Communications on Stochastic Analysis

Volume 8 | Number 4 Article 8

12-1-2014

Higher powers of quantum white noise derivatives

Aymen Ettaieb

Habib Ouerdiane

Hafedh Rguigui

Communications on Stochastic Analysis
Vol. 8, No. 4 (2014) 531-550

Serials Publications
www.serialspublications.com

HIGHER POWERS OF QUANTUM WHITE NOISE DERIVATIVES

AYMEN ETTAIEB, HABIB OUERDIANE, AND HAFEDH RGUIGUI

Abstract. By a Wick differential equation, we characterize the operator $W_{l,m}(f)$ studied in [1, 3, 9] where $l,m \in \mathbb{N} \cup \{0\}$ and $f \in \mathcal{S}(\mathbb{R})$. As an application we give in our setting a new renormalization in order to get the higher powers of white noise. Then, we investigate the commutation relations obtained from the quantum white noise (QWN) derivatives in order to introduce two operators acting on white noise operators, from which we get the higher powers of quantum white noise derivatives and a $*$-Lie algebra generalizing the renormalized higher power white noise Lie algebra.

1. Introduction

In recent years operator theory over white noise functions has been considerably studied. Motivated by the attempts of developing a satisfactory theory of quantization of gravity and by the attempts to developer nonlinear generalization of stochastic and white noise analysis, the renormalized higher powers of quantum white noise (RHPWN) $*$-lie algebra has been investigated in quantum probability.

The white noise functionals a_t (annihilation density) and a_t^* (creation density) satisfy the Boson commutation relation:

$$[a_t^*, a_s^*] = [a_t, a_s] = 0; \quad [a_t, a_s^*] = \delta(t-s),$$

where $t, s \in \mathbb{R}$, δ is the Dirac delta function, $[x,y] := xy - yx$ is usual operator commutator. Giving meaning to the higher powers of creation and annihilation densities, i.e, to the formal expression a_t^n, a_s^k, where $n,k \in \mathbb{N} \cup \{0\}$, is an old and important problem in quantum field theory. In their work, Accardi, Bockas and Franz (see [1], [3] and [5]) studied the higher powers of quantum white noises:

$$W_{n,k}(f) = \int_{\mathbb{R}} f(t)(a_t^*)^n a_t^k dt. \tag{1.1}$$

It was shown that, using the renormalization

$$\delta^l(t-s) = \delta(s)\delta(t-s), \quad l = 2, 3, ... \tag{1.2}$$

Received 2014-2-10; Communicated by the editors. Article is based on a lecture presented at the International Conference on Stochastic Analysis and Applications, Hammamet, Tunisia, October 14-19, 2013.
2010 *Mathematics Subject Classification.* Primary 60H40; Secondary 46A32, 46F25, 46G20.
Key words and phrases. QWN-derivatives, higher powers of the QWN-derivatives, higher powers white noise Lie algebra.

and choosing test functions that vanish at zero, the renormalized higher power white noise (RHPWN) $*$-Lie algebra is effectively defined by the following commutation relations

$$[W_{n,k}(g), W_{N,K}(f)]_{RHPWN} := (kN - Kn)W_{n+N-1,k+K-1}(gf). \tag{1.3}$$

In [9], Chung, Ji and Obata used a new renormalisation to give the renormalized product obtained from the powers of creation and annihilation operators to overcome the difficulties caused by higher powers of delta functions arising from successive application of the canonical commutation relations.

The purpose of this paper is to investigate the commutation relations obtained from the quantum white noise derivatives D_ζ^+, D_ζ^- (see [14]) and their adjoint $(D_\zeta^+)^*, (D_\zeta^-)^*$ (see [6]) in order to give a commutation relation generalizing (1.3). In fact, instead of (1.1) we consider the operator

$$B_{k_1,k_2}^{n_1,n_2}(f) = \int_{\mathbb{R}} f(t)(D_t^{+*})^{n_1}(D_t^{-*})^{n_2}(D_t^+)^{k_1}(D_t^-)^{k_2}dt, \quad f \in \mathcal{S}(\mathbb{R}),$$

and its approximation by an operator $D_{n,k}^{l,m}(f_\epsilon)$ with a more regular kernel.

For two operators $B_{n,k}^{l,m}(f)$ and $B_{n_1,k_1}^{l_1,m_1}(g)$ we take approximations $D_{n,k}^{l,m}(f_\epsilon)$ and $D_{n_1,k_1}^{l_1,m_1}(g_\epsilon)$, respectively. We prove that the composition $D_{n,k}^{l,m}(f_\epsilon)D_{n_1,k_1}^{l_1,m_1}(g_\epsilon)$ is well defined for $f_\epsilon \in N^{\otimes l+m} \otimes N'^{\otimes n+k}$ and $g_\epsilon \in N^{\otimes l_1+m_1} \otimes N'^{\otimes n_1+k_1}$, while $B_{n,k}^{l,m}(f)B_{n_1,k_1}^{l_1,m_1}(g)$ is not in general. We are then lead to a renormalization similar to those in [9]. As a result, we get

$$\begin{aligned}\left[B_{k_1,k_2}^{n_1,n_2}(f)B_{k_1',k_2'}^{n_1',n_2'}(g)\right]_{ren} \quad = \quad & B_{k_1,k_2}^{n_1,n_2}(f) \diamond B_{k_1',k_2'}^{n_1',n_2'}(g) \\ & + k_1 n_1' B_{k_1+k_1'-1,k_2+k_2'}^{n_1+n_1'-1,n_2+n_2'}(fg) \\ & + k_2 n_2' B_{k_1+k_1',k_2+k_2'-1}^{n_1+n_1',n_2+n_2'-1}(fg) \\ & + k_1 k_2 n_1' n_2' B_{k_1+k_1'-1,k_2+k_2'-1}^{n_1+n_1'-1,n_2+n_2'-1}(fg).\end{aligned}$$

This paper is organized as follows: in section 2, we assemble some basic notations in quantum white noise calculus and we recall the classical case in the space of all entire functions with $\theta-$exponential growth of finite type, where θ is a young function. In section 3, we develop in our setting the renormalization introduced in [9] and we characterize the operator $W_{n,k}(f)$ by a Wick differential equation. In section 4, we give two new operators which generalizes those introduced by Accardi, Boukas and Franz using a renormalization similar to those introduced in [9] to obtain the higher powers of the quantum white noise derivatives.

2. Preliminaries

Let $E = \mathcal{S}(\mathbb{R})$ be the Schwartz space consisting of rapidly decreasing C^∞-functions and $E' = \mathcal{S}'(\mathbb{R})$ the space of tempered distributions. We denote by N the complexification of E, i.e., $N = E + iE$. We start with the following real Gel'fand triple:

$$E \subset L^2(\mathbb{R}, dt) \subset E'. \tag{2.1}$$

The Gel'fand triple (2.1) can be reconstructed in a standard way (see Ref. [18]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and $L^2(\mathbb{R}, dt)$. The eigenvalues of A are $2n+2$, $n = 0, 1, 2, \cdots$, the corresponding eigenfunctions $\{e_n;\ n \geq 0\}$ form an orthonormal basis for $L^2(\mathbb{R}, dt)$ and each e_n is an element of E. In fact, E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0 \,, \qquad \xi \in E, \quad p \in \mathbb{R}$$

and we have

$$E = \operatorname{proj}\lim_{p\to\infty} E_p \,, \qquad E' = \operatorname{ind}\lim_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. The inequality

$$|\xi|_p \leq \rho^q |\xi|_{p+q}, \quad \xi \in E, \quad p \in \mathbb{R}, \quad q \geq 0,$$

holds with $\rho = 1/2$.

Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, convex, increasing function satisfying $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$ and $\theta(0) = 0$. Such a function is called a Young function. The polar function associated to θ denoted by $\theta^*(x)$ given by

$$\theta^*(x) = \sup_{t\geq 0}\{tx - \theta(t)\}$$

is again a Young function and $(\theta^*)^* = \theta$.

For a complex Banach space $(B, \|\cdot\|)$, let $\mathcal{H}(B)$ denotes the space of all entire functions on B. For each $m > 0$ we denote by $\mathrm{Exp}(B, \theta, m)$ the space of all entire functions on B with $\theta-$exponential growth of finite type m, i.e.,

$$\mathrm{Exp}(B, \theta, m) = \Big\{ f \in \mathcal{H}(B); \quad \|f\|_{\theta,m} := \sup_{z\in B} |f(z)| e^{-\theta(m\|z\|)} < \infty \Big\}.$$

The projective system $\{\mathrm{Exp}(N_{-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ give the space

$$\mathcal{F}_\theta(N') = \operatorname{proj}\lim_{p\to\infty; m\downarrow 0} \mathrm{Exp}(N_{-p}, \theta, m) \,. \tag{2.2}$$

On the other hand, $\{\mathrm{Exp}(N_p, \theta, m);\ p \in \mathbb{N}\ , m > 0\}$ becomes an inductive system of Banach spaces and we put

$$\mathcal{G}_\theta(N) = \operatorname{ind}\lim_{p\to\infty; m\downarrow\infty} \mathrm{Exp}(N_p, \theta, m) \,. \tag{2.3}$$

It is known that every $\phi \in \mathcal{F}_\theta(N')$ admits a Taylor expansion of the form:

$$\phi(x) = \sum_{n=0}^{\infty} < x^{\otimes n}, \phi_n >, \quad x \in N', \phi_n \in N^{\widehat{\otimes} n}. \tag{2.4}$$

Let $F_\theta(N')$ be the space of all Taylor coefficients ϕ_n obtained from (2.4). It is known that

$$F_\theta(N) = \operatorname{proj}\lim_{p\to\infty; m\downarrow 0} F_{\theta,m}(N_p),$$

where

$$F_{\theta,m}(N_p) = \Big\{ \phi = (\phi_n);\ \phi_n \in N_p^{\widehat{\otimes} n},\ \|\overrightarrow{\phi}\|_{\theta,p,m} = \sum_{n=0}^{\infty} \theta_n^{-2} m^{-n} |\phi_n|_p^2 < \infty \Big\}$$

and

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}, \quad n = 0, 1, 2, ...$$

Moreover, equipped with the projective limit topology, $F_\theta(N)$ is a nuclear Frchet space and is isomorphic to $\mathcal{F}_\theta(N')$. Let

$$G_\theta(N') = \operatorname*{ind\,lim}_{p\to\infty; m\to\infty} G_{\theta,m}(N_{-p}),$$

where

$$G_{\theta,m}(N_{-p}) = \Big\{\Phi = (F_n);\, F_n \in N_{-p}^{\widehat{\otimes} n},\, \sum_{n=0}^{\infty}(n!\theta_n)^2 m^n |F_n|_{-p}^2 < \infty\Big\}.$$

By definition, $F_\theta(N)$ and $G_\theta(N')$ are dual to each other, for more details see ([8], [12] and [20]). Let Γ and Υ be locally convex spaces. We denote by $\mathcal{L}(\Gamma, \Upsilon)$ the space of continuous linear operators from Γ into Υ. It is a fundamental fact in quantum white noise theory ([8] and [18]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{2.5}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $(N^{\otimes(l+m)})'_{sym(l,m)}$ denote the subspace of $(N^{\otimes(l+m)})'$ consisting of symmetric elements. The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ is a $\mathbb{C}$-valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi, \eta) = \ll \Xi e_\xi, e_\eta \gg e^{-<\xi,\eta>}, \quad \xi, \eta \in N, \tag{2.6}$$

where the exponential function is by definition $e_\xi(z) := e^{<z,\xi>}, \quad z \in N'$. In fact, the integral kernel operator $\Xi_{l,m}(\kappa_{l,m})$ is characterized via the Wick symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = < \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} >, \quad \xi, \eta \in N. \tag{2.7}$$

Let $\mathcal{H}_\theta(N \oplus N)$ be the space of all holomorphic functions g given by $g(x, y) = \sum_{l,m=0}^{\infty} < x^{\otimes l} \otimes y^{\otimes m}, g_{l,m} >$ such that

$$\|\overrightarrow{g}\|^2_{\theta,p,(\gamma_1,\gamma_2)} := \sum_{l,m=0}^{\infty} \theta_l^{-2}\theta_m^{-2}\gamma_1^{-l}\gamma_2^{-m}|g_{l,m}|_p^2 < \infty, \quad \forall p \geq 0, \quad \gamma_1, \gamma_2 > 0,$$

see for more details ([6] and [16]). Then from the topological isomorphism between the two spaces $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ and $\mathcal{H}_\theta(N \oplus N)$ via the symbol map (see [6]), we can define a family of seminorms of operators $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$ in $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ by

$$\|\Xi\|^2_{\theta,p,(\gamma_1,\gamma_2)} = \sum_{l,m=0}^{\infty} \theta_l^{-2}\theta_m^{-2}\gamma_1^{-l}\gamma_2^{-m}|\kappa_{l,m}|_p^2,$$

for all $p \geq 0$ and $\gamma_1, \gamma_2 > 0$.

Theorem 2.1. *(see Ref.* [16]*) The Wick symbol map yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ and $\mathcal{G}_{\theta^*}(N \oplus N)$, where $\mathcal{G}_{\theta^*}(N \oplus N)$ denotes the nuclear space obtained as in (2.3).*

The operator $\Xi_{l,m}(\kappa_{l,m})$ formally expressed as

$$\begin{aligned}\Xi_{l,m}(\kappa_{l,m}) &= \int_{\mathbb{R}^{l+m}} \kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m) \\ &\quad \times a^*_{s_1}\cdots a^*_{s_l} a_{t_1}\cdots a_{t_m} ds_1\cdots ds_l dt_1\cdots dt_m,\end{aligned}$$

where a_t and a^*_t are, respectively, the annihilation and creation operators. In this way $\Xi_{l,m}(\kappa_{l,m})$ can be considered as the operator polynomials of degree $l+m$ associated to the distribution $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ as coefficient and therefore every white noise operator is a function of the quantum white noise. This gives a natural idea for defining the derivatives of an operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$ with respect to the quantum white noise coordinate system $\{a_t,\, a^*_t;\ t \in \mathbb{R}\}$. From Refs. [13] and [14], we summarize the novel formalism of quantum white noise derivatives. For $\zeta \in E$, ∂_ζ denotes the holomorphic derivative in the direction ζ and ∂^*_ζ is its adjoint operator. Both ∂_ζ and ∂^*_ζ belong to $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')) \cap \mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}^*_\theta(N'))$. Thus, for any white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$, the commutators

$$[\partial_\zeta, \Xi] = \partial_\zeta \Xi - \Xi\, \partial_\zeta, \quad [\partial^*_\zeta, \Xi] = \partial^*_\zeta \Xi - \Xi\, \partial^*_\zeta,$$

are well defined white noise operators in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$. The *quantum white noise derivatives* are defined by

$$D^+_\zeta \Xi = [\partial_\zeta, \Xi]\,, \quad D^-_\zeta \Xi = -[\partial^*_\zeta, \Xi]. \tag{2.8}$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. The adjoint operators of D^+_ζ and D^-_ζ denoted by $(D^+_\zeta)^*$ and $(D^-_\zeta)^*$ are introduced in [6]. For $\xi, \eta, \zeta \in N$, we define the partial derivatives $\partial_{1,\zeta}$ and $\partial_{2,\zeta}$ in the direction ζ as follows

$$\begin{aligned}(\partial_{1,\zeta} f)(\xi,\eta) &= \lim_{\epsilon\to 0} \frac{f(\xi+\epsilon\zeta,\eta) - f(\xi,\eta)}{\epsilon}, \\ (\partial_{2,\zeta} f)(\xi,\eta) &= \lim_{\epsilon\to 0} \frac{f(\xi,\eta+\epsilon\zeta) - f(\xi,\eta)}{\epsilon},\end{aligned}$$

where $f \in \mathcal{G}_{\theta^*}(N \oplus N)$. The adjoint of $\partial_{1,\zeta}$ and $\partial_{2,\zeta}$ denoted by $\partial^*_{1,\zeta}$ and $\partial^*_{2,\zeta}$ are defined by

$$\ll \partial^*_{1,\zeta} f, g \gg = \ll f, \partial_{1,\zeta} g \gg, \quad \ll \partial^*_{2,\zeta} f, g \gg = \ll f, \partial_{2,\zeta} g \gg,$$

where $f \in \mathcal{G}_{\theta^*}(N \oplus N)$ and $g \in \mathcal{H}_\theta(N \oplus N)$.

Proposition 2.2. [6] *For $\zeta \in N$, the creation and the annihilation derivatives of $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$ are given by*

$$D^-_\zeta \Xi = \sigma^{-1} \partial_{1,\zeta} \sigma(\Xi) \quad \text{and} \quad D^+_\zeta \Xi = \sigma^{-1} \partial_{2,\zeta} \sigma(\Xi). \tag{2.9}$$

Moreover, their dual adjoint are given by

$$(D^-_\zeta)^* \Xi = \sigma^{-1} \partial^*_{1,\zeta} \sigma(\Xi) \quad \text{and} \quad (D^+_\zeta)^* \Xi = \sigma^{-1} \partial^*_{2,\zeta} \sigma(\Xi). \tag{2.10}$$

For $x, y, u, v \in N$, we have the following equalities

$$D^+_x \Xi_{l,m}(\kappa_{l,m}) = l \Xi_{l-1,m}(x \widehat{\otimes}^1 \kappa_{l,m}) \tag{2.11}$$

$$D^-_y \Xi_{l,m}(\kappa_{l,m}) = m \Xi_{l,m-1}(\kappa_{l,m} \widehat{\otimes}_1 y) \tag{2.12}$$

$$(D_u^+)^*\Xi_{l,m}(\kappa_{l,m}) = \Xi_{l+1,m}(u\widehat{\otimes}\kappa_{l,m}) \tag{2.13}$$

$$(D_v^-)^*\Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m+1}(\kappa_{l,m}\widehat{\otimes}v), \tag{2.14}$$

where, for $z_p \in (N^{\otimes p})'$, and $\xi_{l-p+m} \in N^{\otimes l-p+m}$, $p \leq l+m$, the contractions $z_p \otimes_p \kappa_{l,m}$ and $z_p \otimes^p \kappa_{l,m}$ are defined by

$$\begin{aligned} < z_p \otimes^p \kappa_{l,m}, \xi_{l-p+m} > &= < \kappa_{l,m}, z_p \otimes \xi_{l-p+m} > \\ < z_p \otimes_p \kappa_{l,m}, \xi_{l-p+m} > &= < \kappa_{l,m}, \xi_{l-p+m} \otimes z_p > . \end{aligned}$$

For $z \in N$, the QWN-derivatives $D_z^{\pm}$ and their adjoins $(D_z^{\pm})^*$ are respectively a continuous linear operators from $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ into itself and from $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ into itself, (see [6] and [22]).

3. Higher Powers of White Noise Derivatives

Let $\varphi \in \mathcal{F}_\theta(N')$ given by

$$\varphi(z) = \sum_{n=0}^{\infty} < z^{\otimes n}, \varphi_n >, \quad z \in N', \quad \varphi_n \in N^{\widehat{\otimes} n}. \tag{3.1}$$

For $x \in N'$, the holomorphic derivative of φ at $z \in N'$ in the direction x is defined by

$$\partial_x \varphi(z) = \lim_{t \to 0} \frac{\varphi(z+tx) - \varphi(z)}{t}.$$

Therefore, we get

$$\partial_x \varphi(z) = \sum_{n=1}^{\infty} n < z^{\otimes(n-1)}, x\widehat{\otimes}_1 \varphi_n > .$$

Recall that $\delta_t \in N'$ is the Dirac function at t. Then $\partial_{\delta_t} := a_t$ is called Hida's differential operator. The adjoint operator ∂_y^* of ∂_y is defined by duality, i.e.,

$$\ll \partial_y^* \Phi, \varphi \gg = \ll \Phi, \partial_y \varphi \gg, \quad \Phi \in \mathcal{F}_\theta^*(N'),$$

from which we get $\partial_y^* \varphi(z) = \sum_{n=0}^{\infty} < z^{\otimes(n+1)}, x\widehat{\otimes} f_n >$. Then, for $x, y \in N$ we have

$$\left[\partial_x, \partial_y^*\right] = < x, y > I, \tag{3.2}$$

where I is the identity operator. For a later use, let $\{e_i\}_{i \geq 0}$ be the complete orthonormal basis of $L^2(\mathbb{R}, dt)$ and put

$$e_{\overrightarrow{i}} = e_{i_1} \otimes ... \otimes e_{i_l}, \quad e_{\overrightarrow{j}} = e_{j_1} \otimes ... \otimes e_{j_m}, \quad \overrightarrow{i} = (i_1, ..., i_l), \quad \overrightarrow{j} = (j_1, ..., j_m)$$

Lemma 3.1. *For $l, m \in \mathbb{N} \cup \{0\}$ and $\kappa \in (N^{\otimes(l+m)})'$, the integral kernel operator $\Xi_{l,m}(\kappa)$ admits the following expression:*

$$\Xi_{l,m}(\kappa) = \sum_{\overrightarrow{i}, \overrightarrow{j}=0}^{\infty} < \kappa, e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} > \partial_{e_{i_1}}^* ... \partial_{e_{i_l}}^* \partial_{e_{j_1}} ... \partial_{e_{j_m}}. \tag{3.3}$$

Proof. We denote by $T_{l,m}(\kappa)$ the righthand side of (3.3). Then we have

$$\begin{aligned}\sigma(T_{l,m}(\kappa))(\xi,\eta) &= \sum_{\vec{i},\vec{j}=0}^{\infty} <\kappa, e_{\vec{i}}\otimes e_{\vec{j}}><\xi^{\otimes m}, \vec{e_i}><\eta^{\otimes l}, \vec{e_j}> \\ &= <\kappa, \eta^{\otimes l}\otimes\xi^{\otimes m}>, \quad \xi,\eta\in N\end{aligned}$$

which is equal to (2.7). □

Let $l, m \geq 0$ and $g \in (N^{\otimes(l+m)})'$. We put

$$|g|_{l,m;p,q} = \left(\sum_{\vec{i},\vec{j}=0}^{\infty} |<g, e_{\vec{i}}\otimes e_{\vec{j}}>|^2 |e_{\vec{i}}|_p^2 |e_{\vec{j}}|_q^2\right)^{1/2}, \quad p,q\in\mathbb{R}\cdot$$

This is always finite for $g \in N^{\otimes(l+m)}$. Obviously $|g|_{l,m;p,p} = |g|_p$, see [18].

Proposition 3.2. *Let $l, m \geq 0$ and $\kappa \in N^{\otimes l}\otimes(N^{\otimes m})'$. Then, $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$. Moreover, for any $p \geq 0$ and $q > 0$, there exist a constant $c(l, m, \gamma) > 0$ such that*

$$\|\overrightarrow{\Xi_{l,m}(\kappa)\varphi}\|_{\theta,p,\gamma} \leq c(l,m,\gamma)|\kappa|_{l,m;p,-(p+q)}\|\overrightarrow{\varphi}\|_{\theta,p+q,\frac{\gamma}{16\rho^q}}, \quad \forall\gamma>0. \tag{3.4}$$

Proof. Let $\varphi \in \mathcal{F}_\theta(N')$ represented by $(\varphi_n)_{n\geq 0}$. Then, we obtain

$$\begin{aligned}\Xi_{l,m}(\kappa)\varphi(x) &= \sum_{n=0}^{\infty}\sum_{\vec{i},\vec{j}=0}^{\infty} <\kappa, e_{\vec{i}}\otimes e_{\vec{j}}> \frac{(n+m)!}{n!} \\ &\quad \times <x^{\otimes(n+l)}, \widehat{e_{\vec{i}}}\widehat{\otimes}\widehat{e_{\vec{j}}}\widehat{\otimes}_m\varphi_{n+m}>\cdot\end{aligned}$$

Therefore, for $p \geq 0$ and $\gamma > 0$, we have

$$\begin{aligned}\|\overrightarrow{\Xi_{l,m}(\kappa)\varphi}\|_{\theta,p,\gamma}^2 &= \sum_{n=0}^{\infty}\sum_{\vec{i},\vec{j}=0}^{\infty} |<\kappa, e_{\vec{i}}\otimes e_{\vec{j}}>|^2\left(\frac{(n+m)!}{n!}\right)^2\theta_{n+l}^{-2} \\ &\quad \times\gamma^{-l-n}|e_{\vec{i}}|_p^2|\widehat{e_{\vec{j}}}\widehat{\otimes}_m\varphi_{n+m}|_p^2.\end{aligned}$$

Using inequalities

$$\left(\frac{m!}{(m-n)!}\right)^2 \leq 4^m(n!)^2, \quad \theta_{m-n}^{-2} \leq 4^m\theta_m^{-2}\theta_n^2, \quad m\geq n, \tag{3.5}$$

we obtain

$$\begin{aligned}\|\overrightarrow{\Xi_{l,m}(\kappa)\varphi}\|_{\theta,p,\gamma}^2 &\leq \sum_{n=0}^{\infty}\sum_{\vec{i},\vec{j}=0}^{\infty} |<\kappa, e_{\vec{i}}\otimes e_{\vec{j}}>|^2 4^{n+m}4^{n+l}(m!)^2 \\ &\quad \times\theta_n^{-2}\theta_l^{-2}\gamma^{-n}\gamma^{-l}|e_{\vec{i}}|_p^2|\widehat{e_{\vec{j}}}\widehat{\otimes}_m\varphi_{n+m}|_p^2.\end{aligned}$$

From the fact that, see [18],

$$|\widehat{e_{\vec{j}}}\widehat{\otimes}_m\varphi_{n+m}|_p \leq \rho^{qn}|e_{\vec{j}}|_{-(p+q)}|\varphi_{n+m}|_{p+q}, \quad q\geq 0,$$

we get

$$\|\overrightarrow{\Xi_{l,m}(\kappa)\varphi}\|_{\theta,p,\gamma} \leq c(l,m,\gamma)|\kappa|_{l,m;p,-(p+q)}\|\overrightarrow{\varphi}\|_{\theta,p+q,\frac{\gamma}{16\rho^q}},$$

where $c(l,m,\gamma)=2^{m+l}\theta_l^{-1}\gamma^{-l}m!$. This completes the proof. □

In [9], Chung, Ji and Obata studied an operator which admits the following formal expression:

$$W_{l,m}(f)=\Xi_{l,m}(\tau_{l+m}(f))=\int_{\mathbb{R}} f(t)(a_t^*)^l a_t^m dt, \tag{3.6}$$

where $f\in N'$ and the operator $\tau_k(f): N'\to (N^{\otimes k})'$ is defined by

$$<\tau_k(f),\xi_1\otimes...\otimes\xi_k>=<f,\xi_1...\xi_k>, \qquad \xi_1,...,\xi_k\in N, \quad k\in\mathbb{N},$$

where $\xi_1...\xi_k$ is a pointwise product. The operator $\tau_k(f)$ is called the distribution concentrated on the diagonal induced from f. It is noticeable that $W_{l,m}(f)\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$, for any $f\in N'$. Then there is no meaning of composition $W_{l,m}(f)W_{l',m'}(g)$. To overcome this problem, Chung, Ji and Obata approximate τ_{l+m} in equation (3.6) by sufficiently regular functions $(f_\epsilon)_{\epsilon>0}\subset N^{\otimes(l+m)}$ and defined the renormalized product by eliminating a certain divergence terms.

Recall that (see [16] and [14]) for $\Xi_1,\Xi_2\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$ there exists a unique operator $\Xi\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$ such that

$$\sigma(\Xi)(\xi,\eta)=\sigma(\Xi_1)(\xi,\eta)\sigma(\Xi_2)(\xi,\eta), \quad \xi,\eta\in N.$$

The operator Ξ is called the Wick product of white noise operators Ξ_1 and Ξ_2 and is denoted by

$$\Xi=\Xi_1\diamond\Xi_2.$$

It is noteworthy that, equipped with the Wick product $\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$ becomes a commutative $*$-algebra. A continuous linear map

$$\mathcal{D}:\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))\to\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$$

is called a Wick derivation if

$$\mathcal{D}(\Xi_1\diamond\Xi_2)=\mathcal{D}(\Xi_1)\diamond\Xi_2+\Xi_1\diamond\mathcal{D}(\Xi_2), \quad \Xi_1,\Xi_2\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))\cdot$$

For more detail see [15]. Recall that from [22], for $B_1,B_2\in\mathcal{L}(N',N')$, the QWN-conservation operator admits the following integral representation

$$N^Q_{B_1,B_2}=\int_{\mathbb{R}^2}\tau_{B_1}(s,t)a_s^*\diamond D_t^+dsdt+\int_{\mathbb{R}^2}\tau_{B_2}(s,t)a_s\diamond D_t^-dsdt$$

on $\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$. In particular, if we take $B_1=B_2=I$, we get

$$N^Q=\int_{\mathbb{R}}a_t^*\diamond D_t^+dt+\int_{\mathbb{R}}a_t\diamond D_t^-dt\cdot \tag{3.7}$$

It is shown that N^Q is a Wick derivation.

Theorem 3.3. *For $f\in N$, the operator $W_{l,m}(f)$ is given by*

$$W_{l,m}(f)=(\frac{\partial}{\partial t_1})^l(\frac{\partial}{\partial t_2})^m\Xi_{t_1,t_2}(f)|_{t_1=t_2=0},$$

where $\Xi_{t_1,t_2}(f)=\int_{\mathbb{R}}f(s)e^{\diamond t_1a_s^*}e^{\diamond t_2a_s}ds$ and $\Xi_{t_1,t_2}(s)=f(s)e^{\diamond t_1a_s^*}e^{\diamond t_2a_s}$ is the unique solution of the following Wick differential equation

$$N^Q(\Xi)=(t_1a_s^*+t_2a_s)\diamond\Xi, \;\; \textit{if}\; t_1,t_2\neq 0 \tag{3.8}$$

and $\Xi_{0,0}(s)=f(s)I$, where I is the identity operator.

Proof. The unique solution of (3.8), is given by

$$\Xi_{t_1,t_2}(s) = F \diamond e^{\diamond Y},$$

where $N^Q(F) = 0$ and $N^Q(Y) = t_1 a_s^* + t_2 a_s$. Let $Y = t_1 a_s^* + t_2 a_s$. Applying N^Q to Y, we get

$$\begin{aligned} N^Q(Y) &= \int_{\mathbb{R}} a_t^* \diamond D_t^+(t_1 a_s^* + t_2 a_s)dt + \int_{\mathbb{R}} a_t \diamond D_t^-(t_1 a_s^* + t_2 a_s)dt \\ &= \int_{\mathbb{R}} a_t^* \diamond (t_1 \delta(t-s))dt + \int_{\mathbb{R}} a_t \diamond (t_2 \delta(t-s))dt \\ &= t_1 a_s^* + t_2 a_s. \end{aligned}$$

From [22], we have

$$\sigma\left(N^Q(\Xi^{x,y})\right)(\xi,\eta) = (< x,\eta > + < y,\xi >)\exp(< x,\eta > + < y,\xi >),$$

where $x, y, \xi, \eta \in N$ and the operator $\Xi^{x,y}$ is defined by

$$\Xi^{x,y} = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\frac{x^{\otimes l}}{l!} \otimes \frac{y^{\otimes m}}{m!}).$$

It is noteworthy that $\{\Xi^{x,y},\ x, y \in N\}$ spans a dense subset of $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$. On the other hand

$$\begin{aligned} &\sigma\left(\sum_{l,m=0}^{\infty}(l+m)\Xi_{l,m}(\frac{x^{\otimes l}}{l!} \otimes \frac{y^{\otimes m}}{m!})\right)(\xi,\eta) \\ &= \sum_{l=1,m=0}^{\infty} \frac{1}{(l-1)!}\frac{1}{m!}\sigma(\Xi_{l,m}(x^{\otimes l} \otimes y^{\otimes m}))(\xi,\eta) \\ &\quad + \sum_{l=0,m=1}^{\infty} \frac{1}{l!}\frac{1}{(m-1)!}\sigma(\Xi_{l,m}(x^{\otimes l} \otimes y^{\otimes m}))(\xi,\eta) \\ &= \sum_{l=1,m=0}^{\infty} \frac{1}{(l-1)!}\frac{1}{m!}(< x,\eta >)^l(< y,\xi >)^m \\ &\quad + \sum_{l=0,m=1}^{\infty} \frac{1}{l!}\frac{1}{(m-1)!}(< x,\eta >)^l(< y,\xi >)^m \\ &= (< x,\eta > + < y,\xi >)\exp(< x,\eta > + < y,\xi >). \end{aligned}$$

By a density argument, for $F = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$, we deduce that

$$N^Q(F) = \sum_{l,m=0}^{\infty}(l+m)\Xi_{l,m}(\kappa_{l,m}).$$

Then, $N^Q(F) = 0$ gives $l + m = 0$ or $\kappa_{l,m} = 0$ for all $l, m \geq 0$. Therefore, $F = \Xi_{0,0}(\kappa_{0,0}) = cI$ where $c \in \mathbb{C}$ and I is the identity operator. But, for $t_1 = t_2 = 0$, we have $c = f(s)$. Which gives

$$\begin{aligned}\Xi_{t_1,t_2}(s) &= f(s)e^{\diamond(t_1a_s^*+t_2a_s)} \\ &= f(s)e^{\diamond t_1a_s^*}e^{\diamond t_2a_s}.\end{aligned}$$

Therefore,

$$\Xi_{t_1,t_2}(f) = \int_{\mathbb{R}} f(s)e^{\diamond t_1a_s^*}e^{\diamond t_2a_s}ds.$$

Hence, we obtain

$$(\frac{\partial}{\partial t_1})^l(\frac{\partial}{\partial t_2})^m\Xi_{t_1,t_2}(f) = \int_{\mathbb{R}} f(s)(a_s^*)^l \diamond (a_s)^m \diamond e^{\diamond t_1a_s^*} \diamond e^{\diamond t_2a_s}ds \tag{3.9}$$

For $t_1 = t_2 = 0$, the equation (3.9) gives

$$\begin{aligned}(\frac{\partial}{\partial t_1})^l(\frac{\partial}{\partial t_2})^m\Xi_{t_1,t_2}(f)|_{t_1=t_2=0} &= \int_{\mathbb{R}} f(s)(a_s^*)^l \diamond (a_s)^m ds \\ &= W_{l,m}(f),\end{aligned}$$

which completes the proof. □

Its obvious that

$$\begin{aligned}\Xi_{t_1,t_2}(f) &= \sum_{l,m=0}^{\infty}\frac{t_1^lt_2^m}{l!m!}W_{l,m}(f) \\ &= \sum_{l,m=0}^{\infty}\frac{t_1^lt_2^m}{l!m!}\Xi_{l,m}(\tau_{l+m}(f)).\end{aligned}$$

Recall from [9] that, for $f \in N$ there exists a family of functions $\{\kappa_\epsilon^{l,m}\}_{\epsilon>0} \subset N^{\otimes l+m}$ such that

$$\lim_{\epsilon\to 0}|\tau_{l+m}(f) - \kappa_\epsilon^{l,m}|_{-p} = 0,$$

for some $p \geq 0$ and for all $l, m \geq 0$. Let $f, g \in N$ and $l, m, n, k \in \mathbb{N}\cup\{0\}$. We approximate $\tau_{l+m}(f)$ and $\tau_{n+k}(g)$ by sufficiently regular functions $\{\kappa_{\epsilon>0}^{l,m}\}_\epsilon \subset N^{\otimes l+m}$ and $\{\lambda_\epsilon^{n,k}\}_{\epsilon>0} \subset N^{\otimes n+k}$, respectively. By virtue of the regularity of integral kernels $\Xi_{l,m}(\kappa_\epsilon^{l,m})$ and $\Xi_{n,k}(\lambda_\epsilon^{n,k})$, the composition $\Xi_{l,m}(\kappa_\epsilon^{l,m})\Xi_{n,k}(\lambda_\epsilon^{n,k})$ is defined in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$. For $j \leq \min(m+l, n+k)$, we define the inner contraction by

$$\begin{aligned}&\kappa_\epsilon^{l,m} \circ_j \lambda_\epsilon^{n,k}(s_1, ..., s_{m+l-j}, t_1, ..., t_{n+k-j}) \\ &= \int_{\mathbb{R}^j} \kappa_\epsilon^{l,m}(s_1, ..., s_{m+l-j}, u_j, ..., u_1)\lambda_\epsilon^{n,k}(u_1, ..., u_j, t_1, ..., t_{n+k-j})du_1...du_j.\end{aligned}$$

For $l, m, n, k \in \mathbb{N}\cup\{0\}$, we define the coordinate permutation

$$\begin{aligned}&S_{n,k}^{l,m}h(s_1, ..., s_l, t_1, ..., t_m, u_1, ..., u_n, v_1, ..., v_k) \\ &= h(s_1, ..., s_l, u_1, ..., u_n, t_1, ..., t_m, v_1, ..., v_k).\end{aligned}$$

By a simple modification of Proposition 4.1 in [9], we get

Proposition 3.4. *Let* $\kappa = (\kappa^{l,m})_{l,m\geq 0}$ *and* $\lambda = (\lambda^{n,k})_{n,k\geq 0}$*, where* $\kappa^{l,m} \in N^{\otimes l+m}$ *and* $\lambda^{n,k} \in N^{\otimes n+k}$ *for all* $l, m, n, k \in \mathbb{N}\cup\{0\}$*. Then, for* $t_1, t_2, s_1, s_2 \in \mathbb{R}$*, we have*

$$\begin{aligned}\Xi_{t_1,t_2}(\kappa)\Xi_{s_1,s_2}(\lambda) &= \sum_{l,m,n,k=0}^{\infty} \frac{t_1^l t_2^m s_1^n s_2^k}{l!m!n!k!} \sum_{j=0}^{m\wedge n} j! \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \\ &\quad \times \Xi_{l+n-j,m+k-j}(S^{l,n-j}_{m-j,k}(\kappa^{l,m} \circ_j \lambda^{n,k})).\end{aligned}$$

Let $f, g \in N$ and $t_1, t_2, s_1, s_2 \in \mathbb{R}$. Let $\{\kappa^{l,m}_{\epsilon>0}\}_\epsilon \subset N^{\otimes l+m}$ and $\{\lambda^{n,k}_\epsilon\}_{\epsilon>0} \subset N^{\otimes n+k}$ approximate $\tau_{l+m}(f)$ and $\tau_{n+k}(g)$, respectively, for all $l, m, n, k \in \mathbb{N}\cup\{0\}$. Then, we introduce the following renormalized product

$$\begin{aligned}&[W_{l,m}(f)W_{n,k}(g)]_{ren} \\ &= \lim_{\epsilon\to 0}(\frac{\partial}{\partial t_1})^l(\frac{\partial}{\partial t_2})^m(\frac{\partial}{\partial s_1})^n(\frac{\partial}{\partial s_2})^k Z_\epsilon(t_1, t_2, s_1, s_2)/_{t_1,t_2,s_1,s_2=0},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}Z_\epsilon(t_1, t_2, s_1, s_2) &= \Xi_{t_1,t_2}(\kappa_\epsilon)\Xi_{s_1,s_2}(\lambda_\epsilon) - \sum_{l,m,n,k=0}^{\infty} \frac{t_1^l t_2^m s_1^n s_2^k}{l!m!n!k!} \sum_{j=2}^{m\wedge n} j! \begin{pmatrix} m \\ j \end{pmatrix} \\ &\quad \times \begin{pmatrix} n \\ j \end{pmatrix} \Xi_{l+n-j,m+k-j}(S^{l,n-j}_{m-j,k}(\kappa^{l,m}_\epsilon \circ_j \lambda^{n,k}_\epsilon)).\end{aligned}$$

Theorem 3.5. *Let* $f, g \in N$ *and* $l, m, n, k \in \mathbb{N}\cup\{0\}$*. Then, we have*

$$\Big[W_{l,m}(f)W_{l',m'}(g)\Big]_{ren} = W_{l,m}(f) \diamond W_{l',m'}(g) + ml'W_{l+l'-1,m+m'-1}(fg).$$

Proof. By a simple calculus, we have

$$\begin{aligned}Z_\epsilon(t_1, t_2, s_1, s_2) &= \sum_{l,m,n,k=0}^{\infty} \frac{t_1^l t_2^m s_1^n s_2^k}{l!m!n!k!}\{\Xi_{l+n,m+k}(S^{l,n}_{m,k}(\kappa^{l,m}_\epsilon \circ_0 \lambda^{n,k}_\epsilon)) \\ &\quad + mn\Xi_{l+n-1,m+k-1}(S^{l,n-1}_{m-1,k}(\kappa^{l,m}_\epsilon \circ_1 \lambda^{n,k}_\epsilon))\}.\end{aligned}$$

Therefore

$$\begin{aligned}&(\frac{\partial}{\partial t_1})^l(\frac{\partial}{\partial t_2})^m(\frac{\partial}{\partial s_1})^n(\frac{\partial}{\partial s_2})^k Z_\epsilon(t_1, t_2, s_1, s_2)/_{t_1,t_2,s_1,s_2=0} \\ &= \Xi_{l+n,m+k}(S^{l,n}_{m,k}(\kappa^{l,m}_\epsilon \circ_0 \lambda^{n,k}_\epsilon)) + mn\Xi_{l+n-1,m+k-1}(S^{l,n-1}_{m-1,k}(\kappa^{l,m}_\epsilon \circ_1 \lambda^{n,k}_\epsilon)).\end{aligned}$$

But we know (see [9]) that

$$\lim_{\epsilon\to 0} \kappa^{l,m}_\epsilon \circ_0 \lambda^{n,k}_\epsilon = \tau_{l+m}(f) \otimes \tau_{n+k}(g)$$

in $N'^{\otimes(l+m+n+k)}$ and

$$\lim_{\epsilon\to 0} \kappa^{l,m}_\epsilon \circ_1 \lambda^{n,k}_\epsilon = \tau_{l+m+n+k-2}(fg)$$

in $N'^{\otimes(l+m+n+k-2)}$. Hence we obtain

$$\begin{aligned}[W_{l,m}(f)W_{n,k}(g)]_{ren} &= \Xi_{l+n,m+k}(S^{l,n}_{m,k}(\tau_{l+m}(f) \otimes \tau_{n+k}(g))) \\ &\quad + mn\Xi_{l+n-1,m+k-1}(S^{l,n-1}_{m-1,k}(\tau_{l+m+n+k-2}(fg))).\end{aligned}$$

On the other hand,

$$\Xi_{l+n-1,m+k-1}(S^{l,n-1}_{m-1,k}(\tau_{l+m+n+k-2}(fg))) = W_{l+n-1,m+k-}(fg)$$

and

$$\begin{aligned}\sigma\Big(\Xi_{l+n,m+k}(S^{l,n}_{m,k}(\tau_{l+m}(f)\otimes\tau_{n+k}(g)))\Big)(\xi,\eta) \\ &= \Big\langle \tau_{l+m}(f)\otimes\tau_{n+k}(g), \eta^{\otimes(l+n)}\otimes\xi^{\otimes(m+k)}\Big\rangle \\ &= \big\langle \tau_{l+m}(f), \eta^{\otimes l}\otimes\xi^{\otimes m}\big\rangle \big\langle \tau_{n+k}(g), \eta^{\otimes n}\otimes\xi^{\otimes k}\big\rangle \\ &= \sigma\left(W_{l,m}(f)\right)(\xi,\eta)\sigma\left(W_{n,k}(g)\right)(\xi,\eta).\end{aligned}$$

Therefore, we obtain

$$\Xi_{l+n,m+k}(S^{l,n}_{m,k}(\tau_{l+m}(f)\otimes\tau_{n+k}(g))) = W_{l,m}(f)\diamond W_{n,k}(g),$$

which completes the proof. □

4. Higher Powers of QWN-derivatives

In the following proposition we give an important commutation relations using the operators D^+_x, $(D^+_y)^*$, D^-_x and $(D^-_y)^*$ for $x, y \in N$ on the nuclear algebra $\mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}_\theta(N'))$.

Proposition 4.1. [11] *For all $x, y \in N$, we have*

(1) $\left[D^+_x, D^+_y\right] = \left[D^-_x, D^-_y\right] = \left[D^+_x, D^-_y\right] = \left[D^-_x, D^+_y\right] = 0$
(2) $\left[(D^\pm_x)^*, (D^\mp_y)^*\right] = \left[(D^\pm_x)^*, (D^\pm_y)^*\right] = 0$
(3) $\left[D^\pm_x, (D^\mp_y)^*\right] = 0$
(4) $\left[D^\pm_x, (D^\pm_y)^*\right] =< x, y > I.$

Using a similar notation:

$$e_{\overrightarrow{i}} = e_{i_1}\otimes...\otimes e_{i_{n_1}},\ e_{\overrightarrow{j}} = e_{j_1}\otimes...\otimes e_{j_{n_2}},\ e_{\overrightarrow{u}} = e_{u_1}\otimes...\otimes e_{u_{k_1}},\ e_{\overrightarrow{v}} = e_{v_1}\otimes...\otimes e_{v_{k_2}},$$

$$\overrightarrow{i} = (i_1, ..., i_{n_1}),\ \overrightarrow{j} = (j_1, ..., j_{n_2}),\ \overrightarrow{u} = (u_1, ..., u_{k_1}),\ \overrightarrow{v} = (v_1, ..., j_{k_2}),$$

where $n_1, n_2, k_1, k_2 \in \mathbb{N}\cup\{0\}$. For $f \in (N^{\otimes(n_1+n_2+k_1+k_2)})'$ and $p, p', q, q' \in \mathbb{R}$, we put

$$\begin{aligned}|f|^2_{n_1,n_2,k_1,k_2;p,p',q,q'} &= \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} |< f, e_{\overrightarrow{i}}\otimes e_{\overrightarrow{j}}\otimes e_{\overrightarrow{u}}\otimes e_{\overrightarrow{v}} >|^2 \\ &\quad\times |e_{\overrightarrow{i}}|^2_p |e_{\overrightarrow{j}}|^2_{p'} |e_{\overrightarrow{u}}|^2_q |e_{\overrightarrow{v}}|^2_{q'}.\end{aligned}$$

This is always finite for $f \in N^{\otimes(n_1+n_2+k_1+k_2)}$. However, this is possibly infinite. In fact, for $p, p', q, q' \in \mathbb{R}$ and $r, s, r', s' \geq 0$,

$$|f|^2_{n_1,n_2,k_1,k_2;p,p',q,q'} \leq \rho^{2(rn_1+sn_2+r'k_1+s'k_2)}|f|^2_{n_1,n_2,k_1,k_2;p+r,p'+s,q+r',q'+s'}.$$

Definition 4.2. Let $n_1, n_2, k_1, k_2 \in \mathbb{N} \cup \{0\}$ and $f \in (N^{\otimes(n_1+n_2+k_1+k_2)})'$. We introduce the operator

$$\begin{aligned} D^{n_1,n_2}_{k_1,k_2}(f) \quad := & \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} < f, e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} \otimes e_{\overrightarrow{u}} \otimes e_{\overrightarrow{v}} > \\ & \times (D^{+*}_{e_{i_1}})...(D^{+*}_{e_{i_{n_1}}})(D^{-*}_{e_{j_1}})...(D^{-*}_{e_{j_{n_2}}}) \\ & \times (D^{+}_{e_{u_1}})...(D^{+}_{e_{u_{k_1}}})(D^{-}_{e_{v_1}})...(D^{-}_{e_{v_{k_2}}}). \end{aligned}$$

Proposition 4.3. *For $f \in N^{\otimes(n_1+n_2)} \otimes (N^{\otimes(k_1+k_2)})'$, The operator $D^{n_1,n_2}_{k_1,k_2}(f)$ is continuous from $\mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}_\theta(N'))$ into itself. Moreover, for any $p \in \mathbb{R}$, choosing $q, q' \geq 0$ satisfying $|f|_{n_1,n_2,k_1,k_2;p,p,-(p+q),-(p+q+q')} < \infty$, we have*

$$\begin{aligned} \|D^{n_1,n_2}_{k_1,k_2}(f)\Xi\|_{\theta,p,(\gamma_1,\gamma_2)} \quad \leq \quad & c(k_1,k_2,q,q',\gamma_1,\gamma_2)|f|_{n_1,n_2,k_1,k_2;p,-(p+q),-(p+q+q')} \\ & \times \|\Xi\|_{\theta,p+q+q',(\frac{\gamma_1}{16\rho^{q+q'}},\frac{\gamma_2}{16\rho^{q+q'}})}, \end{aligned} \tag{4.1}$$

where $c(k_1,k_2,q,q',\gamma_1,\gamma_2) > 0$ and $\Xi \in \mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}_\theta(N'))$.

Proof. Let $\Xi \in \mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}_\theta(N'))$ with expansion $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$. Then using the equalities (2.14), (2.11), (2.14) and (2.13) respectively, we get

$$\begin{aligned} D^{n_1,n_2}_{k_1,k_2}(f)\Xi \quad = & \sum_{l=k_1,m=k_2}^{\infty} \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} < f, e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} \otimes e_{\overrightarrow{u}} \otimes e_{\overrightarrow{v}} > \frac{m!}{(m-k_2)!} \\ & \times \frac{l!}{(l-k_1)!}\Xi_{l+n_1-k_1,m+n_2-k_2}(\widehat{e_{\overrightarrow{i}}}\widehat{\otimes}\widehat{e_{\overrightarrow{u}}}\widehat{\otimes}^{k_1}\kappa_{l,m}\widehat{\otimes}_{k_2}\widehat{e_{\overrightarrow{v}}}\widehat{\otimes}\widehat{e_{\overrightarrow{j}}}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|D^{n_1,n_2}_{k_1,k_2}(f)\Xi\|^2_{\theta,p,(\gamma_1,\gamma_2)} \quad = & \sum_{l=k_1,m=k_2}^{\infty} \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} | < f, e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} \otimes e_{\overrightarrow{u}} \otimes e_{\overrightarrow{v}} > |^2 \\ & \times |\widehat{e_{\overrightarrow{i}}}\widehat{\otimes}\widehat{e_{\overrightarrow{u}}}\widehat{\otimes}^{k_1}\kappa_{l,m}\widehat{\otimes}_{k_2}\widehat{e_{\overrightarrow{v}}}\widehat{\otimes}\widehat{e_{\overrightarrow{j}}}|^2_p \theta^{-2}_{l+n_1-k_1}\theta^{-2}_{m+n_2-k_2} \\ & \times \gamma_1^{-l-n_1+k_1}\gamma_2^{-m-n_2+k_2}\left(\frac{l!m!}{(l-k_1)!(m-k_2)!}\right)^2. \end{aligned}$$

Using inequalities in (3.5), we obtain

$$\begin{aligned} \|D^{n_1,n_2}_{k_1,k_2}(f)\Xi\|^2_{\theta,p,(\gamma_1,\gamma_2)} \quad = & \sum_{l=k_1,m=k_2}^{\infty} \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} \theta_l^{-2}\theta_{n_1}^{-2}\theta_{k_1}^2\theta_m^{-2}\theta_{n_2}^{-2}\theta_{k_2}^2(k_1!k_2!)^2 \\ & \times 16^{l+m}4^{n_1+n_2-k_1-k_2}\gamma_1^{-l-n_1+k_1}\gamma_2^{-m-n_2+k_2} \\ & \times | < f, e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} \otimes e_{\overrightarrow{u}} \otimes e_{\overrightarrow{v}} > |^2 \\ & \times |\widehat{e_{\overrightarrow{i}}}\widehat{\otimes}\widehat{e_{\overrightarrow{u}}}\widehat{\otimes}^{k_1}\kappa_{l,m}\widehat{\otimes}_{k_2}\widehat{e_{\overrightarrow{v}}}\widehat{\otimes}\widehat{e_{\overrightarrow{j}}}|^2_p. \end{aligned}$$

On the other hand, for any $q, q' \geq 0$, we have

$$
\begin{aligned}
|\widehat{e_{\overrightarrow{i}}}\widehat{\otimes}\widehat{e_{\overrightarrow{u}}}\widehat{\otimes}^{k_1}\kappa_{l,m}\widehat{\otimes}_{k_2}\widehat{e_{\overrightarrow{v}}}\widehat{\otimes}\widehat{e_{\overrightarrow{j}}}|_p^2 \quad \leq \quad & \rho^{2q(l+m-k_1-k_2)+2q'(l+m-k_2)}|e_{\overrightarrow{i}}|_p^2 \\
& \times|e_{\overrightarrow{j}}|_p^2|e_{\overrightarrow{u}}|^2_{-(p+q)}|e_{\overrightarrow{i}}|^2_{-(p+q+q')}|\kappa_{l,m}|^2_{p+q+q'}.
\end{aligned}
$$

This gives the desired statement. □

For $f \in N^{\otimes(n_1+n_2+k_1+k_2)}$ and $g \in N^{\otimes(k_1+k_2+k_1'+k_2')}$, we define the contraction by

$$
\begin{aligned}
f \circ_{k_1,k_2} g \quad = \quad & \sum_{\overrightarrow{\alpha},\overrightarrow{\beta},\overrightarrow{\eta},\overrightarrow{\lambda}}\ \sum_{\overrightarrow{\alpha'},\overrightarrow{\beta'},\overrightarrow{\eta'},\overrightarrow{\lambda'}}\ \sum_{\overrightarrow{\gamma},\overrightarrow{\gamma'}} < f, e_{\overrightarrow{\alpha}} \otimes e_{\overrightarrow{\beta}} \otimes e_{\overrightarrow{\gamma}} \otimes e_{\overrightarrow{\lambda}} > \\
& \times < g, e_{\overrightarrow{\lambda}} \otimes e_{\overrightarrow{\gamma}} \otimes e_{\overrightarrow{\gamma'}} \otimes e_{\overrightarrow{\lambda'}} > \\
& \times e_{\overrightarrow{\alpha}} \otimes e_{\overrightarrow{\beta}} \otimes e_{\overrightarrow{\gamma'}} \otimes e_{\overrightarrow{\lambda'}}.
\end{aligned}
$$

By definition $f \circ_{0,0} g = f \otimes g$. The coordinate permutation S is defined by

$$
\begin{aligned}
& S(n_1, n_1', n_2, n_2', k_1, k_1', k_2, k_2')f(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{\gamma}, \overrightarrow{\lambda}, , \overrightarrow{\alpha'}, \overrightarrow{\beta'}, \overrightarrow{\gamma'}, \overrightarrow{\lambda'}) \\
& \qquad = f(\overrightarrow{\alpha}, \overrightarrow{\alpha'}, \overrightarrow{\beta}, \overrightarrow{\beta'}, \overrightarrow{\gamma}, \overrightarrow{\gamma'}, \overrightarrow{\lambda}, \overrightarrow{\lambda'}),
\end{aligned}
$$

where

$$
\begin{aligned}
& \overrightarrow{\alpha} = (\alpha_1, ..., \alpha_{n_1}),\ \overrightarrow{\beta} = (\beta_1, ..., \beta_{n_2}),\ \overrightarrow{\gamma} = (\gamma_1, ..., \gamma_{k_1}),\ \overrightarrow{\lambda} = (\lambda_1, ..., \lambda_{k_2}), \\
& \overrightarrow{\alpha'} = (\alpha_1', ..., \alpha_{n_1'}'),\ \overrightarrow{\beta'} = (\beta_1', ..., \beta_{n_2'}'),\ \overrightarrow{\gamma'} = (\gamma_1', ..., \gamma_{k_1'}'),\ \overrightarrow{\lambda'} = (\lambda_1', ..., \lambda_{k_2'}').
\end{aligned}
$$

From [9], for $f \in N$ and $m \geq 0$, there exists $\{f_\epsilon\}_{\epsilon\geq 0} \subset N^{\otimes m}$ such that

$$\lim_{\epsilon\to 0}|f_\epsilon - \tau_m(f)|_{-p} = 0,$$

for some $p \geq 0$. By a simple modification of Lemma (4.2) in [9], we get

Lemma 4.4. *Let $\{f_\epsilon\}$ and $\{g_\epsilon\}$ approximate $\tau_{m+2}(f)$ and $\tau_{n+2}(g)$, respectively, where $f, g \in N$ and $m, n \geq 0$. Then*

$$\lim_{\epsilon\to 0} f_\epsilon \circ_{0,0} g_\epsilon = \tau_{m+n+1}(fg) \tag{4.2}$$

in $(N^{\otimes m+n+1})'$,

$$\lim_{\epsilon\to 0} f_\epsilon \circ_{1,0} g_\epsilon = \lim_{\epsilon\to 0} f_\epsilon \circ_{0,1} g_\epsilon = \tau_{m+n+1}(fg) \tag{4.3}$$

in $(N^{\otimes m+n+1})'$ and

$$\lim_{\epsilon\to 0} f_\epsilon \circ_{1,1} g_\epsilon = \tau_{n_1+n_2}(fg) \tag{4.4}$$

in $(N^{\otimes n_1+n_2})'$.

Proposition 4.5. *For $f \in N^{\otimes(n_1+n_2+k_1+k_2)}$ and $g \in N^{\otimes(n_1'+n_2'+k_1'+k_2')}$, we have*

$$
\begin{aligned}
D^{n_1,n_2}_{k_1,k_2}(f)D^{n_1',n_2'}_{k_1',k_2'}(g) \quad = \quad & \sum_{n,k=0}^{(k_2\wedge n_2')\vee(k_1\wedge n_1')} n!k! \begin{pmatrix} k_1 \\ k \end{pmatrix}\begin{pmatrix} n_1' \\ k \end{pmatrix}\begin{pmatrix} k_2 \\ n \end{pmatrix}\begin{pmatrix} n_2' \\ n \end{pmatrix} \\
& \times D^{n_1+n_1'-k,n_2+n_2'-n}_{k_1+k_1'-k,k_2+k_2'-n}(S(f\circ_{k,n} g)),
\end{aligned}
$$

where $S = S(n_1, n_1' - k, n_2, n_2' - n, k_1 - k, k_1', k_2 - n, k_2')$.

Definition 4.6. For $n_1, n_2, k_1, k_2 \in \mathbb{N} \cup \{0\}$ such that $n_1 + n_2 + k_1 + k_2 \geq 1$ and $f \in (N^{\otimes(n_1+n_2+k_1+k_2)})'$, we introduce the operator

$$B_{k_1,k_2}^{n_1,n_2}(f) := D_{k_1,k_2}^{n_1,n_2}(\tau_{n_1+n_2+k_1+k_2}(f))\cdot \tag{4.5}$$

Lemma 4.7. *The operator in equation (4.5) admits the following representation*

$$B_{k_1,k_2}^{n_1,n_2}(f) = \int_{\mathbb{R}} f(t)(D_t^{+*})^{n_1}(D_t^{-*})^{n_2}(D_t^{+})^{k_1}(D_t^{-})^{k_2}dt.$$

Proof. Applying the wick symbol map to righthand side of (4.5), we get

$$\sigma\left(D_{k_1,k_2}^{n_1,n_2}(\tau_{n_1+n_2+k_1+k_2}(f))\Xi^{a,b}\right)(\xi,\eta)$$

$$\begin{aligned}
&= \sum_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{u},\overrightarrow{v}=0}^{\infty} < \tau_{n_1+n_2+k_1+k_2}(f), e_{\overrightarrow{i}} \otimes e_{\overrightarrow{j}} \otimes e_{\overrightarrow{u}} \otimes e_{\overrightarrow{v}} > \\
&\quad \times < e_{\overrightarrow{v}}, b^{\otimes k_2} >< e_{\overrightarrow{u}}, a^{\otimes k_1} >< e_{\overrightarrow{j}}, \xi^{\otimes n_2} >< e_{\overrightarrow{i}}, \eta^{\otimes n_1} > \sigma(\Xi^{a,b})(\xi,\eta) \\
&= < \tau_{n_1+n_2+k_1+k_2}(f), b^{\otimes k_2} \otimes a^{\otimes k_1} \otimes \xi^{\otimes n_2} \otimes \eta^{\otimes n_1} > \sigma(\Xi^{a,b})(\xi,\eta) \\
&= < f, a^{k_1} b^{k_2} \xi^{n_2} \eta^{n_1} > \sigma(\Xi^{a,b})(\xi,\eta).
\end{aligned}$$

On the other hand, we have

$$\sigma\left(\int_{\mathbb{R}} f(t)(D_t^{+*})^{n_1}(D_t^{-*})^{n_2}(D_t^{+})^{k_1}(D_t^{-})^{k_2}dt\Xi^{a,b}\right)(\xi,\eta)$$

$$\begin{aligned}
&= \int_{\mathbb{R}} f(t)\sigma\left((D_t^{+*})^{n_1}(D_t^{-*})^{n_2}(D_t^{+})^{k_1}(D_t^{-})^{k_2}\Xi^{a,b}\right)(\xi,\eta)dt \\
&= \int_{\mathbb{R}} f(t)b(t)^{k_2}a(t)^{k_1}\xi(t)^{n_2}\eta(t)^{n_1}dt \\
&= < f, a^{k_1} b^{k_2} \xi^{n_2} \eta^{n_1} > .
\end{aligned}$$

By density argument, we complete the proof. □

Proposition 4.8. *For $f \in N$ and $l, m \in \mathbb{N} \cup \{0\}$, we have*

$$B_{0,0}^{l,m}(f)\Xi = W_{l,m}(f) \diamond \Xi. \tag{4.6}$$

In particular,

$$B_{0,0}^{l,m}(f)\Xi^{0,0} = W_{l,m}(f). \tag{4.7}$$

Proof. By definition

$$B_{0,0}^{l,m}(f) = \int_{\mathbb{R}} f(t)(D_t^{+*})^{l}(D_t^{-*})^{m}dt.$$

But in [22], the adjoint of the QWN-derivatives are given by

$$D_t^{+*}\Xi = a_t^* \diamond \Xi \tag{4.8}$$

and

$$D_t^{-*}\Xi = a_t \diamond \Xi, \tag{4.9}$$

where $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$. Which gives

$$\begin{aligned} B^{l,m}_{0,0}(f)\Xi &= \int_{\mathbb{R}} f(t)(a^*_t)^{\diamond l} \diamond (a_t)^{\diamond m} \diamond \Xi dt \\ &= \int_{\mathbb{R}} f(t)(a^*_t)^l (a_t)^m dt \diamond \Xi \\ &= W_{l,m}(f) \diamond \Xi. \end{aligned}$$

This completes the proof. □

Proposition 4.9. *For $l, m \in \mathbb{N} \cup \{0\}$ and $f \in N$, we have*

$$B^{l,0}_{m,0}(f) = \sigma^{-1}(W_{l,m}(f) \otimes I)\sigma \tag{4.10}$$

and

$$B^{0,l}_{0,m}(f) = \sigma^{-1}(I \otimes W_{l,m}(f))\sigma \tag{4.11}$$

Proof. Let $x, y, \xi, \eta \in N$. Then, we get

$$\begin{aligned} \sigma\left(B^{l,0}_{m,0}(f)\Xi^{x,y}\right)(\xi,\eta) &= \ll B^{l,0}_{m,0}\Xi^{x,y} e_\xi, e_\eta \gg e^{-<\xi,\eta>} \\ &= \int_{\mathbb{R}} f(s) \ll (D^{+*}_s)^l (D^+_s)^m \Xi^{x,y} e_\xi, e_\eta \gg e^{-<\xi,\eta>} ds \\ &= \int_{\mathbb{R}} f(s) x(s)^m \ll (a^*_s)^l \diamond \Xi^{x,y} e_\xi, e_\eta \gg e^{-<\xi,\eta>} ds \\ &= \sigma\left(\Xi^{x,y}\right)(\xi,\eta) \int_{\mathbb{R}} f(s) x(s)^m \eta(s)^l ds. \end{aligned}$$

On the other hand,

$$\sigma(\sigma^{-1}(W_{l,m}(f) \otimes I)\sigma(\Xi^{x,y}))(\xi,\eta) = (W_{l,m}(f) \otimes I)\sigma(\Xi^{x,y})(\xi,\eta). \tag{4.12}$$

The right hand side of equation (4.12) gives

$$\begin{aligned} (W_{l,m}(f) \otimes I)\sigma(\Xi^{x,y})(\xi,\eta) &= (W_{l,m}(f) \otimes I)(e_x \otimes e_y)(\eta,\xi) \\ &= (W_{l,m}(f)e_x)(\eta) e_y(\xi) \\ &= e^{<y,\xi>} \int_{\mathbb{R}} f(s)((a^*_s)^l (a_s)^m e_x)(\eta) ds \\ &= e^{<y,\xi>} \int_{\mathbb{R}} f(s) x(s)^m ((a^*_s)^l e_x)(\eta) ds. \end{aligned}$$

One can show that

$$(a^*_s)^l e_x \sim \left(\delta_s^{\widehat{\otimes} l} \widehat{\otimes} \frac{x^{\otimes n-l}}{(n-l)!}\right)_{n \geq l}.$$

Therefore, we obtain

$$((a^*_s)^l e_x)(\eta) = \eta(s)^l e^{<x,\eta>}.$$

Which gives

$$(W_{l,m}(f) \otimes I)\sigma(\Xi^{x,y})(\xi,\eta) = \sigma(\Xi^{x,y})(\xi,\eta) \int_{\mathbb{R}} f(s) x(s)^m \eta(s)^l ds.$$

By identification and a density argument, we deduce (4.10). Similarly, we get (4.11). □

Definition 4.10. For two operators $T_1, T_2 \in \mathcal{L}\left(\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))\right)$ there exists a unique $T \in \mathcal{L}\left(\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N')), \mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))\right)$ such that

$$\sigma\left(T\Xi^{a,b}\right)(\xi,\eta) = \sigma\left(T_1\Xi^{a,b}\right)(\xi,\eta)\sigma\left(T_2\Xi^{a,b}\right)(\xi,\eta).$$

This operator T is called the wick product of T_1 and T_2 and it denoted by

$$T = T_1 \diamond T_2.$$

Lemma 4.11. *For $f \in N$, we have $B^{0,0}_{k_1,k_2}(f) \in \mathcal{L}\left(\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))\right)$ and hence $\left[B^{0,0}_{1,1}(f), B^{n_1,n_2}_{k_1,k_2}(g)\right]$ is well-defined in $\mathcal{L}\left(\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))\right)$ for any $g \in N'$. In this case*

$$\left[B^{0,0}_{1,1}(f), B^{n_1,n_2}_{k_1,k_2}(g)\right] = n_1 n_2 B^{n_1-1,n_2-1}_{k_1,k_2}(fg). \tag{4.13}$$

In connection with the wick products, we get

$$B^{0,0}_{1,1}(f)B^{n_1,n_2}_{k_1,k_2}(g) = B^{0,0}_{1,1}(f) \diamond B^{n_1,n_2}_{k_1,k_2}(g) + n_1 n_2 B^{n_1-1,n_2-1}_{k_1,k_2}(fg). \tag{4.14}$$

Proof. We are going to prove that

$$B^{0,0}_{1,1}(f)B^{n_1,n_2}_{k_1,k_2}(g) = B^{0,0}_{1,1}(f) \diamond B^{n_1,n_2}_{k_1,k_2}(g). \tag{4.15}$$

For $a, b, \xi, \eta \in E$, we have

$$\sigma\left(B^{0,0}_{1,1}(f)B^{n_1,n_2}_{k_1,k_2}(g)\Xi^{a,b}\right)(\xi,\eta) = < g, a^{k_1}b^{k_2}\eta^{n_1}\xi^{n_2} >< f, \xi\eta > .$$

On the other hand, we know that

$$\sigma\left(B^{0,0}_{1,1}(f)\Xi^{a,b}\right)(\xi,\eta) = < f, \xi\eta >$$

and

$$\sigma\left(B^{n_1,n_2}_{k_1,k_2}(g)\Xi^{a,b}\right)(\xi,\eta) = < g, a^{k_1}b^{k_2}\eta^{n_1}\xi^{n_2} > .$$

By definition of wick product, we prove (4.15). □

Remark 4.12. By a similar calculus, we remark that

$$B^{n_1,n_2}_{k_1,k_2}(f)B^{0,0}_{k_1',k_2'}(g) = B^{n_1,n_2}_{k_1,k_2}(f) \diamond B^{0,0}_{k_1',k_2'}(g). \tag{4.16}$$

Let us introduce the renormalized product

$$\left[B^{n_1,n_2}_{k_1,k_2}(f)B^{n_1',n_2'}_{k_1',k_2'}(g)\right]_{ren}$$

$$\begin{aligned} = \; & \lim_{\epsilon\to 0}\{D^{n_1,n_2}_{k_1,k_2}(f_\epsilon)D^{n_1',n_2'}_{k_1',k_2'}(g_\epsilon) \\ & - \sum_{n,k=2}^{(k_2\wedge n_2')\vee(k_1\wedge n_1')} n!k! \begin{pmatrix} k_1 \\ k \end{pmatrix}\begin{pmatrix} n_1' \\ k \end{pmatrix}\begin{pmatrix} k_2 \\ n \end{pmatrix}\begin{pmatrix} n_2' \\ n \end{pmatrix} \\ & \times D^{n_1+n_1'-k,n_2+n_2'-n}_{k_1+k_1'-k,k_2+k_2'-n}(S(f_\epsilon \circ_{k,n} g_\epsilon))\}, \end{aligned}$$

where $S = S(n_1, n_1'-k, n_2, n_2'-n, k_1-k, k_1', k_2-n, k_2')$. From Proposition 4.5 and the renormalization condition, we get

Theorem 4.13. *For $f,g\in N$ and $n_1,n_2,n_1',n_2',k_1,k_2,k_1',k_2'\in\mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned}\Big[B^{n_1,n_2}_{k_1,k_2}(f)B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{ren} &= B^{n_1,n_2}_{k_1,k_2}(f)\diamond B^{n_1',n_2'}_{k_1',k_2'}(g)\\ &+k_1n_1'B^{n_1+n_1'-1,n_2+n_2'}_{k_1+k_1'-1,k_2+k_2'}(fg)\\ &+k_2n_2'B^{n_1+n_1',n_2+n_2'-1}_{k_1+k_1',k_2+k_2'-1}(fg)\\ &+k_1k_2n_1'n_2'B^{n_1+n_1'-1,n_2+n_2'-1}_{k_1+k_1'-1,k_2+k_2'-1}(fg).\end{aligned}$$

We define the commutator

$$\begin{aligned}\Big[B^{n_1,n_2}_{k_1,k_2}(f),B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{RHP} &= \Big[B^{n_1,n_2}_{k_1,k_2}(f)B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{ren}\\ &-\Big[B^{n_1',n_2'}_{k_1',k_2'}(g)B^{n_1,n_2}_{k_1,k_2}(f)\Big]_{ren}.\end{aligned}$$

Theorem 4.14. *There exists a $*$-Lie algebra with generators*

$$\{B^{n_1,n_2}_{k_1,k_2}(f):\ k_1,k_2,n_1,n_2\in\mathbb{N},\ f\in N\},$$

involution given by:

$$\Big(B^{n_1,n_2}_{k_1,k_2}(f)\Big)^*=B^{k_1,k_2}_{n_1,n_2}(\overline{f}) \tag{4.17}$$

and brackets given by:

$$\begin{aligned}\Big[B^{n_1,n_2}_{k_1,k_2}(f),B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{RHP} &= (k_1n_1'-k_1'n_1)B^{n_1+n_1'-1,n_2+n_2'}_{k_1+k_1'-1,k_2+k_2'}(fg)\\ &+(k_2n_2'-k_2'n_2)B^{n_1+n_1',n_2+n_2'-1}_{k_1+k_1',k_2+k_2'-1}(fg)\\ &+(k_1k_2n_1'n_2'-k_1'k_2'n_1n_2)B^{n_1+n_1'-1,n_2+n_2'-1}_{k_1+k_1'-1,k_2+k_2'-1}(fg).\end{aligned}$$

Proof. For all $f,g\in N$ and $k_1,k_2,n_1,n_2,k_1',k_2',n_1',n_2'\in\mathbb{N}$,

$$\Big[B^{n_1,n_2}_{k_1,k_2}(f),B^{n_1,n_2}_{k_1,k_2}(f)\Big]_{RHP}=0$$

and

$$\Big[B^{n_1,n_2}_{k_1,k_2}(f),B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{RHP}=-\Big[B^{n_1',n_2'}_{k_1',k_2'}(g),B^{n_1,n_2}_{k_1,k_2}(f)\Big]_{RHP}.$$

Using Theorem 4.13, we show that commutation relations $\Big[,\Big]_{RHP}$ satisfy the Jacobi identity:

$$\begin{aligned}&\Big[B^{n_1,n_2}_{k_1,k_2}(f),\Big[B^{n_1',n_2'}_{k_1',k_2'}(g),B^{n"_1,n"_2}_{k"_1,k"_2}(h)\Big]_{RHP}\Big]_{RHP}\\ &+\Big[B^{n_1',n_2'}_{k_1',k_2'}(g),\Big[B^{n"_1,n"_2}_{k"_1,k"_2}(h),B^{n_1,n_2}_{k_1,k_2}(f)\Big]_{RHP}\Big]_{RHP}\\ &+\Big[B^{n"_1,n"_2}_{k"_1,k"_2}(h),\Big[B^{n_1,n_2}_{k_1,k_2}(f),B^{n_1',n_2'}_{k_1',k_2'}(g)\Big]_{RHP}\Big]_{RHP}=0,\end{aligned}$$

which completes the proof. □

Remark 4.15. Note that for $n_2=k_2=n_2'=k_2'=0$, we get

$$\Big[B^{n_1,0}_{k_1,0}(f),B^{n_1',0}_{k_1',0}(g)\Big]_{RHP}=(k_1n_1'-k_1'n_1)B^{n_1+n_1'-1,0}_{k_1+k_1'-1,0}(fg)$$

and for $n_1 = k_1 = n'_1 = k'_1 = 0$, we obtain

$$\Big[B^{0,n_2}_{0,k_2}(f), B^{0,n'_2}_{0,k'_2}(g)\Big]_{RHP} = (k_2n'_2 - k'_2n_2)B^{0,n_2+n'_2-1}_{0,k_2+k'_2-1}(fg).$$

This gives two copies of representation of renormalized higher powers of quantum white noise $*$-Lie algebra on $\mathcal{L}(\mathcal{F}^*_\theta(N'), \mathcal{F}_\theta(N'))$.

Proposition 4.16. *For $l, m, l', m' \geq 0$, we have*

$$\Big[B^{l,m}_{0,0}(f)\Xi^{0,0}, B^{l',m'}_{0,0}(g)\Xi^{0,0}\Big]_{RHPWN} = (ml' - l'm)B^{l+l'-1,m+m'-1}_{0,0}(fg)\Xi^{0,0} \tag{4.18}$$

on $\mathcal{F}_\theta(N')$, which gives a representation of (RHPWN) $*$-Lie algebra on $\mathcal{F}_\theta(N')$.

Proof. From Proposition 4.8, we get

$$\Big[B^{l,m}_{0,0}(f)\Xi^{0,0}, B^{l',m'}_{0,0}(g)\Xi^{0,0}\Big]_{RHPWN} = \Big[W_{l,m}(f), W_{l',m'}(g)\Big]_{RHPWN}.$$

Using the renormalization condition, we obtain

$$\begin{aligned} &\Big[W_{l,m}(f)W_{l',m'}(g)\Big]_{RHPWN} \\ = \;& \lim_{\epsilon\to 0}\{\Xi_{l,m}(f_\epsilon)\Xi_{l',m'}(g_\epsilon) \\ & -\sum_{k=1}^{m\wedge l'} k! \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} l' \\ k \end{pmatrix} \Xi_{l+l'-k,m+m'-k}(S(f_\epsilon \circ_k g_\epsilon))\} \\ = \;& W_{l,m}(f) \diamond W_{l',m'}(g) + ml'W_{l+l'-1,m+m'-1}(fg). \end{aligned}$$

By a similar calculus,

$$\Big[W_{l',m'}(g)W_{l,m}(f)\Big]_{RHPWN} = W_{l,m}(f) \diamond W_{l',m'}(g) + m'lW_{l+l'-1,m+m'-1}(fg).$$

Obviously, the equation (4.18) is equivalent to (1.3), which completes the proof. □

References

1. Accardi, L. and Boukas, A.: Renormalized hight powers of white noise (RHPWN) and conformal field theory, *Infinite Dimensional Anal. Quantum Probab. Related topics* **9**, No. 3 (2006), 353–360.
2. Accardi, L. and Boukas, A.: The emergence of the Virasoro and ω_∞ Lie algebra through the renormalized higher powers of quantum white noise, *International Journal of Mathematics and computer Sciences*, **1**, No. 3 (2006), 315–342.
3. Accardi, L. and Boukas, A.: Renormalized hight powers of white noise and the Virasoro-Zamolodchikov-ω_∞ *Algebra*, (2006).
4. Accardi, L. and Boukas, A.: Lie Algebras associated with the renormalized hight powers of white noise, *Communications on Stochastic Analysis* **1**, No. 1 (2007), 57–69.
5. Accardi, L., Boukas, A., and Franz, U.: Renormalized powers of quantum white noise, *Infinite Dimensional Analysis Quantum Probability and related topics*, **9** (1) (2006), 129–147.
6. Barhoumi, A., Ouerdiane, H., and Rguigui, H.: QWN-Euler operator and associated Cauchy problem, *Infinite Dimensional Analysis Quantum Probability and Related Topics*, **15**, No. 1, 1250004 (2012).
7. Barhoumi, A., Ouerdiane, H., and Rguigui, H.: Stochastic heat equation on algebra of generalized functions, *Infinite Dimensional Analysis Quantum Probability and Related Topics*, **15**, No. 4, 1250026 (2012).

8. Ben Chrouda, M. and Ouerdiane, H.: Algebras of operators on Holomorphic Functions and Applications, *Mathematical Physics, Analysis and Geometry*, **5**, No. 1 (2002), 65–76.
9. Chung, D. M., Ji, U. C., and Obata, N.: Higher powers of quantum white noises in terms of integral kernel operators, *Infinite dimensional analysis, quantum probability and related topics*, **1**, No. 4 (1998), 533–559.
10. Ettaieb, A., Ouerdiane, H., and Rguigui, H.: Cauchy Problem and Integral Representation Associated to the Power of The QWN-Euler Operator, *Communication on Stochastic Analysis*, **6**, No. 4 (2012), 615–627.
11. Ettaieb, A., Ouerdiane, H., and Rguigui, H.: Powers of quantum white noise derivatives, *Infinite Dimensional Analysis Quantum Probability and Related Topics*, **17**, 1450018 (2014) [16 pages].
12. Gannoun, R., Hachaichi, R., Ouerdiane, H., and Rezgi, A.: un théorème de dualité entre espace de fonction holomorphes à croissance exponentielle, *J. Funct. Anal.*, **171**, (2000), 1–14.
13. Ji, U. C. and Obata, N.: Generalized white noise operator fields and quantum white noise derivatives, *Séminaires & Congrès*, **16**, (2007), 17–33.
14. Ji, U. C. and Obata, N.: Annihilation-derivative, creation-derivative and representation of quantum martingales, *Commun. Math. Phys.*, **286**, (2009), 751–775.
15. Ji, U. C. and Obata, N.: Implementation problem for the canonical commutation relation in terms of quantum white noise derivatives, *Jornal of mathematical physics*, **51**, 123507 (2010).
16. Ji, U. C. Obata, N., and Ouerdiane, H: Analytic characterization of generalized Fock space operators as two-variable entire functions with growth condition, *Infinite dimensional analysis, quantum probability and related topics*, **5**, No. 3 (2002), 395–407.
17. Kuo, H.-H.: *White Noise Distribution Theory*, CRC press, Boca Raton 1996.
18. Obata, N.: *White Noise Calculus and Fock Spaces*, Lecture notes in Mathematics 1577, Spriger-Verlag 1994.
19. Obata, N.: Quantum white noise calculus based on nuclear algebras of entire function, *Trends in Infinite Dimensional Analysis and Quantum Probability* (Kyoto 2001), RIMS No. 1278, 130–157.
20. Ouerdiane, H.: Noyaux et symboles d'opérateurs sur des fonctionelles analytiques gaussiennes, *Japan. J. Math.*, **21**, No 1 (1995), 223–234.
21. Ouerdiane, H.: Algébres nucléaires de fonctions entiéres et équations aux derivées partielles stochastiques, *Nagoya Math. J.*, **151**, (1998), 107–127.
22. Ouerdiane, H. and Rguigui, H.: QWN-conservation operator and associated Wick differential equation, *Communications on Stochastic Analysis*, **6**, No. 3 (2012) 437–450.

Aymen Ettaieb: Department of Mathematics, Faculty of Sciences of Tunis University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail address: ettaieb97134@gmail.com

Habib Ouerdiane: Department of Mathematics, Faculty of Sciences of Tunis University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail address: habib.ouerdiane@fst.rnu.tn

Hafedh Rguigui: High School of Sciences and Technology of Hammam Sousse,, University of Sousse, Rue Lamine Abassi,, 4011 Hammam Sousse, Tunisia.
E-mail address: hafedh.rguigui@yahoo.fr