Communications on Stochastic Analysis

Volume 8 | Number 3 Article 4

9-1-2014

An analysis of stochastic flows

A A Dorogovtsev

I. I. Nishchenko

Communications on Stochastic Analysis
Vol. 8, No. 3 (2014) 331-342

Serials Publications
www.serialspublications.com

AN ANALYSIS OF STOCHASTIC FLOWS

A. A. DOROGOVTSEV* AND I. I. NISHCHENKO

Abstract. This is a survey article devoted to the stochastic flows with singular interaction. It presents the recent results of the authors which show how the basic for the smooth stochastic differential equations statements can be obtained for the flow with coalescence. The proposed approaches allows to get the large deviations for the Arratia flow, Krylov–Veretennikov expansion for the general stochastic semi-group, discrete time approximation for the Harris flows. Also the transformation of the compacts by the semi-group of the finite-dimensional projections is considered.

1. Introduction

A stochastic flow is a versatile object. On the one hand, stochastic flows are similar to solutions of stochastic differential equations with smooth coefficients. On the other hand, when the interaction in a flow is singular, it acquires completely new features. For example, a stochastic flow may consist of discontinuous with respect to a spatial variable maps. Thus, the study of stochastic flows may be based partly on the techniques and methods developed in the study of solutions to stochastic differential equations, and partly requires new ideas. The paper is divided into several parts. The first part contains the necessary definitions and facts, as well as the wording of the questions being explored further in the paper. The second part shows how we can get results that are typical for the theory of stochastic differential equations, in the case of a singular interaction. The third part is devoted to stochastic semi-groups associated with stochastic flows. In the last part we present a discrete approximation for the Harris flow.

It should be noted that the article is based mainly on the results of the authors and does not reflect all the possible methods of investigation of stochastic flows. For example, we do not consider here the Tsirelson's results about the structure of the filtration generated by a flow [21, 20], and we do not consider flows of kernels studied by Le Jan [12, 13].

2. Some Properties of Stochastic Flows

Let $(\mathfrak{X}, \rho)$ be a complete separable metric space and $(\Omega, \mathcal{F}, P)$ a complete probability space.

Received 2013-10-19; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60J65, 60K35, 60H10; Secondary 60G15.
Key words and phrases. Stochastic flows, Arratia flow, random semi-group, large deviations.
*This research is supported by the Presidium of National Academy of Sciences of Ukraine as part of the joint scientific project with the Russian foundation of fundamental research, project number 09-01-14.

Definition 2.1. A measurable map $\varphi : \mathfrak{X} \times \Omega \to \mathfrak{X}$ is called a *random map* in $\mathfrak{X}$.

Definition 2.2. A family $\{\varphi_{s,t}; 0 \le s \le t < +\infty\}$ of random maps in $\mathfrak{X}$ is referred to as a *stochastic flow* if the following conditions hold:
1) for any $0 \le s_1 \le s_2 \le \ldots \le s_n$, maps $\varphi_{s_1,s_2}, \ldots, \varphi_{s_{n-1},s_n}$ are independent;
2) for any $s, t, r \ge 0$, maps $\varphi_{s,t}$ and $\varphi_{s+r,t+r}$ are equidistributed;
3) for any $0 \le s_1 \le s_2 \le s_3$ and $u \in \mathfrak{X}$ holds:

$$\varphi_{s_1,s_3}(u) = \varphi_{s_2,s_3}\varphi_{s_1,s_2}(u);$$

4) for any $s \ge 0$ and $u \in \mathfrak{X}$, $\varphi_{s,s}(u) = u$.

Remark 2.3. Note that due to the separability of $\mathfrak{X}$ the superposition of random maps is defined correctly. Furthermore, a superposition of independent random maps does not depend on the choice of their modifications (up to a modification).

Below we consider point motions or trajectories of individual particles in a stochastic flow, i.e., random processes of the form $x(u,t) = \varphi_{0,t}(u), u \in \mathfrak{X}, t \ge 0$.

If this does not lead to confusion, sometimes x will also be called a stochastic flow. One of the most famous examples of a stochastic flow is the flow of diffeomorphisms corresponding to a stochastic differential equation (SDE) with smooth coefficients. Let $\mathfrak{X} = \mathbb{R}^n$ with the usual metric. Consider the equation

$$dz(t) = a(z(t))dt + b(z(t))dw(t), \tag{2.1}$$

where w is a standard $\mathbb{R}^n$-valued Wiener process, $a : \mathbb{R}^n \to \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are continuously differentiable functions with bounded derivatives. It is well known [11], that this equation generates the stochastic flow $\{\varphi_{s,t}; 0 \le s \le t < +\infty\}$ in $\mathbb{R}^n$, consisting of random diffeomorphisms, such that for any $u \in \mathbb{R}^n$, $s \ge 0$, $\varphi_{s,t}(u)$ is the solution of the Cauchy problem for (2.1), which started at the point u at time s. In addition to the smoothness of its component maps, the flow corresponding to Equation (2.1) has a number of "good" properties. For example, for the solutions of Equation (2.1) are known large deviations, Girsanov's theorem and the presentation of Krylov–Veretennikov or related Chen–Shrishart's formula [1]. All of these properties are due to the fact that the flow is obtained from the Wiener process via Itô's map, generated by the vector fields corresponding to the coefficients a and b. Thus, random maps forming the flow inherit properties of the Wiener process. In general it is not the case. The stochastic flow can be organized in a more complicated way. As an example of the flow with more rich structure one can consider the Harris flows.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous positive definite function.

Definition 2.4 ([8, 14])**.** The *Harris flow* with ψ being its local characteristic is a family $\{x(u,t); u \in \mathbb{R}\}$ of Wiener martingales with respect to the joint filtration such that
1) for every $u \in \mathbb{R}$, $x(u,0) = u$;
2) for any $u \le v$ and $t \ge 0$, $x(u,t) \le x(v,t)$;
3) the joint characteristic of $x(u,\cdot)$ and $x(v,\cdot)$ has the form

$$d\langle x(u,\cdot), x(v,\cdot)\rangle(t) = \psi(x(u,t) - x(v,t))dt.$$

Here we do not consider the family $\{\varphi_{s,t}; 0 \le s \le t\}$, but a set of the one-point motions $\{x(u,t); u \in \mathbb{R}, t \ge 0\}$. Roughly speaking, the Harris flow describes the joint evolution of ordered family of Brownian particles interacting with each other.

Remark 2.5. For a smooth function ψ the Harris flow can be constructed as a solution to SDE. Let W be a Wiener sheet on $\mathbb{R} \times \mathbb{R}_+$ (random Gaussian measure with mean zero, independent values on disjoint sets and structural measure equal to the Lebesgue measure). For a function from the Schwartz space of rapidly decreasing infinitely differentiable functions, consider the following equation

$$dx(u,t) = \int_{\mathbb{R}} f(x(u,t) - p) W(dp, dt). \tag{2.2}$$

This equation generates a flow of random diffeomorphisms in $\mathbb{R}$ [11]. In addition, for fixed u, $x(u, \cdot)$ is a continuous square-integrable martingale with respect to the filtration generated by W, and

$$d\langle x(u,\cdot), x(v,\cdot)\rangle(t) = \int_{\mathbb{R}} f(x(u,t) - p) f(x(v,t) - p) dp dt =$$

$$= \int_{\mathbb{R}} f(x(u,t) - x(v,t) - q) f(-q) dq dt = \psi(x(u,t) - x(v,t)) dt.$$

If $\psi(0) = 1$, then for every u, $x(u, \cdot)$ is a Wiener process due to Levy's theorem [9]. The ordering of x with respect to the spatial variable follows from the smoothness of x. Thus x is a Harris flow. Note that the function ψ is smooth now.

One can, however, establish the existence of the Harris flow for a broader class of functions ψ. For example, it is known [8, 14], that the Harris flow exists for a function ψ, which is continuous on $\mathbb{R}$ and satisfies the Lipschitz condition outside any neighborhood of 0. The resulting flow may already have novel properties. In particular, it is known that under the condition

$$\int_0^{\varepsilon} (1 - \psi(u))^{-1} u du < +\infty$$

particles started from different points of the line, coalesce with each other. In this case, the maps $x(\cdot, t)$ are not smooth in the space variable. Noted property is shown most clearly in the Arratia flow.

The Arratia flow is a Harris flow corresponding to the discontinuous function

$$\psi = 1\!\!\mathrm{I}_{\{0\}}.$$

Roughly speaking, the Arratia flow consists of independent Wiener processes, coalescing at the time of the meeting. It is known [13, 14, 3] that in every positive moment of time, any interval in the Arratia flow turns into a finite number of points. Thus, the maps $x(\cdot, t)$ are step functions with a finite number of jumps on each interval.

From the above examples, we can draw the following conclusions. First, under the same distributions of one-point motions (they can all be Wiener processes) flows can have completely different properties with respect to the spatial variable. Secondly, in some cases, the flow is arranged by an external Gaussian noise and almost automatically inherits its properties, and in other cases only one-point motions are diffusion processes and the entire flow no longer generates a Gaussian noise. Since the Arratia flow delivers one of the most striking examples of the latter situation, the next question is interesting. How the properties of the Gaussian white noise and, therefore, solutions of stochastic differential equations with smooth coefficients are inherent in the Arratia flow? The second paragraph of the article is devoted to answering this question.

3. Gaussian Properties of the Arratia Flow

Let $\{x(u,t); u \in \mathbb{R}, t \geq 0\}$ be an Arratia flow. As mentioned in the previous paragraph, this flow is composed of independent Wiener processes, coalescing at the time of the meeting. The construction of the stochastic integral with respect to the Arratia flow is based on the fact that the processes started from a certain interval, coalesce during a finite time. Formally, this property can be described as follows. Let $\lambda = \{0 = u_0 < \ldots < u_n = 1\}$ be a partition of $[0;1]$. Denote $\tau_0 = 1$, $\tau_k = \min\{1, s : x(u_k, s) = x(u_{k-1}, s)\}, k = 1, \ldots, n$. The sum

$$S_\lambda = \sum_{k=0}^{n} \tau_k$$

can be regarded as a total time of a free motion of particles that started from the points of the partition λ.

The following statement is true.

Theorem 3.1 ([2]). *With probability one*

$$\sup_\lambda S_\lambda = \lim_{|\lambda| \to 0} S_\lambda < +\infty.$$

Here $|\lambda| = \max_{k=\overline{0,n-1}}(u_{k+1} - u_k)$.

Thus, in the Arratia flow the total time of free motion of particles that started from the interval $[0;1]$ (or any other interval) is finite. This allows us to build a stochastic integral by pieces of free trajectories in the flow. Let $a : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function.

Theorem 3.2 ([2]). *There exist the following limits:*

$$\int_0^1 \int_0^{\tau_u} a(x(u,s))ds := P\text{-}\lim_{|\lambda| \to 0} \sum_{k=0}^{n} \int_0^{\tau_k} a(x(u_k, s))ds,$$

$$\int_0^1 \int_0^{\tau_u} a(x(u,s))dx(u,s) := L_2\text{-}\lim_{|\lambda| \to 0} \sum_{k=0}^{n} \int_0^{\tau_k} a(x(u_k, s))dx(u_k, s).$$

In the left-hand side of these equalities, we use two signs of integral and only one differential because the role of the second differential are played by the moments τ_u – times of free motions of the particles.

Built integrals allow us to formulate an analogue of Girsanov's theorem for the Arratia flow. Let a be a bounded measurable function.

Definition 3.3 ([3]). An *Arratia's flow* with a drift a is a stochastic process $\{x_a(u,t); u \in \mathbb{R}, t \geq 0\}$ such that

1) for a fixed u, $x_a(u,\cdot)$ is a diffusion process with a drift a and diffusion 1;
2) for any $u \leq v$ and $t \geq 0$

$$x_a(u,t) \leq x_a(v,t);$$

3) for any $u_1 < u_2 < \ldots < u_n$ and $t \geq 0$ the restriction of the distribution of the stochastic processes $x_a(u_1,\cdot),\ldots,x_a(u_n,\cdot)$ on the set

$$\{f \in C([0;t],\mathbb{R}^n) : f_i(0) = u_i, f_1(s) < \ldots < f_n(s), s \in [0;t]\}$$

coincides with the restriction to this set of the distribution of an n-dimensional diffusion process with the standard Wiener process with the drift $(a,\ldots,a)$;

4) for any u_1, u_2 $x_a(u_1,t) = x_a(u_2,t)$ when $t \geq \tau_{u_1u_2}$, where

$$\tau_{u_1u_2} = \min\{s : x_a(u_1,s) = x_a(u_2,s)\}.$$

The existence of such a flow is proved in [4,11,12].

Note that for a fixed $t > 0$ there exists a modification of x_a in the space $D([0;1], C([0;t]))$. Denote by μ_a the distribution of x_a in this space (μ_0 is the distribution of the Arratia flow) .

Theorem 3.4 ([2]). *The measure μ_a is absolutely continuous with respect to the measure μ_0 and the density has the form*

$$\frac{d\mu_a}{d\mu_0}(x) =$$

$$= \exp\{\int_0^1 \int_0^{\tau_u} a(x(u,s))dx(u,s) - \frac{1}{2}\int_0^1 \int_0^{\tau_u} a(x(u,s))^2 ds\}.$$

This result shows that under smooth perturbations of the motion of individual particles, the Arratia flow behaves like a Wiener process. This is because the flow is composed of "independent pieces" of Wiener processes, and the total time of free motion is finite. In view of the same reason the Arratia flow satisfies the large deviations principle in an appropriate space. Let us denote by $\mathcal{M}$ the space of functions acting from $[0;1]^2$ to $[0;1]$ and having the following properties:

1) for every $u \in [0;1]$, $y(u,\cdot) \in C([0;1])$;
2) for all $u_1 \leq u_2, t \in [0;1]$:

$$y(u_1,t) \leq y(u_2,t);$$

3) for any $t \in [0;1]$, $y(\cdot,t)$ is right-continuous;
4) for all $u_1, u_2 \in [0;1]$

$$y(u_1,t) = y(u_2,t),\ t \geq \tau_{u_1u_2},$$

$$\tau_{u_1u_2} = \inf\{s : y(u_1,s) = y(u_2,s)\};$$

5) for any $u \in [0;1]$

$$y(u,0) = u.$$

Each element of $\mathcal{M}$ can be called a continual forest [13]. Let us introduce the metric in $\mathcal{M}$

$$\rho(y_1, y_2) = \max_{[0;1]} \sigma(y_1(\cdot, t), y_2(\cdot, t)),$$

where σ is the Lévy-Prokhorov distance. For an Arratia flow $\{x(u,t); u \in [0;1], t \in [0;1]\}$ let us define new random fields x_ε via time change

$$x_\varepsilon(u,t) = x(u, \varepsilon t),\ \varepsilon \in (0;1).$$

The following theorem holds true.

Theorem 3.5 ([6]). *The family x_ε under $\varepsilon \to 0+$ satisfies the LDP in the space $\mathcal{M}$ with the speed function*

$$I(x) = \inf_{i(h)=x} I_0(h).$$

Here h ranges over the set of real-valued functions defined on $\mathbb{Q} \cap [0;1] \times [0;1]$ and with the above properties 1)-5), and

$$I_0(h) = \frac{1}{2} \sum_{r \in \mathbb{Q} \cap [0;1]} \int_0^{\tau_r} h'(r,t)^2 dt,$$

$$i(h)(u,t) = \inf_{r>u} h(r,t).$$

Like the previous theorem, this result shows that in the study of the asymptotic behavior of large deviations of the Arratia flow a major role is played by the fact that it is made up of Wiener trajectories.

Emerging as the Radon-Nikodym densities, the stochastic exponents are known [16] to form the total set of all square- integrable functionals of the Wiener process. It turns out that a similar property holds true for the Arratia flow. The following statement holds.

Theorem 3.6 ([4]). *The linear combinations of random variables of the form*

$$\exp\{\int_0^1 \int_0^{\tau_u} f(u,s) dx(u,s) - \frac{1}{2} \int_0^1 \int_0^{\tau_u} f(u,s)^2 ds\}, f \in C([0;1]^2)$$

are dense in the space of square-integrable functionals of the Arratia flow $\{x(u,t); u,t \in [0;1]\}$.

4. Stochastic Semi-groups and Widths

In this part of the paper we make an attempt to find a common approach to the study of the geometry of stochastic flows both for smooth and non-smooth cases. It is well-known [1] that the shift of functions or differential forms by solutions to SDE with smooth coefficients is described in terms of the Lie algebra generated by vector fields, which are the coefficients of the equation. If, however, a flow is composed of discontinuous maps, then such flow may not preserve the continuity. Instead, we propose to consider how the flow transforms finite-dimensional subspaces of functions, and calculate the widths of functional compacts with respect to these subspaces. We consider in detail here a model example of a stochastic semi-group consisting of finite-dimensional projections.

We start with a definition of a strong random operator. Let H be a real separable Hilbert space.

Definition 4.1 ([18])**.** A *strong random operator* in H is a continuous in probability linear map from H to the set of all random elements in H.

Below there is an appropriate example of a strong random operator.

Example 4.2. Let $\{x(u,t), u \in \mathbb{R}, t \geq 0\}$ be the Harris flow, $H = L_2(\mathbb{R})$. Define a strong random operator in H by the equality

$$Af(u) = f(x(u,t)).$$

Since

$$E\int_{\mathbb{R}} f(x(u,t))^2 du = \int_{\mathbb{R}}\int_{\mathbb{R}} f(v)^2 p_1(u-v)dudv =$$
$$= \int_{\mathbb{R}} f(v)^2 \int_{\mathbb{R}} p_1(u-v)dudv = \int_{\mathbb{R}} f(v)^2 dv,$$

where p_1 is the density of the standard Gaussian distribution, A is continuous in the square mean.

It can be checked that in the case of the Arratia flow, the strong random operator constructed in the example has the following property. For this flow does not exist a set of bounded operators $\{\widetilde{A}_\omega, \omega \in \Omega\}$ in H such that

$$\forall f \in H \quad Af = \widetilde{A}_\omega f \ \text{ a.s.}$$

Indeed, suppose that there exists such a set, that is, that A is a bounded random operator in terms of Skorokhod. Let $\{\widetilde{f}_n; n \geq 1\}$ be the Rademacher system of functions on $[0;1]$. For $n \geq 1$ denote $f_n(u) = \widetilde{\widetilde{f}}_n(u), u \in [0;1]$, $f_n(u) = 0, u \notin [0;1]$. Then f_n converges weakly to 0 in H. From the other side the sequence $\{f_n; n \geq 1\}$ does not converge almost everywhere on $[0;1]$. As we have already noted, in the Arratia flow points of any finite interval turn into a finite number of points during any positive time interval. Therefore, with probability 1, there exists an interval $\Delta = \{v : x(v,t) = x(0,t)\}$ of positive Lebesgue measure. Then

$$\int_\Delta (Af_n)(u)du = (f_n, A^*_\omega 1\!\!\mathrm{I}_\Delta) \underset{n\to\infty}{\to} 0 \text{ a.s.}$$

On the other hand,

$$\int_\Delta (Af_n)(u)du = |\Delta| f_n(x(0,t)).$$

It means that the sequence $\{\int_\Delta (Af_n)(u)du, n \geq 1\}$ does not converge to 0 on the set of those ω for which $x(0,t) \in [0;1]$. This contradiction proves that our assumption was incorrect.

Despite the fact that strong random operators often are unlimited, their superpositions can be determined in the usual way for independent operators. Therefore, a stochastic flow in $\mathbb{R}^d$ can sometimes be associated with a semi-group of strong random operators in $L_2(\mathbb{R}^d)$. Let us formulate a precise definition.

Definition 4.3. A set $\{G_{s,t}; 0 \leq s \leq t < +\infty\}$ of strong random operators in H is called a *stochastic semi-group* if

1) the distribution of $G_{s,t}$ depends only on $t - s$;
2) for all $t_1 \leq t_2 \leq \ldots \leq t_n$, operators $G_{t_1,t_2}, \ldots, G_{t_{n-1},t_n}$ are independent;

3) for all $r \le s \le t$, $G_{r,t} = G_{s,t}G_{r,s}$,
$G_{r,r}$ is the identity operator.

Let $\{\varphi_{s,t}; 0 \le s \le t < +\infty\}$ be a stochastic flow (see Definition 2.1) in $\mathbb{R}^d$ whose one-point motions are standard Wiener processes. Then the operators $G_{s,t}$ in $L_2(\mathbb{R}^d)$ defined as

$$G_{s,t}f(u) = f(\varphi_{s,t}(u)),\ u \in \mathbb{R}^d$$

form a stochastic semi-group.

Note that the consideration of stochastic semi-groups associated with the flow, allows one to approach the study of the properties of flows with smooth and singular interaction in a unified way. The notion of a strong random operator was introduced by A.V. Skorokhod [18]. He also began to consider the semi-groups of such operators and received sufficient conditions for the representation of these semi-groups as the solutions of a linear SDE with operator-valued Wiener process [17]. This presentation is made possible, in part, because the noise generated by the semi-group, is Gaussian. We present here two theorems about the structure of semi-groups of strong random operators. One of them gives a description of the multiplicative operator functionals of Gaussian noise. The second one is devoted to semi-groups of random finite-dimensional projections (here a Poisson noise arises).

We start with a Gaussian case. Let $\{w(t); t \ge 0\}$ be a standard one-dimensional Wiener process. Suppose that the semi-group $\{G_{s,t}; 0 \le s \le t < +\infty\}$ of strong random operators is a multiplicative homogeneous functional of w, i.e. the following conditions hold:

1) $G_{s,t}$ is measurable with respect to $\mathcal{F}_{s,t} = \sigma(w(r) - w(s); r \in [s;t])$;
2) $\theta_r G_{s,t} = G_{s+r,t+r}, r \ge 0, s \le t$.

Here θ_r is the shift operator along the trajectories of w. Assume that $G_{s,t}$ are square-integrable, i.e.

$$\forall u \in H:\ \forall s \le t:\ E\|G_{s,t}u\|^2 < +\infty.$$

Define for all t, the mathematical expectation of $G_{0,t}$ as continuous linear operator in H, acting by the rule

$$\forall u \in H:\ T_t u = EG_{0,t}u.$$

Note that for the continuous in the square-mean semi-group $\{G_{s,t}\}$ the family $\{T_t\}$ is a strongly continuous semi-group of operators in H and, thus, is uniquely determined by its infinitesimal operator. It turns out, that in order to describe the semi-group $\{G_{s,t}\}$ we also need a "stochastic" infinitesimal operator. Let us define it in the following way. For $f \in H$ consider

$$Bf := \lim_{t \to 0+} \frac{1}{t} EG_{0,t}fw(t).$$

To make sure that B is densely defined, we consider the family of operators $\{F_t; t \ge 0\}$ defined by the relation

$$F_t f = EG_{0,t}fw(t) = EG_{s,s+t}f(w(t+s) - w(s)).$$

It is easy to check that the following equalities hold

$$F_t G_{0,s} = G_{0,s}F_t,\ F_{t+s} = F_s T_t + T_s F_t.$$

Using these equalities, it is possible to check in a standard way that all elements of H of the form

$$\int_0^s T_r g dr,\ g \in H,\ s > 0,$$

belong to the domain of B. The following theorem is true.

Theorem 4.4. *Suppose that for any $t > 0$, $T_t(H) \subset D(B)$ and the kernels of the Itô-Wiener expansion for $G_{0,t}$ are continuous in time variables. Then for any $f \in H$ the equality holds*

$$G_{0,t}f = T_t f +$$

$$+\sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k} B T_{\tau_k - \tau_{k-1}} B \ldots B T_{\tau_1} f dw(\tau_1) \ldots dw(\tau_k),$$

where $\Delta_k(0;t) = \{0 \le \tau_1 \le \ldots \le \tau_k \le t\}$.

In the case when the semi-group $\{G_{s,t}\}$ is generated by SDE, the statement of Theorem 4.4 is the well-known Krylov-Veretennikov expansion [10].

Example 4.5. Consider the following SDE in $\mathbb{R}$

$$dx(t) = a(x(t))dw(t),$$

where w is a standard Wiener process, $a \in C^{\infty}(\mathbb{R})$ is bounded together with its derivative. To this equation corresponds a stochastic flow $\{\varphi_{s,t}; 0 \le s \le t\}$ of diffeomorphisms in $\mathbb{R}$. One can check that the operators defined by the relation

$$(G_{s,t}f)(u) = f(\varphi_{s,t}(u))$$

on square-integrable functions, form a stochastic semi-group, which is a multiplicative functional on w. And for $f \in C_0^2(\mathbb{R})$

$$Bf(u) = a(u)f'(u).$$

Let $\inf a > 0$. Then the operator T_t for $t > 0$ is an integral operator with an infinitely differentiable kernel and thus the condition of Theorem 4.4 is satisfied. The claim of Theorem 4.4 now takes the form

$$f(\varphi_{0,t}(u)) = T_t f(u) + \sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k}$$

$$a\frac{d}{d\nu_k} T_{\tau_k - \tau_{k-1}} \ldots a\frac{d}{d\nu_1} T_{\tau_1} f(u) dw(\tau_1) \ldots dw(\tau_k).$$

Here $\nu_1, \ldots, \nu_k$ are variables on which the integration is performed by the action of $\{T_t\}$. The last formula coincides with the Krylov-Veretennikov representation [10].

It is not always the case that the noise associated with a stochastic semi-group is given by a Wiener process. In addition, semi-groups corresponding to flows with coalescence can be composed of operators with values in finite-dimensional spaces. Revealing in terms of describing the structure and the asymptotic behavior is an example of a stochastic semi-group consisting of random finite-dimensional projections in a Hilbert space.

Let H be a real Hilbert space. Under the projector in H, we understand the orthogonal projection on a subspace in H. A projector is called finite if the corresponding subspace is finite-dimensional.

Definition 4.6. A *random finite-dimensional projection* G in H is a set of finite-dimensional projections $\{G_\omega, \omega \in \Omega\}$ such that for any $u \in H$, $G_\omega u$ is a random element in H.

It is evident that a random finite-dimensional projector is a strong random operator continuous in the square mean. The following theorem gives a complete description of the mean-square continuous stochastic semi-group consisting of random finite-dimensional projectors.

Theorem 4.7 ([5]). *Let $\{G_{s,t}; 0 \le s \le t\}$ be a mean-square continuous stochastic semi-group consisting of random finite-dimensional projectors in H. Then there exist an orthonormal basis $\{e_n; n \ge 1\}$ in H and a sequence of Poisson, with regard to the general filtration, random processes $\{\nu_n; n \ge 1\}$ such that*

1) $\sum_{n=1}^{\infty} P\{\nu_n(t) = 0\} < +\infty, t > 0,$

2) $G_{s,t} = \sum_{n=1}^{\infty} 1\!\!1_{\{\nu_n(t)-\nu_n(s)=0\}} e_n \otimes e_n.$

Remark 4.8. This theorem states that all projectors of the semi-group have a common basis, the elements of which "are killed" in accordance with the Poisson regulation. If one postulates the existence of a common basis in advance, then the theorem is simple.

Remark 4.9. The object discussed in Theorem 4.7 is significantly stochastic. It is easy to see that there is no deterministic strongly continuous semi-group consisting of finite-dimensional projectors in the infinite-dimensional space H.

Since stochastic semi-groups can be built on stochastic flows, it is natural to expect that the geometric properties of the maps that make up the flow affect the properties of the operators of the semi-group. For the characterization of such properties it seems promising to use the notion of width. Here is the definition. Let $K \subset H$ be a compact and $L \subset H$ a subspace in H.

Definition 4.10 ([19]). The *width* of K with respect to L is the value

$$\max_{u \in K} \rho(u, L),$$

where $\rho(u, L) = \inf_{v \in L} \|u - v\|$.

Further, as an example, we consider the widths of some compacts with respect to subspaces of the form $G_{0,t}(H), t > 0$, where $\{G_{s,t}\}$ is a stochastic semi-group of finite-dimensional projectors.

Example 4.11 ([5]). Let the Poisson processes from the description of the semi-group $\{G_{s,t}\}$ in Theorem 4.7 be independent and their intensities equal $\lambda_n = n, n \ge 1$. Consider the following compacts in H

$$K_1 = \{u : (u, e_n)^2 \le \frac{1}{n^2}, n \ge 1\},$$

$$K_2 = \{u : \sum_{n=1}^{\infty} n^2 (x, e_n)^2 \le 1\}.$$

Define the widths

$$\alpha_i(t) = \max_{u \in K_i} \|u - G_{0,t}u\|,\ i = 1, 2,$$

and the value

$$d(t) = \dim G_{0,t}(H).$$

Then

$$\frac{\alpha_1(t)}{\sqrt{t|\ln t|}} \xrightarrow{P} 1, t \to 0+,$$

$$\varliminf_{t\to 0+} \alpha_2(t)\sqrt{\frac{2}{t}\,\mathrm{lln}\, t} \geq 1,\ \text{a.s.},$$

$$\varlimsup_{t\to 0+} \frac{td(t)}{2|\ln t|} \leq 1,\ \text{a.s.},$$

$$\varliminf_{t\to 0+} 2t|\ln t|d(t) \geq 1,\ \text{a.s.}$$

The example shows the dependence of the asymptotic behavior of the widths on the structure of the compacts and the semi-group.

Thus, it can be expected that the proposed approach will give an opportunity to explore the geometry of both smooth and non-smooth stochastic flows.

5. Discrete Time Approximation of Coalescing Stochastic Flows

It is known that a solution to SDE with smooth coefficients may be obtained via a discrete time approximation. It appears that a discrete time approximation can also be built for coalescing stochastic flows which may not be generated by SDE's. Here we present a discrete time approximation for the Harris flow.

Consider a sequence of independent stationary Gaussian processes $\{\xi_n(u); u \in \mathbb{R}, n \geq 1\}$ with zero mean and covariation function Γ. Suppose, that Γ is continuous. Define a sequence of random mappings $\{x_n; n \geq 0\}$ by the rule

$$x_0(u) = u,\ x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)),\ u \in \mathbb{R}.$$

Note that the continuity of Γ implies that the processes $\{\xi_n; n \geq 1\}$ have measurable modifications. This allows substituting x_n into ξ_{n+1}. The independence of $\{\xi_n; n \geq 1\}$ guarantees that $\xi_{n+1}(x_n(u))$ does not depend on the choice of these modifications. Let us also define the random functions

$$y_n(u,t) = n\Big(\frac{k+1}{n} - t\Big)x_k(u) + n\Big(t - \frac{k}{n}\Big)x_{k+1}(u),$$
$$u \in \mathbb{R},\ t \in \Big[\frac{k}{n}; \frac{k+1}{n}\Big],\ k = 0, \ldots, n-1.$$

Note that the functions y_n for $n \geq 1$ are not monotone. In spite of this any m-point motion of $\{y_n; n \geq 0\}$ approximates the m-point motion of the Harris flow. Namely the following statement is true.

Theorem 5.1 ([15]). *Let Γ be a continuous positive definite function on $\mathbb{R}$ such that $\Gamma(0) = 1$ and Γ has two continuous bounded derivatives. Suppose that y_n is built upon a sequence $\{\xi_k; k \geq 1\}$ with covariance $\frac{1}{\sqrt{n}}\Gamma$. Then for every $u_1, \ldots, u_l \in \mathbb{R}$ the random processes $\{y_n(u_j, \cdot), j = 1, \ldots, l\}$ weakly converge in $C([0;1], \mathbb{R}^l)$ to the l-point motion of the Harris flow with the local characteristic Γ.*

References

1. Baudoin, F.: *An introduction to the geometry of stochastic flows*, Imperial College Press, 2004.
2. Dorogovtsev, A. A.: A stochastic integral with respect to Arratia's flow, *Dokl. Akad. Nauk.* **410** (2006) 156–157.
3. Dorogovtsev, A. A.: *Measure-valued processes and stochastic flows* (in Russian), Institute of Mathematics NAS of Ukraine, 2007.
4. Dorogovtsev, A. A.: Fourier–Wiener Transform of Functionals of Arratia Flow. *Ukrainian Math. Bulletin.* **4:3** (2007) 329–350.
5. Dorogovtsev, A. A.: Semigroups of finite-dimensional random projections, *Lithuanian Math. Journ.* **2** (2011).
6. Dorogovtsev, A. A. and Ostapenko, O. V.: Large deviations for flows of interacting Brownian motions, *Stochastics and Dynamics* **10:3** (2010) 315–339.
7. Evans, S. N.: Probability and Real Trees, in: *Ecole d'Eté de Probabilités de Saint-Flour XXXV-2005*, Springer, 2006.
8. Harris, T. E.: Coalescing and noncoalescing stochastic flows in $\mathbb{R}$, *Stochastic Processes and their Applications* **17** (1984) 187–210.
9. Kallenberg, O.: *Foundations of Modern Pobability*, Springer, 1997.
10. Krylov, N. N. and Veretennikov, A. Yu.: Explicit formulae for the solutions of the stochastic differential equations, *Math. USSA Sb.* **29:2** (1976) 239–256.
11. Kunita, H.: *Stochastic flows and stochastic differential equations*, Cambridge Univ. Press, Cambridge, 1990.
12. Le Jan, Y. and Raimond, O.: Sticky flows on the cirle and their noises, *Probab.Theory Related Fields* **129:1** (2004) 63–82.
13. Le Jan, Y. and Raimond, O.: Flows, coalescence and noise, *Ann. Probab.* **32:2** (2004) 1247–1315.
14. Matsumoto, H.: Coalescing stochastic flows on the real line, *Osaka J.Math.* **26** (1989) 139–158.
15. Nishchenko, I. I.: Discrete time approximation of coalescing stochastic flows on the real line, *Theory of Stochastic Processes*, **17(33):1** (2011) 70–78.
16. Simon, B.:$P(\varphi)_2$ *model of Euclidean quantum field theory*, Moscow:"Mir", 1976.
17. Skorokhod, A. V.: Operator-valued stochastic differential equations and stochastic semigroups, *Uspekhi Mat.Nauk* **37, No. 6(228)** (1982) 157–183.
18. Skorokhod, A. V.: *Random Linear Operators*, D.Reidel Publishing Company, Dordrecht, Holland, 1983.
19. Tikhomirov, V. M.: *Some questions in approximation theory* (Russian), Izdat. Moskov. Univ., 1976.
20. Tsirelson, B.: Spectral densities desribing off-white noises, *Ann. Inst. H. Poincare Probab.Statist.* **38:6** (2002) 1059–1069.
21. Tsirelson, B.: Nonclassical stochastic flows and continuous products, *Probability surweys* **1** (2004) 173–298.
22. Watanabe, S.: *Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels* **15:1** (1987) 1–39.

A. A. Dorogovtsev: Institute of Mathematics, National Academy of Sciences of Ukraine, Ukraine
E-mail address: adoro@imath.kiev.ua

I. I. Nishchenko: National Technical University of Ukraine "KPI", Ukraine
E-mail address: irynan@gmail.com