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Robust Control of Uncertain Time -Delay Systems.

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ROBUST CONTROL OF UNCERTAIN TIME-DELAY SYSTEMS

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Abstract

Time-delay systems are common in industries. Direct analysis and synthesis of control systems with time delays are complicated and approximation methods such as Pade approximation are usually applied. However, the issues of control system robustness with respect to model uncertainties and approximation errors have not been sufficiently addressed.

This dissertation focus on robustness of time-delay systems, especially robustness with respect to time delays, which has been discussed extensively using Lyapunov second method. We propose two methods in this dissertation to reformulate the problems into standard μ or $\mathcal{H}_\infty$ problems. The first method involves representing the systems in linear functional transformation (LFT) framework and approximating delays by rational transfer functions. The approximation errors are then treated as uncertainties. We show that all the well-known techniques of $\mathcal{H}_\infty$ control theory can be applied to this framework. Consequently, controller design becomes a routine process. We also show that the conventional Lyapunov method is a special case in our proposed framework and our proposed method offers less conservative results. In the second method, we treat uncertain delays as uncertainties with restricted phase angles and extend structured singular value to include phase information. We show that the extended small-μ theorem can be applied to analyze stability and performance of uncertain delay systems with many other type of uncertainties, such as plant model uncertainties and parametric uncertainties.

Finally, we generalize the above techniques to linear systems with feedback connected nonlinear elements. Both time invariant and time-varying nonlinearities are discussed by incorporating circle/Popov criterion with small-μ theorem.

Chapter 1

Introduction

Comparing with classical control, modern $\mathcal{H}_\infty$ control theory is a much more powerful tool to deal with system robustness issues, for instance, noise rejection and system perturbations problems. However, there are still many difficulties in applying modern control theories to industrial processes. One of the difficulties comes from time-delays in industrial systems. Indeed, not only many physical systems include time delays in their mechanism, but also signal transmission or transport delays are common in industrial processes. Many applications can be found in neural network, in control of satellite devices, in chemical reactors, in technological systems (wind tunnel, automatic steering of high velocity aircraft, antirolling stabilization systems in ships) etc. ([4, 58]).

Although many advanced control theories have been developed in the past fifty years, control engineers still don't have many tools when they face time-delay systems. Since most existing control theories are developed for finite dimensional systems, they are not valid to infinite dimensional time-delay systems. Even if they are, the computation involved is usually very complicated ([13, 16, 34, 47, 50, 51, 56, 67, 73]). Hence, control engineers usually have no choices but to use classical methods, such as Pade approximation or Smith predictor to perform system design, which can't tackle many important robustness issues for time-delay systems.

Motivated by the need of improving time-delay control systems design, we focus on the stability and performance analysis of uncertain time-delay systems in this dissertation.

Time delays may introduce some interesting features into systems. Consider a simple time-delay system shown in following Figure 1.1 (see [49]).

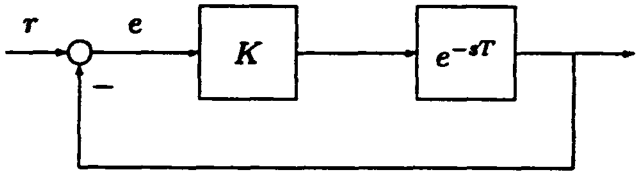


Figure 1.1: A Pure Delay Feedback System

Suppose that a unit step input is applied. If $T = 0$, then internal signal $e(t)$ is identical to $1/(1+K)$, which is stable. However, if $T > 0$, the signal $e(t)$ depends on value of K.

$$e(t) = \begin{cases} 1 & ,0 < t < T \\ 1-K & ,T < t < 2T \\ 1-K(1-K) & ,2T < t < 3T \\ \vdots & \end{cases}.$$

If $K > 1$, the signal $e(t)$ goes unstable. If $K = 1$, $e(t)$ jumps between 0 and 1 back and forth. When $0 < K < 1$, $e(t)$ approaches to $1/(1+K)$ as time goes to infinity.

This example shows that time delays may complicate the systems design. It also illustrates that ignoring delays in the system design may destabilize systems, or drag the systems to poor performance. For analysis and synthesis with fixed known delays, frequency domain stability analysis techniques, namely, Root-locus method, Nyquist criterion, and Bode plot can be directly applied ([48, 49]) even though the computation is very complicated. However, if systems involved with uncertainties,

for instance, delay T in Figure 1.1 is uncertain, these methods can't be effectively used.

Consider a time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau),$$

where $x(t) \in \mathcal{R}^n$, and $\tau > 0$. First, assume that the system is stable without delayed states, i.e., $\dot{x}(t) = (A + A_d)x(t)$ is stable. Then, intuitively, we may conjecture that there exists an interval $[0, h)$ such that the time-delay system $\dot{x}(t) = Ax(t) + A_d x(t-\tau)$, $\tau \in [0, h)$ is stable but unstable when $\tau \geq h$. This property can be seen as robustness property with respect to delays.

The system is said to be delay-dependent stable, if it is stable for delay $\tau \in [0, h)$, where h is a finite real number, and delay-independent stable if it is stable for all $\tau \in \mathcal{R}^+$. More generally, a delay system may be unstable for delay $\tau \in [0, h_1] \cup [h_2, \infty)$, but stable for $\tau \in (h_1, h_2)$. Stability region of a multiple time-delay systems may be arbitrary and the characterization of the stability regions is still an open problem.

The problems we are interested in this dissertation can be stated as

Problem 1.1 *Given an uncertain time-delay system*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\tau_i), \tag{1.1}$$

with an appropriate initial condition:

$$x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\bar{\tau}, 0], \quad \bar{\tau} = \max_{i=1,\ldots,\ell} \tau_i,$$

where A is stable, and $\tau_i > 0$ are unknown numbers. Find the largest h_i's in some sense, such that the system is stable for all $\tau_i \in [0, h_i)$.

Problem 1.2 *Given an uncertain time-delay system (1.1), and suppose that the system is stable for* $\tau_i = \underline{h}_i$. *Find the largest* $\bar{h}_i$*'s in some sense, such that system is stable for all* $\tau_i \in [\underline{h}_i, \bar{h}_i)$.

Much research have been done for stability analysis of uncertain time-delay systems. Both frequency and time domain approaches have been applied. In [10], Chen and Latchman stated the necessary and sufficient condition for delay-independent stability problems via small μ theorem. On the other hand, if the system is not delay-independent stable, only sufficient conditions are available to estimate the stability margins. These conditions are usually derived from time domain approach using Lyapunov second method. Since they are only sufficient, conservativeness of the sufficient conditions is another important issue. Much work have been done to derive the least conservative sufficient conditions, see [7, 8],[21]-[28], [31, 32], [35]-[40], [42]-[44], [53, 55, 57, 63, 69, 75, 76]. Exact stability margins for commensurate time delay systems can be obtained in [9].

In this dissertation, we use frequency domain approach to explore the stability problems. Uncertain delays are first approximated by stable rational transfer functions and the approximation errors are then treated as model uncertainties. Hence, the problems are converted into standard μ-analysis and synthesis problems for which well-known results can be applied. We also investigate one of existing results which derived from Lyapunov second method, and show that the results can be reformulated into our proposed framework. It's then easy to see the conservativeness.

Our second approach doesn't involve approximating uncertain delays, instead we take the phase information into account when we apply small μ theorem. Structured

singular values with phase information is derived first, and computational issues are considered also.

In the sections to follow, we shall review some basic system concept and some analysis tools.

1.1 Linear Dynamic Systems

In this dissertation, a linear time invariant dynamical system which we consider can be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $y(t) \in \mathcal{R}^p$ are the state, the input, and the output, respectively. The transfer function matrix from $u(t)$ to $y(t)$ is defined by $Y(s) = G(s)U(s)$, where

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

We say that $G(s)$ belongs to $\mathcal{H}_\infty$ if $G(s)$ is analytic and bounded in the open right-half plane, i.e., a stable transfer function matrix. $G(s)$ belongs to $\mathcal{RH}_\infty$ if $G(s)$ is a rational transfer function matrix and stable.

Let $G(s)$ be a $p \times p$ proper transfer function matrix, and stable. Then, $G(s)$ is called positive-real if

$$G(j\omega) + G^*(j\omega) \geq 0, \quad \text{for all } \omega \in \mathcal{R},$$

and strictly positive-real if

$$G(j\omega) + G^*(j\omega) > 0, \quad \text{for all } \omega \in \mathcal{R} \cup \{\infty\}.$$

In the following chapters, a system $G(s)$ may be represented in a general feedback configuration, as shown in Figure 1.2. Let $G(s)$ be the transfer function matrix from w to z. Suppose M can be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

and M has compatible dimensions with Δ. Then we can define upper linear fractional transformation (upper LFT),

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta\,(I - M_{11}\Delta)^{-1}\,M_{12}.$$

The transfer matrix from w to z is $G(s) = \mathcal{F}_u(M, \Delta)$. Similarly, we can define lower linear fractional transformation, $\mathcal{F}_l(M, \Delta)$, see [78].

$$\mathcal{F}_l(M, \Delta) = M_{11} + M_{12}\Delta\,(I - M_{22}\Delta)^{-1}\,M_{21}.$$

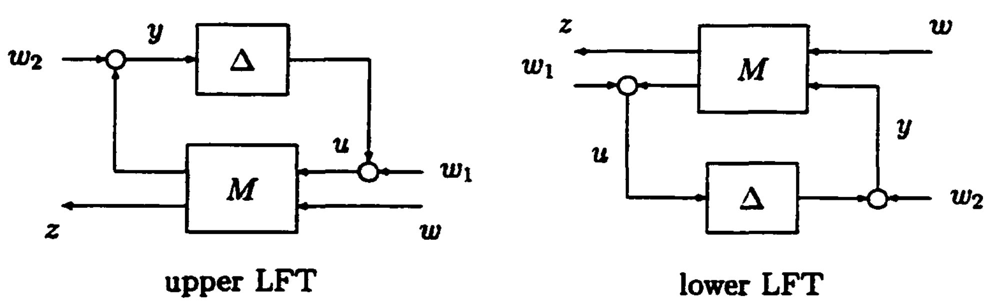


Figure 1.2: General Control Configuration

1.2 Lyapunov Second Method

We are now going to introduce some existing results on delay-independent/ dependent stability derived from Lyapunov second method. For simplicity, we only

consider a single delay in the system (1.1). Results can be easily extended to multiple delay cases. Note that many other Lyapunov functionals have been applied to reduce the conservativeness. However, it's not our concern here. The purpose of this section is to state a simple and clear result to introduce the concept of Lyapunov second method, so readers can compare the existing results to our results later.

1. **(Delay independent stability)** Consider the system (1.1), if the functional

$$V(x,t) = x^T(t)Px(t) + \int_0^\tau x^T(t-\theta)P_1x(t-\theta)d\theta,$$

where $P > 0,\ P_1 > 0$, satisfies $dV(x,t)/dt < 0$, then the system (1.1) is stable. It is easy to show that $dV(x,t)/dt = y(t)^TWy(t)$, where

$$W = \begin{bmatrix} A^TP + PA + P_1 & PA_d \\ A_d^TP & -P_1 \end{bmatrix}, \quad y(t) = \begin{bmatrix} x(t) \\ x(t-\tau_1) \end{bmatrix}.$$

Then a sufficient condition for stability is $W < 0$.

2. **(Delay dependent stability)** System (1.1) can be rewritten as

$$\dot{x}(t) = (A + A_d)\,x(t) + A_d \int_{t-\tau}^{t} (Ax(\theta) + A_dx(\theta-\tau))\,d\theta,$$

where $\tau \in [0, h)$ is uncertain. Consider the Lyapunov functional

$$V(x,t) = x^T(t)Px(t) + \int_{-\tau}^{0}\left(\int_{t+\xi}^{t} x^T(\xi)S_0x(\xi)d\xi\right)d\theta$$
$$+ \int_{-2\tau}^{-\tau}\left(\int_{t+\xi}^{t} x^T(\xi)S_1x(\xi)d\xi\right)d\theta.$$

Similar to delay-independent stability, system is delay-dependent stable for $\tau \in [0, h)$ if there exist $P > 0,\ S_0 > 0,\ S_1 > 0$, such that

$$W = \begin{bmatrix} (A + A_d)^T P + P(A + A_d) + h(S_0 + S_1) & hPA_dA & hPA_dA_d \\ hA^TA_d^TP & -hS_0 & 0 \\ hA_d^TA_dP & 0 & -hS_1 \end{bmatrix} < 0.$$

3. **(Robust stability)** In addition to uncertain delays, other types of uncertainties may also be included in the systems. For example, consider a system with parametric uncertainties

$$\dot{x}(t) = (A + \Delta A)\, x(t) + (A_d + \Delta A_d)\, x(t - \tau),$$

where uncertainties $\Delta A,\ \Delta A_d$ are bounded. Then, robust delay-independent/ dependent stability criteria with respect to uncertain delays and parametric uncertainties can be derived by Lyapunov second method. Various situations have been considered in literature, e.g., parametric uncertainties, time-varying uncertain delays and nonlinear time-delay systems. In each case, Lyapunov functionals may need to be modified to fit the specific condition to draw the conclusion.

1.3 SSV and Small-μ Theorem

Let $r = k_1 + \cdots + k_m + \cdots k_{m+n} + n_1 + \cdots + n_p$. Define

$$\mathbf{\Delta} := \{ \operatorname{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_m I_{k_m}, \delta_{m+1} I_{k_{m+1}}, \ldots, \delta_{m+n} I_{k_{m+n}}, \Delta_1, \ldots, \Delta_p) :$$

$$\gamma_i \in \mathbf{C},\ \delta_i \in \mathcal{R},\ \Delta \in \mathbf{C}^{n_i \times n_i} \}.$$

The structured singular value (SSV) at each frequency of a matrix $M(j\omega) \in \mathbf{C}^{r \times r}$ with respect to a block structure $\mathbf{\Delta}$ is defined to be $\mu_{\mathbf{\Delta}}(M(j\omega)) = 0$ if there is no

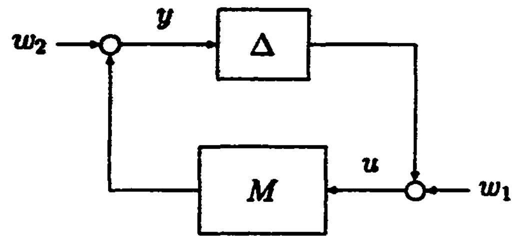


Figure 1.3: Feedback Interconnected System

$\Delta \in \mathbf{\Delta}$ such that $\det(I - \Delta M(j\omega)) = 0$, and

$$\mu_{\mathbf{\Delta}}(M) = \left(\min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) : \ \det(I - \Delta M(j\omega)) = 0\}\right)^{-1} \tag{1.2}$$

otherwise.

Consider a simple feedback interconnected system, in Figure 1.3. Let Δ and M be two real proper transfer matrices. The feedback interconnected system is said to be well-posed, if and only if the transfer matrix from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ exists and is proper. Moreover, the system is said to be internally stable, if the transfer matrix from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ belong to $\mathcal{H}_\infty$.

Theorem 1.1 [78] *Let $\beta > 0$. The feedback loop in Figure 1.2 is well-posed and internally stable for all $\Delta \in \mathbf{\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ if and only if*

$$\sup_{\omega \in \mathcal{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq \beta.$$

In most cases, only upper bounds of structured singular value can be computed. To compute an upper bound of structured singular value, we define:

$$\mathcal{D} = \left\{\operatorname{diag}\left[D_{k_1}, \ldots, D_{k_{m+n}}, d_1 I_{n_1}, \ldots, d_{n-1} I_{n_{p-1}}, I_{n_p}\right] \ : \ D_i \in \mathbf{C}^{k_i \times k_i},\right.$$

$$D_i = D_i^* > 0,\ d_i \in \mathcal{R},\ d_i > 0\},$$

and

$$\mathcal{G} = \left\{\text{diag}\,[0,\ldots,0,G_1,\ldots,G_n,0,\ldots,0]\ :\ G_i = G_i^* \in \mathbf{C}^{k_i \times k_i}\right\}.$$

Then, an upper bound of structured singular value of system M at each frequency can be computed by ([5])

$$\mu_\Delta(M) \leq \inf_{D\in\mathcal{D}} \min_{\beta} \left\{\beta\ :\ M^*DM + j(GM - M^*G) - \beta^2 D \leq 0\right\}.$$

1.4 Main Contributions

We have made contributions in several aspects. We first rewrite the uncertain time-delay system into LFT configuration. Under LFT, we can easily explain Chen's ([10]) result for delay-independent stability.

For delay-dependent stability, uncertain delays are approximated by rational transfer functions and the approximation errors are then treated as model uncertainties. Then Problem (1.1) and (1.2) are converted into μ-analysis problem where the well-known small-μ theorem can be applied. Examples show that our results are much less conservative than existing results which are derived by Lyapunov second method. We also reformulate one existing result into LFT configuration and explain the conservativeness.

We also show that robust controllers for uncertain time-delay systems can be obtained by standard $\mathcal{H}_\infty$ controller design since the nominal systems can now be represented by rational transfer functions.

Next, we consider the structured singular value with phase information and application to stability and performance analysis of uncertain time-delay systems. Robust stability and performance criteria are derived. The results are demonstrated

with two examples: an uncertain time-delay system with real constant parametric uncertainties and with time-varying parametric uncertainties.

Lastly, the structured singular value with phase information is applied to a specific nonlinear system, namely, a linear uncertain time-delay system with feedback connected nonlinear elements. Absolute delay-dependent/independent stability are discussed by incorporating circle/Popov criterion and small-μ theorem.

Chapter 2

Robust Stability Analysis and Synthesis by Using Approximation Method

In this chapter, we consider the robust stability analysis and controller design of uncertain time-delay systems by using approximation method. We shall first formulate the problem in LFT framework and then apply small-μ theorem to derive delay-independent/dependent stability criteria. Finally, controller designs are considered.

2.1 Delay-Independent Stability

Consider a time-delay system:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{\ell} A_i x(t - \tau_i), \tag{2.1}$$

where $\tau_i \in [0, h_i], i = 1, 2, \ldots, \ell$ are uncertain constants, such that $h_i \geq 0$ and $h_i > h_j$ for $i > j$, with initial condition $x(t_0 + \theta) = \phi(\theta)$, $\forall \theta \in [-\bar{\tau}, 0]$, $\bar{\tau} = \max_i\{\tau_i\}$.

For convenience, we shall denote $\mathcal{D}_\tau$ as the delay operator such that

$$\mathcal{D}_\tau \phi(t) = \phi(t - \tau)$$

for any scalar function $\phi(t)$.

We shall also assume that there is a $B_i \in \mathcal{R}^{n \times r_i}$ and a $C_i \in \mathcal{R}^{r_i \times n}$ such that

$$A_i = B_i C_i.$$

In particular, B_i and C_i can be chosen to have full rank so that $r_i = \text{rank}(A_i)$. Of course, results presented in this chapter do not necessarily require these factorizations be full rank. For example, one can always use a trivial factorization: $B_i = A_i$

and $C_i = I$. However, results may become computationally more difficult to apply when $r_i > \text{rank}(A_i)$. Moreover, denote

$$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_\ell \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_\ell \end{bmatrix}, \quad D = \text{diag}\{h_1 I_{r_1}, h_2 I_{r_2}, \ldots, h_\ell I_{r_\ell}\}. \tag{2.2}$$

It is obvious that the uncertain delay system (2.1) can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{\ell} B_i v_i(t) \\ z_i(t) &= C_i x(t) \\ v_i(t) &= \mathcal{D}_{\tau_i} I_{r_i} z_i(t) \end{aligned}$$

This system interconnection diagram is shown in Figure 2.1, where

$$G(s) = C\,(sI - A)^{-1}\,B.$$

Applying small μ theorem, a precise condition of delay-independent stability has been obtained in [10].

Theorem 2.1 *The uncertain delay system (2.1) is stable independent of delay if and only if A is stable, $\mu_\Delta(G(j\omega)) < 1, \forall\omega > 0$, and either (a) $\mu_\Delta(G(0)) < 1$ or (b) $\mu_\Delta(G(0)) = 1$, $\det(I - G(0)) \neq 0$ where $G(s) = C(sI - A)^{-1}B$, and $\Delta =$ $\text{diag}\{\gamma_1 I_{r_1}, \gamma_2 I_{r_2}, \ldots, \gamma_\ell I_{r_\ell}\}$.*

Example 2.1 Consider the uncertain time-delay system,

$$\dot{x}(t) = Ax(t) + BCx(t - \tau),$$

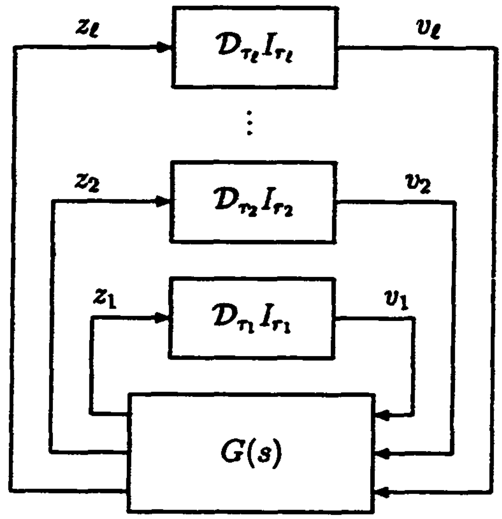


Figure 2.1: Interconnected Representation of the Delay System

where $\tau \in [0, h)$ and A is stable. Let $G(s) = C(sI - A)^{-1}B$, and suppose $\det(I - G(0)) \neq 0$. Then, it is clear that the system is stable if and only if

$$\det\left(I - G(s)e^{-s\tau}\right) = 0$$

has no solution in the closed right half plane for all $\tau \in [0, h)$. It's also clear that the system is stable if and only if $\lambda\left(G(j\omega)\right)e^{-j\omega\tau} \neq 1$, for all $\tau \in [0, h)$, and all ω.

Therefore, the system is delay-independent stable if $\rho\left(G(j\omega)\right) < 1$ for all ω. If $\rho\left(G(j\omega)\right) \geq 1$ at some frequency, the system will not be delay-independent stable. We can find the largest h, such that the system is stable for $\tau \in [0, h)$. Suppose $|\lambda_i\left(G(j\omega)\right)| = 1$ at some frequency $\omega = \omega_i > 0$. Then, the largest delay h is given by

$$h = \min_i \frac{2\pi + \angle\lambda_i\left(G(j\omega_i)\right)}{\omega_i}.$$

2.2 Delay-Dependent Stability

If an uncertain time-delay system is not delay-independent stable. Then, it's useful to find the maximal interval of delays that the system is stable. To that end, we need an alternative interconnected representation.

Notice that system (2.1) can be rewritten as

$$\begin{aligned}\dot{x}(t) &= \left(A+\sum_{i=1}^{\ell}A_i\right)x(t)+\sum_{i=1}^{\ell}A_i\left(x(t-\tau_i)-x(t)\right)\\ &= \left(A+\sum_{i=1}^{\ell}A_i\right)x(t)+\sum_{i=1}^{\ell}B_iC_i\left(\mathcal{D}_{\tau_i}-I\right)x(t).\end{aligned}$$

Hence, the system can be represented by

$$\begin{aligned}\dot{x}(t) &= \left(A+\sum_{i=1}^{\ell}A_i\right)x(t)+\sum_{i=1}^{\ell}B_iu_i(t)\\ y_i(t) &= C_ix(t)\\ u_i(t) &= \left(\mathcal{D}_{\tau_i}-1\right)I_{r_i}y_i(t)\end{aligned}$$

and shown in Figure 2.2.

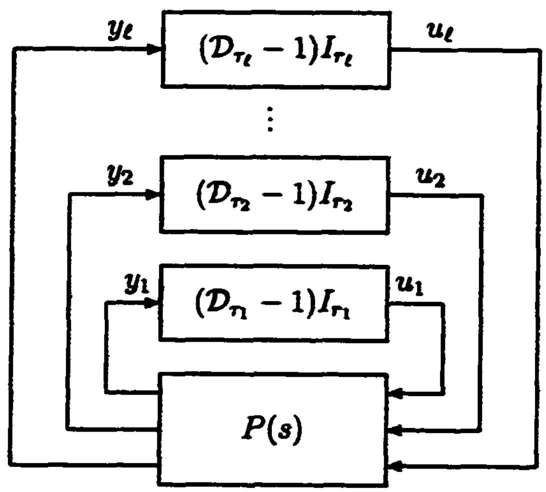


Figure 2.2: Reformulated Uncertain Delay System

Define

$$P(s) = C(sI - A - \sum_{i=1}^{\ell} A_i)^{-1}B.$$

Since the magnitude of uncertainty

$$\Delta = \text{diag}\left((e^{-\tau_1 s} - 1)I_{r_1}, \cdots, (e^{-\tau_\ell s} - 1)I_{r_\ell}\right)$$

is bounded by

$$L(w) = \text{diag}\left\{\ell_1(\omega)I_{r_1}, \cdots, \ell_\ell(\omega)I_{r_\ell}\right\},$$

where $\ell_i(\omega)$ bounds the magnitude of $e^{-j\omega\tau_i} - 1$.

$$\ell_i(\omega) \; := \; \max_{\tau_i \in [0,h_i]} \left| e^{-j\omega h_i} - 1 \right| \; = \; \max_{\tau_i \in [0,h_i]} 2 \left| \sin \frac{\omega h_i}{2} \right|.$$

Then,

$$\ell_i(\omega) = \begin{cases} 2 \sin \frac{h_i \omega}{2}, & \forall \, 0 \leq \omega \leq \pi/h_i \\ 2, & \forall \, \omega \geq \pi/h_i \end{cases} \tag{2.3}$$

which is plotted in Figure 2.3. Moreover, it is shown in [74] that all of the following transfer functions satisfy $\ell_i(\omega) \leq |v_{ij}(j\omega)|$, $j = 1, 2, 3, 4, 5$:

$$\begin{aligned} v_{i1}(s) &= h_i s \\ v_{i2}(s) &= \frac{h_i s}{h_i s/3.465 + 1} \\ v_{i3}(s) &= \frac{1.216 h_i s}{h_i s/2 + 1} \\ v_{i4}(s) &= \frac{h_i s(2 \times 0.2152^2 h_i s + 1)}{(0.2152 h_i s + 1)^2} \\ v_{i5}(s) &= \frac{h_i s}{h_i s/2 + 1} \frac{(h_i s/2.363)^2 + 1.676(h_i s/2.363) + 1}{(h_i s/2.363)^2 + 1.370(h_i s/2.363) + 1}. \end{aligned}$$

The frequency response of these transfer functions are shown in Figure 2.3.

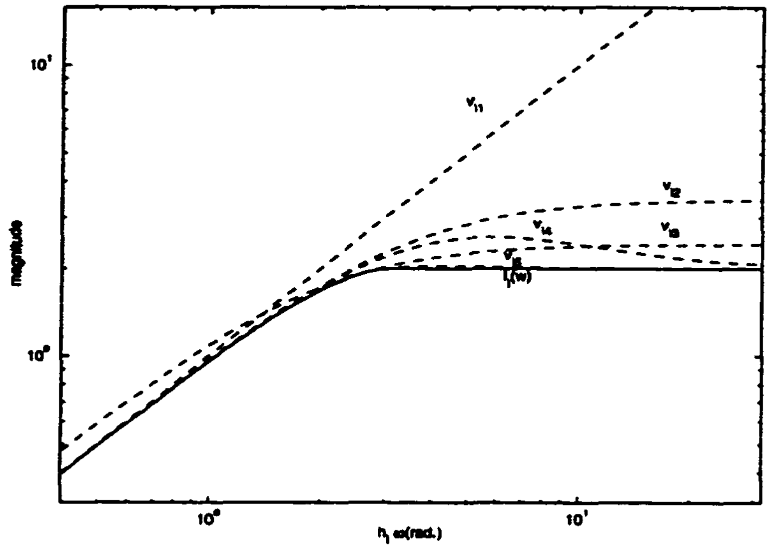


Figure 2.3: Frequency Response of $\ell_i(\omega)$ and v_{ij}

Lemma 2.2 *The uncertain delay system (2.1) is stable for $\tau \in [0, h)$, if the following system is robustly stable for all scalar $\delta_i(s) \in \mathcal{H}_\infty$ with $|\delta_i(j\omega)| \le \ell_i(\omega)$ such that*

$$
\begin{aligned}
\dot{x}(t) &= \left(A + \sum_{i=1}^{\ell} A_i\right) x(t) + \sum_{i=1}^{\ell} B_i u_i(t) \\
y_i(t) &= C_i x(t) \\
u_i(t) &= \delta_i(t) I_{r_i} y_i(t).
\end{aligned}
$$

The next theorem follows immediately by applying the small μ theorem for systems with frequency dependent uncertainty bounds to the system in Lemma 2.2.

Theorem 2.3 *The uncertain delay system (2.1) is robustly stable for all $\tau_i \in [0, h_i)$ if $\dot{x}(t) = \left(A + \sum_{i=1}^{\ell} A_i\right) x(t)$ is stable and either one of the following holds*

(a) $\mu_\Delta (L(\omega)P(j\omega)) < 1, \;\; \forall\, \omega$,

(b) $\mu_\Delta (L(\omega)P(j\omega)) < 1, \forall \omega > 0,$ *and* $\mu_\Delta (L(0)P(0)) = 1, \;\; \det (I - L(0)P(0)) \neq 0$

.

where

$$L(\omega) = \text{diag}\left\{\ell_1(\omega)I_{r_1}, \ell_2(\omega)I_{r_2}, \ldots, \ell_\ell(\omega)I_{r_\ell}\right\}.$$

Corollary 2.4 *Let $w_i(s)$, $i = 1, \ldots, \ell$ be stable rational transfer functions such that*

$$\ell_i(\omega) \leq |w_i(j\omega)|, \quad i = 1, \ldots, \ell.$$

Then the uncertain delay system (2.1) is stable for all $\tau_i \in [0, h_i)$ if $\left(A + \sum_{i=1}^{\ell} A_i\right)$ is stable and either one of the following conditions holds:

(a) $\mu_\Delta\left(W(j\omega)P(j\omega)\right) < 1, \quad \forall\ \omega;$

(b) $\mu_\Delta\left(W(j\omega)P(j\omega)\right) < 1, \forall\omega > 0,$ *and* $\mu_\Delta\left(W(0)P(0)\right) = 1,\ \det\left(I - W(0)P(0)\right) \neq 0$

(c) There exists a transfer matrix

$$T(s) = \text{diag}(T_1(s), T_2(s), \ldots, T_\ell(s)), \quad T_i^{-1}(s), \quad T_i(s) \in (\mathcal{RH}_\infty)^{r_i \times r_i}$$

such that

$$\left\|T^{-1}(s)W(s)P(s)T(s)\right\|_\infty < 1$$

where

$$W(s) = \text{diag}\{w_1(s)I_{r_1}, w_2(s)I_{r_2}, \ldots, w_\ell(s)I_{r_\ell}\}.$$

Proof.

Part (a) and (b) are obvious and part (c) follows by noting the fact that

$$\mu_\Delta\left(W(j\omega)P(j\omega)\right) \leq \min_{T(s)} \left\|T^{-1}(s)W(s)P(s)T(s)\right\|_\infty.$$

□

2.3 Stability with General Time Delays

For a more general case, $\tau_i \in [\underline{\tau}_i,\ \bar{\tau}_i)$, where $\underline{\tau}_i$ may be nonzero, the results in the last section can also be applied with some minor variations of system representations.

The uncertain time-delay system (2.1) can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\tau_i) \\
&= Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\underline{\tau}_i) + \sum_{i=1}^{\ell} A_i \left(x(t-\tau_i) - x(t-\underline{\tau}_i)\right) \\
&= Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\underline{\tau}_i) + \sum_{i=1}^{\ell} B_i u_i(t-\underline{\tau}_i)
\end{aligned}$$

where $A_i = B_i C_i$ and $\tau_i \in [\underline{\tau}_i, \bar{\tau}_i)$

$$u_i(t) = C_i\left(x(t-\tau_i+\underline{\tau}_i) - x(t)\right) = \left(\mathcal{D}_{\tau_i-\underline{\tau}_i} - 1\right) I_{r_i} C_i x(t) = \left(\mathcal{D}_{\tau_i-\underline{\tau}_i} - 1\right) I_{r_i} y_i(t).$$

The system equation becomes

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\underline{\tau}_i) + \sum_{i=1}^{\ell} B_i u_i(t-\underline{\tau}_i) \\
y_i(t) &= C_i x(t) \\
u_i(t) &= \left(\mathcal{D}_{\tau_i-\underline{\tau}_i} - 1\right) I_{r_i} y_i(t)
\end{aligned}$$

Define $h_i = \bar{\tau}_i - \underline{\tau}_i$,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_\ell(t) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_\ell(t) \end{bmatrix},$$

and denote $P(s)$

$$P(s) = C\left(sI - A - \sum_{i=1}^{\ell} A_i e^{-\underline{\tau}_i s}\right)^{-1} B \ \operatorname{diag}\left(e^{-\underline{\tau}_1 s} I_{r_1}, e^{-\underline{\tau}_2 s} I_{r_2}, \ldots, e^{-\underline{\tau}_\ell s} I_{r_\ell}\right). \quad (2.4)$$

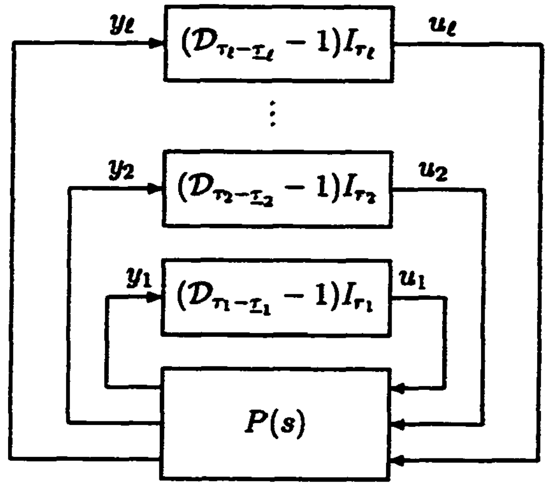


Figure 2.4: Reformulated Uncertain Delay System for $\tau_i \in [\underline{\tau}_i, \bar{\tau}_i]$

Assume that $\dot{x}(t) = Ax(t) + \sum_{i=1}^{\ell} A_i x(t - \underline{\tau}_i)$ is stable. Then, the results in the last section can be applied.

Example 2.2 Consider a second order oscillatory system which can be described by

$$\ddot{y} + \omega_0^2 y = u.$$

It can be stabilized by choosing $u(t) = -k\dot{y}(t)$, $k > 0$. The closed loop system becomes $\ddot{y} + k\dot{y} + \omega_0^2 y = 0$. However, differentiation of output $y(t)$ may not be desirable. Alternatively, One may choose input $u(t) = ky(t-\tau)$ to stabilize the systems ([1]). Then we end up with a closed loop system:

$$\ddot{y} + \omega_0^2 y - ky(t-\tau) = 0,$$

where τ is a fixed unknown number. Given ω_0 and k, we want to know what interval τ can be, such that the system is stable.

Denote $x_1 = y$, $x_2 = \dot{x}_1$. Rewrite the system as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}.$$

Since we know that system is unstable when $\tau = 0$, we expect to find an interval of τ such that the system is stable for $\tau \in [\underline{\tau}, \bar{\tau})$.

Consider system (2.4), and the associated uncertainty $\Delta = \{e^{-s(\bar{\tau}-\underline{\tau})} - 1\}$. Applying Theorem 2.3 for the case of $w_0 = 1$, $k = 0.5$, the result is as shown in Figure 2.5. It can be checked that the system is stable for a fixed $\tau = 0.25$. We first choose $\underline{\tau} = 0.25$, and obtain $\bar{\tau} = 0.5$. Hence, the system is delay-dependent stable for a fixed τ, $\tau \in [0.25, 0.50)$. Next, we extend $\underline{\tau} = 0.49$ and find out that the system is stable for $\tau \in [0.5, 0.96)$. Repeat this procedure by extending $\underline{\tau}$ and finding the maximum $\bar{\tau}$, and we can show that the system is stable for $\tau \in [0.25, 3.00)$.

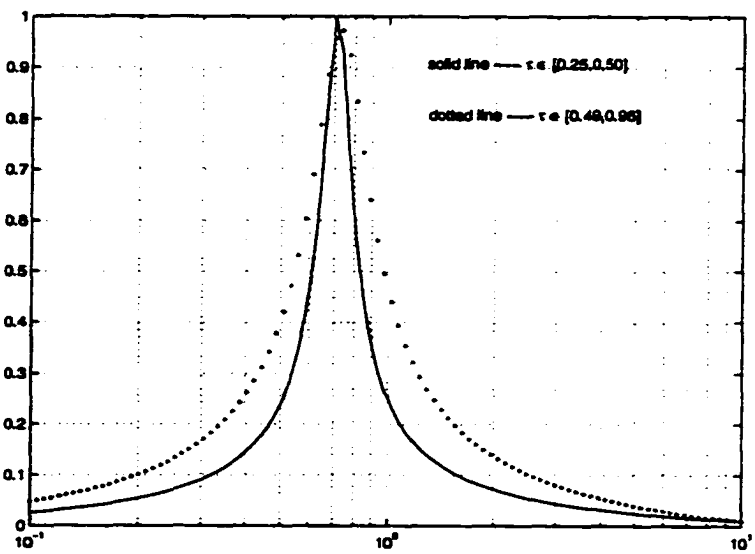


Figure 2.5: Structured singular value of delay dependent stability problems

2.4 Special Cases

It's seen that different $w_i(s)$ functions in Corollary 2.4 will lead to different sufficient conditions. With some special choice of $w_i(s)$, it's easy to explain relation between the results of time domain Lyapunov method (See [7, 8, 14, 18, 20, 29, 30, 38, 45, 46, 58, 59, 64, 66, 75]) and Theorem 2.3.

Corollary 2.5 *The uncertain delay system (2.1) is robustly stable for all $\tau_i \in [0, h_i)$ if $\dot{x}(t) = \left(A + \sum_{i=1}^{\ell} A_i\right) x(t)$ is stable and either one of the following conditions holds:*

(a) $\mu_\Delta\left(j\omega DP(j\omega)\right) < 1 \;\; \forall\, \omega$;

(b) There exists a transfer matrix

$$T(s) = \mathrm{diag}(T_1(s), T_2(s), \ldots, T_\ell(s)), \quad T_i^{-1}(s), \quad T_i(s) \in (\mathcal{RH}_\infty)^{r_i \times r_i}$$

such that

$$\left\| T^{-1}(s) DsP(s) T(s) \right\|_\infty < 1$$

where

$$D = \mathrm{diag}\{h_1 I_{r_1}, h_2 I_{r_2}, \ldots, h_\ell I_{r_\ell}\}.$$

Proof.

Let $w_i(s) = h_i s$. Then

$$W(s) = \mathrm{diag}\{w_1(s) I_{r_1}, w_2(s) I_{r_2}, \ldots, w_\ell(s) I_{r_\ell}\} = sD.$$

Hence it follows from Corollary 2.4 that the uncertain delay system (2.1) is robustly stable for all $\tau_i \in [0, h_i)$ if either

$$\mu_\Delta\left(j\omega DP(j\omega)\right) < 1 \;\; \forall\, \omega$$

or

$$\left\| T^{-1}(s) D s P(s) T(s) \right\|_\infty < 1.$$

□

As a special case, take $T(s)$ as a constant matrix.

Corollary 2.6 *The uncertain delay system (2.1) is robustly stable for all $\tau_i \in [0, h_i)$ if $A + \sum_{i=1}^{\ell} A_i$ is stable and there is a constant matrix*

$$T = \mathrm{diag}(T_1, T_2, \ldots, T_\ell), \quad T_i \in \mathcal{R}^{r_i \times r_i}$$

such that

$$\left\| T^{-1} D s P_0(s) T \right\|_\infty < 1$$

where

$$P_0(s) = C\left(sI - A - \sum_{i=1}^{\ell} A_i\right)^{-1} B, \quad D = \mathrm{diag}\{h_1 I_{r_1}, h_2 I_{r_2}, \ldots, h_\ell I_{r_\ell}\}.$$

It is clear that the result in Corollary 2.6 can be much more conservative than that in Theorem 2.3. We claim that most results in the literature such as the ones in [17, 36, 37, 41, 55, 57, 58, 63] are more conservative than that of Corollary 2.6. To see that, we note that

$$|e^{-j\omega\tau_i} - 1| \le \omega h_i, \quad \forall \omega \ge 0.$$

Hence the uncertainty due to delay can be written as

$$e^{-s\tau_i} - 1 = s h_i \gamma_i(s)$$

for some stable $\gamma_i(s)$ with $\|\gamma_i(s)\|_\infty \le 1$. Then the uncertain delay system can be written as follows:

$$\dot{x}(t) = (A + \sum_{i=1}^{\ell} A_i) x(t) + B v(t)$$

$$
\begin{aligned}
z(t) &= DC\dot{x}(t) \\
v(t) &= \Delta(s)z(t)
\end{aligned}
$$

where

$$
\Delta = \mathrm{diag}(\gamma_1 I_{r_1}, \gamma_2 I_{r_2}, \ldots, \gamma_\ell I_{r_\ell}).
$$

Note that $DsP_0(s)$ is the transfer matrix from v to z in Figure 2.6.

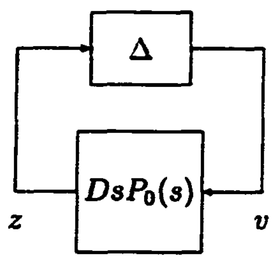


Figure 2.6: Uncertain Delay System

The system can also be written as

$$
\begin{aligned}
\dot{x}(t) &= (A + \sum_{i=1}^{\ell} A_i)x(t) + B\Delta DC\dot{x}(t) \\
&= (A + \sum_{i=1}^{\ell} A_i)x(t) + B\Delta DC\left(Ax(t) + \sum_{i=1}^{\ell} A_i x(t-\tau_i)\right) \\
&= (A + \sum_{i=1}^{\ell} A_i)x(t) + B\Delta DCAx(t) + B\Delta DC\sum_{i=1}^{\ell} A_i x(t-\tau_i)
\end{aligned}
$$

Define

$$
\begin{aligned}
\tilde{z}_0 &= DCAx(t), \quad \tilde{u}_0(t) = \Delta\tilde{z}_0(t) \\
\tilde{z}_1 &= DCA_1x(t), \quad \tilde{u}_1(t) = \Delta e^{-\tau_1 s}\, z_1(t) \\
&\vdots \\
\tilde{z}_\ell &= DCA_\ell x(t), \quad \tilde{u}_\ell(t) = \Delta e^{-\tau_\ell s}\, z_\ell(t) \\
\tilde{\Delta} &= \mathrm{diag}(\Delta e^{-\tau_1 s}, \Delta e^{-\tau_2 s}, \ldots, \Delta e^{-\tau_\ell s})
\end{aligned}
$$

Then $\|\tilde{\Delta}\|_\infty \leq 1$ and

$$\dot{x}(t) = (A + \sum_{i=1}^{m} A_i)x(t) + B\tilde{u}_0(t) + B\tilde{u}_1(t) + \cdots + B\tilde{u}_\ell(t)$$

which is shown in Figure 2.6. Note that this is the same system as in Figure 2.7. Naturally, the system is robustly stable if

$$\|\tilde{T}^{-1}F(s)\tilde{T}\|_\infty < 1 \tag{2.5}$$

where

$$\tilde{T} = \mathrm{diag}(T_{01}, T_{02}, \ldots, T_{0\ell}, T_{11}, T_{12}, \ldots, T_{1\ell}, \ldots, T_{i1}, T_{12}, \ldots, T_{i\ell}), \quad T_{ij} \in \mathcal{R}^{r_j \times r_j}$$

$$F(s) = \begin{bmatrix} DCA \\ DCA_1 \\ \vdots \\ DCA_\ell \end{bmatrix} (sI - A - \sum_{i=1}^{\ell} A_i)^{-1} \begin{bmatrix} B & B & \ldots & B \end{bmatrix}.$$

It is now straightforward to verify by using bounded real lemma ([78]) that those conditions given in [17, 36, 37, 41, 55, 57, 58, 63] are either exactly the condition given in (2.5) or more conservative than this condition. Obviously, the condition given in (2.5) is more conservative than the condition given in Corollary 2.6, which uses the system interconnection in Figure 2.2, while the condition given in (2.5) does not take advantage of the structure of the uncertainties in Figure 2.7.

2.5 Examples

Consider a simple delay system

$$\dot{x}(t) = -10x(t) - 15x(t - \tau).$$

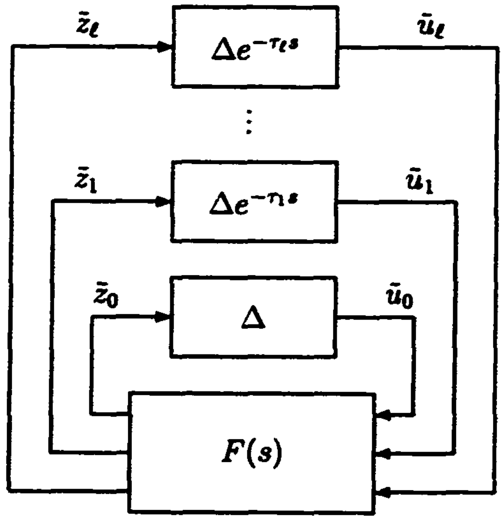


Figure 2.7: Expanded Representation of the Uncertain Delay System in Figure 2.2

It's clear that $G(s) = \frac{-15}{s+10}$. From the example at the beginning of the chapter, we can get the analytic solution of $h = 0.2057$, for which system is stable for $\tau \in [0, h)$. However, using the $L(\omega)$ approximation function, we have:

$$L(\omega) = \begin{cases} 2\sin\frac{h\omega}{2}, & \forall\omega\ :\ 0 \le \omega \le \pi/h \\ 2, & \forall\omega\ :\ \omega \ge \pi/h \end{cases},$$

with $P(s) = C\,(sI - A - BC)^{-1}\,B = \frac{-15}{s+25}$. Then,

$$|LP| = \begin{cases} \dfrac{30\sin(h\omega/2)}{\sqrt{25^2+\omega^2}}, & \forall\omega\ :\ 0 \le \omega \le \pi/h \\ \dfrac{30}{\sqrt{25^2+\omega^2}}, & \forall\omega\ :\ \omega \ge \pi/h \end{cases}$$

The result gives $|LP| < 1$, for all ω, if $h < 0.177$.

This example shows clearly that our method can also be very conservative. In fact, our method can fail for a system which is stable independent of delay. (Of course, many other methods given in the literature such as the ones given in [17, 36,

37, 41, 55, 57, 58, 63] will also fail since those methods are much more conservative than ours.) Take an example,

$$\dot{x}(t) = -10x(t) + 9x(t-\tau)$$

where $\tau \in [0, h)$. Let

$$G(s) = C(sI - A)^{-1}B = \frac{9}{s+10}$$

and $\|G(s)\| = 9/10 < 1$. So the uncertain delay system is delay-independent stable.

Now use our method and note that

$$P(s) = C(sI - A - BC)^{-1}B = \frac{9}{s+1}.$$

Then

$$|LP| = \begin{cases} \dfrac{18\sin(h\omega/2)}{\sqrt{1+\omega^2}}, & \forall\omega \ : \ 0 \le \omega \le \pi/h \\ \\ \dfrac{18}{\sqrt{1+\omega^2}}, & \forall\omega \ : \ \omega \ge \pi/h \end{cases}$$

It can be shown that $\|LP\|_\infty > 1$ for $h > 0.115$. Hence the test fails to detect delay-independent stability.

2.6 Stability with Parametric Uncertainties

Consider the following uncertain time-delay system:

$$\dot{x}(t) = (A + \Delta A)\, x(t) + \sum_{i=1}^{\ell} (E_i + \Delta E_i)\, x(t - \tau_i),$$

where ΔA, ΔE_i are parametric uncertainties. Let $E_i + \Delta E_i$ be factorized as $B_i(\Delta_i)\, C_i(\Delta_i)$ where $B_i \in \mathcal{R}^{n \times r_i}$, $C_i \in \mathcal{R}^{r_i \times n}$. Then, system can be rewritten as,

$$\dot{x}(t) \;=\; \left(A + \Delta A + \sum_{i=1}^{\ell}(E_i + \Delta E_i)\right) x + \sum_{i=1}^{\ell} B_i v_i(t),$$

$$
\begin{aligned}
z_i(t) &= C_i x(t), \\
v_i(t) &= (\mathcal{D}_{\tau_i} - 1)\, I_{k_i} z_i(t),
\end{aligned}
$$

which can be represented in LFT framework with constant matrix $\tilde{P}$, as shown in Figure 2.8:

$$
\tilde{P} = \begin{bmatrix} A + \Delta A + \sum_{i=1}^{\ell} (E_i + \Delta E_i) & B_1 & \cdots & B_m \\ C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ C_m & 0 & \cdots & 0 \end{bmatrix} := \begin{bmatrix} \tilde{A} + \Delta \tilde{A} & \tilde{B}(\Delta) \\ \tilde{C}(\Delta) & 0 \end{bmatrix}.
$$

Without loss of generality, we assume $\tilde{P}$ can be written as

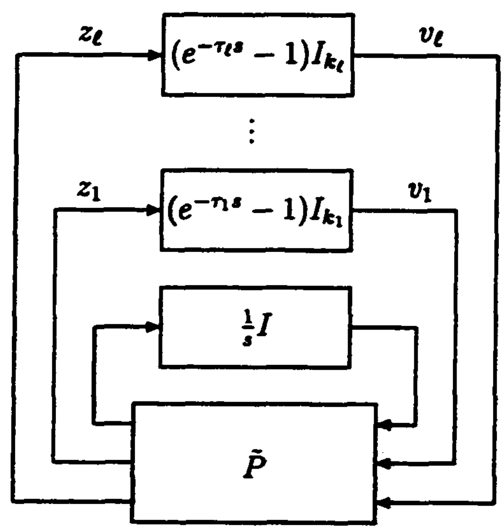


Figure 2.8: Interconnected Representation of the Delay System

$$
\tilde{P} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 \end{bmatrix}
$$

for some block diagonal uncertainty parameter Δ. The uncertain system can be put in the LFT form as shown in Figure 2.2 with $P(s)$,

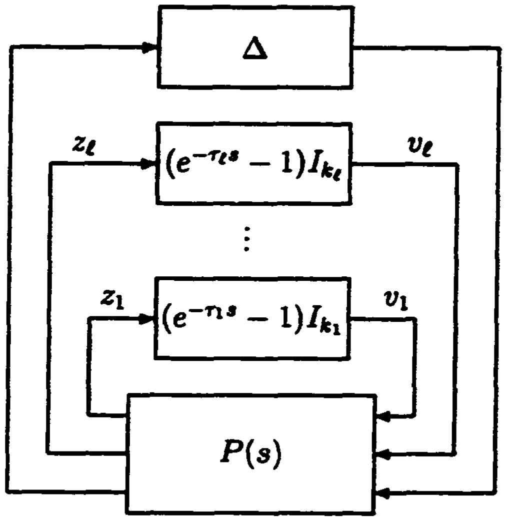


Figure 2.9: Interconnected Representation of the Delay System

$$P(s) = \begin{pmatrix} C_0 \\ N_1 \end{pmatrix} (sI - A_0)^{-1} \begin{pmatrix} B_0 & M_1 \end{pmatrix} + \begin{pmatrix} D_0 & M_2 \\ N_2 & 0 \end{pmatrix}.$$

Denote $\mathcal{D}(s) = \operatorname{diag}((e^{-\tau_1 s} - 1)I_{k_1}, \ldots, (e^{-\tau_\ell s} - 1)I_{k_\ell})$. Then the robust stability problem of this uncertain delay system can be converted into robust stability problem of system $\mathcal{F}_u\left(P, \begin{bmatrix} \mathcal{D}(s) & 0 \\ 0 & \Delta \end{bmatrix}\right)$, to which Theorem 2.3 can be applied directly.

Example 2.3 (Real Parametric Uncertainties)

Consider the following uncertain system from [38],

$$\dot{x}(t) = \begin{bmatrix} -2+1.6\delta_1 & 0 \\ 0 & -1+0.05\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1+0.1\delta_3 & 0 \\ -1 & -1+0.3\delta_4 \end{bmatrix} x(t-h)$$

where uncertain parameters δ_i, $i = 1, \cdots, 4$ are real constant values and satisfy $|\delta_i| \leq 1$. To find out the maximum of h such that system stays stable, we rewrite the system as:

$$\dot{x} = \begin{bmatrix} -3+1.6\delta_1+0.1\delta_3 & 0 \\ -1 & -2+0.05\delta_2+0.3\delta_4 \end{bmatrix} x(t) + \begin{bmatrix} -1+0.1\delta_3 & 0 \\ -1 & -1+0.3\delta_4 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t).$$

Let $\tilde{P}$ be the representation of system in Figure 2.8 :

$$\begin{aligned} \tilde{P} &= \begin{bmatrix} -3+1.6\delta_1+0.1\delta_3 & 0 & -1+0.1\delta_3 & 0 \\ -1 & -2+0.05\delta_2+0.3\delta_4 & -1 & -1+0.3\delta_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -3 & 0 & -1 & 0 \\ -1 & -2 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1.6 & 0 & 0.1 & 0 \\ 0 & 0.05 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \delta_3 & \\ & & & \delta_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Uncertain delay system can be rearranged as in Figure 2.9,

$$P(s) = \left[\begin{array}{cc|cccccc} -3 & 0 & -1 & 0 & 1.6 & 0 & 0.1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0.05 & 0 & 0.3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

with uncertainty $\Gamma = \operatorname{diag}\left((e^{-jh\omega} - 1)I_2, \delta_1, \delta_2, \delta_3, \delta_4\right)$, where $\delta_i \in \mathcal{R}$, $|\delta_i| \leq 1$, $i = 1, \ldots, 4$. Let

$$L(\omega) = \begin{cases} \operatorname{diag}\left(2\sin\frac{h\omega}{2} I_2, I_4\right), & 0 \leq \omega \leq \pi/h \\ \operatorname{diag}\left(2I_2, I_4\right), & \omega \geq \pi/h \end{cases},$$

and denote $M(\omega) = LP(\omega)$. By Theorem 2.3, we need the structured singular value of system LP, i.e. $\mu_\Delta(LP)$, less than one for all frequency to conclude the robust stability.

The associated uncertainty Γ consists of real and repeated complex numbers. For such so-called "mixed" μ value, we can obtain an upper bound by

$$\mu_\Delta(M) \le \inf_{R\in\mathcal{D},\, G\in\mathcal{G}} \min_{\beta} \left\{\beta :\ M^*RM + j(GM - M^*G) - \beta^2 R \le 0\right\},$$

where

$$\mathcal{D} = \left\{\operatorname{diag}(D, d_1, d_2, d_3, d_4) \ :\ D \in C^{2\times 2}, D = D^* > 0,\ d_i \in \mathcal{R}, i = 1, \ldots, 4\right\}$$

$$\mathcal{G} = \left\{\operatorname{diag}(0_2, g_1, g_2, g_3, g_4) \ :\ g_i \in \mathcal{R}, i = 1, \ldots, 4\right\}.$$

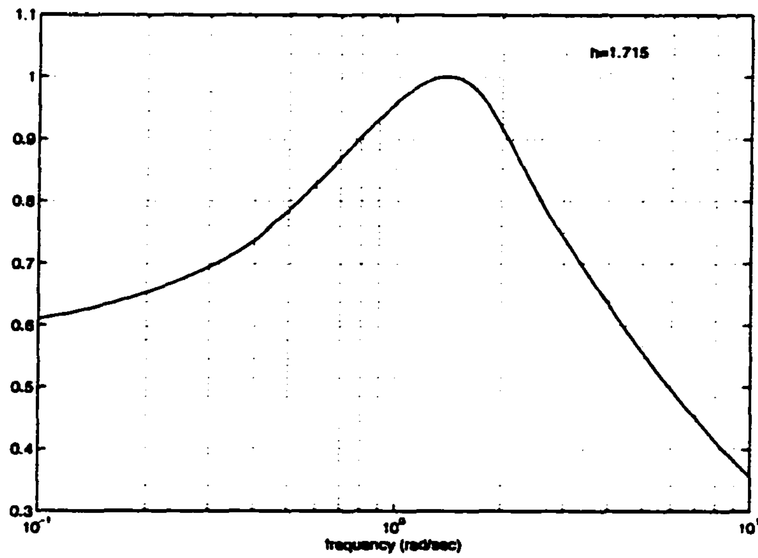


Figure 2.10: An upper bound of $\mu_\Delta(LP)$

The result is shown in Figure 2.10. By Theorem 2.3, the uncertain time-delayed system is stable if $h < 1.715$, in contrast with $h < 0.689$ given in [38].

To improve this result, we can apply system (2.4) to test stability for general delay $h \in [h_1, h_2)$, since delay h is a fixed unknown number belonging to the interval. Rewrite the system:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -2+1.6\delta_1 & 0 \\ 0 & -1+0.05\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1+0.1\delta_3 & 0 \\ -1 & -1+0.3\delta_4 \end{bmatrix} x(t-h_1) \\
&\quad + \begin{bmatrix} -1+0.1\delta_3 & 0 \\ -1 & -1+0.3\delta_4 \end{bmatrix} u(t-h_1), \\
y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t), \\
u(t) &= x(t-h+h_1) - x(t),
\end{aligned}
$$

which can be rearranged as in Figure 2.9,

$$
P(s) = \left[\begin{array}{cc|cccccc}
-2-e^{-h_1 s} & 0 & -e^{-h_1 s} & 0 & 1.6 & 0 & 0.1e^{-sh_1} & 0 \\
-e^{-h_1 s} & -1-e^{-h_1 s} & -e^{-h_1 s} & -e^{h_1 s} & 0 & 0.05 & 0 & 0.3e^{-sh_1} \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

with uncertainty $\Gamma = \mathrm{diag}\left((e^{-s\tau}-1)I_2, \delta_1, \delta_2, \delta_3, \delta_4\right)$, where $\tau = h - h_1 \in [0, h_2 - h_1)$. Applying Theorem 2.3, we can show that the system is stable if delay h belongs to intervals: $[1.70, 1.75)$,$[1.74, 1.79)$,$[1.78, 1.82)$,$[1.81, 1.86)$,$[1.85, 1.89)$, $[1.885, 1.895)$,

[1.89, 1.91) and [1.910, 1.921). Results are shown in Figure 2.11. Then, we can conclude that the system is stable for $h \in [0, 1.921)$.

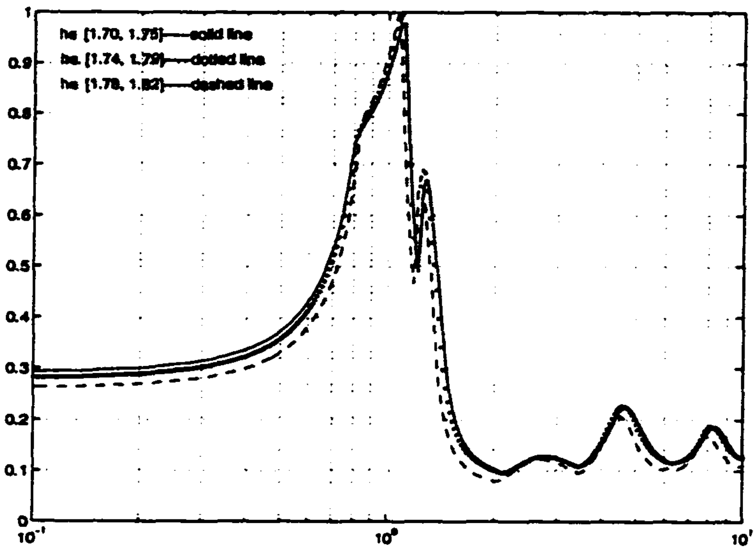


Figure 2.11: Upper bounds of $\mu_\Delta(LP)$ for various delay intervals

Remark 2.1 For a multiple delay system, enlarging the stability regions by testing stability for general delays intervals will involve more testing. For instance, considering a system with two delays h_1 and h_2, assume the system is stable if $h_1 \in [0, \underline{h}_1)$, $h_2 \in [0, \underline{h}_2)$. To expand the stability region, say the system is stable if $h_1 \in [0, \bar{h}_1)$, $h_2 \in [0, \bar{h}_2)$, we need to check the stability for $h_1 \in [0, \underline{h}_1)$, $h_2 \in [\underline{h}_2, \bar{h}_2)$ $h_1 \in [\underline{h}_1, \bar{h}_1)$, $h_2 \in [\underline{h}_2, \bar{h}_2)$, and $h_1 \in [\underline{h}_1, \bar{h}_1)$, $h_2 \in [0, \underline{h}_2)$. This is shown in Figure 2.12. The uncertain delay space is two dimension for h_1, h_2, and we need to test for area A, B and C.

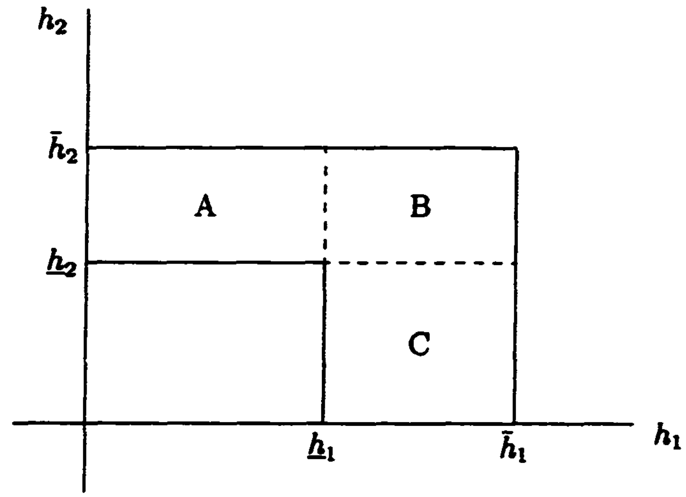


Figure 2.12: Uncertain delay space

2.7 Controller Synthesis

All results presented in the previous sections can be generalized to synthesis of uncertain delay systems. In fact, we can also include other types of uncertainties and disturbances easily using the general linear fractional framework [78]. It will also be clear that the problems considered in [17, 41, 44, 40, 52] are special cases in our framework. We shall only illustrate the basic idea here since it is relatively straightforward once the problem is set up in a suitable linear fractional transformation form.

Consider a general uncertain system as shown in Figure 2.13 where $G(s)$ is the general interconnection of the system, K is the controller, Δ_d is the block of uncertain delays, and Δ_u is the block of model uncertainties. Our objective is to design a controller K so that it stabilizes the uncertain system and at same time rejects optimally the disturbance d. Without loss of generality, we can assume that $G(s)$ is a finite dimensional system with a state space realization

$$G(s) = \left[\begin{array}{c|cccc} A & B_1 & B_2 & B_3 & B_4 \\ \hline C_1 & D_{11} & D_{12} & D_{13} & D_{14} \\ C_2 & D_{21} & D_{22} & D_{23} & D_{24} \\ C_3 & D_{31} & D_{32} & D_{33} & D_{34} \\ C_4 & D_{41} & D_{42} & D_{43} & D_{44} \end{array}\right]$$

with uncertainties

$$\Delta_d = \mathrm{diag}\{\mathcal{D}_{\tau_1} I_{r_1}, \mathcal{D}_{\tau_2} I_{r_2}, \ldots, \mathcal{D}_{\tau_\ell} I_{r_\ell}\}$$

and

$$\Delta_u = \mathrm{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_k\}.$$

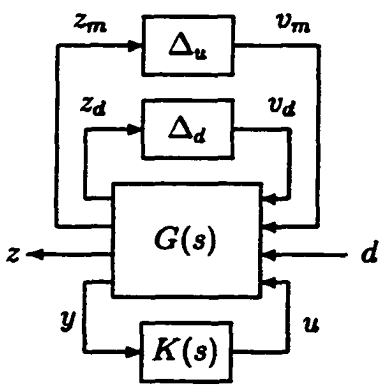


Figure 2.13: General Synthesis Framework

It is clear that all problems considered in the existing literature are special cases of this framework. Take, for example, $\Delta_d = \mathcal{D}_\tau I_r$, $\Delta_u = \Delta_1$, and $D_{11} = 0, D_{13} = 0, D_{21} = 0, D_{22} = 0, D_{23} = 0, D_{31} = 0$, $D_{32} = 0, D_{33} = 0$. Then the system equations

can be written as

$$
\begin{aligned}
\dot{x}(t) &= (A + B_2\Delta_1 C_2)x(t) + B_1(C_1 + D_{12}\Delta_1 C_2)x(t-\tau) \\
&\quad +(B_4 + B_2\Delta_1 D_{24})u(t) + B_1(D_{14} + D_{12}\Delta_1 D_{24})u(t-\tau) \\
z(t) &= C_3 x(t) + D_{34}u(t) \\
y(t) &= (C_4 + D_{41}\Delta_1 C_2)x(t) + D_{41}(C_1 + D_{12}\Delta_1 C_2)x(t-\tau) \\
&\quad +D_{41}(D_{14} + D_{12}\Delta_1 D_{24})u(t-\tau) + D_{43}d(t) + (D_{44} + D_{42}\Delta_1 D_{24})u(t)
\end{aligned}
$$

It is clear that one can apply directly the μ-synthesis techniques ([78]) to the system in Figure 2.13 to synthesis delay-independent controllers.

To use the delay dependent analysis results in the previous sections for synthesis, we need first to modify the framework by subtracting an I and adding an I to the delay block Δ_d as shown in Figure 2.14. Now apply the μ-synthesis techniques to the system in Figure 2.14 with the generalized plant $\tilde{G}(s)$:

$$
\begin{bmatrix} \tilde{z}_d^T & z_m^T & z^T & y^T \end{bmatrix} = \begin{bmatrix} \tilde{v}_d^T & v_m^T & d^T & u^T \end{bmatrix} \tilde{G}^T(s),
$$

$$
\tilde{G}(s) = \left[\begin{array}{c|cccc}
\tilde{A} & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 & \tilde{B}_4 \\
\hline
\tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} \\
\tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} & \tilde{D}_{24} \\
\tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} & \tilde{D}_{34} \\
\tilde{C}_4 & \tilde{D}_{41} & \tilde{D}_{42} & \tilde{D}_{43} & \tilde{D}_{44}
\end{array}\right],
$$

where the corresponding matrices can be obtained by

$$\begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 & \tilde{B}_4 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} & \tilde{D}_{24} \\ \tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} & \tilde{D}_{34} \\ \tilde{C}_4 & \tilde{D}_{41} & \tilde{D}_{42} & \tilde{D}_{43} & \tilde{D}_{44} \end{bmatrix} =$$

$$\begin{bmatrix} A & B_1 & B_2 & B_3 & B_4 \\ C_1 & D_{11} & D_{12} & D_{13} & D_{14} \\ C_2 & D_{21} & D_{22} & D_{23} & D_{24} \\ C_3 & D_{31} & D_{32} & D_{33} & D_{34} \\ C_4 & D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & (I - D_{11})^{-1} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ C_1 & I & D_{12} & D_{13} & D_{14} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

We would also like to point out that many other control problems, such as the guaranteed cost control problem considered in [14, 40], can be easily handled using this framework.

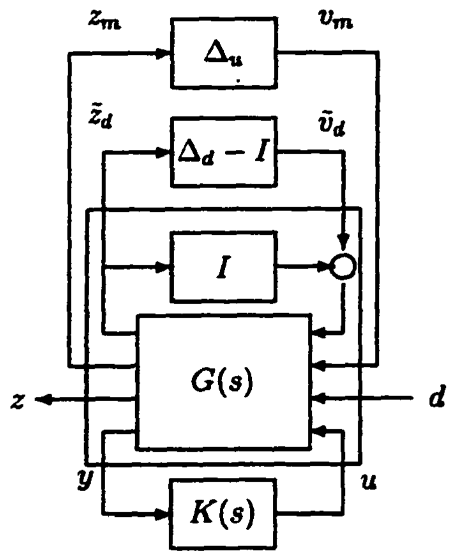


Figure 2.14: Framework for Synthesis Using Delay-dependent Results

Chapter 3

Structured Singular Value with Phase Information

We have seen in the last chapter that the structured singular value (SSV) plays the key role in the robust stability and performance analysis of an uncertain time-delay system. In the case when an uncertain time-delay system is not delay-independent stable, the possible phase variation range of an delay term $e^{-s\tau}$ is crucial in determining the delay stability margin. In this chapter, we shall give an extension of the structured singular value to include phase information of the uncertainties. The results presented here are minor extension of [71, 72] for repeated scalar uncertainties case.

3.1 SSV with Phase Information

Let $r = k_1 + \cdots + k_{\ell+m+n} + n_1 + \cdots + n_p$ and $\ell,\ m,\ n,\ p\ \geq 0$. Define

$$\begin{aligned}
\mathbf{\Gamma} &:= \Big\{\text{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \\
&\qquad \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \delta_i \in \mathcal{R}, \gamma_i \in \mathbf{C}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Gamma}_{\Theta_\ell} &:= \Big\{\text{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \\
&\qquad \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \delta_i \in \mathcal{R},\ \ \gamma_i \in \mathbf{C};\ |\angle\gamma_j| \leq \theta_j, j = 1, \ldots, \ell, \\
&\qquad \Delta_i \in \mathbf{C}^{n_i \times n_i}\ \Big\},
\end{aligned}$$

where $\Theta_\ell = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i \in [0, \pi/2]$ for $i = 1, \ldots, \ell$.

The structured singular value of a matrix $M \in \mathbf{C}^{r \times r}$ with respect to a block structure $\mathbf{\Gamma}_{\Theta_\ell}$ is defined to be $\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M) = 0$ if there is no $\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}$ such that

$\det(I - \Gamma M) = 0$, and

$$\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M) = \left(\min_{\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}} \{ \bar{\sigma}(\Gamma) : \ \det(I - \Gamma M) = 0 \} \right)^{-1}$$

otherwise.

Define

$$\begin{aligned} \mathbf{T} \ &:= \ \Big\{ T : \ T = \mathrm{diag}(T_1, \ldots, T_\ell, \ldots, T_{\ell+m}, \ldots, T_{\ell+m+n}, d_1 I_{n_1}, \ldots, d_{p-1} I_{n_{p-1}}, I_{n_p}), \\ &\qquad 0 < T_i^* = T_i \in \mathbf{C}^{k_i \times k_i}, \ d_i \in \mathcal{R}, \ d_i > 0 \Big\}, \end{aligned}$$

$$\mathbf{S}_\ell := \Big\{ S : \ \ S = \mathrm{diag}(S_1, \ldots, S_\ell, 0, \ldots, 0), \ \ 0 \le S_i^* = S_i \in \mathbf{C}^{k_i \times k_i} \Big\},$$

$$\mathbf{B}_{\Theta_\ell} := \{ B : \ \ B = \mathrm{diag}(\beta_1 I_{k_1}, \ldots, \beta_\ell I_{k_\ell}, 0, \ldots, 0) : \ \beta_i \in [-\cot\theta_i, \cot\theta_i] \ \},$$

$$\mathbf{G}_{\mathbf{n}} := \Big\{ G : \ \ G = \mathrm{diag}(0, \ldots, 0, G_{\ell+m+1}, \ldots, G_{\ell+m+n}, 0, \ldots, 0), \ \ G_i^* = G_i \in \mathbf{C}^{k_i \times k_i} \Big\}.$$

We have the following results, which are generalizations of the corresponding results in [71] for block structured case.

Lemma 3.1 *Let $\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}$, with $\Gamma^*\Gamma \le \alpha^{-2} I$. Then Γ satisfies*

$$\begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -\alpha^2 R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} \ge 0,$$

for every $R \in \mathbf{T}, S \in \mathbf{S}_\ell$, and $B \in \mathbf{B}_{\Theta_\ell}$.

Proof.

Consider any $\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}$ satisfying $\Gamma^*\Gamma \le \alpha^{-2} I$. Since Γ commutes with every $R \in \mathbf{T}$, we have

$$R - \alpha^2 \Gamma^* R \Gamma = R(I - \alpha^2 \Gamma^* \Gamma) = R^{1/2}(I - \alpha^2 \Gamma^* \Gamma) R^{1/2} \ge 0.$$

Next, since every $\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}$, every $S \in \mathbf{S}_\ell$, and every $B \in \mathbf{B}_{\Theta_\ell}$ commute with each other, and since $|\beta_i| \le \cot\theta_i = \frac{\mathrm{Re}(\gamma_i)}{\mathrm{Im}(\gamma_i)}$, we have

$$\gamma_i + \gamma_i^* - j\beta_i(\gamma_i^* - \gamma_i) \ge 0$$

for $i = 1, \ldots, \ell$. Since $B \in \mathbf{B}_{\Theta_\ell}$, we then have

$$S\Gamma + \Gamma^* S - j(BS\Gamma - \Gamma^* SB) = S(\Gamma + \Gamma^* + jB(\Gamma^* - \Gamma)) \ge 0.$$

Hence we conclude that

$$R - \alpha^2\Gamma^* R\Gamma + S\Gamma + \Gamma^* S - j(BS\Gamma - \Gamma^* SB) \ge 0,$$

which is exactly what we need. $\square$

Using the above lemma, we can show the following theorem which is a generalization of the result in [71].

Theorem 3.2 *Let $\Gamma \in \mathbf{\Gamma}_{\Theta_\ell}$, with $\Gamma^*\Gamma \le \alpha^{-2}I$. If there exist some $R \in \mathbf{T}, S \in \mathbf{S}_\ell$, $B \in \mathbf{B}_{\Theta_\ell}$, and $G \in \mathbf{G}_n$ such that*

$$M^* RM - \alpha^2 R + (S(I + jB)M + M^*(I - jB)S) + j(GM - M^*G) < 0, \tag{3.1}$$

then $\det(I - \Gamma M) \ne 0$.

Proof.

Rewrite equation (3.1) as

$$\begin{bmatrix} M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S - jG \\ S(I + jB) + jG & -\alpha^2 R \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix} < 0. \tag{3.2}$$

We proceed by contradiction. Suppose that $\det(I - \Gamma M) = 0$. Then for some nonzero $v \in \mathbf{C}^r$, we have $(I - \Gamma M)v = 0$. Defining $u = Mv$, we have $v = \Gamma u$. Now, from equation (3.2), we have

$$v^* \begin{bmatrix} M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S - jG \\ S(I + jB) + jG & -\alpha^2 R \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix} v < 0,$$

i.e,

$$\begin{bmatrix} u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S - jG \\ S(I + jB) + jG & -\alpha^2 R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} < 0.$$

Next, let $G \in \mathbf{G}_n$ and note that $\Gamma^* G = G\Gamma$, $v = \Gamma M v$, then $v^* M^* G v = v^* M^* G \Gamma M v = v^* M^* \Gamma^* G M v = v^* G M v$, i.e.,

$$v^* \left(GM - M^* G\right) v = 0.$$

But from Lemma 3.1, we must have

$$\begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -\alpha^2 R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} \geq 0,$$

which yields

$$u^* \begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -\alpha^2 R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} u \geq 0.$$

Adding $v^* \left(GM - M^* G\right) v = 0$, we get

$$\begin{bmatrix} u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S - jG \\ S(I + jB) + jG & -\alpha^2 R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0,$$

which is a contradiction. $\square$

From the definition of $\mu_{\Gamma_{\Theta_\ell}}(M)$, we have the following result.

Corollary 3.3 *Let $M \in \mathbf{C}^{r\times r}$. Then $\mu_{\Gamma_{\Theta_\ell}}(M) \le \hat{\mu}_{\Gamma_{\Theta_\ell}}(M)$, where $\hat{\mu}_{\Gamma_{\Theta_\ell}}(M) =$*

$$\inf\{\,\alpha \;:\; M^*RM - \alpha^2 R + (S(I+jB)M + M^*(I-jB)S) + j\,(GM - M^*G) < 0,$$
$$\alpha > 0, R \in \mathbf{T}, S \in \mathbf{S}_\ell, B \in \mathbf{B}_{\Theta_\ell}, G \in \mathbf{G_n}\,\}.$$

3.2 Computational Issues

It is noted that the optimization involved in Corollary 3.3 is in general non-convex. However, several special cases listed below are convex problems.

1. If $M \in \mathbf{C}$ is a scalar and $\ell = 1$, we have

$$\mu_{\Gamma_{\Theta_\ell}}(M) = \begin{cases} 0, & \text{if } \mathrm{Re}(M) - \cot\theta|\mathrm{Im}(M)| < 0 \\ |M|, & \text{if } \mathrm{Re}(M) - \cot\theta|\mathrm{Im}(M)| \ge 0 \end{cases}.$$

2. If $\theta_i = \pi/2, i = 1, \ldots, \ell$, i.e., if $\mathrm{Re}(\gamma_i) \ge 0, i = 1, \ldots, \ell$, then $B = 0$ and the optimization becomes a convex optimization.

$$\hat{\mu}_{\Gamma_{\Theta_\ell}}(M) = \inf\Big\{\alpha : \; M^*RM - \alpha^2 R + SM + M^*S + j\,(GM - M^*G) < 0,$$
$$\alpha > 0, R \in \mathbf{T}, S \in \mathbf{S}_\ell, G \in \mathbf{G_n}\Big\}.$$

3. As pointed out in [71], if $k_j = 1, j = 1, \ldots, \ell$, then the optimization in Corollary 3.3 can be converted to a convex one by setting $\hat{B} = SB$ and computing

$$\inf\{\alpha : M^*RM - \alpha^2 R + SM + M^*S + j(\hat{B}M - M^*\hat{B}) + j(GM - M^*G) < 0\} \quad (3.3)$$

with convex constraints

$$-SB \le \hat{B} \le SB$$

where $B = \mathrm{diag}(\cot\theta_1 I_{k_1}, \ldots, \cot\theta_\ell I_{k_\ell})$.

It is also possible to compute an upper bound of $\mu_{\mathbf{\Gamma}_{\Theta_\ell}}(M)$ by solving equation (3.3), even when $k_j \neq 1, j = 1, \ldots, \ell$. In this case, we will get more conservative result. In general, a suboptimal $\hat{\mu}_{\mathbf{\Gamma}_{\Theta_\ell}}(M)$ can be obtained through an iterative algorithm: First solving R, S and G with a fixed B, then solving B with R, S and G obtained in the previous step, repeat the process until a satisfactory solution is found.

Chapter 4

Robust Stability and Performance Analysis Using Phase Information

In this chapter, we shall see how the structured singular value with phase information can be used for robust stability and performance analysis of uncertain time delay systems with possibly structured uncertainties.

4.1 Analysis Using Phase Information

Let's start with robust stability problem to get some intuitive idea. Consider the uncertain time-delay system represented in Figure 2.1. We are interested in determining the largest possible range of delays so that the system is robustly stable. Consider the time-delay system (2.1):

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{\ell} A_i x(t - \tau_i).$$

In Figure 2.1, denote

$$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_\ell \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_\ell \end{bmatrix},$$

and let

$$P(s) = C\,(sI - A)^{-1}\,B, \tag{4.1}$$

$$\Delta = \operatorname{diag}(e^{-\tau_1 s} I_{k_1}, e^{-\tau_2 s} I_{k_2}, \cdots, e^{-\tau_\ell s} I_{k_\ell}), \tag{4.2}$$

where $\tau_i \in [0, h_i)$ is an uncertain constant.

Lemma 4.1 *Suppose A is stable. Then the uncertain delay system is stable if*

$$\det\left(I - \lambda P(j\omega)\Delta(j\omega)\right) \neq 0, \ \forall \omega \geq 0, \ \tau_i \in [0, h_i), \lambda \in [0, 1].$$

Proof.

It is clear that $\det(I - \lambda P(s)\Delta(s))$ has no unstable zeros for a sufficiently small $\lambda > 0$. Since $\det(I - \lambda P(s)\Delta(s))$ is a continuous function of λ, there must be a $\lambda \in [0, 1]$ and a ω such that $\det(I - \lambda P(j\omega)\Delta(j\omega)) = 0$ if $\det(I - P(s)\Delta(s))$ has any unstable zeros (i.e., the uncertain delay system is unstable). □

Note that the phase of each uncertain delay term is given by

$$\angle e^{-j\tau_i\omega} = -\tau_i\omega \in [-h_i\omega, 0],$$

and $\Delta(j\omega) \in \mathbf{\Gamma}_{\Theta_\ell}$ with $\mathbf{\Gamma}_{\Theta_\ell} = \{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}) \ : \ \gamma_i \in \mathbf{C}, \ |\angle\gamma_i| \leq h_i\omega\}$, if $0 \leq h_\ell \leq \pi/2$ (since $h_1\omega < h_2\omega < \ldots < h_\ell\omega$). We have:

Theorem 4.2 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Then the uncertain delay system is stable for $\tau_i \in [0, \ h_i)$, $i = 1, \ldots, \ell$ if the following conditions hold*

(a) $\nu(\omega) = \mu_{\mathbf{\Gamma}_{\Theta_\ell}}(P(j\omega)) < 1$ *for* $0 \leq \omega \leq \pi/2h_\ell$ *with* $\theta_i = h_i\omega, \ i = 1, \ldots, \ell$;

(b) For each $n = 1, \ldots, \ell - 1$, *we have* $\nu(\omega) = \mu_{\mathbf{\Gamma}_{\Theta_n}}(P(j\omega)) < 1$ *for* $\pi/2h_{n+1} < \omega \leq \pi/2h_n$, *with* $\theta_i = h_i\omega, \ i = 1, \ldots, n$ *and*

$$\mathbf{\Gamma}_{\Theta_n} = \{ \ \mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \gamma_{n+1} I_{k_{n+1}}, \ldots, \gamma_\ell I_{k_\ell}) :$$
$$\gamma_i \in \mathbf{C}, |\angle\gamma_j| \leq h_j\omega, j = 1, 2, \ldots, n \ \};$$

(c) $\nu(\omega) = \mu_{\mathbf{\Gamma}}(P(j\omega)) < 1$ *for* $\omega > \pi/2h_1$ *with* $\mathbf{\Gamma} = \{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}) : \gamma_i \in \mathbf{C}\}$.

Proof.

The proof follows from Lemma 4.1. □

Since the structured singular value using the phase information is always no greater than the structured singular value without using the phase information for any complex matrix M, the stability condition in the above theorem can be much less conservative than the delay independent stability test in [10] using the structured singular value without using the phase information.

Example 4.1 To illustrate above theorem, consider a first order delay system.

$$\dot{x}(t) = -10x(t) - 15x(t-\tau)$$

where $\tau \in [0, h)$. Now let

$$P(s) = \frac{-15}{s+10},$$

and define $\nu(P(j\omega))$ as

$$\begin{aligned}
\nu\left(P(j\omega)\right) &= \begin{cases} \mu_{\Gamma_\theta}\left(P(j\omega)\right), & 0 \le \omega \le \pi/2h \text{ with } \theta = h\omega, \\ |P(j\omega)|, & \omega > \pi/2h \end{cases} \\
&= \begin{cases} 0, & \forall\omega \text{ s.t. } 0 \le \omega \le \pi/2h \text{ and } \mathrm{Re}(P) - \cot\theta|\mathrm{Im}(P)| < 0 \\ |P(j\omega)|, & \text{otherwise} \end{cases} .
\end{aligned}$$

The $\nu(P(j\omega))$ is plotted in Figure 4.1. By Theorem 4.2, the delay system is robustly stable if $\nu(P(j\omega)) < 1$. Hence we conclude that the system is stable if $h \le 0.142$ since $\nu(P(j\omega))$ plotted in Figure 4.1 for $h = 0.142$ is no greater than 1.

It is noted that $\nu(\omega)$ in Theorem 4.2 is computed with $-h_i\omega \le \angle\gamma_i \le h_i\omega$, but in fact $\angle\gamma_i e^{-j\tau_i\omega} \in [-h_i\omega, 0]$. Hence, the stability condition given in Theorem 4.2

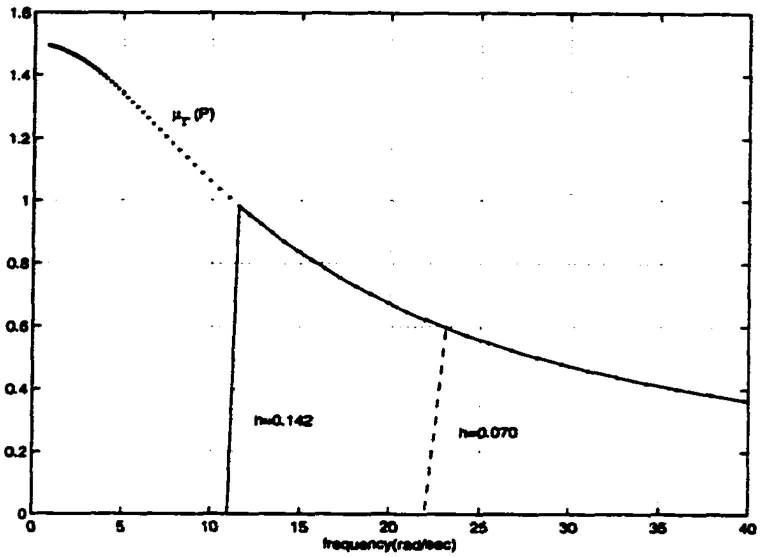


Figure 4.1: An upper bound of $\nu(P(j\omega))$ for $h = 0.142$ (solid line), $h = 0.07$ (dashed line), and $\mu_\Gamma(P)$ (doted line)

may be conservative. To reduce the conservativeness, we now define an angle shifted system $M(s)$:

$$M(s) = P(s)\mathrm{diag}(e^{-h_1 s/2}I_{k_1}, e^{-h_2 s/2}I_{k_2}, \ldots, e^{-h_\ell s/2}I_{k_\ell}), \tag{4.3}$$

with the associated uncertainty

$$\begin{aligned}\Gamma(s) &= \mathrm{diag}(e^{-(\tau_1-\frac{h_1}{2})s}I_{k_1}, e^{-(\tau_2-\frac{h_2}{2})s}I_{k_2}, \ldots, e^{-(\tau_\ell-\frac{h_\ell}{2})s}I_{k_\ell}), \quad \tau_i \in [0, h_i] \\ &= \mathrm{diag}(e^{-\phi_1 s}I_{k_1}, e^{-\phi_2 s}I_{k_2}, \cdots, e^{-\phi_\ell s}I_{k_\ell}), \quad \phi_i \in [-h_i/2, h_i/2].\end{aligned}$$

Then

$$\angle e^{-j\phi_i\omega} = -\phi_i\omega \in [-h_i\omega/2, h_i\omega/2],$$

and $\Gamma(j\omega) \in \mathbf{\Gamma}_{\Theta_\ell}$ with $\mathbf{\Gamma}_{\Theta_\ell} = \{\mathrm{diag}(\gamma_1 I_{k_1}, \gamma_2 I_{k_2}, \ldots, \gamma_\ell I_{k_\ell}) : \gamma_i \in \mathbf{C}, |\angle\gamma_i| \le h_i\omega/2\}$, if $h_\ell\omega \le \pi$ (since $h_1\omega < h_2\omega < \ldots < h_\ell\omega$).

Theorem 4.3 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Then the uncertain delay system is stable for $\tau_i \in [0, h_i)$, $i = 1, \ldots, \ell$, if the following conditions hold*

(a) $\nu(\omega) = \mu_{\Gamma_{\Theta_\ell}}(M(j\omega)) < 1$ *for* $0 \le \omega \le \pi/h_\ell$ *with* $\theta_i = h_i\omega/2,\ i = 1, \ldots, \ell;$

(b) For each $n = 1, \ldots, \ell - 1$, *we have* $\nu(\omega) = \mu_{\Gamma_{\Theta_n}}(M(j\omega)) < 1$ *for* $\pi/h_{n+1} < \omega \le \pi/h_n$, *with* $\theta_i = h_i\omega/2,\ i = 1, \ldots, n$ *and*

$$\Gamma_{\Theta_n} = \{ \ \mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \gamma_{n+1} I_{k_{n+1}}, \ldots, \gamma_\ell I_{k_\ell}) : \\ \gamma_i \in \mathbf{C}, |\angle\gamma_j| \le h_j\omega/2, j = 1, \ldots, n \ \};$$

(c) $\nu(\omega) = \mu_\Gamma(M(j\omega)) < 1$ *for* $\omega > \pi/h_1$ *with* $\Gamma = \{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}) : \gamma_i \in \mathbf{C}\}$.

Proof.

From Lemma 4.1, the system is stable if

$$\det(I - \lambda P(j\omega)\Delta(j\omega)) \ne 0,\ \forall\omega \ge 0,\ \lambda \in [0, 1].$$

Now note that

$$P(j\omega)\Delta(j\omega) = M(j\omega)\Gamma(j\omega)$$

and hence we only need to make sure that

$$\det(I - \lambda M(j\omega)\Gamma(j\omega)) \ne 0,\ \forall\omega \ge 0,\ \lambda \in [0, 1].$$

For $0 \le \omega \le \pi/h_\ell$, we have $|\angle e^{-j\phi_i\omega}| = |\phi_i\omega| \le h_i\omega/2 \le \pi/2$ since $h_1\omega < h_2\omega < \ldots < h_\ell\omega \le \pi$. Hence $\lambda\Gamma \in \Gamma_{\Theta_\ell}$ with $\theta_i = h_i\omega/2$ for $i = 1, \ldots, \ell$, and

$$\det(I - \lambda M(j\omega)\Gamma(j\omega)) \ne 0,\ \forall\ 0 \le \omega \le \pi/h_\ell,\ \lambda \in [0, 1],$$

if part (a) is true. Similarly, we can show that for any given n, $n = 1, \ldots, \ell - 1$,

$$\det(I - \lambda M(j\omega)\Gamma(j\omega)) \neq 0, \ \forall\, \pi/h_{n+1} < \omega \leq \pi/h_n, \ \lambda \in [0, 1]$$

if $\mu_{\mathbf{\Gamma}_{\Theta_n}}(M(j\omega)) < 1$. Finally, when $\omega > \pi/h_1$, the μ with phase information cannot be applied, so we would require the plain μ condition (i.e., part (c) without any phase information) holds. □

Now consider again the above example:

$$\dot{x}(t) = -10x(t) - 15x(t - \tau)$$

where $\tau \in [0, h)$. Now let

$$P(s) = \frac{-15}{s + 10}, \qquad M(s) = P(s)e^{-hs/2} = \frac{-15e^{-hs/2}}{s + 10}$$

and define $\nu(M(j\omega))$ as

$$\begin{aligned}
\nu(M(j\omega)) &= \begin{cases} \mu_{\mathbf{\Gamma}_{\Theta}}(M(j\omega)), & 0 \leq \omega \leq \pi/h \text{ with } \theta = h\omega/2, \\ |M(j\omega)|, & \omega > \pi/h \end{cases} \\
&= \begin{cases} 0, & \forall\omega \text{ s.t. } 0 \leq \omega \leq \pi/h \text{ and } \mathrm{Re}(M) - \cot\theta|\mathrm{Im}(M)| < 0 \\ |M(j\omega)|, & \text{otherwise} \end{cases}.
\end{aligned}$$

The $\nu(M(j\omega))$ is plotted in Figure 4.2. By Theorem 4.3, the delay system is stable if $\nu(M(j\omega)) < 1$. Hence we conclude that the system is stable if $h \leq 0.2057$ since $\nu(M(j\omega))$ plotted in Figure 4.2 for $h = 0.2057$ is no greater than 1. It should be pointed out that the estimate obtained using Theorem 4.2 is $h \leq 0.142$, and the estimate from the last chapter is $h \leq 0.06667$, while the estimate using the existing methods in [17, 37, 41] are much smaller.

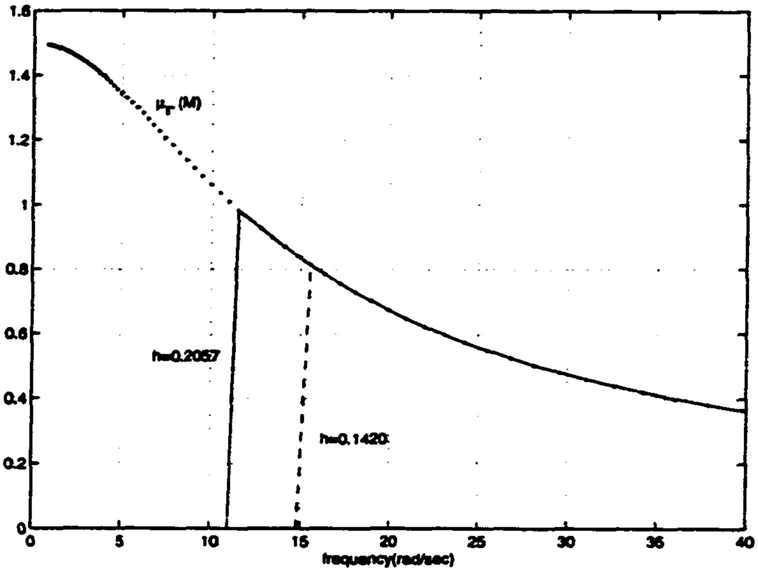


Figure 4.2: An upper bound of $\nu(M(j\omega))$ for $h = 0.2057$ (solid line), $h = 0.1420$ (dashed line), and $\mu_\Gamma(M)$ (doted line)

Hence to test the robust stability of an uncertain time-delay system with a given time-delay bound h, we can compute the structured singular value with phase information for system $F_u(P(s), e^{-hs})$, or for $F_u(P(s)e^{-\frac{hs}{2}}, e^{-\frac{hs}{2}})$. In general, it is not clear which one gives the less conservative results. At specified frequency, the structured singular value with phase information of the angle shifted system $M(s)$ may be greater or smaller than that of the system $P(s)$.

Theorem 4.4 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Then the uncertain delay system is stable for $\tau_i \in [0, h_i)$, $i = 1, \ldots, \ell$, if*

$$\nu(\omega) = \min\left\{\nu\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right), \quad \nu\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right)\right\} < 1,$$

for all ω, *where* $\nu\left(P(j\omega),\Gamma^{P}_{\Theta_\ell}\right)$ *and* $\nu\left(M(j\omega),\Gamma^{M}_{\Theta_\ell}\right)$ *are defined as*

$$\nu\left(P(j\omega),\Gamma^{P}_{\Theta_\ell}\right)=\begin{cases}\mu_{\Gamma_{\Theta_\ell}}(P(j\omega)) & 0\le\omega\le\pi/2h_\ell,\ \ \theta_i=h_i\omega,\ \ i=1,\ldots,\ell;\\ \mu_{\Gamma_{\Theta_n}}(P(j\omega)) & \pi/2h_{n+1}<\omega\le\pi/2h_n,\ \ n=1,\ldots,\ell-1,\\ & \qquad\theta_i=h_i\omega,\ \ i=1,\ldots,n\\ \mu_{\Gamma}(P(j\omega)), & \qquad\omega>\pi/2h_1\end{cases}$$

where the corresponding Γ_{Θ_n} *is defined as*

$$\Gamma_{\Theta_n}=\left\{\mathrm{diag}\left(\gamma_1 I_{k_1},\ldots,\gamma_n I_{k_n},\ldots,\gamma_\ell I_{k_\ell}\right)\ :\gamma_i\in\mathbf{C},|\angle\gamma_j|\le h_j\omega,j=1,2,\ldots,n,\right\};$$

$$\Gamma=\left\{\mathrm{diag}\left(\gamma_1 I_{k_1},\ldots,\gamma_\ell I_{k_\ell},\right)\ :\gamma_i\in\mathbf{C},i=1,\ldots,\ell\right\}.$$

And,

$$\nu\left(M(j\omega),\Gamma^{M}_{\Theta_\ell}\right)=\begin{cases}\mu_{\Gamma_{\Theta_\ell}}(M(j\omega)), & 0\le\omega\le\pi/h_\ell,\ \theta_i=h_i\omega/2,\ \ i=1,\ldots,\ell;\\ \mu_{\Gamma_{\Theta_n}}(M(j\omega)), & \pi/h_{n+1}<\omega\le\pi/h_n,\ \ n=1,\ldots,\ell-1,\\ & \qquad\theta_i=h_i\omega/2,\ i=1,\ldots,n\\ \mu_{\Gamma}(M(j\omega)), & \qquad\omega>\pi/h_1\end{cases}$$

where the corresponding Γ_{Θ_n} *is defined as*

$$\Gamma_{\Theta_n}=\left\{\mathrm{diag}\left(\gamma_1 I_{k_1},\ldots,\gamma_n I_{k_n},\ldots,\gamma_\ell I_{k_\ell}\right)\ :\ \gamma_i\in\mathbf{C},|\angle\gamma_j|\le h_j\omega/2,j=1,2,\ldots,n\right\}.$$

$$\Gamma=\left\{\mathrm{diag}\left(\gamma_1 I_{k_1},\ldots,\gamma_\ell I_{k_\ell}\right)\gamma_i\in\mathbf{C},i=1,\ldots,\ell\right\}.$$

4.2 Stability with General Time Delays

We have discussed the stability analysis using phase information formulated as $\mathcal{F}_u(G(s),\mathcal{D})$, where $G(s)$ is the generalized system and $\tau_i\in[0,h_i]$, uncertainty $\mathcal{D}=\mathrm{diag}\left(e^{-s\tau_1}I_{r_1},\ldots,e^{-s\tau_\ell}I_{r_\ell}\right)$, as shown in Figure 2.1.

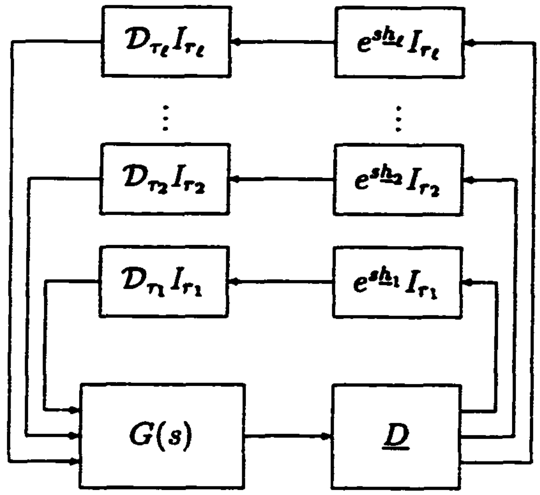


Figure 4.3: Phase-shifted Delay System

For general time delays $\tau_i \in [\underline{h}_i, \bar{h}_i]$, the phase information provided by uncertainties becomes $[-\bar{h}_i\omega, -\underline{h}_i\omega]$. By shifting phase of the system and uncertainty, we can analyze the stability by applying Theorem 4.4.

Shown in Figure 4.3, assume $G(s)$ stable and let

$$\underline{D} = \text{diag}\left(e^{-s\underline{h}_1} I_{r_1}, e^{-s\underline{h}_2} I_{r_2}, \ldots, e^{-s\underline{h}_\ell} I_{r_\ell}\right).$$

$\tilde{G}(s) = G(s)\underline{D}$. Note that the phase of corresponding uncertainty $\tilde{\mathcal{D}}$,

$$\begin{aligned} \tilde{\mathcal{D}} &= \text{diag}\left(e^{-s(\tau_1 - \underline{h}_1)} I_{r_1}, \ldots, e^{-s(\tau_\ell - \underline{h}_\ell)} I_{r_\ell}\right), \\ &= \text{diag}\left(e^{-s\tilde{\tau}_1} I_{r_1}, \ldots, e^{-s\tilde{\tau}_\ell} I_{r_\ell}\right), \qquad \tilde{\tau}_i \in [0, \bar{h}_i - \underline{h}_i], \end{aligned}$$

turns out to be $\left[-(\bar{h}_i - \underline{h}_i)\omega, 0\right]$. Then, Theorem 4.4 can be applied.

If $G(s)$ is an unstable system, stability analysis for general delays using phase information can not apply, since $G(s)\underline{\mathbf{D}}$ is unstable. For instance, consider a second order oscillatory system as we have done in chapter 2:

$$\ddot{y} + \omega_0^2 y - ky(t - \tau) = 0.$$

Rewrite the system $G(s)$ as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k \end{pmatrix} u(t),$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with input $u(t) = y(t-\tau)$. To find out the delay which stabilizes the system, we consider the system $\mathcal{F}_u\left(G(s), e^{-s\tau}\right)$ for $\tau \in [\underline{h}, \bar{h}), \underline{h} > 0$ since system is unstable for $\underline{h} = 0$. Analysis methods described in this chapter doesn't apply then, since $G(s)\underline{D}$ is unstable for all $\underline{h} > 0$.

4.3 Robust Stability and Performance Analysis

Now, we are ready to consider the general robust stability and robust performance problem of uncertain time delay systems with possibly structured uncertainties using Corollary 3.3. To that end, it is noted that without loss of generality, we can assume that the uncertain delay system have been arranged in the standard linear fractional transformation form as shown in Figure 4.4. We shall also assume that the system matrix $P(s)$ is a rational stable transfer matrix with suitable dimensions, and is denoted by $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI-A)^{-1}B + D.$

We assume that $\mathcal{D}(s) = \operatorname{diag}\{e^{-\tau_1 s}I_{k_1}, \ldots, e^{-\tau_\ell s}I_{k_\ell}\}$ includes all uncertain delays such that $\tau_i \in [\,0, h_i\,)$, $i = 1, \ldots, \ell$ and $h_1 < h_2 < \ldots < h_\ell$, and Δ is a block structured uncertainty includes all real, complex scalar and full block uncertainties and those blocks associated with robust performance criteria. With an appropriate

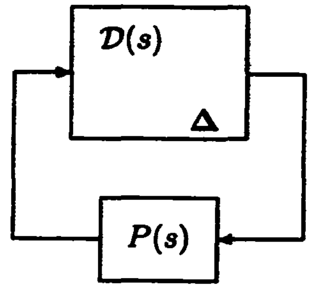


Figure 4.4: LFT Form of an Uncertain Delay System

arrangement, uncertainty $\Gamma = \text{diag}\{\mathcal{D}(s), \Delta\}$ can be written as

$$\begin{aligned} \text{diag}\{\mathcal{D}(s), \Delta\} &:= \Big\{\text{diag}(e^{-\tau_1 s} I_{k_1}, \ldots, e^{-\tau_\ell s} I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \\ &\qquad \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \\ &\qquad\quad \tau_i \in [\,0, h_i\,), \gamma_i \in \mathbf{C}, \delta_i \in \mathcal{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\}, \end{aligned}$$

where ℓ, m, n, $p \geq 0$. Then, the uncertainty with phase information is given by

$$\begin{aligned} \Gamma_{\Theta_\ell}{}^{P} &:= \Big\{\text{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \\ &\qquad \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + m; \ |\angle\gamma_j| \leq h_j\omega, \\ &\qquad j = 1, \ldots, \ell, \delta_i \in \mathcal{R}, \ \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\}, \end{aligned}$$

for ω such that $h_\ell\omega \leq \pi/2$.

Next, define

$$M(s) = P(s)\text{diag}(e^{-h_1 s/2} I_{k_1}, e^{-h_2 s/2} I_{k_2}, \ldots, e^{-h_\ell s/2} I_{k_\ell}, I_{k_{\ell+1}}, \ldots, I_p)$$

and the corresponding uncertainty blocks as

$$\begin{aligned} \Gamma_{\Theta_\ell}{}^{M} &:= \Big\{\text{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1} I_{k_{\ell+1}}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \\ &\qquad \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p) : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + m; \ |\angle\gamma_j| \leq h_j\omega/2, \\ &\qquad j = 1, \ldots, \ell, \delta_i \in \mathcal{R}, \ \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\}, \end{aligned}$$

if $h_\ell \omega \leq \pi$ (since $h_1\omega < h_2\omega < \ldots < h_\ell\omega$).

Theorem 4.5 *Suppose A is stable. Then the uncertain delay system is robustly stable if the following conditions hold*

$$\nu(\omega) = \min\left\{\nu\left(P(j\omega), \Gamma^{P}_{\Theta_\ell}\right), \quad \nu\left(M(j\omega), \Gamma^{M}_{\Theta_\ell}\right)\right\} < 1,$$

for all ω, where $\nu\left(P(j\omega), \Gamma^{P}_{\Theta_\ell}\right)$ and $\nu\left(M(j\omega), \Gamma^{M}_{\Theta_\ell}\right)$ are defined as

$$\nu\left(P(j\omega), \Gamma^{P}_{\Theta_\ell}\right) = \begin{cases} \mu_{\Gamma_{\Theta_\ell}}\left(P(j\omega)\right) & 0 \leq \omega \leq \pi/2h_\ell, \;\; \theta_i = h_i\omega, \;\; i = 1, \ldots, \ell, \\ \mu_{\Gamma_{\Theta_n}}\left(P(j\omega)\right) & \pi/2h_{n+1} < \omega \leq \pi/2h_n, \;\; n = 1, \ldots, \ell - 1, \\ & \qquad \theta_i = h_i\omega, \;\; i = 1, \ldots, n \\ \mu_{\Gamma}\left(P(j\omega)\right), & \qquad \omega > \pi/2h_1 \end{cases}$$

where the corresponding Γ_{Θ_n} is defined as

$$\Gamma_{\Theta_n} = \Big\{\mathrm{diag}\left(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}},\right.$$

$$\left.\Delta_1, \ldots, \Delta_p\right) \; : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell, |\angle\gamma_j| \leq h_j\omega, j = 1, \ldots, n, \delta \in \mathcal{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\},$$

$$\Gamma = \Big\{\mathrm{diag}\left(\gamma_1 I_{k_1}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p\right) \;\; :$$

$$\gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + m; \delta_i \in \mathcal{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\},$$

and

$$\nu\left(M(j\omega), \Gamma^{M}_{\Theta_\ell}\right) = \begin{cases} \mu_{\Gamma_{\Theta_\ell}}\left(M(j\omega)\right), & 0 \leq \omega \leq \pi/h_\ell, \;\; \theta_i = h_i\omega/2, \;\; i = 1, \ldots, \ell, \\ \mu_{\Gamma_{\Theta_n}}\left(M(j\omega)\right), & \pi/h_{n+1} < \omega \leq \pi/h_n, \;\; n = 1, \ldots, \ell - 1, \\ & \qquad \theta_i = h_i\omega/2, \;\; i = 1, \ldots, n \\ \mu_{\Gamma}\left(M(j\omega)\right), & \qquad \omega > \pi/h_1 \end{cases}$$

where the corresponding Γ_{Θ_n} *is defined as*

$$\Gamma_{\Theta_n} = \Big\{\mathrm{diag}\Big(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}},$$

$$\Delta_1, \ldots, \Delta_p\Big) : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell, |\angle\gamma_j| \le h_j\omega/2, j = 1, \ldots, n, \delta_i \in \mathcal{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\},$$

$$\Gamma = \Big\{ \mathrm{diag}\Big(\gamma_1 I_{k_1}, \ldots, \gamma_{\ell+m} I_{k_{\ell+m}}, \delta_{\ell+m+1} I_{k_{\ell+m+1}}, \ldots, \delta_{\ell+m+n} I_{k_{\ell+m+n}}, \Delta_1, \ldots, \Delta_p\Big) :$$

$$\gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + m;\ \delta_i \in \mathcal{R}, \Delta_i \in \mathbf{C}^{n_i \times n_i}\Big\}.$$

The proof of this theorem is similar to the proof of Theorem 4.3 and an upper bound of $\nu(\omega)$ can be computed by applying Corollary 3.3 .

4.4 Examples

Consider the following uncertain delay system:

$$\dot{x}(t) = (A + \Delta A)\, x(t) + \sum_{i=1}^{\ell} (E_i + \Delta E_i)\, x(t - \tau_i).$$

Let $E_i + \Delta E_i$ be factorized as $B_i(\Delta_i)\, C_i(\Delta_i)$ where $B_i \in \mathcal{R}^{n \times r_i}$, $C_i \in \mathcal{R}^{r_i \times n}$. Then, the system can be rewritten as,

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)\, x + \sum_{i=1}^{\ell} B_i v_i(t) \\ z_i(t) &= C_i x(t) \\ v_i(t) &= \mathcal{D}_{\tau_i} I_{k_i} z_i(t), \end{aligned}$$

which can be represented in LFT framework with the constant matrix $\tilde{P}$, as shown in Figure 4.5:

$$\tilde{P} = \begin{bmatrix} A + \Delta A & B_1 & \cdots & B_m \\ C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ C_m & 0 & \cdots & 0 \end{bmatrix} := \begin{bmatrix} A + \Delta A & \tilde{B}(\Delta) \\ \tilde{C}(\Delta) & 0 \end{bmatrix}.$$

Suppose $\tilde{P}$ can be written as

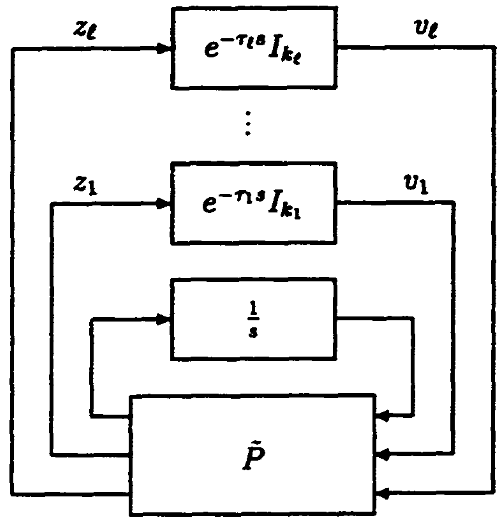


Figure 4.5: Interconnected Representation of the Delay System

$$\tilde{P} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

for some block diagonal uncertain parameter Δ. Again, the uncertain system can be put in the LFT form as shown in Figure 4.6 with $P(s)$,

$$P(s) = \begin{pmatrix} C_0 \\ N_1 \end{pmatrix} (sI - A_0)^{-1} \begin{pmatrix} B_0 & M_1 \end{pmatrix} + \begin{pmatrix} D_0 & M_2 \\ N_2 & 0 \end{pmatrix}.$$

Denote $\mathcal{D}(s) = \operatorname{diag}(e^{-\tau_1 s} I_{k_1}, \ldots, e^{-\tau_\ell s} I_{k_\ell})$. Then the robust stability of this uncertain delay system can be converted into stability of system $\mathcal{F}_u \left(P, \begin{bmatrix} \mathcal{D}(s) & 0 \\ 0 & \Delta \end{bmatrix} \right)$, to which Theorem 4.5 can be applied directly.

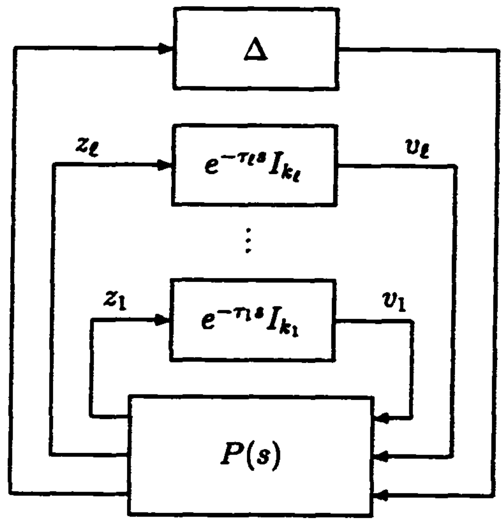


Figure 4.6: Interconnected Representation of the Delay System

Example 4.2 (Stability with Real Constant Parametric Uncertainties)

Consider the following uncertain system from [38],

$$\dot{x}(t) = \begin{bmatrix} -2 + 1.6\delta_1 & 0 \\ 0 & -1 + 0.05\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1 + 0.1\delta_3 & 0 \\ -1 & -1 + 0.3\delta_4 \end{bmatrix} x(t-h)$$

where uncertain parameters δ_i, $i = 1, \cdots, 4$ are real constants and satisfy $|\delta_i| \leq 1$. To find the maximum of h such that the system stays stable for $\tau \in [0, h)$, we compute the system's structure singular value with/without the phase information.

Rewrite the original system equation:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2 + 1.6\delta_1 & 0 \\ 0 & -1 + 0.05\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1 + 0.1\delta_3 & 0 \\ -1 & -1 + 0.3\delta_4 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t). \end{aligned}$$

Let $\tilde{P}$ be the representation of system in Figure 4.5 :

$$\tilde{P} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1.6 & 0 & 0.1 & 0 \\ 0 & 0.05 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \delta_3 & \\ & & & \delta_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The uncertain delay system can be rearranged as in Figure 4.6,

$$P(s) = \left[\begin{array}{cc|cccccc} -2 & 0 & -1 & 0 & 1.6 & 0 & 0.1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0.05 & 0 & 0.3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

with $\Gamma = \mathrm{diag}(e^{-jh\omega} I_2, \delta_1, \delta_2, \delta_3, \delta_4)$, where $|e^{-jh\omega}| \leq 1$, $\forall \omega \in \mathcal{R}$, $\delta_i \in \mathcal{R}$, $|\delta_i| \leq 1$, $i = 1, \ldots, 4$.

1. Consider system P associated with uncertainty Γ. The structured singular value, $\mu_\Gamma(P)$, can be obtained as following. Given a frequency ω, denote the

complex matrix $M = P(j\omega)$. Define $\mathbf{T}$, $\mathbf{G}$ as,

$$\mathbf{T} = \left\{T : T = \operatorname{diag}(T_1, T_2, T_3, T_4, T_5), 0 < T_1^* = T_1,\ T_1 \in \mathbf{C}^{2\times 2},\ T_2, \ldots, T_5 \in \mathcal{R}\right\},$$

$$\mathbf{G} = \{G\ :\ G = \operatorname{diag}(0_2, G_2, G_3, G_4, G_5), G_i \in \mathcal{R}\}.$$

Then, an upper bound of $\mu_\Gamma(M)$ is obtained by solving

$$\inf\left\{\alpha\ :\ M^* RM - \alpha^2 R + j(GM - M^* G) < 0\right\},$$

where $R \in \mathbf{T}$, $G \in \mathbf{G}$.

2. Consider system P associated with uncertainty Γ. The structured singular value with phase information, $\mu_{\Gamma_\Theta}(P)$, can be obtained as following. Given a frequency ω and delay h, denote the complex matrix $M = P(j\omega)$. Define $\mathbf{T}$, $\mathbf{G}$, $\mathbf{S}$, and $\hat{\mathbf{B}}$ as

$$\mathbf{T} = \left\{T : T = \operatorname{diag}(T_1, T_2, T_3, T_4, T_5), 0 < T_1^* = T_1,\ T_1 \in \mathbf{C}^{2\times 2},\ T_2, \ldots, T_5 \in \mathcal{R}\right\},$$

$$\mathbf{G} = \{G\ :\ G = \operatorname{diag}(0_2, G_2, G_3, G_4, G_5), G_i \in \mathcal{R}\},$$

$$\mathbf{S} = \left\{S\ :\ S = \operatorname{diag}(S_1, 0_4),\ 0 \le S_1^* = S_1 \in \mathbf{C}^{2\times 2}\right\},$$

$$\hat{\mathbf{B}} = \left\{\operatorname{diag}\left(\hat{B}_1, 0_4\right),\ \hat{B}_1^* = \hat{B}_1\right\}.$$

Then, an upper bound of $\mu_{\Gamma_\Theta}(M)$ is obtained by solving

$$\inf\left\{\alpha : M^* RM - \alpha^2 R + SM + M^* S + j\left(\hat{B}M - M^*\hat{B}\right) + j(GM - M^* G) < 0\right\},$$

subject to

$$-S\bar{B} \le \hat{B} \le S\bar{B},$$

where $\bar{B} = \operatorname{diag}(\beta_1 I_2, 0_4)$, $\beta_1 \in [-\cot\theta_i, \cot\theta_i]$ and $R \in \mathbf{T}$, $S \in \mathbf{S}$, $\hat{B} \in \hat{\mathbf{B}}$, $G \in \mathbf{G}$.

3. Next, consider the angle shifted system. Given a frequency ω, denote the complex matrix $M = P(j\omega)\text{diag}\left(e^{-j\frac{h\omega}{2}} I_2, I_4\right)$. The associated uncertainty is

$$\Gamma = \text{diag}(e^{-jh\omega/2} I_2, \delta_1, \delta_2, \delta_3, \delta_4).$$

Define $\mathbf{T}$, $\mathbf{G}$ as

$$\mathbf{T} = \left\{T : T = \text{diag}\left(T_1, T_2, T_3, T_4, T_5\right), 0 < T_1^* = T_1,\ T_1 \in \mathbf{C}^{2\times 2},\ T_2, \ldots, T_5 \in \mathcal{R}\right\},$$

$$\mathbf{G} = \left\{G \ :\ G = \text{diag}\left(0_2, G_2, G_3, G_4, G_5\right), G_i \in \mathcal{R}\right\}.$$

Then, an upper bound of $\mu_\Gamma(M)$ is obtained by solving

$$\inf\left\{\alpha \ :\ M^* R M - \alpha^2 R + j\left(GM - M^* G\right) < 0\right\},$$

where $R \in \mathbf{T}$, $G \in \mathbf{G}$.

4. An upper bound of $\mu_{\Gamma_\Theta}(M)$, denoted by $\nu(\omega)$, can also be computed. Define $\mathbf{T}$, $\mathbf{G}$ as

$$\mathbf{T} = \left\{T : T = \text{diag}\left(T_1, T_2, T_3, T_4, T_5\right), 0 < T_1^* = T_1,\ T_1 \in \mathbf{C}^{2\times 2},\ T_2, \ldots, T_5 \in \mathcal{R}\right\},$$

$$\mathbf{G} = \left\{G \ :\ G = \text{diag}\left(0_2, G_2, G_3, G_4, G_5\right), G_i \in \mathcal{R}\right\}.$$

$$\mathbf{S} = \left\{S \ :\ S = \text{diag}\left(S_1, 0_4\right), 0 \le S_1^* = S_1 \in \mathbf{C}^{2\times 2}\right\},$$

(a) $\forall\omega \ :\ 0 \le \omega \le \pi/h$, solving

$$\nu(\omega) = \inf\{\ \alpha :\ M^* R M - \alpha^2 R + SM + M^* S + j(\hat{B}M - M^*\hat{B})$$

$$+ j(GM - M^* G) < 0\ \}$$

with the constrain,

$$-S\bar{B} \le \hat{B} \le S\bar{B},$$

where $\bar{B} = \text{diag}\left(\cot\left(h\omega/2\right) I_2, 0_4, \right)$.

(b) $\forall\omega\ :\ \omega > \pi/h$, then $\nu(\omega) = \mu_\Gamma(M)$.

For a given h, the results are plotted in Figure 4.7, Figure 4.8, Figure 4.9 and Figure 4.10.

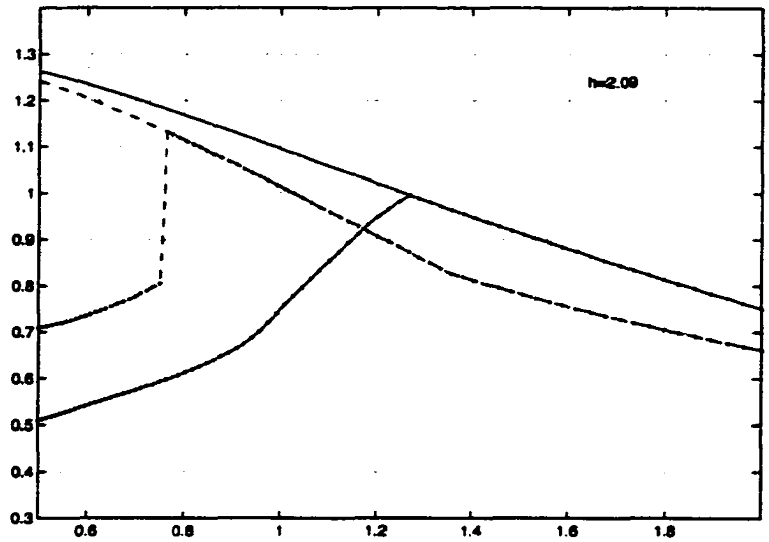


Figure 4.7: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 2.09$: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

By Theorem 4.5, the uncertain time-delayed system is stable if $h \leq 2.57$, in contrast to the stability conclusion in the last chapter ([25]) if $h \leq 1.715$, and by [38], if $h \leq 0.689$.

Next, to improve the result, we would like to apply stability test on $[\underline{h}, \bar{h}]$, where $0 < \underline{h} < 2.57$. However, we can't get $\bar{h} > 2.57$ in this case.

Remark 4.1 Since $\mu_{\Gamma_\Theta}(M)$ is associated with the phase uncertainty $[-\frac{h_i\omega}{2}, \frac{h_i\omega}{2}]$ due to delays, and in contrast to the phase uncertainty $[0, h_i\omega]$ of $\mu_{\Gamma_\Theta}(P)$, we would expect $\mu_{\Gamma_\Theta}(M)$ give us better results. This is actually not necessarily true, as shown in Figure 4.8, the (upper bound of) structure singular value $\nu(\omega)$ is taken

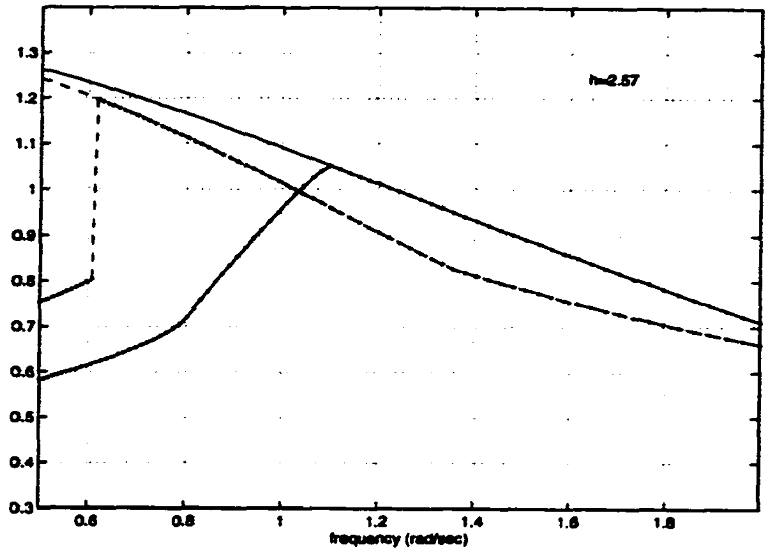


Figure 4.8: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 2.57$: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

from $\mu_{\Gamma_\Theta}(M)$ at low frequency, and from $\mu_{\Gamma_\Theta}(P)$ at higher frequency. However, this switch will not happen for a single scalar time-delay system.

Example 4.3 (Stability with Time-Varying Parametric Uncertainties)

This example is taken from [23]. Consider an uncertain time delay system with time-varying uncertainties

$$\dot{x}(t) = \begin{bmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{bmatrix} x(t-h)$$

where the uncertain parameter $|\rho(t)| \leq 0.1$.

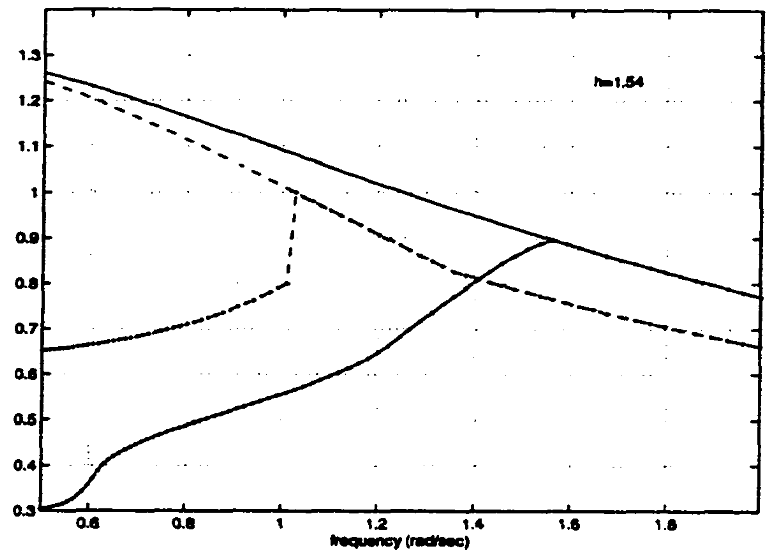


Figure 4.9: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 1.54$: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

First, rewrite the original system equation:

$$\dot{x}(t) = \begin{bmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t).$$

Let $\tilde{P}$ be the representation of the system in Figure 4.5 :

$$\tilde{P} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

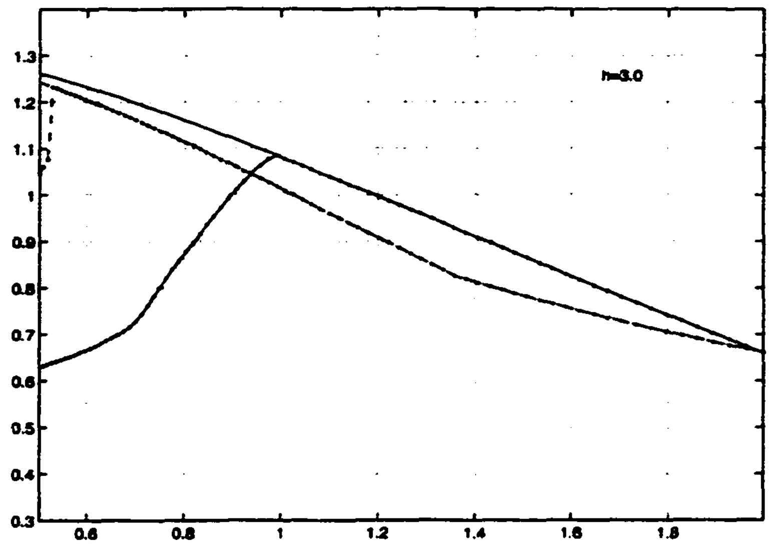


Figure 4.10: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 1.54$: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

$$= \begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & -0.9 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \rho(t) & \\ & \rho(t) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & -0.9 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 10\rho(t) & \\ & 10\rho(t) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}.$$

The uncertain delay system can be rearranged as in Figure 4.6 with

$$P(s) = \left[\begin{array}{cc|cccc} -2 & 0 & -1 & 0 & 0.1 & 0 \\ 0 & -0.9 & -1 & -1 & 0 & 0.1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \end{array}\right]$$

and $\Gamma = \mathrm{diag}(e^{-hs}I_2, 10\rho(t)I_2)$, $|\Gamma| \leq 1$.

1. Consider the system $P(s)$ associated with Γ first. For a given ω, let $M = P(j\omega)$. For time-varying uncertainty, the corresponding scaling matrix should be kept the same for all frequencies. Therefore, an upper bound of $\mu_\Gamma(M)$ can be obtained from Equation (3.3). Define

$$\mathbf{T} = \left\{T \; : \; T = \mathrm{diag}(T_1, T_2), 0 < T_1^* = T_1 \in \mathbf{C}^{2\times 2}\right\},$$

$$\inf\left\{\alpha \; : \; M^*RM - \alpha^2 R < 0 \;, R \in \mathbf{T}\right\},$$

where T_2 is a real constant matrix for all frequencies. For simplicity, we take $T_2 = I_2$.

2. An upper bound of the structured singular value with phase information $\mu_{\Gamma_\Theta}(M(\omega))$ is then computed.

   (a) $\forall \omega \; : \; 0 \leq \omega \leq \pi/2h$, $\nu(\omega) = \hat{\mu}_{\Gamma_\theta}(M)$. Define $\mathbf{T}$, $\mathbf{S}$:

$$\mathbf{T} = \left\{T \; : \; T = \mathrm{diag}(T_1, I_2), 0 < T_1^* = T_1, T_1 \in \mathbf{C}^{2\times 2}\right\},$$

$$\mathbf{S} = \left\{S \; : \; S = \mathrm{diag}(S_1, 0_2), 0 \leq S_1^* = S_1 \in \mathbf{C}^{2\times 2}\right\},$$

Solve

$$\inf\left\{\alpha \ :\ M^*RM-\alpha^2R+SM+M^*S+j\left(\hat{B}M-M^*\hat{B}\right)<0,\ \alpha>0\right\}$$

where $R\in\mathbf{T},\ S\in\mathbf{S}$ and the convex constrains satisfying

$$-S\bar{B}\leq\hat{B}\leq S\bar{B},$$

where $\bar{B}=\operatorname{diag}\left(\cot\left(h\omega\right)I_2,0_2\right)$.

(b) $\forall\omega\ :\ \omega>\pi/2h$, then $\nu(\omega)=\mu_\Gamma(M)$.

3. Next, define the angle shifted system $M(j\omega)=P(j\omega)\operatorname{diag}\left(e^{-jh\omega/2}I_2,I_2\right)$. The uncertainty is structured as $\Gamma=\operatorname{diag}(\gamma I_2,10\rho(t)I_2)$, where $\gamma\in\mathbf{C},\ |\gamma|\leq 1,\ \rho(t)\in\mathcal{R},\ |10\rho(t)|\leq 1$. Again, define

$$\mathbf{T}=\left\{T\ :\ T=\operatorname{diag}(T_1,I_2),0<T_1^*=T_1\in\mathbf{C}^{2\times2}\right\}.$$

Let $R\in\mathbf{T}$. We can obtain $\hat{\mu}_\Gamma(M)$ by solving

$$\inf\left\{\alpha\ :\ M^*RM-\alpha^2R<0,\ \alpha>0\right\}.$$

4. An upper bound of the structured singular value with phase information $\nu(\omega)$ is then computed. M and Γ are as above, and

$$\Gamma_\Theta(M)=\{\operatorname{diag}(\gamma I_2,10\rho(t)I_2),\ |\gamma|\leq1,\ \angle\gamma\leq h\omega/2,\ |10\rho(t)|\leq1\}.$$

(a) $\forall\omega\ :\ 0\leq\omega\leq\pi/h$, $\nu(\omega)=\hat{\mu}_{\Gamma_\theta}(M)$. Define $\mathbf{T},\ \mathbf{S}$:

$$\mathbf{T}=\left\{T\ :\ T=\operatorname{diag}\left(T_1,I_2\right),0<T_1^*=T_1,T_1\in\mathbf{C}^{2\times2}\right\},$$

$$\mathbf{S}=\left\{S\ :\ S=\operatorname{diag}\left(S_1,0_2\right),0\leq S_1^*=S_1\in\mathbf{C}^{2\times2}\right\}.$$

Solve

$$\inf\left\{\alpha:\ M^*RM-\alpha^2R+SM+M^*S+j(\hat{B}M-M^*\hat{B})<0\right\},$$

with the convex constrains satisfying

$$-S\bar{B} \leq \hat{B} \leq S\bar{B},$$

where $\bar{B} = \mathrm{diag}\,(\cot(h\omega/2)\, I_2, 0_2)$.

(b) $\forall\omega\ :\ \omega > \pi/h$, then $\nu(\omega) = \mu_\Gamma(M)$.

The result is plotted in Figure 4.11, Figure 4.12. Figure 4.13 shows that the stability test result on delay interval $h \in [2.69, 2.71)$.

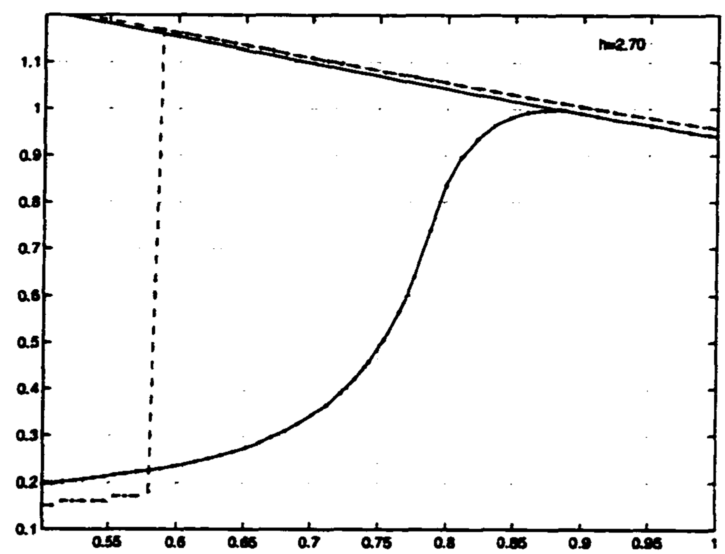


Figure 4.11: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 2.70$: $\mu_{\Gamma_e}(M)$ (dot-solid line), $\mu_{\Gamma_e}(P)$ (dot-dashed line)

By Theorem 14, the uncertain system is robustly stable if $h \leq 2.71$ since an upper bound of $\nu(\omega) < 1$ for all ω. Note that the stability conditions obtained in [23] and [38] are respectively $h \leq 2.61$, and $h \leq 0.72$.

In the following, we are going to investigate the effect of scaling matrix corresponding to time-varying uncertainty.

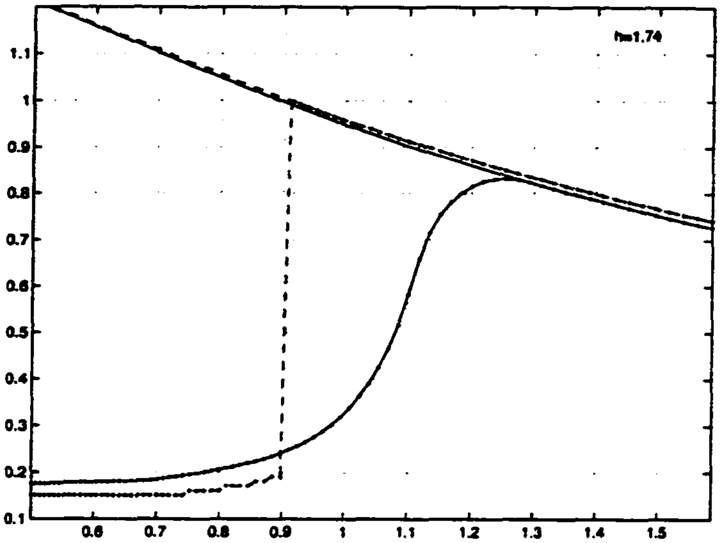


Figure 4.12: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 1.74$: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

Case I:

To claim that there exists an $h > 2.70$, such that system is stable, we basically want to show that $\mu_{\Gamma_\Theta}(M)$ is smaller than what we have computed in Figure 4.11. For h=2.70, at the peak frequency $\omega = 0.8848(rad)$, we can solve R_1, S_1, Q_1 by

$$M^*\begin{bmatrix} R_1 & \\ & I_2 \end{bmatrix} M - \alpha^2 \begin{bmatrix} R_1 & \\ & I_2 \end{bmatrix} + \begin{bmatrix} S_1 & \\ & 0_2 \end{bmatrix} M + M^* \begin{bmatrix} S_1 & \\ & 0_2 \end{bmatrix} +$$

$$\begin{bmatrix} Q_1 & \\ & 0_2 \end{bmatrix} (jM) + (jM)^* \begin{bmatrix} Q_1 & \\ & 0_2 \end{bmatrix} < 0,$$

where $M = P(j\omega)\text{diag}(e^{-jh\omega/2}I_2, I_2)$, $\alpha = \nu(\omega)$ obtained from above example. Then, given α, M, R_1, S_1 and Q_1, the scaling matrix associated with time-varying uncer-

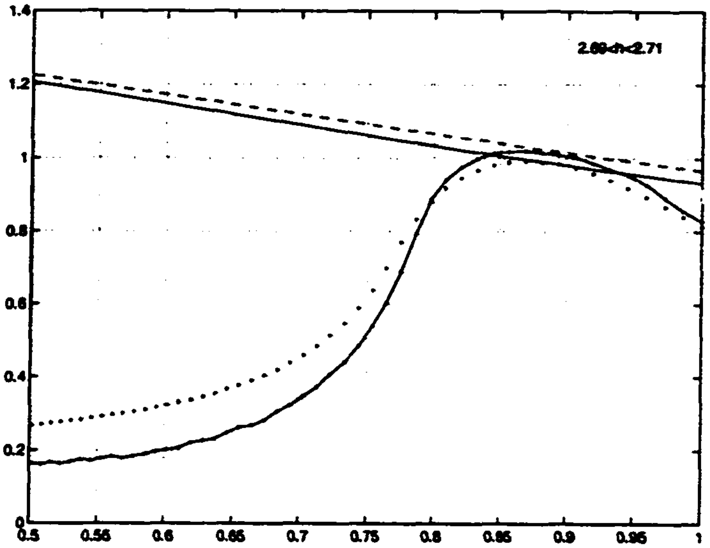


Figure 4.13: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $2.69 < h < 2.71$: $\mu_{\Gamma_\Theta}(M)$ (dot line), $\mu_{\Gamma_\Theta}(P)$ (dot-solid line)

tainty can be obtained by solving

$$M^* \begin{bmatrix} R_1 & \\ & Scal \end{bmatrix} M - \alpha^2 \begin{bmatrix} R_1 & \\ & Scal \end{bmatrix} + \begin{bmatrix} S_1 & \\ & 0_2 \end{bmatrix} M + M^* \begin{bmatrix} S_1 & \\ & 0_2 \end{bmatrix} +$$

$$\begin{bmatrix} Q_1 & \\ & 0_2 \end{bmatrix} (jM) + (jM)^* \begin{bmatrix} Q_1 & \\ & 0_2 \end{bmatrix} < 0.$$

And,

$$Scal = \begin{bmatrix} 1.1759 & -0.0393 \\ -0.0393 & 0.8371 \end{bmatrix}.$$

We then compute the structured singular value with/without phase information with the above scaling matrix. The algorithm is repeated as above with scaling matrix "*Scal*" instead of I_2. The result is plotted in Figure 4.14.

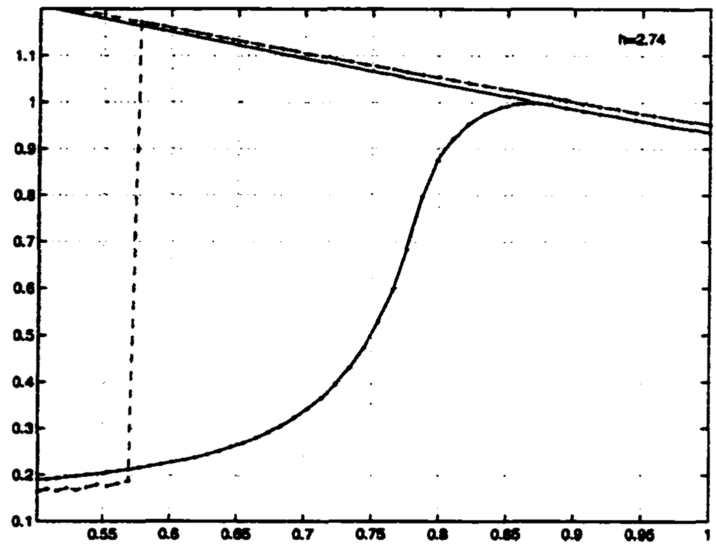


Figure 4.14: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 2.74$ with nonidentity scaling matrix: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

Comparing Figure 4.11, Figure 4.12 and Figure 4.14, $\nu(\omega)$ is pushed down a little bit with non-identity scaling matrix. For the case of $h = 2.74$, the peak value of $\nu(\omega)$ obtained with identity scaling matrix is 1.0030, but drops to 0.9990 if the scaling matrix is chosen as $Scal$.

To complete, we show $\mu_\Gamma(M)$, $\mu_{\Gamma_\Theta}(M)$ for $h = 2.74$ in Figure 4.15.

Case II

Next, we are going to find a scaling matrix from $\hat{\mu}_\Gamma(M)$, instead of from $\hat{\mu}_{\Gamma_\Theta}(\omega)$. As we have done in Case I, at the same frequency $\omega = 0.8848(rad)$, consider the structured singular value without phase information. Solve R_1, and minimum α by

$$M^* \begin{bmatrix} R_1 & \\ & I_2 \end{bmatrix} M - \alpha^2 \begin{bmatrix} R_1 & \\ & I_2 \end{bmatrix} < 0.$$

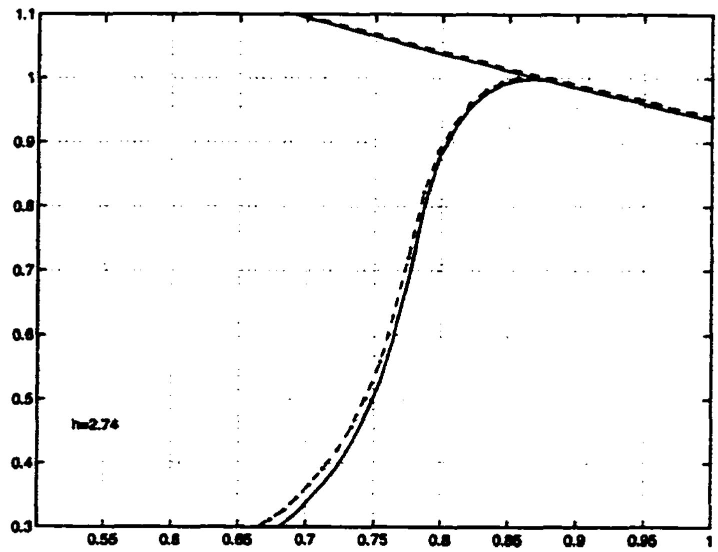


Figure 4.15: Upper bounds of $\mu_\Gamma(M)$, $\mu_{\Gamma_\Theta}(M)$ for $h = 2.74$ with scaling matrix $= I_2$ (dot-dashed line), and $\mu_\Gamma(M)$ (dashed line), with scaling matrix $= Scal$ (solid line)

Then, given R_1 and α solve the scaling matrix $Scal$ by:

$$M^* \begin{bmatrix} R_1 & \\ & Scal \end{bmatrix} M - \alpha^2 \begin{bmatrix} R_1 & \\ & Scal \end{bmatrix} < 0.$$

The scaling matrix $Scal$ is:

$$Scal = \begin{bmatrix} 1.9005 & -0.4120 \\ -0.4120 & 1.0865 \end{bmatrix}.$$

Use this scaling matrix to compute upper bounds $\hat{\mu}_\Gamma(\omega)$ and $\hat{\mu}_{\Gamma_\Theta}(\omega)$. The result is plotted in Figure 4.16. The maximum of $\mu_{\Gamma_\Theta}(M)$ is 1.000. As we have seen in CASE I, the scaling matrix does make the $\mu_\Gamma(\omega)$ slightly less around the frequency $\omega = 0.8848$ also.

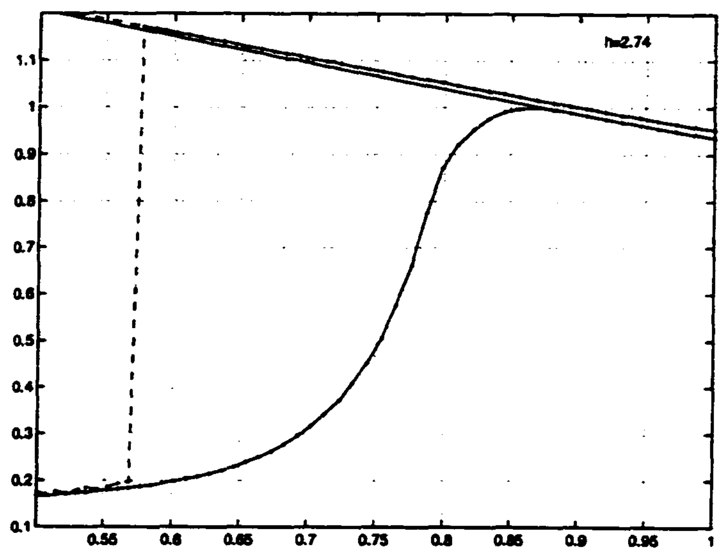


Figure 4.16: Upper bounds of $\mu_\Gamma(M)$ (solid line), $\mu_\Gamma(P)$ (dashed line) and $\nu(\omega) = \min\{\nu(M), \nu(P)\}$ for $h = 2.74$ with nonidentity scaling matrix: $\mu_{\Gamma_\Theta}(M)$ (dot-solid line), $\mu_{\Gamma_\Theta}(P)$ (dot-dashed line)

Chapter 5

Nonlinear Time-Delay Systems

It is well-known that stability analysis and controller design for a time-delay system with nonlinear components are very challenging problems, see [6, 11, 24, 52, 54, 62, 68, 70] and references therein. Here, we are interested in analysis of a nonlinear time-delay system which can be represented as a feedback connection of a linear dynamical system and a nonlinear element, shown below:

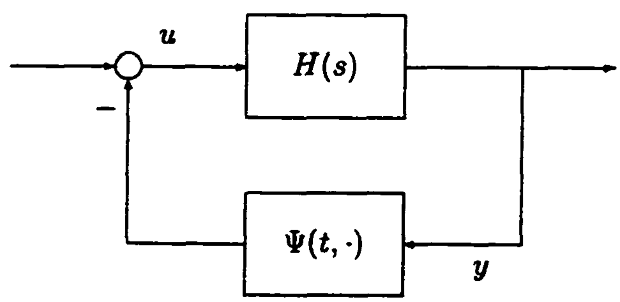


Figure 5.1: Feedback Connection of a Linear System and a Nonlinear Element

To motivate our presentation, let us consider a time-delay system represented as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + \sum_i A_i x(t-\tau_i) + Bu(t), \\
y(t) &= Cx(t) + \sum_i C_i x(t-\tau_i),
\end{aligned} \tag{5.1}
$$

with feedback input $u(t) = -\Psi(t, y)$. Let the nonlinearity $\Psi(t, \cdot)$

$$\Psi(t, y) = \begin{bmatrix} \psi_1(t, y_1) \\ \psi_2(t, y_2) \\ \vdots \\ \psi_p(t, y_p) \end{bmatrix},$$

satisfy sector condition $[0, K]$,

$$\psi_i(t, y_i)[\psi_i(t, y_i) - k_i y_i] \leq 0, \quad \forall t \geq 0,$$

where $K = \text{diag}\,(k_1, k_2, \ldots, k_p)\,, k_i > 0$. We call the closed loop system absolutely stable, if the origin $x = 0$ is asymptotically stable for all nonlinearities in the sector.

Absolute stability problems have been investigated thoroughly in the literature for a linear dynamical system without delayed states connected with a nonlinearity. Let a dynamical system $G(s) = C\,(sI - A)^{-1}\,B + D$ be feedback connected with a nonlinear element $\Psi(t, \cdot)$ satisfying sector condition $[0, K]$, where $K = \text{diag}\,(k_1, k_2, \ldots, k_p)\,, k_i > 0$. Then the well-known circle criterion says that the feedback system is absolutely stable if $G(s)$ is stable, and $[I + KG(s)]$ is strictly positive real. Moreover, if the nonlinearity is time invariant, Popov criterion gives less conservative result. Suppose there exists a matrix $N = \text{diag}(n_1, n_2, \ldots, n_p)$, where $n_i > 0$. The system is absolutely stable if $I + (I + Ns)KG(s)$ is strictly positive real, where $\Psi(\cdot)$ satisfies sector condition $[0, K]\ :\ \psi_i(y_i)[\psi(y_i) - k_i y_i] \leq 0$.

These results have been extended to time-delay systems [3]. Let a time-delay dynamical system be described by

$$H(s) = \left(C + \sum_{i=1}^{\ell} C_i e^{-\tau_i s}\right)\left(sI - A - \sum_{i=1}^{\ell} A_i e^{-\tau_i s}\right)^{-1} B$$

and connected with a nonlinear element $\Psi(t,\cdot)$. Then, the system is absolute stable if all roots of $\det(sI - A - \sum_i A_i e^{-\tau_i s}) = 0$ lie in the left half plane, and $[I + KH(s)]$ is strictly positive real, where $\Psi(t,\cdot)$ satisfies sector condition $[0, K]$. Moreover, if the nonlinearity is time invariant, applying Lur's type Lyapunov functions leads to the conclusion : the system is absolute stable if all roots of $\det(sI - A - \sum_i A_i e^{-\tau_i s}) = 0$ lie in the left half plane, and $I + (I + Ns)KH(s)$ is strictly positive real, where $\Psi(\cdot)$ satisfies sector condition $[0, K]$, $N = \mathrm{diag}(n_1, n_2, \ldots, n_p)$, where $n_i > 0$. More extensions can be found in [2, 3, 19, 27, 52].

In this chapter, we are interested in stability analysis for a linear uncertain time-delay system with a feedback connected nonlinear element. Absolute delay-dependent stability is referred to the fact that the system is robustly stable with respect to time delays $\tau_i \in [0, h_i)$ and nonlinearity satisfying sector condition $[0, K]$. And absolute delay-independent stability is robustly stable for delays $\tau_i \in \mathcal{R}^+$, and nonlinearity satisfying sector condition $[0, K]$. By incorporating structures of delay and nonlinearity in LFT framework, we can easily employ structured singular value with/without phase information on delay uncertainty and circle/Popov criterion on nonlinear uncertainty. Then, conclusion can be obtained by small-μ theorem.

5.1 Absolute Delay Independent Stability for Time-Varying Nonlinearity

Consider the uncertain time-delay system described by Equation (5.1) and shown in Figure 5.1. Let

$$H(s) = \left(C + \sum_i C_i e^{-s\tau_i}\right)\left(sI - A - \sum_i A_i e^{-s\tau_i}\right)^{-1} B,$$

for some fixed but unknown τ_i's and $x \in \mathcal{R}^n$, $u, y \in \mathcal{R}^p$. Then, we have

Theorem 5.1 (Circle Criterion) [33] *The feedback system in Figure 5.1 is absolute stable if*

$$I + KH(s)$$

is strictly positive real.

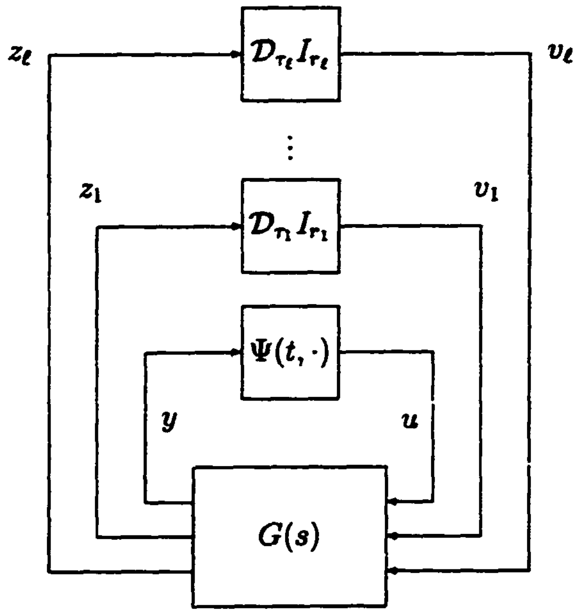


Figure 5.2: LFT Framework of Time Delay Nonlinear Systems

Let $A_i = A_i^l A_i^r$, $C_i = C_i^l A_i^r$, $i = 1, \ldots, \ell$, and suppose the nonlinearity $\psi_i(t,\cdot)$ satisfies sector condition $[0, k_i]$. Denote $A_G = A$, $B_G = \begin{bmatrix} A_1^l & \ldots & A_\ell^l & -B \end{bmatrix}$,

$$C_G = \begin{bmatrix} A_1^r \\ \vdots \\ A_\ell^r \\ C \end{bmatrix}, \qquad D_G = \begin{bmatrix} 0 & & 0 \\ & \ddots & \vdots \\ & 0 & 0 \\ C_1^l & \ldots \; C_\ell^l & 0 \end{bmatrix},$$

Then, the system $H(s)$ can be rewritten as

$$H(s) = \mathcal{F}_u\left(G(s), \Delta\right),$$

where $G(s) = G_C\left(sI - G_A\right)^{-1} G_B + G_D$, $\Delta = \text{diag}\left(\mathcal{D}_{\tau_1} I_{r_1}, \ldots, \mathcal{D}_{\tau_\ell} I_{r_\ell}, \Psi(t,\cdot)\right)$, as shown in Figure 5.2.

Next, we perform loop transformation to normalize nonlinearity. Let $\psi_i^-(t, y_i) = \psi_i(t, y_i) - \frac{k_i}{2} y_i$. Then,

$$\left(\psi_i^-(t, y_i) + \frac{k_i}{2} y_i\right)\left(\psi_i^-(t, y_i) - \frac{k_i}{2} y_i\right) \leq 0.$$

i.e., $\psi_i^-(t, y_i)$ satisfies sector condition $[-k_i/2, k_i/2]$, and $\|\psi_i^-(t, y_i)\|_\infty \leq k_i/2$. Moreover, let $\tilde{\Psi}(t,\cdot) = \text{diag}\left(2\psi_1^-(t, y_1)/k_1, \ldots, 2\psi_p^-(t, y_p)/k_p\right)$. Then, $\|\tilde{\Psi}(t,\cdot)\|_\infty \leq 1$. Let the corresponding system matrix be $\tilde{G}(s)$,

$$\begin{aligned}
\tilde{G}(s) &= \left(I - G(s)\begin{bmatrix} 0_{r_1+\ldots+r_\ell} & \\ & K/2 \end{bmatrix}\right)^{-1} G(s)\begin{bmatrix} I_{r_1+\ldots+r_\ell} & \\ & K/2 \end{bmatrix} \\
&:= \tilde{G}_C\left(sI - \tilde{G}_A\right)^{-1} \tilde{G}_B + \tilde{G}_D.
\end{aligned}$$

Denote the structured singular value of the system as $\tilde{\mu}_\Delta\left(\tilde{G}(j\omega)\right)$, which can be computed as follows. Let $M = \tilde{G}(j\omega)$ and suppose there exist fixed constants d_j, $j = \ell+1, \ldots, \ell+p-1$.

$$\mathcal{D} := \{\, \text{diag}\left(D_1, \ldots, D_\ell, d_{\ell+1}, \ldots, d_{\ell+p-1}, I_1\right) \,:\, D_i \in \mathbf{C}^{r_i \times r_i}, D_i = D_i^* > 0, i = 1, \ldots, \ell,$$

$$d_j \in \mathcal{R},\ d_j > 0, j = \ell+1, \ldots, \ell+p-1 \,\}$$

Then $\tilde{\mu}_\Delta\left(\tilde{G}(j\omega)\right) \leq \hat{\tilde{\mu}}_\Delta\left(\tilde{G}(j\omega)\right)$, where

$$\hat{\tilde{\mu}}_\Delta(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right),$$

which can be computed by

$$\hat{\bar{\mu}}_{\Delta}(M) = \inf_{D \in \mathcal{D}} \min_{\beta} \left\{ \beta \ : \ M^* D M - \beta^2 D \leq 0 \right\}. \tag{5.2}$$

Note that d_j, $j = \ell + 1, \ldots, \ell + p - 1$ are fixed constants and don't vary with frequencies.

Lemma 5.2 *In Figure 5.2, the feedback system is absolute delay-independent stable if $\tilde{G}(s)$ is stable and*

$$\bar{\mu}_{\Delta}\left(\tilde{G}(j\omega)\right) < 1, \qquad \forall \omega,$$

where

$$\Delta = \{\mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ : \gamma_i \in \mathbf{C}, \ |\gamma_i| \leq 1, \ i = 1, \ldots, \ell + p\}.$$

5.2 Absolute Delay Dependent Stability for Time-Varying Nonlinearity

We have shown that system can be represented as $\mathcal{F}_u\left(\tilde{G}(s), \Delta\right)$, where

$$\Delta = \{\mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ : \gamma_i \in \mathbf{C}, \ |\gamma_i| \leq 1, \ i = 1, \ldots, \ell + p\}.$$

For a delay dependent stability problem, $\tau_i \in [0, h_i]$, $h_1 < h_2 < \ldots < h_\ell$, we have

$$\begin{aligned} \Delta_\Theta = \{ \ \mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ &: \ \gamma_i \in \mathbf{C}, \ |\gamma_i| \leq 1, \ i = 1, \ldots, \ell + p, \\ &\quad |\angle \gamma_i| \leq \theta_i, \ i = 1, \ldots, \ell\}. \end{aligned}$$

Then, structured singular value with phase information in Theorem 4.2 can be applied with some modification.

Lemma 5.3 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Denote $P(j\omega) = \tilde{G}(j\omega)$. Then the uncertain delay system is stable for $\tau_i \in [0, \ h_i)$, $i = 1, \ldots, \ell$ if the following conditions hold*

(a) $\tilde{\nu}(\omega) = \tilde{\mu}_{\Gamma_{\Theta_\ell}}(P(j\omega)) < 1$ *for* $0 \leq \omega \leq \pi/2h_\ell$ *with* $\theta_i = h_i\omega,\ i = 1, \ldots, \ell$;

(b) For each $n = 1, \ldots, \ell - 1$, *we have* $\tilde{\nu}(\omega) = \tilde{\mu}_{\Gamma_{\Theta_n}}(P(j\omega)) < 1$ *for* $\pi/2h_{n+1} < \omega \leq \pi/2h_n$, *with* $\theta_i = h_i\omega,\ i = 1, \ldots, n$ *and*

$$\Gamma_{\Theta_n} = \{\ \mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \gamma_{n+1} I_{k_{n+1}}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) :$$
$$\gamma_i \in \mathbf{C}, |\angle\gamma_j| \leq h_j\omega, j = 1, 2, \ldots, n\ \};$$

(c) $\tilde{\nu}(\omega) = \tilde{\mu}_\Gamma(P(j\omega)) < 1$ *for* $\omega > \pi/2h_1$ *with*

$$\Gamma = \{\mathrm{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) : \gamma_i \in \mathbf{C}\}.$$

Furthermore, we define the angle shifted system $M(s)$:

$$M(s) = P(s)\mathrm{diag}(e^{-h_1 s/2} I_{r_1}, e^{-h_2 s/2} I_{r_2}, \ldots, e^{-h_\ell s/2} I_{r_\ell}, I_p), \tag{5.3}$$

with the associated uncertainty

$$\begin{aligned}\Gamma(s) &= \mathrm{diag}(e^{-(\tau_1 - \frac{h_1}{2})s} I_{r_1}, e^{-(\tau_2 - \frac{h_2}{2})s} I_{r_2}, \ldots, e^{-(\tau_\ell - \frac{h_\ell}{2})s} I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}), \quad \tau_i \in [0, h_i] \\ &= \mathrm{diag}(e^{-\phi_1 s} I_{r_1}, e^{-\phi_2 s} I_{r_2}, \cdots, e^{-\phi_\ell s} I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}), \quad \phi_i \in [-h_i/2, h_i/2].\end{aligned}$$

Then, we have

Lemma 5.4 *Suppose* A *is stable and assume* $h_1 < h_2 < \ldots < h_\ell$. *Then the uncertain delay system is stable for* $\tau_i \in [0, h_i)$, $i = 1, \ldots, \ell$, *if the following conditions hold*

(a) $\tilde{\nu}(\omega) = \tilde{\mu}_{\Gamma_{\Theta_\ell}}(M(j\omega)) < 1$ *for* $0 \leq \omega \leq \pi/h_\ell$ *with* $\theta_i = h_i\omega/2,\ i = 1, \ldots, \ell$;

(b) For each $n = 1, \ldots, \ell - 1$, *we have* $\tilde{\nu}(\omega) = \tilde{\mu}_{\Gamma_{\Theta_n}}(M(j\omega)) < 1$ *for* $\pi/h_{n+1} < \omega \leq \pi/h_n$, *with* $\theta_i = h_i\omega/2,\ i = 1, \ldots, n$ *and*

$$\Gamma_{\Theta_n} = \{ \operatorname{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_n I_{k_n}, \gamma_{n+1} I_{k_{n+1}}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) :$$

$$\gamma_i \in \mathbf{C}, |\angle\gamma_j| \leq h_j\omega/2, j = 1, \ldots, n \};$$

(c) $\tilde{\nu}(\omega) = \tilde{\mu}_\Gamma (M(j\omega)) < 1$ *for* $\omega > \pi/h_1$ *with*

$$\Gamma = \{\operatorname{diag}(\gamma_1 I_{k_1}, \ldots, \gamma_\ell I_{k_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) : \gamma_i \in \mathbf{C}\}.$$

Combining Lemma 5.3 and Lemma 5.4, we get:

Lemma 5.5 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Then the uncertain delay system is stable for $\tau_i \in [0, h_i)$, $i = 1, \ldots, \ell$, if*

$$\tilde{\nu}(\omega) = \min\left\{\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right), \quad \tilde{\nu}\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right)\right\} < 1,$$

for all ω, where $\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right)$ and $\tilde{\nu}\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right)$ are defined as

$$\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right) = \begin{cases} \tilde{\mu}_{\Gamma_{\Theta_\ell}}(P(j\omega)) & 0 \leq \omega \leq \pi/2h_\ell, \quad \theta_i = h_i\omega, \ i = 1, \ldots, \ell; \\ \tilde{\mu}_{\Gamma_{\Theta_n}}(P(j\omega)) & \pi/2h_{n+1} < \omega \leq \pi/2h_n, \quad n = 1, \ldots, \ell-1, \\ & \qquad \theta_i = h_i\omega, \quad i = 1, \ldots, n \\ \tilde{\mu}_\Gamma(P(j\omega)), & \qquad \omega > \pi/2h_1 \end{cases}$$

where

$$\Gamma_{\Theta_n} = \{ \operatorname{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_n I_{r_n}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ : \gamma_i \in \mathbf{C}, |\angle\gamma_j| \leq h_j\omega,$$

$$i = 1, \ldots, p, \ j = 1, 2, \ldots, n \};$$

$$\Gamma = \{\operatorname{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + p\}.$$

And,

$$\tilde{\nu}\left(M(j\omega),\Gamma^{M}_{\Theta_\ell}\right)=\begin{cases}\tilde{\mu}_{\Gamma_{\Theta_\ell}}(M(j\omega)), & 0\le\omega\le\pi/h_\ell,\ \theta_i=h_i\omega/2,\ i=1,\dots,\ell;\\ \tilde{\mu}_{\Gamma_{\Theta_n}}(M(j\omega)), & \pi/h_{n+1}<\omega\le\pi/h_n,\quad n=1,\dots,\ell-1,\\ & \theta_i=h_i\omega/2,\ i=1,\dots,n\\ \tilde{\mu}_{\Gamma}(M(j\omega)), & \omega>\pi/h_1\end{cases}$$

where

$$\Gamma_{\Theta_n}=\{\ \mathrm{diag}\,(\gamma_1 I_{r_1},\dots,\gamma_n I_{r_n},\dots,\gamma_\ell I_{r_\ell},\gamma_{\ell+1},\dots,\gamma_{\ell+p})\ :\ \gamma_i\in\mathbf{C},|\angle\gamma_j|\le h_j\omega/2,$$

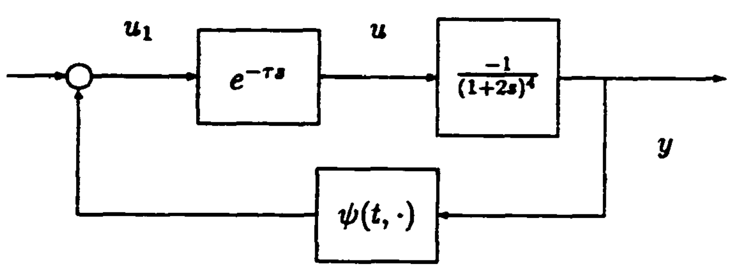
$$i=1,\dots,p,\ j=1,2,\dots,n\ \}.$$

$$\Gamma=\{\mathrm{diag}\,(\gamma_1 I_{r_1},\dots,\gamma_\ell I_{r_\ell},\gamma_{\ell+1},\dots,\gamma_{\ell+p})\,\gamma_i\in\mathbf{C},i=1,\dots,\ell+p\}.$$

Example 5.1 Take an example from [3]. Consider

$$y(t)=\frac{e^{-\tau s}}{(1+2s)^4}u(t),$$

$u(t)=-\psi(t,y)$, where $\psi(t,\cdot)$ satisfies sector condition $[0,k]$. We want to know the largest value of k, such that system is absolute delay independent/dependent stable.



System can be represented as:

$$\begin{pmatrix}u_1\\ y\end{pmatrix}=\begin{pmatrix}0 & 1\\ \frac{-1}{(1+2s)^4} & 0\end{pmatrix}\begin{pmatrix}u\\ u_1\end{pmatrix}.$$

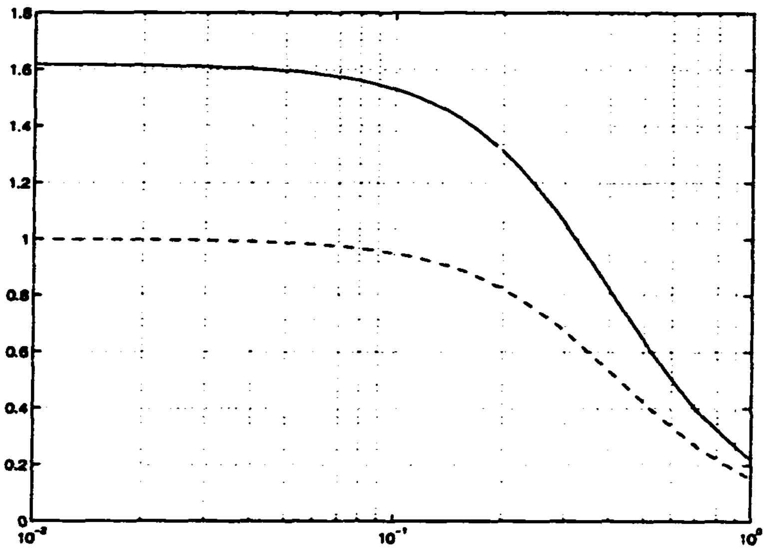


Figure 5.3: Upper bounds of structured singular value of $\bar{\mu}_{\Delta}\left(\tilde{G}(j\omega)\right)$ for $k \in [0, 1.0]$ (dashed line), and $k \in [0, 2.0]$ (solid line)

An upper bound of structured singular value is shown in Figure 5.3 for $k \in [0, 1.0]$ and $k \in [0, 2.0]$. System is delay independent stable for $k \in [0, 1.0]$.

For delay dependent stability, we first find the stability interval $\tau \in [0, 1.7)$ when $k \in [0, 2.0]$,as shown in Figure 5.4. Then, we apply stability test for general delay cases of $\tau \in [1.6, 1.8)$ which is shown in Figure 5.5. Then, we can say system is stable for $\tau \in [0, 1.8)$ when $k \in [0, 2]$.

5.3 Absolute Delay Independent Stability for Time Invariant Nonlinearity

For time invariant nonlinearity case, $u = -\Psi(y)$, all the results in the previous section applies, where $\Psi(\cdot)$ satisfies sector condition $[0, K]$, $K = \text{diag}(k_1, \ldots, k_p)$. However, well-known Popov criterion may give less conservative result for time invariant nonlinearity cases.

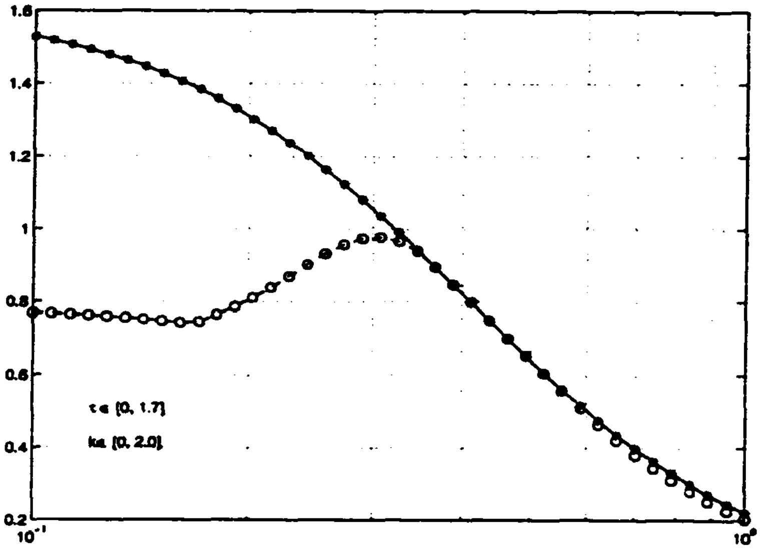


Figure 5.4: Upper bounds of $\tilde{\mu}_\Gamma(P)$ (solid line), $\tilde{\mu}_\Gamma(M)$ (star line) and $\tilde{\nu}(\omega) = \min\{\tilde{\nu}(M), \tilde{\nu}(P)\}$ for $h = 1.7$: $\tilde{\mu}_{\Gamma_\Theta}(P)$ (dashed line), $\tilde{\mu}_{\Gamma_\Theta}(M)$ (circle line)

Theorem 5.6 (Popov Criterion) [33] *The feedback system in Figure 5.1 is absolute stable if there exists a matrix* $N = \mathrm{diag}(n_1, n_2, \ldots, n_p)$, *where* $n_i > 0$, *such that*

$$I + (I + Ns)KH(s)$$

is strictly positive real, where

$$H(s) = \left(C + \sum_i C_i e^{-\tau_i s}\right)\left(sI - A - \sum_i A_i e^{-\tau_i s}\right)^{-1} B.$$

Denote $H_T(s) = (I + Ns)H(s)$. Suppose $I + KH_T(s)$ is strictly positive real, i.e.,

$$(I + KH_T(s)) + (I + KH_T(s))^* \geq 0.$$

This is equivalent to say that

$$\left\|\left(\frac{K}{2}H_T(s)\right)\left(I + \frac{K}{2}H_T(s)\right)^{-1}\right\|_\infty < 1,$$

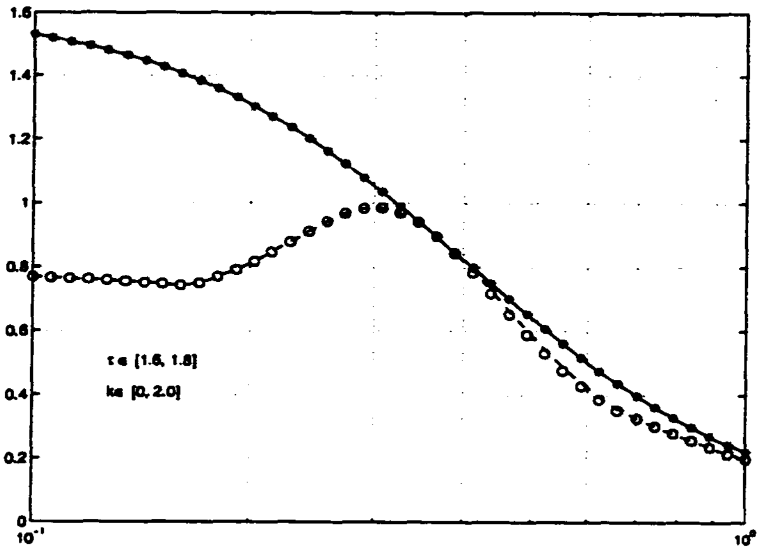


Figure 5.5: Upper bounds of $\tilde{\mu}_\Gamma(P)$ (solid line), $\tilde{\mu}_\Gamma(M)$ (star line) and $\bar{\nu}(\omega) = \min\{\bar{\nu}(M), \bar{\nu}(P)\}$ for $1.6 < h < 1.8$: $\tilde{\mu}_{\Gamma_\Theta}(P)$ (dashed line), $\tilde{\mu}_{\Gamma_\Theta}(M)$ (circle line)

which can be represented as Figure 5.6. Let $\tilde{\Psi}(y) = \text{diag}\left(\tilde{\psi}_1(y_1), \ldots, \tilde{\psi}_p(y_p)\right)$ where $\tilde{\psi}_i(y_i) = \frac{2}{k_1}\left(\psi_i(y_i) - \frac{k_i}{2}y_i\right)$, and $\tilde{H}_T(s) = \frac{K}{2}H_T(s)\left(I + \frac{K}{2}H_T(s)\right)^{-1}$. Then, it's easy to see that Popov Criterion in terms of infinity norm says that the system $\tilde{H}_T(s)$ is robustly stable with respect to $\|\tilde{\Psi}(\cdot)\|_\infty \leq 1$, if $\|\tilde{H}_T(s)\|_\infty < 1$. This observation is consistent with circle criterion results, in which we have done in the last section.

We can rewrite $H(s)$ as $\mathcal{F}_u(G, \mathcal{D})$, where $\mathcal{D} = \text{diag}(e^{-s\tau_1}I_{r_1}, \ldots, e^{-s\tau_\ell}I_{r_\ell})$, as shown in Figure 5.7. Next, we can rearrange system in Figure 5.7 into LFT framework, shown in Figure 5.8. It's easy to see that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I + Ns \end{pmatrix} G(s) \begin{pmatrix} v \\ u \end{pmatrix}.$$

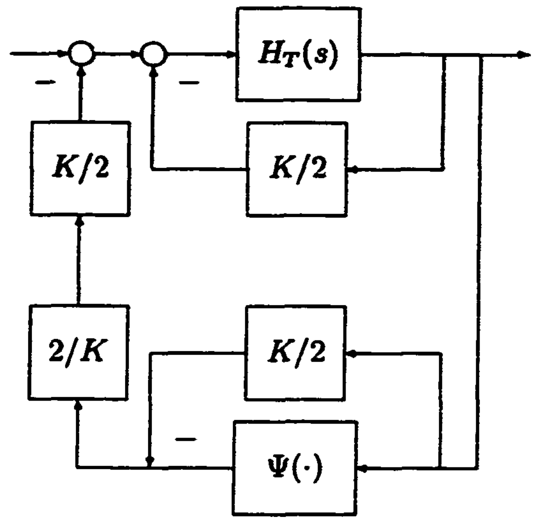


Figure 5.6: Loop Transformation and Scaling

After applying loop transformation and normalization on nonlinearity elements, let $\tilde{G}(s)$ be the generalized plant of $\begin{pmatrix} I & 0 \\ 0 & I+Ns \end{pmatrix} G(s)$.

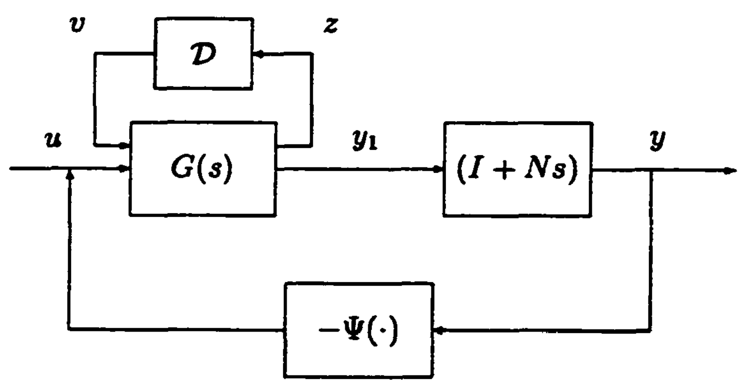


Figure 5.7: Popov Criterion

Lemma 5.7 *In Figure 5.8, assume there exists a matrix* $N = \mathrm{diag}(n_1, n_2, \ldots, n_p)$, *wherer* $n_i > 0$, *and* $\tilde{G}(s)$ *stable, then the feedback system is absolute delay-independent*

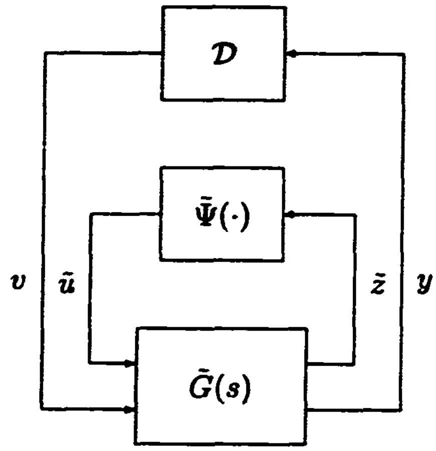


Figure 5.8: Interconnected Representation of System

stable if

$$\tilde{\mu}_{\Delta}\left(\tilde{G}(j\omega)\right) < 1, \qquad \forall\omega,$$

where $\Delta \in \mathbf{\Delta}$,

$$\mathbf{\Delta} = \{\mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \ : \gamma_i \in \mathbf{C},\ |\gamma_i| \leq 1,\ i = 1, \ldots, \ell + p\}.$$

The upper bound $\tilde{\mu}_{\Delta}\left(\tilde{G}(j\omega)\right) \leq \hat{\tilde{\mu}}_{\Delta}\left(\tilde{G}(j\omega)\right)$ can be obtained by equation (5.2).

5.4 Absolute Delay Dependent Stability for Time Invariant Nonlinearity

Consider a system shown in Figure 5.8 and uncertainty Δ,

$$\Delta = \left\{\mathrm{diag}\left(e^{-s\tau_1} I_{r_1}, \ldots, e^{-s\tau_\ell} I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}\right) \ : \ \tau_i \in [0, h_i], \gamma_i \in \mathbf{C}\right\}.$$

Assume $h_1 < h_2, \ldots < h_\ell$. Define

$$\mathbf{\Gamma} = \{\mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots \gamma_{\ell+p}) \ : \ \gamma_i \in \mathbf{C},\ i = 1, \ldots, \ell + p\},$$

Then, $\Delta \in \mathbf{\Gamma}$.

Let $P(s) = \tilde{G}(s)$. Then, the uncertainty with phase information is given by

$$\Gamma_{\Theta_\ell}{}^P = \{\ \mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots \gamma_{\ell+p})\ :\ \gamma_i \in \mathbf{C},\ |\angle\gamma_j| \leq h_j\omega,$$

$$i = 1, \ldots, \ell + p,\ j = 1, \ldots, \ell\,\},$$

for ω such that $h_i\omega < \pi$.

Next, define

$$M(s) = P(s)\mathrm{diag}(e^{-h_1 s/2} I_{r_1}, e^{-h_2 s/2} I_{r_2}, \ldots, e^{-h_\ell s/2} I_{r_\ell}, I_p)$$

and the corresponding uncertainty blocks as

$$\Gamma_{\Theta_\ell}{}^M = \{\ \mathrm{diag}(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots \gamma_{\ell+p})\ :\ \gamma_i \in \mathbf{C},\ |\angle\gamma_j| \leq h_j\omega/2,$$

$$i = 1, \ldots, \ell + p,\ j = 1, \ldots, \ell\,\},$$

for ω such that $h_i\omega \leq 2\pi$.

Lemma 5.8 *Suppose A is stable and assume $h_1 < h_2 < \ldots < h_\ell$. Then the uncertain delay system is stable for $\tau_i \in [0, h_i)$, $i = 1, \ldots, \ell$, if*

$$\tilde{\nu}(\omega) = \min\left\{\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right),\ \ \tilde{\nu}\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right)\right\} < 1,$$

for all ω, where $\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right)$ and $\tilde{\nu}\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right)$ are defined as

$$\tilde{\nu}\left(P(j\omega), \Gamma^P_{\Theta_\ell}\right) = \begin{cases} \tilde{\mu}_{\Gamma_{\Theta_\ell}}(P(j\omega)) & 0 \leq \omega \leq \pi/2h_\ell,\ \ \theta_i = h_i\omega,\ i = 1, \ldots, \ell; \\ \tilde{\mu}_{\Gamma_{\Theta_n}}(P(j\omega)) & \pi/2h_{n+1} < \omega \leq \pi/2h_n,\ \ n = 1, \ldots, \ell - 1, \\ & \qquad\qquad \theta_i = h_i\omega,\ \ i = 1, \ldots, n \\ \tilde{\mu}_{\Gamma}(P(j\omega)), & \qquad\qquad \omega > \pi/2h_1 \end{cases}$$

where

$$\Gamma_{\Theta_n} = \{\text{diag}\,(\gamma_1 I_{r_1}, \ldots, \gamma_n I_{r_n}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \;: |\angle\gamma_j| \leq h_j\omega, j = 1, 2, \ldots, n\};$$

$$\Gamma = \{\text{diag}\,(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \;\; : \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + p\}.$$

And,

$$\tilde{\nu}\left(M(j\omega), \Gamma^M_{\Theta_\ell}\right) = \begin{cases} \tilde{\mu}_{\Gamma_{\Theta_\ell}}(M(j\omega)), & 0 \leq \omega \leq \pi/h_\ell,\; \theta_i = h_i\omega/2,\; i = 1, \ldots, \ell; \\ \tilde{\mu}_{\Gamma_{\Theta_n}}(M(j\omega)), & \pi/h_{n+1} < \omega \leq \pi/h_n,\;\; n = 1, \ldots, \ell - 1, \\ & \qquad \theta_i = h_i\omega/2,\; i = 1, \ldots, n \\ \tilde{\mu}_{\Gamma}(M(j\omega)), & \qquad \omega > \pi/h_1 \end{cases}$$

where

$$\Gamma_{\Theta_n} = \{\, \text{diag}\,(\gamma_1 I_{r_1}, \ldots, \gamma_n I_{r_n}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p}) \;:\; \gamma_i \in \mathbf{C}, |\angle\gamma_j| \leq h_j\omega/2,$$

$$i = 1, \ldots, \ell + p,\; j = 1, 2, \ldots, n\,\}.$$

$$\Gamma = \{\text{diag}\,(\gamma_1 I_{r_1}, \ldots, \gamma_\ell I_{r_\ell}, \gamma_{\ell+1}, \ldots, \gamma_{\ell+p})\, \gamma_i \in \mathbf{C}, i = 1, \ldots, \ell + p\}.$$

Example 5.2 Consider

$$y(t) = \frac{e^{-\tau s}}{(1 + 2s)^4} u(t),$$

$u(t) = -\psi(y)$, where $\psi(\cdot)$ satisfies sector condition $[0, k]$. We want to know the largest value of k, such that system is absolute delay independent/dependent stable.

Rewrite the system as shown in Figure 5.7. $G(s)$ can be found the example in the previous section. Then, the generalized plant $\tilde{G}(s)$ can be got from applying loop transformation and normalization on nonlinearity part of diag$(1, 1 + \eta s)G(s)$.

Figure 5.9 shows that the upper bound of structured singular value via various η when $k \in [0, 1.0]$. It shows that any choice of η doesn't help in this case. System

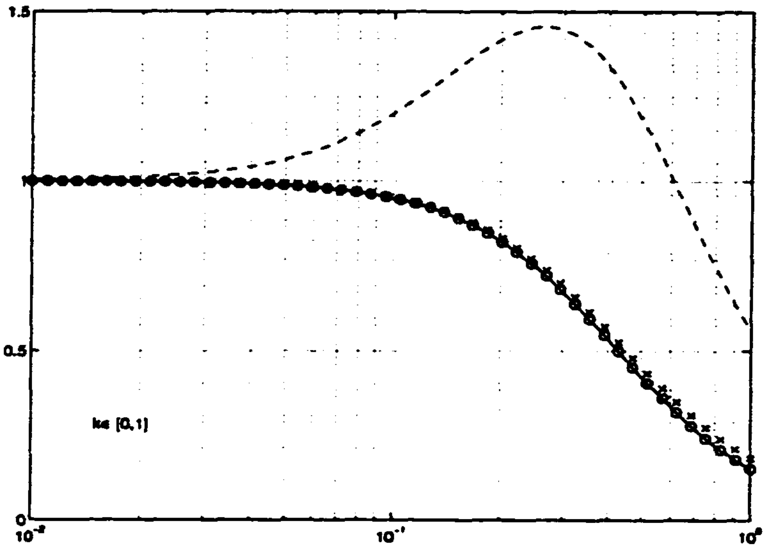


Figure 5.9: Upper bounds of $\mu_{\Delta}\left(\tilde{G}(j\omega)\right)$: $\eta = 0.0$ (solid line), $\eta = 0.1$ (circle line), $\eta = 1.0$ (x-mark line), $\eta = 10.0$ (dashed line)

is absolute delay independent stable for time varying nonlinearity or time invariant nonlinearity satisfying sector condition $[0, 1.0]$.

Figure 5.10 shows that system is not absolute delay independent stable when time invariant nonlinearity satisfying sector condition $[0, 2.0]$, but it is absolute delay dependent stable for $\tau \in [0, 2.2)$. Since the upper bound of structured singular value with phase information is less than one. For completeness, Figure 5.11 shows that upper bounds when $\eta = 0.0$.

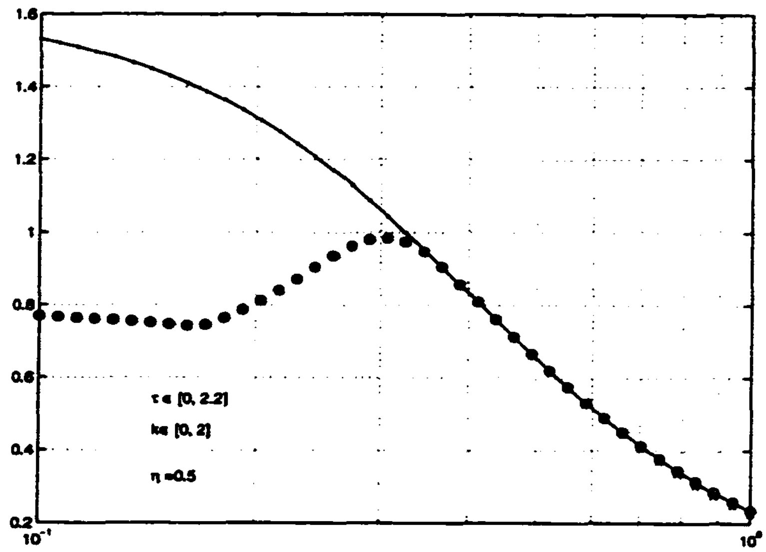


Figure 5.10: Upper bounds of $\mu_\Gamma(P)$ (solid line), $\mu_\Gamma(M)$ (dot line) and for $h = 2.2$: $\mu_{\Gamma_\Theta}(M)$ (x-mark line), $\mu_{\Gamma_\Theta}(P)$ (circle line)

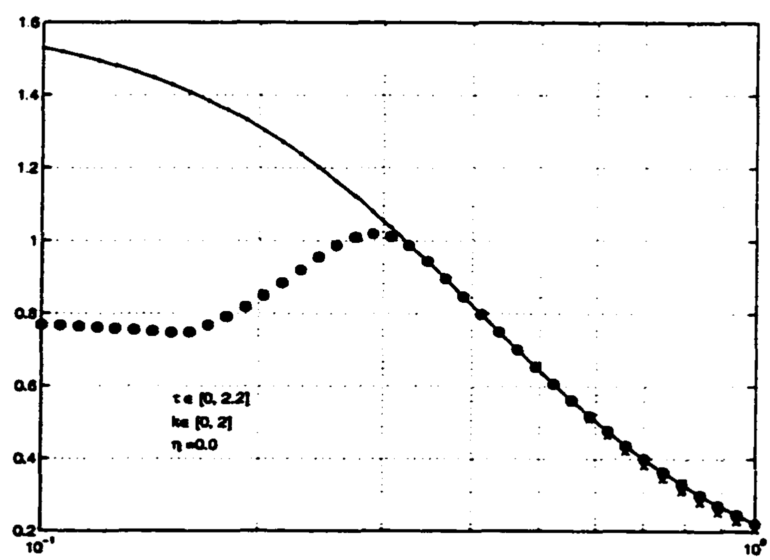


Figure 5.11: Upper bounds of $\mu_\Gamma(P)$ (solid line), $\mu_\Gamma(M)$ (dot line) and for $h = 2.2$: $\mu_{\Gamma_\Theta}(M)$ (x-mark line), $\mu_{\Gamma_\Theta}(P)$ (circle line)

Chapter 6

Conclusion

We have adopted the linear fractional transformation (LFT) framework for the analysis and synthesis of uncertain time-delay systems. Consequently, stability analysis with respect to uncertain delays are just an application of the standard small-μ theorem. We have also shown that existing results which are derived by Lyapunov method can be converted into our proposed framework and explain their conservativeness.

Under LFT framework, we can handle various robustness problems systematically. The framework allows to incorporate easily other uncertainties, such as parametric and model uncertainties, etc as we have demonstrated in examples. Absolute stability analysis of nonlinear time-delay systems is another example which shows that our proposed method integrity, while Lyapunov method needs to find specific Lyapunov functionals to each case.

We have mainly treated the problems in two ways. In the first case, uncertainty $(e^{-s\tau_i} - 1)$ was approximated by a rational transfer function, where τ_i is uncertain. Computation involved is simple since system is represented by a rational transfer function. Standard $\mathcal{H}_\infty$ controller design technique can also be applied.

In the second method, uncertainty $e^{-s\tau_i}$ was considered with the phase information. Examples show that results are less conservative than using the first method.

It's hard to say which one is superior to the other. Both offer sufficient conditions only. Hence, we can say nothing about the system if the test fails.

We can extend the result by applying stability analysis on the case of $\tau_i \in [\underline{\tau}_i, \bar{\tau}_i]$, $\underline{\tau}_i \neq 0$. For example, if we can tell system $G(s)$ is stable for $\tau_i \in [0, h_{i,1})$ by one of proposed methods, next we can examine the stability of the system on $\tau_i \in [h_{i,1}, h_{i,2})$. Both proposed methods can handle stability analysis for general delays easily.

We have shown that controller design for uncertain time-delay systems can be done by standard $\mathcal{H}_\infty$ design technique. This controller may be conservative since we actually cover the uncertainties to guarantee the stability. Alternatively, we may seek an $\mathcal{H}_\infty$ controller by parameterizing all stabilization controllers for uncertain time-delay systems and then solve the $\mathcal{H}_\infty$ norm minimization. Actually, this has been done in $\mathcal{H}_2$ norm minimization. For $\mathcal{H}_\infty$ minimization, we may solve with a suitable LFT problem formulation. More further study is needed.

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Vita

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Title of Dissertation: Robust Control of Uncertain Time-Delay Systems

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

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Date of Examination:

June 13, 2001