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AN ENTROPY APPROACH TO BOSE-EINSTEIN CONDENSATION

FRANCESCO DE VECCHI AND STEFANIA UGOLINI

Abstract. A proper *one-particle relative entropy* is introduced for the ground state of trapped Bose particles and its asymptotic behavior under the GP scaling limit is investigated. Some peculiar properties of the relative entropy allow to prove an existence theorem for the probability measure associated to the GP functional and a related weak convergence result.

1. Introduction

The Bose-Einstein Condensation (BEC) of a dilute gas of Bose-particles is a quantum phenomenon which qualitatively consists in reaching, when the temperatures are very close to the absolute zero under certain physical conditions, a well-defined state of matter, called BE Condensate. The BE Condensate is mainly characterized by the fact that a large fraction of bosons occupies the *same* lowest quantum state.

In Quantum Mechanics this quantum state is systematically described by a one-particle wave function which is the minimizer of the Gross-Pitaevski (GP) functional , usually denoted order parameter. As a consequence it solves a stationary cubic non linear equation, noted as GP equation ([14],[26]).

From the mathematical point of view, the GP model of the BE Condensate has been rigorously proved starting from the ground state of N interacting bosons and exploiting a peculiar scaling limit of infinite particles known as GP limit ([19],[18],[17]). In particular it has been proved an *Energy Theorem* which states that the mean ground state quantum energy of the N interacting bosons converges to the GP ground state energy in weak L^1 sense.

A recent stochastic approach to BEC ([20],[22],[23],[30]), based on Nelson's Stochastic Mechanics ([24],[25]), allows to describe the N interacting bosons with a system of N interacting Nelson-Carlen diffusions, having a symmetric stationary joint probability density.

On the other hand, being the GP functional a non linear one, a Nelson-Carlen diffusion cannot be rigorously associated to it. Therefore the problem of finding

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a stochastic process able to correctly describe the motion of the single particle in the BE Condensate is not trivial.

Anyway in [22], according to the Nelson map, a diffusion with a drift coefficient of gradient type uniquely associated to the order parameter has been introduced, denoted GP process, and it has been verified that it is a very good approximation of the true one.

A transition to chaos result for the probability measure corresponding to the N interacting Nelson-Carlen diffusions under the GP scaling limit has been stated in [30]. Asymptotically the n-particles marginal density ($n \geq 1$) weakly converges to the n product of the one-particle GP probability density given by the square of the order parameter.

In [22] it has also been introduced the relative entropy between the probability measure associated to the N interacting Nelson-Carlen diffusions and the probability measure given by the product measure of N times the well-defined single GP probability measure. Moreover it has been shown that the total relative entropy is given by the sum of N copies of a sort of one-particle relative entropy.

In this paper we introduced a well-defined *one particle relative entropy* and, exploiting a well-known chain-rule for relative entropy particularized for our case , we prove a useful upper bound estimate for our proper *one-particle relative entropy.*

From this estimate and the *Energy Theorem*([18]) we deduce an existence result for a probability measure which is rigorously associated to the GP functional through the GP probability density. Moreover we individuate for it a peculiar characterization: it is absolute continuous with respect to the relevant probability measure which can also be associated to the GP process. In addition, our existing probability measure admits as probability density exactly the GP density.

Finally, exploiting the deep connection between relative entropy and weak convergence, we prove the existence of a weakly converging subsequence of the one-particle marginal of our N interacting Nelson-Carlen diffusions.

2. Nelson-Carlen Diffusions and Bose-Einstein Condensation

Nelson's Stochastic Mechanics allows to study quantum phenomena using diffusion processes instead of the standard analytical tools of Quantum Mechanics ([24],[25]). See [6] for a recent review on Stochastic Mechanics.

We will briefly introduce the class of *Nelson* diffusions which are associated to a solution of a Schrödinger equation.

Let $\psi(x,t)$ be a solution of the equation:

$$i\partial_t \psi(x,t) = H\psi(x,t) \tag{2.1}$$

with $\psi(x,0) = \psi_0(x)$, corresponding to the Hamiltonian operator:

$$H = -\frac{\hbar^2}{2m}\triangle + V(x),$$

where m denotes the mass of a particle, and V is some scalar potential.

Denoting by:

$$u(x,t) = Re[\frac{\nabla\psi(x,t)}{\psi(x,t)}] \tag{2.2}$$

$$v(x,t) = Im[\frac{\nabla\psi(x,t)}{\psi(x,t)}] \tag{2.3}$$

when $\psi(x,t) \neq 0$ and, otherwise, both $u(x,t)$ and $v(x,t)$ are set equal to zero. Let us put

$$b(x,t) := u(x,t) + v(x,t) \tag{2.4}$$

In a more general approach ([5]) the Nelson diffusions can be seen as contained in the following diffusions class, which is characterized mainly by admitting the probability density and the symmetry for time-reversal.

The space of proper infinitesimal characteristics is the space of the pairs (v_t, ρ), where ρ_t is a time-dependent probability density on $\mathbb{R}^d$ and v_t a time-dependent vector field on $\mathbb{R}^3$ defined $\rho_t(x)dxdt - a.e$ so that:

$$\int_{\mathbb{R}^d} f(x,T)\rho(x,T)dx - \int_{\mathbb{R}^d} f(x,0)\rho(x,0)dx = \int_0^T \int_{\mathbb{R}^d} (v \cdot \nabla f)dx$$

for all $T \geq 0$ and all $f \in C_0^\infty(\mathbb{R}^{d+1})$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t)$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^3)$, be the evaluation stochastic process $X_t(\omega) = \omega(t)$, with $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the natural filtration.

Carlen ([3],[4],[5]) proved that if (v_t, ρ) is a proper infinitesimal characteristic and if the following *finite energy condition* holds:

$$\int_0^T (||\nabla\sqrt{\rho_t}||^2_{L^2} + ||v_t\sqrt{\rho_t}||^2_{L^2})dt < +\infty$$

for all $T \geq 0$ and all $f \in C_0^\infty(\mathbb{R}^{d+1})$, then there exists a unique Borel probability measure $\mathbb{P}$ on Ω such that

i)$(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P})$ is a Markov process;

ii) the image of $\mathbb{P}$ under X_t has density $\rho(t,x)$;

iii) $W_t := X_t - X_0 - \int_0^t b(X_s, s)ds$

is a $(\mathbb{P}, \mathcal{F}_t)$-Brownian Motion.

In this diffusions class the Nelson diffusions are properly those having:

$$\rho_t = \Psi_t \bar{\Psi}_t \qquad v_t = Im[\frac{\nabla\psi_t}{\psi_t}].$$

The continuity problem for the above Nelson-Carlen map is investigated in [9]. For a generalization to the case of Hamiltonian operators with magnetic potential see [27].

We adopt the following notations: capital letters for stochastic processes or, otherwise, we will explicitly specify them, $\hat{Y} = (Y_1, ..., Y_N)$ to denote arrays in $\mathbb{R}^{3N}$ and bold letters for vectors in $\mathbb{R}^3$.

The Hamiltonian introduced to describe the recent experiments ([15], [8]) on BEC is the following N-body Hamiltonian

$$H_N = \sum_{i=1}^N (-\frac{\hbar^2}{2m}\triangle_i + V(\mathbf{r_i})) + \sum_{1 \leq i < j \leq N} v(\mathbf{r_i} - \mathbf{r_j}) \tag{2.5}$$

where V is a confining potential and v a pair-wise repulsive interaction potential. As it is suitable for Bose particles it operates on symmetric wave functions in

$L^2(\mathbb{R}^{3N})$. Being the physical experiments realized at very low temperature, a ground state approach to (2.5) is physically justified.

We consider the mean quantum mechanical energy

$$E[\Psi] = T_\Psi + \Phi_\Psi, \tag{2.6}$$

where

$$T_\Psi = \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} |\nabla_i \Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N$$

is physically called the *kinetic energy* and

$$\Phi_\Psi = \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i)|\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N + \frac{1}{2}\sum_{i=2}^{N} \int v(\mathbf{r}_1 - \mathbf{r}_i)|\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N$$

the *potential energy*. The variational problem associated to H_N consists in minimizing $E[\Psi]$ with respect to the complex-valued function Ψ in $L^2(\mathbb{R}^{3N})$ subject to the constrain $\|\Psi\|_2 = 1$. If such a minimizing function Ψ_N^0 exists it is called a *ground state*. The corresponding energy $E_0[\Psi_N^0]$ given by

$$E_0[\Psi_N^0] := \inf\{E(\Psi) : \int |\Psi|^2 = 1\}$$

is known as *ground state energy*.

Under suitable assumptions on the potentials V and v one can prove the existence of the ground state Ψ_N^0 of (2.5). As concerns uniqueness of the ground state we mean that it is unique apart from an *overall phase*. For our purpose we need a strictly positive and continuous differentiable ground state. See [28] (Thm.XIII.46 and XIII.47) for the regularities conditions on the potentials V and v implying the strictly positivity and (XIII.11) for those implying the differentiability of the ground state wave function.

In the particular case of the ground state solution Ψ_N^0 the pairs of proper infinitesimal characteristics are of the form

$$(\rho_N, 0), \qquad \rho_N := |\Psi_N^0|^2, \qquad \sqrt{\rho_N} \in H_1(\mathbb{R}^{3N}).$$

Introducing the probability space $(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t)$, with $\hat{Y}_t(\omega) = \omega(t)$ the evaluation stochastic process, with $\mathcal{F}_t^N = \sigma(Y_s, s \leq t)$ the natural filtration, then by Carlen's Theorem there exists a unique Borel probability measure $\mathbb{P}_N$ such that

i)$(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t, \mathbb{P}_N)$ is a Markov process;

ii) the image of $\mathbb{P}_N$ under $\hat{Y}_t$ has density $\rho_N(\mathbf{r})$;

iii) $\hat{W}_t := \hat{Y}_t - \hat{Y}_0 - \int_0^t b_N(\hat{Y}_s)ds$,

where

$$b_N(\hat{Y}_t) = \frac{\nabla^{(N)}\Psi_N^0}{\Psi_N^0} = \frac{1}{2}\frac{\nabla^{(N)}\rho_N}{\rho_N}.$$

The stationary probability measure $\mathbb{P}_N$ with density ρ_N can be alternatively defined as the one associated to the Dirichlet form ([13]):

$$\epsilon_{\rho_N}(f,g) := \frac{1}{2}\int_{\mathbb{R}^{3N}} \nabla f(\mathbf{r}) \cdot \nabla g(\mathbf{r}) \rho_N d\mathbf{r}^{3N}, \qquad f, g \in C_c^\infty(\mathbb{R}^{3N}) \tag{2.7}$$

When Bose-Einstein condensation occurs, the condensate is usually described by the order parameter $\phi_{GP} \in L^2(\mathbb{R}^3)$, also called wave function of the condensate, which is the minimizer of the Gross-Pitaevskii functional

$$E^{GP}[\phi] = \int (\frac{\hbar^2}{2m}|\nabla\phi(r)|^2 + V(r)|\phi(r)|^2 + g|\phi(r)|^4)d\mathbf{r} \tag{2.8}$$

under the L^2-normalization condition

$$\int_{\mathbb{R}^3} |\phi^{GP}|^2 d\mathbf{r} = 1$$

and where $g > 0$ is a parameter depending on the interaction potential v (see also next assumption h3)). Therefore ϕ_{GP} solves the stationary cubic non-linear equation (in this context called Gross-Pitaevskii equation)

$$-\frac{\hbar^2}{2m}\triangle\phi + V\phi + 2g|\phi|^2\phi = \lambda\phi, \tag{2.9}$$

λ, the Lagrange multiplier of the normalization constraint, calling chemical potential. One can prove that ϕ_{GP} is continuously differentiable and strictly positive ([18]).

In [20] the stochastic quantization approach for the system of N interacting Bose particles has been exploited for the first time.

It is proved in [22] that the Stochastic Mechanics of the N-body problem associated to H_N uniquely determines a well defined stochastic process which describes the motion of the single particle in the condensate, in the case of the Gross-Pitaevskii scaling limit as introduced in [18], which allows to prove the existence of an exact Bose-Einstein condensation for the ground state of H^N ([18][16]). For the time-dependent derivation of the Gross-Pitaevskii equation see [1] and [11].

3. Mean Energy Rescaling According to the GP Limit

For simplicity of notations, we will put $\hbar = 2m = 1$. Let us consider the mean energy (2.6) expressed in terms of the joint probability density of our $3N-$dimensional process $\hat{Y}$ as:

$$E[\rho_N] = E\{\sum_{i=1}^{N}[b_i^2(\hat{Y}) + V(Y_i(t))] + \sum_{1\le i<j\le N} v(Y_i(t) - Y_j(t))\}$$

b_i being the drift of the interacting i-th particle, whose position is given by the process Y_i.

Following [18], we assume

h1) $V(|\mathbf{r_i}|)$ locally bounded, positive and going to infinity when $|\mathbf{r_i}|$ goes to infinity.

h2) v smooth, compactly supported, non negative, spherically symmetric, with finite *scattering length* a ([17] Appendix C).

We perform the following scaling, known as Gross-Pitaevskii (GP) scaling [18], writing

h3)

$$v(r) = v_1(\frac{r}{a})/a^2$$

$$a = \frac{g}{4\pi N}$$

where v_1 has scattering length equal to 1 and remains fixed while $N \uparrow +\infty$. Moreover $g > 0$ as a consequence of our assumptions on v.

The GP limit is a *dynamical* one, where the kinetic and potential energies remain comparable ([21]).

In [18] an important theorem is proven. We denote it as *Energy Theorem.*

Theorem 3.1 (Energy [18])**.** *Under the previous hypothesis h1),h2) h3) then*

$$\lim_{N\to\infty} \frac{E[\rho_N]}{N} = E[\rho_{GP}] \tag{3.1}$$

and

$$\lim_{N\to\infty} \int \rho_N d\mathbf{r}_2 \cdots \mathbf{r}_N = \rho_{GP} \tag{3.2}$$

where $\rho_{GP} := |\phi_{GP}|^2$, with ϕ_{GP} the minimizer of the Gross-Pitaevskii functional (2.8) *and the convergence is in weak $L^1(\mathbb{R}^3)$ sense.*

Moreover let ϕ_0 denote the solution of the zero-energy scattering equation for v (i.e. $-\triangle\phi_0(\mathbf{r}) + \frac{1}{2}v(\mathbf{r})\phi_0(\mathbf{r}) = 0$) under the boundary condition $lim_{|\mathbf{r}|\to+\infty}\phi_0(\mathbf{r}) = 1$ and $s = \int |\nabla\phi_0|^2/(4\pi a)$. Then $s \in (0, 1]$ and

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} |\nabla_1\sqrt{\rho_N}(\mathbf{r}_1, ..., \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^3} |\nabla\sqrt{\rho_{GP}}(\mathbf{r})|^2 d\mathbf{r} + \\ + gs\int_{\mathbb{R}^3} (\rho_{GP}(\mathbf{r}))^2 d\mathbf{r} \tag{3.3}$$

and, moreover,

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} V(\mathbf{r})\rho_N(\mathbf{r}_1, ..., \mathbf{r}_N))d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int V(\mathbf{r})\rho_{GP}(\mathbf{r})d\mathbf{r} \tag{3.4}$$

$$\lim_{N\uparrow\infty} \frac{1}{2}\sum_{j=2}^{N} \int_{\mathbb{R}^{3N}} v(|\mathbf{r}_1 - \mathbf{r}_j|)\rho_N(\mathbf{r}_1, ..., \mathbf{r}_N)d\mathbf{r}_1 \cdots d\mathbf{r}_N = (1-s)g\int (\rho_{GP}(\mathbf{r}))^2 d\mathbf{r} \tag{3.5}$$

4. A Transition to Chaos Result

We recall some useful results concerning the asymptotic behavior of our N interacting diffusions $(Y_1, Y_2, ..., Y_N)$. The fixed time joint probability density of $(Y_1, \ldots, Y_N)$ is given by $\rho_N := |\Psi_N^0|^2$, which is invariant under spatial permutations. In [22] it has been proved that if Ψ_N^0 is the ground state of H_N and it is strictly positive and of class C^1, then the three-dimensional processes $\{Y_i\}_{i=1,\ldots,N}$ are equal in law.

It has been proved in [30] the following:

Corollary 4.1. *For $N \uparrow +\infty$ the n-particle marginal density ($n \geq 1$)*

$$\rho_N^{(n)} := \int \rho_N d\mathbf{r_{n+1}} \cdots d\mathbf{r_N}$$

is such that

$$lim_{N\uparrow+\infty}\rho_N^{(n)} = \rho_{GP}^{\otimes n}$$

in the weak convergence sense.

With the usual notation

$$< \mu, \phi >= \int \phi(x)\mu(dx)$$

for a probability measure μ on Ω and $\phi \in C_b(\Omega)$, we recall the following

Definition 4.2 ([29])**.** Let E be a separable metric space, $\mathbf{u}_N$ a sequence of symmetric probability measures on E^N. We say that $\mathbf{u}_N$ is $\mathbf{u} -$ *chaotic*, $\mathbf{u}$ a probability measure on E, if for $\phi_1, \phi_2, ..., \phi_n \in C_b(E)$

$$lim_{N\uparrow+\infty} < \mathbf{u}_N, \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_n \otimes 1 \cdots 1 >= \prod_{i=1}^{n} < \mathbf{u}, \phi_i > \tag{4.1}$$

The above definition can be reformulated by considering the standard projection map of the $\mathbf{u}_N$ on E^n and saying that the projection converges to $\mathbf{u}^{\otimes n}$ when N goes to $+\infty$ ([2], p.20).

In [30] it has been established

Proposition 4.3. *Under the hypothesis h1),h2),h3), the symmetric law* $\mathbb{P}_N$ *is* $\mathbb{P}_{GP}$*-chaotic according with Definition 4.2.*

Let us now introduced the so called *empirical measure*(see for example [7])

$$Y_N(t) := \frac{\sum_{i=1}^{N} \delta_{Y_i(t)}}{N}$$

where for all i: $\delta_{Y_i(t)}$ is a random measure on $\mathcal{B}(\mathbb{R}^3)$ such that, for all $\phi \in C_0(\mathbb{R}^3)$

$$\int \phi(x)\delta_{Y_i(t)}(dx) = \phi(Y_i(t))$$

Therefore, finally, the *empirical measure* is such that for all $\phi \in C_0(\mathbb{R}^3)$

$$\int \phi(x)[Y_N(t)](dx) = \frac{\sum_{i=1}^{N} \phi(Y_i(t))}{N}$$

In particular, for $A \in \mathcal{B}(\mathbb{R}^3)$

$$[Y_N(t)](A) := \frac{\sharp\{Y_i(t) \in A\}}{N}$$

i.e. the *empirical measure* of a set A is the relative frequency of particles which stay in A at time t.

Following ([29]) it has been shown in [30] that the *chaotic* property in Proposition 4.3 implies a non trivial convergence result for the *empirical measures.*

Proposition 4.4. *The empirical measures* $Y_N(t) = \frac{\sum_{i=1}^{N} \delta_{X_i(t)}}{N}$ *converge in law to the constant random variable* $\mathbb{P}_{GP}$. *In particular one has for* $N \uparrow +\infty$ *and* $\forall \phi \in C_b(E)$*:*

$$E_{\rho_N}[(< Y_N - \mathbb{P}_{GP}, \phi >)^2] \to 0$$

5. One Particle Relative Entropy and Its Asymptotic Behavior

In this section the results contained in [22] and [23] are briefly recalled. Successively we prove some important consequences of the asymptotic behavior of the *one particle relative entropy*.

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω is $C(\mathbb{R}^+ \to \mathbb{R}^3)$, and $\mathcal{F}$ is its Borel sigma-algebra. We denote by $\hat{Y} := (Y_1, \ldots, Y_N)$ the coordinate process and by $\mathcal{F}_t^N$ the natural filtration.

Let us introduce a process X^{GP} with invariant density ρ_{GP}, which is the square of the order parameter, and try to compare it with the generic interacting *non* markovian diffusion $Y_1(t)$.

We assume that X^{GP} is a weak solution of the SDE

$$dX_t^{GP} := u_{GP}(X_t^{GP})dt + (\frac{\hbar}{m})^{\frac{1}{2}} dW_t \tag{5.1}$$

where,

$$u_{GP} := \frac{1}{2} \frac{\nabla \rho_{GP}}{\rho_{GP}}$$

We denote by $\mathbb{P}_N$ and $\mathbb{P}_{GP}^N$ the measures corresponding to the weak solutions of the $3N$- dimensional stochastic differential equations

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t \tag{5.2}$$

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{u}_{GP}(\hat{Y}_s) ds + \hat{W}'_t, \tag{5.3}$$

where

$$\hat{u}_{GP}(\mathbf{r}_1, \cdots, \mathbf{r}_N) = (u_{GP}(\mathbf{r}_1), \cdots, u_{GP}(\mathbf{r}_N)),$$

$\hat{Y}_0$ is a random variable with probability density equal to ρ_N, while $\hat{W}_t$ and $\hat{W}'_t$ are $3N$-dimensional $\mathbb{P}_N$ and $\mathbb{P}_{GP}^N$ standard Brownian Motions, respectively.

In this section we use the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$ and $\hat{u}_s^N =: \hat{u}_{GP}(\hat{Y}_s)$.

Following [22] we now compute the relative entropy between the three-dimensional *one-particle* non markovian diffusion Y_1, first component of the interacting diffusions system given by (5.2), and X^{GP}.

In order to use Girsanov Theorem, we will assume that u_{GP} is bounded. We recall that under our hypothesis on the potentials v and V, ρ_{GP} is strictly positive and in $C^1(\mathbb{R}^3) \bigcap L^\infty(\mathbb{R}^3)$ and therefore $u_{GP} \in L^2(\mathbb{R}^3)$ (see [18], Thm 2.1). Then the following finite energy conditions hold:

$$E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty \tag{5.4}$$

$$E_{\mathbb{P}_N} \int_0^t \| \hat{u}_s^{GP}\|^2 ds < \infty, \tag{5.5}$$

which follow from the fact that Ψ_N^0 is the minimizer of $E^N[\Psi]$, and our hypothesis on u_{GP}. It is well known that these are also *finite entropy conditions* which imply that $\forall t \geq 0$

$$\mathbb{P}_N|_{\mathcal{F}_t} \ll \hat{W}|_{\mathcal{F}_t}, \quad \mathbb{P}^N_{GP}|_{\mathcal{F}_t} \ll \hat{W}'|_{\mathcal{F}_t}$$

Then, by Girsanov's theorem, we have, for all $t > 0$,

$$\frac{d\mathbb{P}_N}{d\mathbb{P}^N_{GP}}|_{\mathcal{F}_t} = \exp\{-\int_0^t (\hat{b}^N_s - \hat{u}^{GP}_s) \cdot d\hat{W}_s + \frac{1}{2}\int_0^t \|\hat{b}^N_s - \hat{u}^{GP}_s\|^2 ds\}, \tag{5.6}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^{3N}$. The relative entropy restricted to $\mathcal{F}_t$ reads

$$\mathcal{H}(\mathbb{P}_N, \mathbb{P}^N_{GP})|_{\mathcal{F}_t} \;=:\; \mathbb{E}_{\mathbb{P}_N}[\log \frac{d\mathbb{P}_N}{d\mathbb{P}^N_{GP}} \;\; |_{\mathcal{F}_t}] \;=\; \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|\hat{b}^N_s \;-\; \hat{u}^{GP}_s\|^2 ds \tag{5.7}$$

Since under $\mathbb{P}_N$ the $3N$-dimensional process $\hat{Y}$ is a solution of (5.2) with invariant probability density ρ_N , we can write, recalling also (5.4) and (5.5),

$$\begin{aligned}\frac{1}{2} E_{\mathbb{P}^N} \int_0^t \|\hat{b}^N_s - \hat{u}^{GP}_s\|^2 ds & \\ &= \frac{1}{2}\int_0^t E_{\mathbb{P}_N} \|\hat{b}^N_s - \hat{u}^{GP}_s\|^2 ds \\ &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \|\hat{b}^N(\mathbf{r}_1, \ldots, \mathbf{r}_N) - \hat{u}_{GP}(\mathbf{r}_1, \ldots, \mathbf{r}_N)\|^2 \rho_N d\mathbf{r}_1 \ldots d\mathbf{r}_N\end{aligned} \tag{5.8}$$

so that we get

$$\begin{aligned}\mathcal{H}(\mathbb{P}_N, \mathbb{P}^N_{GP})|_{\mathcal{F}_t} & \\ &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \sum_{i=1}^N \|b^N_i(\mathbf{r}_1, \ldots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_i)\|^2 \rho_N d\mathbf{r}_1 \ldots d\mathbf{r}_N \\ &= \frac{1}{2} N t \int_{\mathbb{R}^{3N}} \|b^N_1(\mathbf{r}_1, \ldots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 \rho_N d\mathbf{r}_1 \ldots d\mathbf{r}_N \\ &= \frac{1}{2} N E_{\mathbb{P}_N} \int_0^t \|b^N_1(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds,\end{aligned} \tag{5.9}$$

where the symmetry of $\hat{b}^N$ and ρ_N has been exploited.

Finally we get the sum of a sort of N identical one-particle relative entropies, each of them being defined by

$$\begin{aligned}\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}^N_{GP})|_{\mathcal{F}_t} &=: \frac{1}{N}\mathcal{H}(\mathbb{P}_N, \mathbb{P}^N_{GP})|_{\mathcal{F}_t} \\ &= \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|b^N_1(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds\end{aligned} \tag{5.10}$$

By the *Energy Theorem* we can deduce that for any $t > 0$ the one particle relative entropy does not go to zero in the scaling limit but it is asymptotically *finite*. Here we want to take advantage of this non trivial fact.

In particular from (3.3) we obtain that:

$$lim_{N\uparrow+\infty}\bar{\mathcal{H}}(\mathbb{P}_N,\mathbb{P}^N_{GP})|_{\mathcal{F}_t} =: \frac{1}{N}\mathcal{H}(\mathbb{P}_N,\mathbb{P}^N_{GP})|_{\mathcal{F}_t}$$
$$= lim_{N\uparrow+\infty}\frac{1}{2}E_{\mathbb{P}_N}\int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds = C \quad (5.11)$$

where C is a positive constant.

In the next Lemma we extend to our case a useful chain-rule for the relative entropy.

Lemma 5.1. *We consider $M = X \times Y$, where X and Y are Polish spaces. Let $\mathbb{P}$ be a measure on M and $\mathbb{Q}_1$ and $\mathbb{Q}_2$ measures on X and Y respectively. We denote by $\mathbb{Q} = \mathbb{Q}_1 \otimes \mathbb{Q}_2$ the product measure on M of the measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ and we suppose that $\mathbb{P} \ll \mathbb{Q}$. Then we have*

$$\mathcal{H}(\mathbb{P}|\mathbb{Q}) \geq \mathcal{H}(\mathbb{P}_1|\mathbb{Q}_1) + \mathcal{H}(\mathbb{P}_2|\mathbb{Q}_2), \quad (5.12)$$

where $\mathbb{P}_1$ and $\mathbb{P}_2$ are the marginal probabilities of $\mathbb{P}$.

Proof. Following [10] we can prove that:

$$\bar{\mathcal{H}}(\mathbb{P}|\mathbb{Q}) \geq \bar{\mathcal{H}}(\mathbb{P}_1|\mathbb{Q}_1) + \int_X \bar{\mathcal{H}}(\mathbb{P}(\cdot|x)|\mathbb{Q}(\cdot|x)\mathbb{P}_1(dx), \quad (5.13)$$

where $\mathbb{P}(\cdot|x)$ and $\mathbb{Q}(\cdot|x)$ are stochastic kernels on Y given X for which the following decompositions hold:

$$\mathbb{P}(dx \times dy) = \mathbb{P}_1(dx) \otimes \mathbb{P}(dy|x) \quad \mathbb{Q}(dx \times dy) = \mathbb{Q}_1(dx) \otimes \mathbb{Q}(dy|x).$$

Since $\mathbb{P} \ll \mathbb{Q}$ we also have $\mathbb{P}_i \ll \mathbb{Q}_i$ for $i = 1, 2$, and by Radon-Nykodim Theorem:

$$f(x,y) := \frac{d\mathbb{P}}{d\mathbb{Q}}(x,y)$$
$$\Psi(x) := \frac{d\mathbb{P}_1}{d\mathbb{Q}_1} = \int_Y f(x,y)\mathbb{Q}(dy|x)$$
$$\Phi(y) := \frac{d\mathbb{P}_2}{d\mathbb{Q}_2} = \int_Y f(x,y)\mathbb{Q}(dx|y)$$

For any Borel subsets A of X and B of Y:

$$\mathbb{P}(A \times B) = \int_A \mathbb{P}(B|x)\mathbb{P}_1(dx) = \int_A \mathbb{P}(B|x)\Psi(x)\mathbb{Q}_1(dx)$$
$$= \int_A\int_B f(x,y)\mathbb{Q}(dx \times dy) = \int_A(\int_B f(x,y)\mathbb{Q}(dy|x))\mathbb{Q}_1(dx).$$

Therefore there exists a $\mathbb{Q}_1-$null set Γ such that for all $x \in \Gamma^c$

$$\Psi(x)\mathbb{P}(B|x) = \int_B f(x,y)\mathbb{Q}(dy|x).$$

As a consequence for all $x \in \Gamma^c \cap \{\Psi > 0\}$ $\quad \mathbb{P}(\cdot|x) \ll \mathbb{Q}(\cdot|x)$ and

$$\zeta(x,y) := \frac{d\mathbb{P}(\cdot|x)}{d\mathbb{Q}(\cdot|x)} = \frac{f(x,y)}{\Psi(x)}.$$

We have $\mathbb{P}_1(\{\Psi > 0\}) = 1$ and $\mathbb{Q}_1(\{\Gamma^c\}) = 1$ and, by absolute continuity, $\mathbb{P}_1(\{\Gamma^c\}) = 1$. So we obtain:

$$\mathcal{H}(\mathbb{P}|\mathbb{Q}) = \int log(f(x,y))\mathbb{P}(dx \times dy) = \int log(f(x,y))\mathbb{P}_1(dx)\mathbb{P}(dy|x)$$

$$= \int_{(\Gamma^c \cap \{\Psi > 0\} \times Y} log(\zeta(x,y)\Psi(x))\mathbb{P}_1(dx)\mathbb{P}(dy|x)$$

$$= \int_{(\Gamma^c \bigcap \{\Psi>0\}} log(\Psi(x))\mathbb{P}_1(dx) + \int_{(\Gamma^c \cap \{\Psi>0\}} (\int_Y log(\zeta(x,y))\mathbb{P}(dy|x))\mathbb{P}_1(dx)$$

$$= \mathcal{H}(\mathbb{P}_1|\mathbb{Q}_1) + \int_X \mathcal{H}(\mathbb{P}(\cdot|x)|\mathbb{Q}(\cdot|x))\mathbb{P}_1(dx) \quad (5.14)$$

Since in our case $\mathbb{Q}(dy|x) = \mathbb{Q}_2(dy)$ we can rewrite the second integral as:

$$\int_X (\int_Y log(\zeta(x,y))(\zeta(x,y))\mathbb{Q}_2(dy))\mathbb{P}_1(dx)$$

$$= \int_Y (\int_X log(\zeta(x,y))(\zeta(x,y))\mathbb{P}_1(dx))\mathbb{Q}_2(dy))$$

$$\geq \int_Y [log(\int_X \zeta(x,y)\mathbb{P}_1(dx)) \int_X \zeta(x,y)\mathbb{P}_1(dx)]\mathbb{Q}_2(dy)$$

where Jensen's inequality applying to the convex function $xlogx$ has been used. We now observe that:

$$\int_X \zeta(x,y)\mathbb{P}_1(dx) = \int_X \zeta(x,y)\Psi(x)\mathbb{Q}_1(dx)$$

$$= \int_X f(x,y)\mathbb{Q}_1(dx) = \int_X f(x,y)\mathbb{Q}(dx|y)$$

Finally, recognizing that:

$$\int_X f(x,y)\mathbb{Q}(dx|y) = \Phi(y) = \frac{d\mathbb{P}_2}{d\mathbb{Q}_2}$$

the proof is completed. □

We now consider the one-particle marginal $\mathbb{P}_N^1$ of the symmetric measure $\mathbb{P}_N$, defined on Ω^N and we put $\mathbb{P}_{GP} := \mathbb{P}_{GP}^1$ on Ω.

The following theorem holds.

Theorem 5.2 (Marginal Entropy Estimate)**.** *For the relative entropy of the one-particle marginal $\mathbb{P}_N^1$ versus the measure $\mathbb{P}_{GP}$ one has:*

$$\mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) \leq \frac{1}{N}\mathcal{H}(\mathbb{P}_N|\mathbb{P}_{GP}^N) \quad (5.15)$$

Proof. We can use Lemma 5.1. In fact if we decompose $\Omega^N = \Omega \times \Omega^{N-1}$ the hypothesis in Lemma 5.1 are satisfied and we obtain:

$$\mathcal{H}(\mathbb{P}_N|\mathbb{P}_{GP}^N) \geq \mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) + \mathcal{H}(\mathbb{P}_N^{N-1}|\mathbb{P}_{GP}^{N-1}), \quad (5.16)$$

where with $\mathbb{P}_N^{N-1}$ we denote the marginal of $\mathbb{P}_N$ with respect to $N-1$ particles and, of course, for the symmetry of $\mathbb{P}_N$ it does not matter which particles we take. Applying again Lemma 5.1 we have:

$$\mathcal{H}(\mathbb{P}_N^{N-1}|\mathbb{P}_{GP}^{N-1}) \geq \mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) + \mathcal{H}(\mathbb{P}_N^{N-2}|\mathbb{P}_{GP}^{N-2}), \quad (5.17)$$

So with $N-2$ consecutive applications of Lemma 5.1 we obtain the thesis. □

The next theorem states that in the GP scaling limit there exists an asymptotic probability measure which is absolutely continuous with respect to the probability measure $\mathbb{P}_{GP}$, weak solution of the three-dimensional SDE (5.1).

Theorem 5.3 (Existence Theorem). *On the space $(\Omega, \mathcal{F}, \mathbb{P}_{GP})$ there exists a probability measure $\hat{\mathbb{P}}$ such that:*

i) $\mathbb{P}^1_{N_j}$ weakly converges to $\hat{\mathbb{P}}$ for some subsequence $\mathbb{P}^1_{N_j}$ of $\mathbb{P}^1_N$;

ii) $\hat{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}_{GP}$;

iii) $\hat{\mathbb{P}}$ admits a probability density given by ρ_{GP}.

Proof. As a consequence of the non trivial asymptotic behavior described in (5.11) and from Theorem 5.3 we have that there exists a constant $A > 0$ such that for all N one has:

$$\mathcal{H}(\mathbb{P}^1_N | \mathbb{P}_{GP}) < A$$

It is well-known that the relative entropy has the crucial property of the compactness of level sets. More precisely, for each probability measure $\nu \in \mathcal{P}(\Omega)$, with $\mathcal{P}(\Omega)$ denoting the space of all probability measures on Ω, $\mathcal{H}(\cdot|\nu)$ has compact level sets, that is, for each $A < +\infty$ the set $\{\mu : \mathcal{H}(\mu|\nu) \leq A\}$ is a compact subset of $\mathcal{P}(\Omega)$. Therefore from the weak compactness of the sub-levels of the relative entropy (see [10] Lemma 1.4.3 pp.29) there exists a probability measure $\hat{\mathbb{P}}$ and a subsequence $\mathbb{P}^1_{N_j}$ of $\mathbb{P}^1_N$ such that when j goes to infinity

$$\mathbb{P}_{N_j} \longrightarrow \hat{\mathbb{P}}$$

in the weak convergence sense.

Moreover, since:

$$\mathcal{H}(\hat{\mathbb{P}} | \mathbb{P}_{GP}) < liminf \mathcal{H}(\mathbb{P}^1_{N_j} | \mathbb{P}_{GP}) \leq A$$

we have that $\hat{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}_{GP}$.

Finally, being $\hat{\mathbb{P}}$ absolutely continuous with respect to $\mathbb{P}_{GP}$, $\hat{\mathbb{P}}$ is also absolutely continuous with respect to the Lebesgue measure with density ρ_{GP}. □

References

1. Adami, R., Golse F., and Teta, A.: Rigorous derivation of the cubic NLS in dimension one, *Journal of Statistical Physics* **127** (2007), no. 6, 1193–1220.
2. Billingsley, P.: *Convergence of probability measures*, second edition, John Wiley and Sons, New York, 1999.
3. Carlen, E.: Conservative diffusions, *Commun. Math. Phys.* **94** (1984), no. 3, 293–315.
4. Carlen, E.: Existence and Sample path Properties of the Diffusions in Nelson's Stochastic Mechanics, in: *Lecture Notes in Mathematics, Vol. 1158*, (1985) 25–51, Springer, Berlin, Heidelberg, New York.
5. Carlen, E.: Progress and Problems in Stochastic Mechanics, in *Stochastic Methods in Mathematical Physics.*, (1989), Word Scientific, Singapore.
6. Carlen, E.: Stochastic Mechanics: a Look Back and a Look Ahead, in: *Diffusion, Quantum Theory and Radically Elementary Mathematics*, chapter 5 (2006), Princeton University Press, Princeton.

7. Capasso, V. and Bakstein, D.: *An introduction to Continuous-Time Stochastic Processes. Theory, Models, and Applications to Finance, Biology, and Medicine* second edition, Birkhäuser, Boston, 2012.
8. Cornell, E. A. and Wieman, C. E.: Bose-Einstein condensation in a dilute gas: the first 70 years and some recent experiments (Nobel Lecture) *Chemphyschem* **3** (2002), no. 6, 473–493.
9. Dell'Antonio, G. and Posilicano, A.: Convergence of Nelson Diffusions, *Comm. Math. Phys.* **141** (1991), no. 3, 559–576.
10. Dupuis, P. and Ellis, R. S.: 1997 *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley Series in Probability and Statistics, John Wiley & Sons, New York, 1997.
11. Erdos, L., Schlein, B., and Yau H. T.: Rigorous derivation of the Gross-Pitaevskii equation, *Phys. Rev. Lett.* **98** (2007), no. 4, 040404, 1–4.
12. Föllmer, H.: Random Fields and Diffusion Processes, in: *Lecture Notes in Mathematics* **1362**, (1988) 101–203, Springer, Berlin.
13. Fukushima, M.: *Dirichlet Forms and Markov Processes*, North-Holland, Amsterdam, 1980.
14. Gross, E. P.: Structure of a quantized vortex in boson system, *Nuovo Cimento* **20** (1961), no. 3, 454–477.
15. Ketterle, W. and van Druten, N. J.: Evaporative Cooling of Trapped Atoms, in: *Advances in Atomic, Molecular and Optical Physics* **37**, (1996) 181–236, Academic Press, S. Diego.
16. Lieb, E. H. and Seiringer, R.: Proof of Bose-Einstein condensation for dilute trapped gases *Phys. Rev. Lett.* **88** (2002), 170409, 1–4.
17. Lieb, E. H., Seiringer, R., Solovej, J. P., and Yngvason, J.: *The Mathematics of the Bose Gas and its Condensation*, Birkhäuser Verlag, Basel, 2005.
18. Lieb, E. H., Seiringer, R., and Yngvason, J.: Bosons in a trap: a rigorous derivation of the Gross-Pitaevskii energy functional, *Phys. Rev. A* **61** (2000), 043602, 1–13.
19. Lieb, E. H. and Yngvason, J.: Ground State Energy of the Low Density Bose Gas, *Phys. Rev. Lett.* **80** (1998), no. 12, 2504–2507.
20. Loffredo, M. and Morato, L. M.: Stochastic Quantization for a system of N identical interacting Bose particles. *J Phys. A: Math. Theor* **40** (2007), no. 30, 8709.
21. Michelangeli, A.: *Bose-Einstein Condensation: analysis of problems and rigorous results*, Ph.D.Thesis, SISSA, Italy, 2007.
22. Morato, L. M. and Ugolini, S.: Stochastic Description of a Bose-Einstein Condensate *Annales Henry Poincaré* **12** (2011), no. 8, 1601–1612.
23. Morato, L. M. and Ugolini, S.: Localization of relative entropy in Bose-Einstein Condensation of trapped interacting bosons, in:*Seminar on Stochastic Analysis, Random Fields and Applications VII* **67**, (2013) 197–210, Birkhäuser, Basel.
24. Nelson, E.: *Dynamical Theories of Brownian Motion*, Princeton University Press, Princeton, 1967.
25. Nelson, E.: *Quantum Fluctuations*, Princeton University Press, Princeton, 1985.
26. Pitaevskii, L. P.: Vortex lines in an imperfect Bose gas, *Sov. Phys.-JETP* **13** (1961), 451–454.
27. Posilicano, A. and Ugolini, S.: Convergence of Nelson Diffusions with time-dependent Electromagnetic Potentials, *J. Math. Phys.* **34** (1993), 5028–5036.
28. Reed, M. and Simon, B.: *Analysis of Operators, Methods of Modern Mathematical Physics Vol. IV*, Academic Press, S. Diego, 1978.
29. Sznitman, A. S.: Topics in propagation of chaos in: *Lecture notes in mathematics* **1464**, (1991) 164–251, Springer.
30. Ugolini, S.: Bose-Einstein Condensation: a transition to chaos result, *Communications on Stochastic Analysis* **6** (2012), no. 4, 565–587.

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