\bar{e}$ induces a path and, again, the result holds by Lemma 2.1. ■

We also need the next three lemmas.

Lemma 2.3 ([1]). *If x is a vertex of a 4-connected graph G with $d(x) \geq 5$, then G has a contractible edge contained in $E(y)$ for some $y \in N(x)$.*

Lemma 2.4. *Let xy and xz be two edges of a k -connected graph G with $N(x) \subseteq N(y) \cup \{y, z\}$. If xy is k -contractible then so is xz .*

Proof. Suppose xz is non-contractible. Then G has a k -separator T containing both x and z . Notice that $y \notin T$ since xy is contractible. Let A be a T -fragment that contains y . Now, since $N(x) \subseteq N(y) \cup \{y, z\}$, we find that x has no neighbor in $\bar{A}$, contradicting the k -connectivity of G . ■

Lemma 2.5. *Let $e = xy$ be a k -contractible edge of a k -connected graph G . Let e' be an edge of G/e (and hence also an edge of G). If e' is k -contractible in G/e but not in G , then some $z \in \{x, y\}$ has degree k and is such that $N_G(z)$ contains both ends of e' .*

Proof. Let x', y' be the two ends of e' in G . Since e' is not contractible in G , G has a k -separator T that contains both x' and y' . Clearly, $\{x, y\} - T \neq \emptyset$ since e is contractible. Let A be a T -fragment with $A \cap \{x, y\} \neq \emptyset$. By symmetry, we may assume $x \in A$. If $y \in A$ then T is a k -separator of G/e with $T \supseteq \{x', y'\}$, contradicting the contractibility of e' . So we must have $y \in T$. If $|A| \geq 2$ then $T' = (T - \{y\}) \cup \{\bar{e}\}$ is a k -separator of G/e and T' contains both ends of e' in G/e . This is again a contradiction. It follows that $A = \{x\}$ and thus the Lemma holds with $z = x$. ■

3. A key lemma

Let x, y, z be three distinct vertices of a cycle C . Then C has two subpaths with ends x and y . We denote the vertex set of the path that contains z by $C[x, z, y]$, and we denote the vertex set of the other path by $C[x, \bar{z}, y]$. We also define $C(x, z, y) = C[x, z, y] - \{x, y\}$, $C(x, z, y) = C[x, z, y] - \{x\}$, and $C(x, z, y) = C[x, z, y] - \{y\}$. In addition, $C(x, \bar{z}, y)$,

$C(x, \bar{z}, y]$, and $C[x, \bar{z}, y)$ are defined analogously. The purpose of this section is to prove the following.

Lemma 3.1. *If a 4-connected nonplanar graph G does not belong to $\mathcal{C} \cup \mathcal{L}$, then G has an edge e such that G/e remains 4-connected and nonplanar.*

Proof. It is hard to separate our proof into independent lemmas, so this proof will last till the end of this section. To make the proof easier to follow, we divide it into a sequence of claims.

By Theorem 1.1, G has at least one contractible edge. We assume that, for every contractible edge e of G , G/e is planar because otherwise we are done. Let $e = xy$ be a contractible edge of G . It follows that G/e is planar. This implies $|V(G)| \geq 7$ because otherwise G/e would be a 4-connected planar graph on at most five vertices, which is impossible.

Let us consider the unique planar embedding of G/e . This embedding induces an embedding of $(G/e) - \bar{e}$. Notice that this embedding of $(G/e) - \bar{e}$ has a face F such that, in the planar embedding of G/e , all edges of $E_{G/e}(\bar{e})$ are embedded in F . Since G/e is 4-connected, $(G/e) - \bar{e}$ is 3-connected, which implies that F is bounded by a cycle C of $(G/e) - \bar{e}$ and this cycle contains all neighbors of $\bar{e}$. Let $B = G - (V(C) \cup \{x, y\})$. It is easy to see that, as a facial cycle of the 3-connected planar graph $G - x - y = (G/e) - \bar{e}$, C satisfies the following, which, using the language of Tutte [11], says that C is peripheral in $G - x - y$.

Claim 1. *C is an induced cycle of G , B is connected, and $N(V(B)) = V(C)$.*

Let $x_1, x_2, \dots, x_s$ be the neighbors of x (other than y), which are listed in the order they appear on C . Let $N(y) = \{x, y_1, y_2, \dots, y_t\}$. For the purpose of simplifying our notation, we do not require $y_1, y_2, \dots, y_t$ to be listed in a specific order. This setting creates a non-symmetry between x and y . As a result, in the following discussions, some of our statements are only made for one of x, y . We point out that these statements are still valid if we swap x and y , since x and y are indeed symmetric.

A quadruple (x_i, y_j, x_k, y_l) is said to be *crossing* if the four vertices are distinct and y_j, y_l are contained in different components of $C - \{x_i, x_k\}$.

Claim 2. *There exists a crossing quadruple.*

Proof. Suppose $\{x_1, x_2, \dots, x_s\} = \{y_1, y_2, \dots, y_t\}$. Since G/e is 4-connected, we must have $s \geq 4$ and thus Claim 2 follows. Next, we assume by symmetry that $y_1 \notin \{x_1, x_2, \dots, x_s\}$. Choose i such that $C(x_i, y_1, x_{i+1})$ contains no neighbors of x (in this section the indices on the letter x are always taken modulo s). Since G is nonplanar, $C(x_i, \bar{y}_1, x_{i+1})$ must contain a neighbor of y and thus Claim 2 is proved. $\square$

When we say “ y_j is contained in a crossing quadruple” we mean that there exists a crossing quadruple of the form (x_i, y_j, x_k, y_l) . We need to make this clear since in general y_j could be equal to some x_i .

Claim 3. *Every y_j is contained in a crossing quadruple, unless $y_j = x_r$ for some r and $N(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Moreover, there is at most one y_j that is not contained in any crossing quadruple.*

Proof. By Claim 2, G has crossing quadruple (x_m, y_a, x_n, y_b) . If $y_j \notin \{x_m, x_n\}$ then either (x_m, y_a, x_n, y_j) or (x_m, y_j, x_n, y_b) is crossing. So we may assume $y_j = x_m$. If $C(x_{m-1}, \bar{x}_m, x_{m+1})$ contains a neighbor y_l of y , then $(x_{m-1}, y_j, x_{m+1}, y_l)$ is a crossing quadruple. Else $r = m$ satisfies the lemma. Finally, if in addition to y_j , vertex $y_{j'}$ ($j' \neq j$) is not contained in any crossing quadruple either, then the first part of the lemma implies either $y_{j'} = x_{r-1}$ and $N(y) - \{x\} \subseteq C[x_{r-2}, x_{r-1}, x_r]$ or $y_{j'} = x_{r+1}$ and $N(y) - \{x\} \subseteq C[x_r, x_{r+1}, x_{r+2}]$, which in turns implies either $N(y) - \{x\} \subseteq C[x_{r-1}, \bar{x}_{r+1}, x_r]$ or $N(y) - \{x\} \subseteq C[x_r, \bar{x}_{r-1}, x_{r+1}]$, contradicting the non-planarity of G . $\square$

Claim 4. *If (x_i, y_j, x_k, y_l) is a crossing quadruple, then G/yy_j is nonplanar.*

Proof. It is clear that $G/yy_j - V(B)$ has a K_4 minor whose four branch sets each contain exactly one member of $\{x_i, \overline{yy_j}, x_k, y_l\}$. By Claim 1, B is connected and each vertex of C has a neighbor in B . Thus G/yy_j has a K_5 minor, which proves that G/yy_j is nonplanar. $\square$

Since we have assumed that G/e is planar if e is contractible, we deduce the following from Claim 4.

Claim 5. *If (x_i, y_j, x_k, y_l) is a crossing quadruple, then yy_j is not contractible.*

Claim 6. *Suppose T is a 4-separator of G that contains both y and some y_j . Then either $T = N(x)$ or $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$.*

Proof. It is clear that $x \notin T$ since xy is contractible in G . Let A be a T -fragment of G with $x \in A$. If $N(x) = T$ then we are done. So let x have a neighbor $x^* \in A$. Clearly, $x^* \in V(C)$ as $x^* \neq y$. Since $y \in T$, y must have a neighbor $y^* \in \bar{A}$. Observe that $y^* \in V(C)$ as $y^* \neq x$. Therefore, as T separates x^* from y^* , T must contain a vertex $z \in C(x^*, \bar{y}_j, y^*) \subseteq V(C) - \{y_j\}$. On the other hand, since $x^*, y^* \in V(C)$, we deduce from Claim 1 that G has a path P between x^*, y^* such that $V(P) - \{x^*, y^*\} \subseteq V(B)$. Again, since T separates x^* from y^* , T must contain a vertex $z' \in V(P) - \{x^*, y^*\} \subseteq V(B)$. $\square$

Claim 7. Suppose T is a 4-separator of G such that $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$. Then for any distinct x_i, x_k different from y_j , $C(x_i, y_j, x_k)$ contains at least two neighbors of y .

Proof. Let A be a T -fragment of G with $x \in A$. Let P_1, P_2 be the two subpaths of C with ends y_j and z . Since $T \cap V(C) = \{y_j, z\}$, for $i = 1, 2$, $V_i = V(P_i) - \{y_j, z\}$ is entirely contained in A or $\bar{A}$. Notice that x has a neighbor in A since $d(x) \geq 4$ and $xz' \notin E(G)$. It follows that $V(C) \cap A \neq \emptyset$. On the other hand, $V(C) \cap \bar{A} \neq \emptyset$ since y has a neighbor y_l in $\bar{A}$. Hence, we may assume $V_1 \subseteq A$ and $V_2 \subseteq \bar{A}$. Observe that $N(x) - \{y\} \subseteq V(P_1)$, it follows that $C(x_i, y_j, x_k)$ contains y_j and y_l for any distinct x_i, x_k different from y_j . Hence Claim 7 holds. $\square$

Claim 8. $|N(\{x, y\})| \geq 5$.

Proof. Suppose $|N(\{x, y\})| \leq 4$. Then $|N(\{x, y\})| = 4$ since G is 4-connected with $|V(G)| \geq 7$. Choose $z \in N(\{x, y\})$ such that, if possible, z is adjacent to only one of x, y . Without loss of generality, we assume z is adjacent to y and thus $z = y_j$ for some j . Note that $|N(y) - N(x) - \{x\}| \leq 1$ since $|N(\{x, y\})| = 4 \leq d(x)$. Then our choice of z implies $N(y) \subseteq N(x) \cup \{x, z\}$ and thus, by Lemma 2.4, yy_j is contractible. Consequently, by Claim 5, y_j is not contained in any crossing quadruple. By Claim 3, we must have $y_j \in N(x)$. Now the way we choose z implies $N(x) - \{y\} = N(y) - \{x\}$ and in this case y_j is clearly contained in a crossing quadruple. This contradiction proves Claim 8. $\square$

Claim 9. Both $d(x) \geq 5$ and $d(y) \geq 5$ hold.

Proof. By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l) . Suppose Claim 9 is false. Then we may assume by the symmetry between x and y that either $d(x) > d(y) = 4$ holds or $d(x) = d(y) = 4$ with $|\{y_j, y_l\} - N(x)| \geq |\{x_i, x_k\} - N(y)|$ holds. Since $d(y) = 4$ and thus $|N(y) \cap V(C)| = 3$, we may further assume that y_j is the only neighbor of y contained in $C(x_i, y_j, x_k)$.

By Claim 5, G has a 4-separator T containing both y and y_j . Note that $T \neq N(x)$ because otherwise $y_j \in N(x)$, implying $1 \geq |\{y_j, y_l\} - N(x)| \geq |\{x_i, x_k\} - N(y)|$, and thus $\{x_i, x_k\} \cap N(y) \neq \emptyset$. From these observation and $d(x) = |T| = 4 = d(y)$ we deduce that $N(\{x, y\}) = \{x_i, y_j, x_k, y_l\}$, which contradicts Claim 8. Therefore, by Claim 6 and Claim 7, y has at least two neighbors in $C(x_i, y_j, x_k)$, contradicting the choice of y_j , which proves Claim 9. $\square$

Claim 10. Every y_j is contained in a crossing quadruple.

Proof. Suppose there exists y_j that is not contained in any crossing quadruple. By Claim 3, there exists r such that $y_j = x_r$ and $N(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Note that $x_{r+2} \notin C[x_{r-1}, x_r, x_{r+1}]$ since $d(x) \geq 5$, as shown in Claim 9. Choose y_m, y_n such that $N(y) - \{x\} \subseteq C[y_m, \bar{x}_{r+2}, y_n]$. Since G is nonplanar, each of $C[x_{r-1}, \bar{x}_{r+2}, x_r)$

and $C(x_r, \bar{x}_{r+2}, x_{r+1})$ contains one of y_m and y_n . As a result, (y_m, x_r, y_n, x_{r+2}) is a crossing quadruple. By Claim 5, xx_r is not contractible. It follows that there is a 4-separator T containing both x and x_r . Since $d(y) \geq 5$ (by Claim 9), Claim 6 implies that $T = \{x, x_r, z, z'\}$ for some $z \in V(C) - \{x_r\}$ and $z' \in V(B)$. Notice that x_r is the only neighbor of x in $C(y_m, x_r, y_n)$. This contradicts Claim 7 and thus Claim 10 is proved. $\square$

Claim 11. *No edge of C is contractible.*

Proof. Suppose to the contrary that $f \in E(C)$ is a contractible edge of G . By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l) . Let H be the subgraph of G formed by edges in $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$. Then H is a subdivision of $K_{3,3}$. Since we assumed contracting any contractible edge leaves a planar graph, G/f is planar. It follows that H/f is no longer a subdivision of $K_{3,3}$. By symmetry, we assume $f = x_i y_j$.

If $C(x_i, \bar{y}_j, y_l)$ contains a neighbor x_m of x , then $(H + xx_m)/f$ would still contain a subdivision of $K_{3,3}$, which is impossible. Hence $N(x) - \{y\} \subseteq C[x_i, y_j, y_l]$. This implies that $C(y_j, x_i, y_l)$ contains exactly one neighbor of x . However, by Claim 5, G has a 4-separator T that contains $\{x, x_i\}$ since (y_l, x_i, y_j, x_k) is crossing. By Claim 6 and Claim 9, $T = \{x, x_i, z, z'\}$ for some $z \in V(C) - \{x_i\}$ and $z' \in V(B)$. Now, by Claim 7, $C(y_j, x_i, y_l)$ contains at least two neighbors of x . This contradiction proves Claim 11. $\square$

Now we are ready to complete the proof of Lemma 3.1. We apply Lemma 2.3 to $G' = G/xy$. By Claim 8, $d_{G'}(\overline{xy}) \geq 5$. Thus $E_{G'}(v)$ contains a contractible edge e' of G' for some $v \in N_{G'}(\overline{xy})$. By Lemma 2.5 and Claim 9, e' is contractible in G . However, by Claim 10 and Claim 5, no edge of $E(\{x, y\})$ is contractible in G ; by Claim 11, no edge of C is contractible in G ; and by Claim 1, cycle C has no chords. Hence, e' is between C and B . What this means is, for any crossing quadruple (x_i, y_j, x_k, y_l) , the graph formed by $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$ remains a subdivision of $K_{3,3}$ when e' is contracted from G . Thus G/e' is 4-connected and nonplanar. The lemma is proved. $\blacksquare$

4. A proof of the main theorem

In this section we prove Theorem 1.2. Recall that a (G, H) -chain is a sequence $G_0, G_1, \dots, G_n$ of 4-connected graphs such that $G_0 = G$, $G_n = H$, and every G_i ($i < n$) has an edge e_i such that $G_i/e_i = G_{i+1}$.

Proof of Theorem 1.2. Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. By Theorem 1.1, there exists a (G, G_n) -chain $G_0, G_1, \dots, G_n$ such that $G_n \in \mathcal{C} \cup \mathcal{L}$. We choose such a chain as follows:

- (1) if G is planar, we choose the chain with as many terms as possible;
- (2) if G is not planar, we choose the chain with as many nonplanar terms as possible.

If G is planar, we need to show that $G_n = C_6^2$. Suppose otherwise. By applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain an edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is 4-connected but G'_n does not belong to $\mathcal{C} \cup \mathcal{L}$, as it cannot be 4-regular. Now, by Theorem 1.1 again, there exists a (G'_n, G'_m) -chain $G'_n, G'_{n+1}, \dots, G'_m$ with $G'_m \in \mathcal{C} \cup \mathcal{L}$. It follows that $G_0, G_1, \dots, G_{n-1}, G'_n, G'_{n+1}, \dots, G'_m$ is a chain contradicting the choice of (1), which proves the first part of the theorem.

If G is nonplanar, let $G_0, G_1, \dots, G_k$ be all the nonplanar terms. We need to show that $k = n$ and $G_n = K_5$. If $k < n$ then $G_k \notin \mathcal{C} \cup \mathcal{L}$ since no graph in $\mathcal{C} \cup \mathcal{L}$ has a contractible edge while G_k has a contractible edge e_k . By Lemma 3.1, G_k has a contractible edge e'_k such that $G'_{k+1} = G_k/e'_k$ is nonplanar. Like in the planar case, we can extend $G_0, G_1, \dots, G_k, G'_{k+1}$ to a chain that contradicts the choice of (2), which proves that $k = n$. If $G_n \neq K_5$, by applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain a contractible edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is nonplanar and not in $\mathcal{C} \cup \mathcal{L}$. Consequently, by Lemma 3.1, G'_n has an edge e'_n such that $G'_{n+1} = G'_n/e'_n$ is 4-connected and nonplanar. Now, once again, $G_0, G_1, \dots, G_{n-1}, G'_n, G'_{n+1}$ can be extended into a chain. This contradicts the choice of (2), which completes our proof of the theorem. ■

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