Communications on Stochastic Analysis

Volume 8 | Number 2 Article 3

6-1-2014

Quasi-invariance of fermion processes with J-Hermitian kernel

Giovanni Luca Torrisi

Communications on Stochastic Analysis
Vol. 8, No. 2 (2014) 169-190

Serials Publications
www.serialspublications.com

QUASI-INVARIANCE OF FERMION PROCESSES WITH J-HERMITIAN KERNEL

GIOVANNI LUCA TORRISI

Abstract. Fermion (or determinantal) processes with J-Hermitian kernel constitute a large class of random point fields which is of interest in mathematical physics. They generalize the popular family of fermion processes with Hermitian kernel and their existence has been recently established in full generality by Lytvynov [24]. In this paper we prove a quasi-invariance property for fermion processes with J-Hermitian kernel. Our findings extend in various directions the corresponding result in Camilier and Decreusefond [10].

1. Introduction

Fermion (or determinantal) processes with Hermitian kernel have been introduced by Macchi [25] in order to represent configurations of fermions. Subsequently, fermion processes have attracted much interest from various viewpoints. The full existence theorem for these processes is due to Soshnikov [29] who also discussed many examples occurring in mathematics and physics. The theory in [29] has been later on extended by Shirai and Takahashi [28] who introduced the class of α-determinantal processes. The Gibbsianness of fermion processes with Hermitian kernel was investigated by Georgii and Yoo [20]. Camilier and Decreusefond [10] proved a quasi-invariance property, with respect to the action of compactly supported diffeomorphisms, for fermion processes with Hermitian kernel and provided a related integration by parts formula. Fermion processes with J-Hermitian kernel constitute a large class of random point fields which extends the family of determinantal processes with Hermitian kernel. They appeared in the Eighties in works of mathematical physicists on solvable models of systems with positive and negative charged particles, see Alastuey and Forrester [1], Cornu and Jancovici [11] and [12], Forrester [16], [17] and [18], Gaudin [19]. More recently, fermion processes with J-Hermitian kernel occurred in the studies of Borodin, Okounkov and Olshanski [4], [5], [6], [7], [8] and [26] on harmonic analysis of both the infinite symmetric group and the infinite-dimensional unitary group. The full existence theorem for these processes is due to Lytvynov [24]. In this paper, after providing

Received 2013-12-21; Communicated by N. Privault.
2010 *Mathematics Subject Classification.* Primary 60G55, 60K35.
Key words and phrases. Determinantal processes, Fermion processes, mapping theorem, point processes, quasi-invariance, thinning.

a mapping theorem for fermion processes with J-Hermitian kernel, we show a related quasi-invariance property, under the action of diffeomorphisms. Our findings extend the corresponding result in [10].

The paper is structured as follows. In Section 2 we give some preliminaries on point processes, Hilbert-Schmidt and trace-class operators, J-Hermitian integral operators and fermion processes. In Section 3 we provide a mapping theorem for fermion processes with J-Hermitian kernel. In Section 4 we prove a quasi-invariance property for fermion processes with J-Hermitian kernel. In Section 5 we check that the class of fermion processes with J-Hermitian kernel is closed under independent thinning, and finally in Section 6 we discuss some examples.

2. Preliminaries

In this section we give some preliminaries on point processes, Hilbert-Schmidt and trace-class operators, J-Hermitian integral operators and fermion processes. The reader is referred to [13] and [14] for an introduction to point processes theory, to [2] for notions of functional analysis and to [23], [24], [25] and [29] for an introduction to fermion processes.

2.1. Point processes. Let E be any locally compact Polish space serving as the state space of the points, $\mathcal{B}(E)$ the Borel σ-field on E, $\mathcal{B}_0(E)$ the family of relatively compact Borel sets in E, and λ a Radon measure on $(E, \mathcal{B}(E))$. We denote by Γ_E the space of locally finite subsets (configurations) in E, i.e.

$$\Gamma_E = \{\xi \subset E : \ |\xi \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_0(E)\},$$

where the symbol $|\xi|$ denotes the cardinality of the set ξ and by

$$\Gamma_{E,0} = \{\xi \in \Gamma_E : \ |\xi| < \infty\}$$

the set of all finite configurations in E. Let $\Lambda \subset E$ be fixed. We write

$$\Gamma_\Lambda = \{\xi \in \Gamma_E : \ \xi \subset \Lambda\}$$

for the set of all configurations in Λ. The configuration space Γ_E is equipped with the vague topology, which makes it a Polish space. We denote by $\mathcal{B}(\Gamma_E)$ the Borel σ-field on Γ_E. A point process on E is a probability measure μ on $(\Gamma_E, \mathcal{B}(\Gamma_E))$. Given $\Lambda \in \mathcal{B}(E)$, we write μ_Λ for its marginal on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$. On the measurable space $(\Gamma_{E,0}, \mathcal{B}(\Gamma_{E,0}))$ we consider the so-called λ-sample measure L^λ defined by

$$\int_{\Gamma_{E,0}} \varphi(\xi)\, L^\lambda(\mathrm{d}\xi) = \sum_{n\geq 0} \frac{1}{n!} \int_{E^n} \varphi(\{x_1, \dots, x_n\})\, \lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n),$$

for any measurable $\varphi : \Gamma_{E,0} \to [0, \infty)$.

Assumed to exist, the so-called Janossy density $j^{(\mu_\Lambda)}(\xi)$, $\xi \in \Gamma_\Lambda$, $\Lambda \in \mathcal{B}_0(E)$, of a point process μ is the density function of μ_Λ with respect to L^λ_Λ, the restriction on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ of the λ-sample measure.

A point process μ is said to have the correlation function $c_\mu : \Gamma_{E,0} \to [0, \infty)$ if c_μ is measurable and satisfies

$$\int_{\Gamma_E} \left(\sum_{\alpha \in \Gamma_{E,0}:\ \alpha \subset \xi} \varphi(\alpha) \right) \mu(\mathrm{d}\xi) = \int_{\Gamma_{E,0}} \varphi(\xi) c_\mu(\xi)\, L^\lambda(\mathrm{d}\xi)$$

for any measurable $\varphi : \Gamma_{E,0} \to [0, \infty)$.

2.2. Hilbert-Schmidt and trace-class operators. In this section we collect some basic facts of functional analysis.

We denote by $L^2(E, \lambda)$ the space of λ-square integrable $f : E \to \mathbb{C}$ and by $\mathcal{L}(L^2(E, \lambda))$ the space of bounded linear operators on $L^2(E, \lambda)$, i.e. $\mathcal{T} \in \mathcal{L}(L^2(E, \lambda))$ if and only if $\mathcal{T} : L^2(E, \lambda) \to L^2(E, \lambda)$ is linear and there exists $c > 0$ so that $\|\mathcal{T} f\|_{L^2(E,\lambda)} \leq c\|f\|_{L^2(E,\lambda)}$, for any $f \in L^2(E, \lambda)$. Hereafter $\mathcal{S}, \mathcal{T} \in \mathcal{L}(L^2(E, \lambda))$. The usual operator norm and numerical range of $\mathcal{T}$ are defined respectively by

$$\|\mathcal{T}\| = \sup\{|\langle f, \mathcal{T} g\rangle_{L^2(E,\lambda)}| : \; f, g \in L^2(E, \lambda), \, \|f\|_{L^2(E,\lambda)} = \|g\|_{L^2(E,\lambda)} = 1\}$$

and

$$\Theta(\mathcal{T}) = \{\langle f, \mathcal{T} f\rangle_{L^2(E,\lambda)} : \; f \in L^2(E, \lambda), \; \|f\|_{L^2(E,\lambda)} = 1\}.$$

$\mathcal{T}$ is called positive and we write $\mathcal{T} \geq 0$ if $\inf \Theta(\mathcal{T}) \geq 0$. We write $\mathcal{T} \geq \mathcal{S}$ if $\mathcal{T} - \mathcal{S} \geq 0$. $\mathcal{T}$ is called Hilbert-Schmidt operator on $L^2(E, \lambda)$ if there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of $L^2(E, \lambda)$ such that

$$\sum_{n\geq 1} \|\mathcal{T} e_n\|^2_{L^2(E,\lambda)} < \infty.$$

This infinite sum does not depend on the choice of the basis. We recall that $\mathcal{T}$ is a Hilbert-Schmidt operator on $L^2(E, \lambda)$ if and only if there exists $T : E \times E \to \mathbb{C}$, $T \in L^2(E \times E, \lambda^{\otimes 2})$, called kernel of $\mathcal{T}$, such that

$$\mathcal{T} f(x) = \int_E T(x, y) f(y) \, \lambda(\mathrm{d}y).$$

We recall that Hilbert-Schmidt operators are compact operators. We remind that the adjoint of $\mathcal{T}$ is the unique operator $\mathcal{T}^* \in \mathcal{L}(L^2(E, \lambda))$ such that

$$\langle f, \mathcal{T} g\rangle_{L^2(E,\lambda)} = \langle \mathcal{T}^* f, g\rangle_{L^2(E,\lambda)}.$$

To any $\mathcal{T}$ we may associate the positive operator $|\mathcal{T}| \in \mathcal{L}(L^2(E, \lambda))$ defined by $|\mathcal{T}| = \sqrt{\mathcal{T}^*\mathcal{T}}$. If $\mathcal{T} \geq 0$ it holds $|\mathcal{T}| = \mathcal{T}$ and we define the trace of $\mathcal{T}$ by

$$\mathrm{Tr}(\mathcal{T}) = \sum_{n\geq 1} \langle e_n, \mathcal{T} e_n\rangle_{L^2(E,\lambda)}, \tag{2.1}$$

where $\{e_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(E, \lambda)$. This sum makes sense (finite or infinite) and does not depend on the choice of the basis. The operator $\mathcal{T}$ is called of trace-class if $\mathrm{Tr}(|\mathcal{T}|) < \infty$. We recall that trace-class operators are Hilbert-Schmidt operators. If $\mathcal{T}$ is of trace-class (not-necessarily positive) then one defines the trace of $\mathcal{T}$ exactly as in (2.1) (the series converges absolutely and again the sum does not depend on the choice of the basis). The following relations will be useful:

$$\text{For any } \mathcal{T} \text{ of trace-class,} \quad |\mathrm{Tr}(\mathcal{T})| \leq \mathrm{Tr}(|\mathcal{T}|), \tag{2.2}$$

for any $\mathcal{T}_1 \in \mathcal{L}(L^2(E, \lambda))$ and $\mathcal{T}_2$ of trace-class,

$$\mathrm{Tr}(|\mathcal{T}_1\mathcal{T}_2|) \leq \|\mathcal{T}_1\|\mathrm{Tr}(|\mathcal{T}_2|) \tag{2.3}$$

and

$$\mathrm{Tr}(\mathcal{T}_1\mathcal{T}_2) = \mathrm{Tr}(\mathcal{T}_2\mathcal{T}_1). \tag{2.4}$$

For any $\Lambda \in \mathcal{B}(E)$, we define the projection operator $P_\Lambda f = f\mathbb{1}_\Lambda$, $f \in L^2(E,\lambda)$, where $\mathbb{1}_\Lambda$ is the indicator of the set Λ, and the operator $\mathcal{T}_\Lambda = P_\Lambda \mathcal{T} P_\Lambda$. $\mathcal{T}$ is called locally of trace-class if $\mathcal{T}_\Lambda$ is of trace-class for any $\Lambda \in \mathcal{B}_0(E)$.

2.3. J-Hermitian integral operators. From now on, in this paragraph $\mathcal{U}$ will always denote a bounded integral operator on $L^2(E,\lambda)$ with kernel $U : E\times E \to \mathbb{C}$, i.e. $\mathcal{U} : L^2(E,\lambda) \to L^2(E,\lambda)$ is a bounded linear operator defined by

$$\mathcal{U}f(x) = \int_E U(x,y)f(y)\,\lambda(\mathrm{d}y).$$

We say that $\mathcal{U}$ is Hermitian if

$$U(x,y) = \overline{U(y,x)}, \quad \lambda^{\otimes 2}\text{-a.e.},$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We recall that if $\mathcal{U}$ is compact and Hermitian then there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of $L^2(E,\lambda)$ formed by eigenfunctions of $\mathcal{U}$ i.e., letting $\{\alpha_n\}_{n\geq 1}$ denote the corresponding sequence of eigenvalues, $\mathcal{U}e_n = \alpha_n e_n$. We also note that, for any $\Lambda \in \mathcal{B}(E)$, $\mathcal{U}_\Lambda$ is a bounded integral operator on $L^2(E,\lambda)$ with kernel $U_\Lambda(x,y) = \mathbb{1}_\Lambda(x)U(x,y)\mathbb{1}_\Lambda(y)$.

Throughout this paper, we assume that the underlying space E is split into two disjoint parts: $E = E_1 \sqcup E_2$. We then write the space $L^2(E,\lambda)$ as a direct sum of two subspaces: $L^2(E,\lambda) = L^2(E_1,\lambda) \oplus L^2(E_2,\lambda)$. We say that $\mathcal{U}$ is J-Hermitian if

$$U(x,y) = \overline{U(y,x)}, \quad \text{for } \lambda^{\otimes 2}\text{-a.e. } (x,y) \text{ in } E_1^2 \text{ or } E_2^2$$

and

$$U(x,y) = -\overline{U(y,x)}, \quad \text{for } \lambda^{\otimes 2}\text{-a.e. } (x,y) \in E^2 \setminus (E_1^2 \cup E_2^2).$$

Note that the class of J-Hermitian bounded integral operators is more general than the class of Hermitian bounded integral operators. According to the above splitting, we rewrite $\mathcal{U}$ in block form as

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix},$$

where $\mathcal{U}_{ij} = P_{E_i}\mathcal{U}P_{E_j}$, $i,j = 1,2$. We define the even and odd parts of $\mathcal{U}$ as follows:

$$\mathcal{U}_e = \begin{pmatrix} \mathcal{U}_{11} & 0 \\ 0 & \mathcal{U}_{22} \end{pmatrix}$$

and

$$\mathcal{U}_o = \begin{pmatrix} 0 & \mathcal{U}_{12} \\ \mathcal{U}_{21} & 0 \end{pmatrix}.$$

Note that the kernels of $\mathcal{U}_e$ and $\mathcal{U}_o$ are given by

$$U_e(x,y) = \mathbb{1}_{E_1}(x)U(x,y)\mathbb{1}_{E_1}(y) + \mathbb{1}_{E_2}(x)U(x,y)\mathbb{1}_{E_2}(y) \tag{2.5}$$

and

$$U_o(x,y) = \mathbb{1}_{E_1}(x)U(x,y)\mathbb{1}_{E_2}(y) + \mathbb{1}_{E_2}(x)U(x,y)\mathbb{1}_{E_1}(y),$$

respectively.

We denote by $\mathcal{L}_{1|2}(L^2(E,\lambda))$ the collection of bounded integral operators $\mathcal{U}$ on $L^2(E,\lambda)$ such that $\mathcal{U}_e$ is of trace-class and $\mathcal{U}_o$ is Hilbert-Schmidt. The classical definition of Fredholm determinant for trace-class operators may be extended to operators in $\mathcal{L}_{1|2}(L^2(E,\lambda))$ as follows. Hereafter, I denotes the identity operator.

Proposition 2.1. *If $\mathcal{U} \in \mathcal{L}_{1|2}(L^2(E,\lambda))$ and $\mathcal{U}_e$ is positive and Hermitian, then the Fredholm determinant of $I \pm \mathcal{U}$ is given by*

$$\begin{aligned}\mathrm{Det}(I \pm \mathcal{U}) &= \sum_{n\geq 0} \frac{1}{n!} \int_{E^n} \det(\pm U(x_i,x_j))_{1\leq i,j\leq n}\, \lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n)\\ &= 1 + \sum_{n\geq 1} \frac{1}{n!} \int_{E^n} \det(\pm U(x_i,x_j))_{1\leq i,j\leq n}\, \lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n),\end{aligned}$$

where $\det(\pm U(x_i,x_j))_{1\leq i,j\leq n}$ *is the determinant of the $n\times n$ matrix*

$$(\pm U(x_i,x_j))_{1\leq i,j\leq n}.$$

Proof. We only compute the Fredholm determinant of $I - \mathcal{U}$. The Fredholm determinant of $I + \mathcal{U}$ may be calculated similarly. Clearly $-\mathcal{U}$ is a bounded integral operator on $L^2(E,\lambda)$ with kernel $-U$ and $(-\mathcal{U})_o$ is Hilbert-Schmidt. So the claim follows by Proposition 6 in [24] if we check that $(-\mathcal{U})_e$ is of trace-class,

$$\int_E |U(x,x)|\,\lambda(\mathrm{d}x) < \infty$$

and

$$\mathrm{Tr}((-\mathcal{U})_e) = -\int_E U(x,x)\,\lambda(\mathrm{d}x).$$

Since $|\mathcal{U}_e| = |-\mathcal{U}_e| = |(-\mathcal{U})_e|$ and $\mathcal{U}_e$ is of trace-class, we have $\mathrm{Tr}(|(-\mathcal{U})_e|) < \infty$. Since $\mathcal{U}_e$ is positive, Hermitian and of trace-class, letting $\{e_n\}_{n\geq 1}$ denote an orthonormal basis of $L^2(E,\lambda)$ formed by eigenfunctions of $\mathcal{U}_e$ and $\{\alpha_n\}_{n\geq 1}$ the corresponding sequence of eigenvalues, we can choose the kernel U_e of $\mathcal{U}_e$ so that

$$U_e(x,x) \geq 0 \text{ for any } x \in E \text{ and } \mathrm{Tr}(\mathcal{U}_e) = \sum_{n\geq 1} \alpha_n = \int_E U_e(x,x)\,\lambda(\mathrm{d}x)$$

(see the proof of Lemma A.3 in [20]). Since $U(x,x) = U_e(x,x)$, $x \in E$, we then have

$$\int_E |U(x,x)|\,\lambda(\mathrm{d}x) = \int_E U_e(x,x)\,\lambda(\mathrm{d}x) = \mathrm{Tr}(\mathcal{U}_e) = \mathrm{Tr}(|\mathcal{U}_e|) < \infty$$

where the latter term is finite since $\mathcal{U}_e$ is of trace-class. The claim follows noticing that

$$\mathrm{Tr}((-\mathcal{U})_e) = \mathrm{Tr}(-\mathcal{U}_e) = -\sum_{n\geq 1}\alpha_n = -\int_E U_e(x,x)\,\lambda(\mathrm{d}x) = -\int_E U(x,x)\,\lambda(\mathrm{d}x).$$

□

Let $\varphi : E \to [0,\infty)$ be a non-negative bounded measurable function. We denote by $\mathcal{U}[\varphi]$ the bounded integral operator on $L^2(E,\lambda)$ with kernel $U[\varphi](x,y) =$

$\sqrt{\varphi(x)}U(x,y)\sqrt{\varphi(y)}$ and by $\widehat{\mathcal{U}}$ the bounded integral operator on $L^2(E,\lambda)$ defined by

$$\widehat{\mathcal{U}} = \mathcal{U}P_{E_1} + (I - \mathcal{U})P_{E_2}.$$

Note that if $\mathcal{U} \geq 0$, then $\mathcal{U}[\varphi] \geq 0$. Indeed, for any $f \in L^2(E,\lambda)$, setting $g = f\sqrt{\varphi}$, we have

$$\begin{aligned}
\langle f, \mathcal{U}[\varphi]f\rangle_{L^2(E,\lambda)} &= \int_E \mathcal{U}[\varphi]f(x)\overline{f(x)}\,\lambda(\mathrm{d}x) \\
&= \int_E \overline{f(x)}\left(\int_E U[\varphi](x,y)f(y)\,\lambda(\mathrm{d}y)\right)\lambda(\mathrm{d}x) \\
&= \int_E \overline{f(x)\sqrt{\varphi(x)}}\left(\int_E U(x,y)f(y)\sqrt{\varphi(y)}\,\lambda(\mathrm{d}y)\right)\lambda(\mathrm{d}x) \\
&= \int_E \mathcal{U}g(x)\overline{g(x)}\,\lambda(\mathrm{d}x) \\
&= \langle g, \mathcal{U}g\rangle_{L^2(E,\lambda)} \geq 0.
\end{aligned}$$

The following propositions hold.

Proposition 2.2. *If $0 \leq \widehat{\mathcal{U}} \leq I$, then $\mathcal{U}_{11} \geq 0$ and $\mathcal{U}_{22} \geq 0$.*

Proposition 2.3. *Assume that $\mathcal{U}$ is a J-Hermitian bounded integral operator on $L^2(E,\lambda)$, $\mathcal{U}_{11}$ and $\mathcal{U}_{22}$ are locally of trace-class and $0 \leq \widehat{\mathcal{U}} \leq I$. Then, for any non-negative bounded measurable $\varphi : E \to [0,\infty)$ with compact support, we have $\mathcal{U}[\varphi] \in \mathcal{L}_{1|2}(L^2(E,\lambda))$ and $\mathcal{U}[\varphi]_e$ is positive and Hermitian. In particular, under the foregoing assumptions, is defined the Fredholm determinant of $I \pm \mathcal{U}[\varphi]$ (and so, taking $\varphi = 1\!\!1_\Lambda$, $\Lambda \in \mathcal{B}_0(E)$, is defined the Fredholm determinant of $I \pm \mathcal{U}_\Lambda$.)*

Proof. (Proposition 2.2). Since the operators $\widehat{\mathcal{U}}$ and $I - \widehat{\mathcal{U}}$ are positive, we have

$$\mathcal{U}_{11} = P_{E_1}\mathcal{U}P_{E_1} = P_{E_1}\widehat{\mathcal{U}}P_{E_1} \geq 0$$

and

$$\mathcal{U}_{22} = P_{E_2}\mathcal{U}P_{E_2} = P_{E_2}(I - \mathcal{U}P_{E_1} - (I-\mathcal{U})P_{E_2})P_{E_2} = P_{E_2}(I - \widehat{\mathcal{U}})P_{E_2} \geq 0.$$

□

Proof. (Proposition 2.3). Set $\Lambda = \mathrm{supp}(\varphi)$ and note that $\Lambda \in \mathcal{B}_0(E)$ and $\mathcal{U}[\varphi] = \mathcal{U}_\Lambda[\varphi]$. By Proposition 12 in [24] we have $\mathcal{U}_\Lambda \in \mathcal{L}_{1|2}(L^2(E,\lambda))$. We shall check later on that this implies $\mathcal{U}_\Lambda[\varphi]_e$ of trace-class and $\mathcal{U}_\Lambda[\varphi]_o$ Hilbert-Schmidt, i.e. $\mathcal{U}[\varphi] \in \mathcal{L}_{1|2}(L^2(E,\lambda))$. Letting U_e denote the kernel of $\mathcal{U}_e$ and $U[\varphi]_e$ the kernel of $\mathcal{U}[\varphi]_e$, using relation (2.5) one may easily see that $U_e[\varphi] = U[\varphi]_e$. Consequently, $\mathcal{U}_e[\varphi] = \mathcal{U}[\varphi]_e$. By Proposition 2.2 $\mathcal{U}_{ii} \geq 0$, $i = 1,2$, so $\mathcal{U}_e \geq 0$ and therefore $\mathcal{U}[\varphi]_e = \mathcal{U}_e[\varphi] \geq 0$. Using again relation (2.5), one may easily realize that the Hermitianity of $\mathcal{U}[\varphi]_e$ follows by the Hermitianity of $\mathcal{U}_e$, which is implied by the J-Hermitianity of $\mathcal{U}$. It remains to check that $\mathcal{U}_\Lambda[\varphi]_e$ is of trace-class and $\mathcal{U}_\Lambda[\varphi]_o$ is Hilbert-Schmidt. Let $\Pi_{\sqrt{\varphi}}$ be the multiplication operator on $L^2(E,\lambda)$ defined by $\Pi_{\sqrt{\varphi}}g(x) = \sqrt{\varphi(x)}g(x)$. One may easily see that $\Pi_{\sqrt{\varphi}}$ is bounded. Note that

$$\mathcal{U}_\Lambda[\varphi]_e = (\mathcal{U}_\Lambda)_e[\varphi] = \Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e\Pi_{\sqrt{\varphi}}.$$

Since $(\mathcal{U}_\Lambda)_e$ is of trace-class by (2.3) we have

$$\mathrm{Tr}(|\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e|) \leq \|\Pi_{\sqrt{\varphi}}\|\mathrm{Tr}(|(\mathcal{U}_\Lambda)_e|) < \infty$$

and so $\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e$ is of trace-class. Applying again (2.3) one sees that the operator $\Pi_{\sqrt{\varphi}}\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e$ is also of trace-class. Finally, by (2.4) we have

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}_\Lambda[\varphi]_e) &= \mathrm{Tr}(\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e\Pi_{\sqrt{\varphi}}) \\ &= \mathrm{Tr}(\Pi_{\sqrt{\varphi}}\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e) \\ &\leq \mathrm{Tr}(|\Pi_{\sqrt{\varphi}}\Pi_{\sqrt{\varphi}}(\mathcal{U}_\Lambda)_e|) < \infty \end{aligned} \tag{2.6}$$

where the first inequality in (2.6) follows by the positivity of $\mathcal{U}_\Lambda[\varphi]_e$ (which implies $\mathrm{Tr}(\mathcal{U}_\Lambda[\varphi]_e) \geq 0$) and (2.2). Regarding the Hilbert-Schmidt property of $\mathcal{U}_\Lambda[\varphi]_o$, we note that since $(\mathcal{U}_\Lambda)_o$ is Hilbert-Schmidt we have that its kernel $(U_\Lambda)_o$ is $\lambda^{\otimes 2}$-square integrable on E^2, so due to the boundedness of φ we have that $(U_\Lambda[\varphi])_o$ is $\lambda^{\otimes 2}$- square integrable on E^2. The claim follows noticing that $(U_\Lambda[\varphi])_o$ is the kernel of $\mathcal{U}_\Lambda[\varphi]_o$. □

2.4. Fermion processes. Our first standing assumption is as follows.
Condition I $\mathcal{K}$ is a J-Hermitian bounded integral operator on $L^2(E,\lambda)$, $\mathcal{K}_{11}$ and $\mathcal{K}_{22}$ are locally of trace-class and $0 \leq \widehat{\mathcal{K}} \leq I$. We denote by K the kernel of $\mathcal{K}$.
The following theorem is proved in [24] (see Theorem 2 therein).

Theorem 2.4. *Under Condition I we have that there exists a unique point process* $\mu^{(K,\lambda)}$ *on* $(\Gamma_E, \mathcal{B}(\Gamma_E))$ *with correlation function*

$$c_{\mu^{(K,\lambda)}}(\xi) = \det(K(x,y))_{x,y\in\xi}, \quad \xi \in \Gamma_{E,0}.$$

The point process $\mu^{(K,\lambda)}$ is called fermion process with kernel K and reference measure λ.

We shall also consider the following condition:
Condition II Condition I holds and there exists $\Lambda \in \mathcal{B}_0(E)$ such that $\|\mathcal{K}_\Lambda\| < 1$.

Lemma 2.5. *Assume Condition II and define the operator* $\mathcal{J}[\Lambda] = (I-\mathcal{K}_\Lambda)^{-1}\mathcal{K}_\Lambda$. *We have:*
(i) $\mathcal{J}[\Lambda] \in \mathcal{L}_{1|2}(L^2(E,\lambda))$, $\mathcal{J}[\Lambda]$ *is J-Hermitian, the operators* $\mathcal{J}[\Lambda]_{11}$ *and* $\mathcal{J}[\Lambda]_{22}$ *are positive (and so* $\mathcal{J}[\Lambda]_e$ *is positive and Hermitian.)*
(ii) $\det(J[\Lambda](x,y))_{x,y\in\xi} \geq 0$, *for* L^λ*-a.e.* $\xi \in \Gamma_{E,0}$, *where* $J[\Lambda]$ *denotes the kernel of* $\mathcal{J}[\Lambda]$.

Proof. Proof of (i). By Proposition 2.3 we have $\mathcal{K}_\Lambda \in \mathcal{L}_{1|2}(L^2(E,\lambda))$. The claim follows by Proposition 10 in [24]. *Proof of* (ii). By part (i) we have that the operators $\mathcal{J}[\Lambda]_{11}$ and $\mathcal{J}[\Lambda]_{22}$ are positive and of trace-class on $L^2(E,\lambda)$. So, by Lemma A4 in [20] it follows that the kernel of $\mathcal{J}[\Lambda]_{ii}$, denoted by $J[\Lambda]_{ii}$, $i = 1, 2$, can (and will) be chosen in such a way that

$$\det(J[\Lambda]_{ii}(x,y))_{x,y\in\xi} \geq 0, \quad \text{for } L^\lambda\text{-a.e. } \xi \in \Gamma_{E,0}.$$

The claim then follows by the J-Hermitianity of $\mathcal{J}[\Lambda]$ and e.g. Proposition 11 in [24]. □

The following formula for the Janossy density of a fermion process is proved in [24] (see Theorem 2 therein. The strict positivity of $j^{(\mu_\Lambda^{(K,\lambda)})}(\emptyset)$ is indeed contained in its proof).

Theorem 2.6. *Assume Condition II. Then the fermion process $\mu_\Lambda^{(K,\lambda)}$ has Janossy density*

$$j^{(\mu_\Lambda^{(K,\lambda)})}(\emptyset) = \mathrm{Det}(I - \mathcal{K}_\Lambda) > 0,$$

$$j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) = \mathrm{Det}(I - \mathcal{K}_\Lambda)\det(J[\Lambda](x,y))_{x,y\in\xi}, \quad \textit{for } L^\lambda\textit{-a.e. } \xi \in \Gamma_\Lambda \setminus \{\emptyset\}.$$

Remark 2.7. For later purposes, we remark that, under Condition II, we have

$$\det(J[\Lambda](x,y))_{x,y\in\xi} > 0, \quad \text{for } \mu_\Lambda^{(K,\lambda)}\text{-a.e. } \xi \in \Gamma_\Lambda.$$

Indeed, by definition

$$j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) = \frac{\mathrm{d}\mu_\Lambda^{(K,\lambda)}(\xi)}{\mathrm{d}L_\Lambda^\lambda(\xi)}$$

and so

$$\begin{aligned}\mu_\Lambda^{(K,\lambda)}(\{\xi:\ j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) > 0\}) &= \int_{\{\xi:\ j^{(\mu_\Lambda^{(K,\lambda)})}(\xi)>0\}} j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) L_\Lambda^\lambda(\mathrm{d}\xi)\\ &= \int_{\Gamma_\Lambda} j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) L_\Lambda^\lambda(\mathrm{d}\xi) = 1.\end{aligned}$$

Therefore, $j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) > 0$ for $\mu_\Lambda^{(K,\lambda)}$-a.e. $\xi \in \Gamma_\Lambda$. The claim follows by Theorem 2.6.

3. A Mapping Theorem

Let F be a locally compact Polish space (in general different from E) and $\mathrm{Bci}(E,F)$ the set of all measurable bijections $\phi : E \to F$ such that ϕ^{-1} is continuous. Since λ is a Radon measure on $(E, \mathcal{B}(E))$ and ϕ^{-1} is continuous, the push-forward measure $\phi^*\lambda = \lambda \circ \phi^{-1}$ is a Radon measure on $(F, \mathcal{B}(F))$. Any $\phi \in \mathrm{Bci}(E,F)$ transforms a configuration $\xi \in \Gamma_E$ into the configuration $\{\phi(x)\}_{x\in\xi} \in \Gamma_F$, which, with an abuse of notation, we denote by $\phi(\xi)$. Let η be a point process on $(\Gamma_E, \mathcal{B}(\Gamma_E))$. For $\phi \in \mathrm{Bci}(E,F)$, we denote by $\phi^*\eta = \eta \circ \phi^{-1}$ the push-forward measure by ϕ of η. For $\phi \in \mathrm{Bci}(E,F)$, we define the bijective map $\Phi : L^2(F, \phi^*\lambda) \to L^2(E, \lambda)$ by $\Phi f(x) = f \circ \phi(x)$. Note that Φ is well-defined in the sense that, for any $f \in L^2(F, \phi^*\lambda)$, it holds $\Phi f \in L^2(E,\lambda)$. Indeed, for any $f \in L^2(F, \phi^*\lambda)$,

$$\int_E |\Phi f(x)|^2 \, \lambda(\mathrm{d}x) = \int_E |f(\phi(x))|^2 \, \lambda(\mathrm{d}x) = \int_F |f(x)|^2 \, \phi^*\lambda(\mathrm{d}x).$$

One may easily check that $\Phi^{-1}g(x) = g \circ \phi^{-1}(x)$, $g \in L^2(E,\lambda)$, is the inverse of Φ. Note that, for any $f_1, f_2 \in L^2(F, \phi^*\lambda)$,

$$\begin{aligned}\langle \Phi f_1, \Phi f_2\rangle_{L^2(E,\lambda)} &= \int_E \Phi f_1(x)\overline{\Phi f_2(x)}\,\lambda(\mathrm{d}x)\\ &= \int_E f_1(\phi(x))\overline{f_2(\phi(x))}\,\lambda(\mathrm{d}x)\\ &= \int_F f_1(x)\overline{f_2(x)}\,\phi^*\lambda(\mathrm{d}x)\\ &= \langle f_1, f_2\rangle_{L^2(F,\phi^*\lambda)}\end{aligned}$$

and similarly, for any $g_1, g_2 \in L^2(E,\lambda)$, $\langle \Phi^{-1}g_1, \Phi^{-1}g_2\rangle_{L^2(F,\phi^*\lambda)} = \langle g_1, g_2\rangle_{L^2(E,\lambda)}$, i.e. Φ and Φ^{-1} are isometries.

Let $\mathcal{U}$ be a bounded integral operator on $L^2(E,\lambda)$ with kernel U and $\phi \in \mathrm{Bci}(E,F)$. We shall check later on (see Lemma 3.2(i)) that the operator $\mathcal{U}^\phi = \Phi^{-1}\mathcal{U}\Phi$ is a bounded integral operator on $L^2(F,\phi^*\lambda)$ with kernel $U^\phi(x,y) = U(\phi^{-1}(x), \phi^{-1}(y))$. Since E is split into two disjoint parts E_1, E_2, we have $F = F_1 \sqcup F_2$, where $F_i = \phi(E_i)$, $i = 1,2$, and we write the space $L^2(F,\phi^*\lambda)$ as a direct sum of two subspaces: $L^2(F,\phi^*\lambda) = L^2(F_1,\phi^*\lambda) \oplus L^2(F_2,\phi^*\lambda)$. $\mathcal{U}^\phi$ is J-Hermitian if

$$U^\phi(x,y) = \overline{U^\phi(y,x)}, \quad \text{for } (\phi^*\lambda)^{\otimes 2}\text{-a.e. } (x,y) \text{ in } F_1^2 \text{ or } F_2^2$$

and

$$U^\phi(x,y) = -\overline{U^\phi(y,x)}, \quad \text{for } (\phi^*\lambda)^{\otimes 2}\text{-a.e. } (x,y) \in F_1 \times F_2.$$

According to the above splitting, we rewrite $\mathcal{U}^\phi$ in block form as

$$\mathcal{U}^\phi = \begin{pmatrix} (\mathcal{U}^\phi)_{11} & (\mathcal{U}^\phi)_{12}\\ (\mathcal{U}^\phi)_{21} & (\mathcal{U}^\phi)_{22}\end{pmatrix},$$

where $(\mathcal{U}^\phi)_{ij} = P_{F_i}\mathcal{U}^\phi P_{F_j}$, $i,j = 1,2$. We define the even and odd parts of $\mathcal{U}^\phi$ as follows:

$$(\mathcal{U}^\phi)_e = \begin{pmatrix} (\mathcal{U}^\phi)_{11} & 0\\ 0 & (\mathcal{U}^\phi)_{22}\end{pmatrix}$$

and

$$(\mathcal{U}^\phi)_o = \begin{pmatrix} 0 & (\mathcal{U}^\phi)_{12}\\ (\mathcal{U}^\phi)_{21} & 0\end{pmatrix}.$$

Finally, we set $\widehat{\mathcal{U}^\phi} = \mathcal{U}^\phi P_{F_1} + (I - \mathcal{U}^\phi)P_{F_2}$.

The following mapping theorem extends in various directions Theorem 7 in [10] with $\alpha = -1$. Furthermore, our proof, which is based on the computation of the correlation function, is certainly less technical than that one of Theorem 7 in [10], which is based on the computation of the Laplace functional.

Theorem 3.1. *Under Condition I, for any $\phi \in \mathrm{Bci}(E,F)$, we have $\phi^*\mu^{(K,\lambda)} = \mu^{(K^\phi,\phi^*\lambda)}$, i.e. the point process $\phi^*\mu^{(K,\lambda)}$ is a fermion process on F with kernel K^ϕ and reference measure $\phi^*\lambda$.*

The proof of this theorem is based on the following preliminary lemma, whose proof is given below (in fact, at this stage the first part of (ii), (iv) and (vi) are not needed, and they will be used in the next section).

Lemma 3.2. *Let* $\mathcal{U}$ *be a bounded integral operator on* $L^2(E,\lambda)$ *with kernel* U *and* $\phi\in\mathrm{Bci}(E,F)$. *Then:*
(i) $\mathcal{U}^\phi$ *is a bounded integral operator on* $L^2(F,\phi^*\lambda)$ *with kernel*

$$U^\phi(x,y)=U(\phi^{-1}(x),\phi^{-1}(y)).$$

(ii) $\|\mathcal{U}^\phi\|=\|\mathcal{U}\|$; *if* $\mathcal{U}$ *is positive, then* $\mathcal{U}^\phi$ *is positive.*
(iii) *If* $0\le\widehat{\mathcal{U}}\le I$, *then* $0\le\widehat{\mathcal{U}^\phi}\le I$.
(iv) *If* $\mathcal{U}_e\ge 0$ *and* $\mathcal{U}\in\mathcal{L}_{1|2}(L^2(E,\lambda))$, *then* $\mathcal{U}^\phi\in\mathcal{L}_{1|2}(L^2(F,\phi^*\lambda))$.
(v) *If* $\mathcal{U}$ *is* J*-Hermitian, then* $\mathcal{U}^\phi$ *is* J*-Hermitian.*
(vi) *For any* $\Lambda'\in\mathcal{B}(F)$, $(\mathcal{U}^\phi)_{\Lambda'}=(\mathcal{U}_{\phi^{-1}(\Lambda')})^\phi$.
(vii) *If* $\mathcal{U}_{ii}$ *is positive and locally of trace-class, then* $(\mathcal{U}^\phi)_{F_i}$ *is positive and locally of trace-class.*

Proof. (Theorem 3.1). By Lemma 3.2 the operator $\mathcal{K}^\phi$ satisfies the corresponding Condition I, i.e. $\mathcal{K}^\phi$ is a J-Hermitian bounded integral operator on $L^2(F,\phi^*\lambda)$, the operators $(\mathcal{K}^\phi)_{F_i}$, $i=1,2$, are locally of trace-class and $0\le\widehat{\mathcal{K}^\phi}\le I$. On the other hand, for any measurable $\varphi:\Gamma_{F,0}\to[0,\infty)$, we have

$$\begin{aligned}\int_{\Gamma_F}\left(\sum_{\beta\in\Gamma_{F,0}:\ \beta\subset\eta}\varphi(\beta)\right)\phi^*\mu^{(K,\lambda)}(\mathrm{d}\eta)&=\int_{\Gamma_E}\left(\sum_{\alpha\in\Gamma_{E,0}:\ \alpha\subset\xi}\varphi(\phi(\alpha))\right)\mu^{(K,\lambda)}(\mathrm{d}\xi)\\&=\int_{\Gamma_{E,0}}\varphi(\phi(\xi))\det(K(x,y))_{x,y\in\xi}\,L^\lambda(\mathrm{d}\xi)\qquad(3.1)\\&=\int_{\Gamma_{F,0}}\varphi(\eta)\det(K^\phi(x,y))_{x,y\in\eta}\,L^{\phi^*\lambda}(\mathrm{d}\eta),\end{aligned}$$

where in (3.1) we used Theorem 2.4. Applying again Theorem 2.4 we then have that $\phi^*\mu^{(K,\lambda)}$ is a fermion process with kernel K^ϕ and reference measure $\phi^*\lambda$. $\square$

Proof. (Lemma 3.2). *Proof of* (i). For any $f\in L^2(F,\phi^*\lambda)$,

$$\begin{aligned}\mathcal{U}^\phi f(x)=\Phi^{-1}\mathcal{U}\Phi f(x)&=\Phi^{-1}\mathcal{U}f\circ\phi(x)\\&=\Phi^{-1}\int_E U(x,y)f\circ\phi(y)\,\lambda(\mathrm{d}y)\\&=\int_E U(\phi^{-1}(x),y)f\circ\phi(y)\,\lambda(\mathrm{d}y)\\&=\int_F U(\phi^{-1}(x),\phi^{-1}(y))f(y)\,\phi^*\lambda(\mathrm{d}y)\\&=\int_F U^\phi(x,y)f(y)\,\phi^*\lambda(\mathrm{d}y).\end{aligned}$$

Since Φ^{-1} and Φ are isometries and $\mathcal{U}$ is bounded, for some constant $c > 0$ we have

$$\begin{aligned}\|\mathcal{U}^\phi f\|_{L^2(F,\phi^*\lambda)} &= \|\Phi^{-1}\mathcal{U}\Phi f\|_{L^2(F,\phi^*\lambda)}\\ &= \|\mathcal{U}\Phi f\|_{L^2(E,\lambda)}\\ &\leq c\|\Phi f\|_{L^2(E,\lambda)}\\ &= c\|f\|_{L^2(F,\phi^*\lambda)} \quad \text{for any } f \in L^2(F,\phi^*\lambda).\end{aligned}$$

The claim is proved. *Proof of* (ii). Since Φ and Φ^{-1} are isometries, for any $f_1, f_2, f \in L^2(F,\phi^*\lambda)$, we have

$$\begin{aligned}\langle f_1, \mathcal{U}^\phi f_2\rangle_{L^2(F,\phi^*\lambda)} = \langle f_1, \Phi^{-1}\mathcal{U}\Phi f_2\rangle_{L^2(F,\phi^*\lambda)} &= \langle \Phi^{-1}\Phi f_1, \Phi^{-1}\mathcal{U}\Phi f_2\rangle_{L^2(F,\phi^*\lambda)}\\ &= \langle \Phi f_1, \mathcal{U}\Phi f_2\rangle_{L^2(E,\lambda)},\end{aligned}$$

and $\|\Phi f\|_{L^2(E,\lambda)} = \|f\|_{L^2(F,\phi^*\lambda)}$. The claim easily follows by these relations. *Proof of* (iii). Writing $f \in L^2(F,\phi^*\lambda)$ as $f = f_1 \oplus f_2$, $f_i \in L^2(F_i,\phi^*\lambda)$, $i = 1,2$, we deduce

$$\begin{aligned}\langle f, \widehat{\mathcal{U}^\phi} f\rangle_{L^2(F,\phi^*\lambda)} &= \langle f_1 \oplus f_2, \widehat{\mathcal{U}^\phi} f_1 \oplus f_2\rangle_{L^2(F,\phi^*\lambda)}\\ &= \langle f_1, \mathcal{U}^\phi f_1\rangle_{L^2(F,\phi^*\lambda)} + \langle f_2, (I-\mathcal{U}^\phi) f_2\rangle_{L^2(F,\phi^*\lambda)}.\end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned}\langle f_1, \mathcal{U}^\phi f_1\rangle_{L^2(F,\phi^*\lambda)} &= \langle \Phi f_1, \mathcal{U}\Phi f_1\rangle_{L^2(E,\lambda)}\\ &= \langle f_1 \circ \phi, \mathcal{U} f_1 \circ \phi\rangle_{L^2(E,\lambda)}\end{aligned}$$

and similarly

$$\langle f_2, (I-\mathcal{U}^\phi) f_2\rangle_{L^2(F,\phi^*\lambda)} = \langle f_2 \circ \phi, (I-\mathcal{U}) f_2 \circ \phi\rangle_{L^2(E,\lambda)}.$$

Combining this with (3.2) we deduce

$$\begin{aligned}\langle f, \widehat{\mathcal{U}^\phi} f\rangle_{L^2(F,\phi^*\lambda)} &= \langle f_1 \circ \phi, \mathcal{U} f_1 \circ \phi\rangle_{L^2(E,\lambda)} + \langle f_2 \circ \phi, (I-\mathcal{U}) f_2 \circ \phi\rangle_{L^2(E,\lambda)}\\ &= \langle f \circ \phi, \widehat{\mathcal{U}} f \circ \phi\rangle_{L^2(E,\lambda)}.\end{aligned}$$

The claim follows since $0 \leq \widehat{\mathcal{U}} \leq I$ by assumption. *Proof of* (iv). We have to check that $(\mathcal{U}^\phi)_e$ is of trace-class and $(\mathcal{U}^\phi)_o$ is Hilbert-Schmidt. Let $(U_e)^\phi$ be the kernel of $(\mathcal{U}_e)^\phi$. By (i) and (2.5) we have

$$\begin{aligned}(U_e)^\phi(x,y) &= U_e(\phi^{-1}(x), \phi^{-1}(y))\\ &= \mathbb{1}_{E_1}(\phi^{-1}(x)) U(\phi^{-1}(x), \phi^{-1}(y)) \mathbb{1}_{E_1}(\phi^{-1}(y))\\ &\qquad + \mathbb{1}_{E_2}(\phi^{-1}(x)) U(\phi^{-1}(x), \phi^{-1}(y)) \mathbb{1}_{E_2}(\phi^{-1}(y))\\ &= \mathbb{1}_{F_1}(x) U^\phi(x,y) \mathbb{1}_{F_1}(y) + \mathbb{1}_{F_2}(x) U^\phi(x,y) \mathbb{1}_{F_2}(y)\\ &= (U^\phi)_e(x,y).\end{aligned}$$

So $(\mathcal{U}_e)^\phi = (\mathcal{U}^\phi)_e$ and by (ii) we have $(\mathcal{U}^\phi)_e = (\mathcal{U}_e)^\phi \geq 0$ since $\mathcal{U}_e \geq 0$. Now, let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $L^2(E,\lambda)$. Since Φ^{-1} is an isometry we have that $\{\Phi^{-1}e_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(F,\phi^*\lambda)$ and it holds

$$\begin{aligned}\mathrm{Tr}((\mathcal{U}^\phi)_e) &= \sum_{n\geq 1}\langle \Phi^{-1}e_n, (\mathcal{U}^\phi)_e\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\langle \Phi^{-1}e_n, (\mathcal{U}_e)^\phi\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\langle \Phi^{-1}e_n, \Phi^{-1}\mathcal{U}_e\Phi\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\langle \Phi^{-1}e_n, \Phi^{-1}\mathcal{U}_e e_n\rangle_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\langle e_n, \mathcal{U}_e e_n\rangle_{L^2(E,\lambda)}\\ &= \mathrm{Tr}(\mathcal{U}_e) < \infty,\end{aligned}$$

where the latter inequality follows by the trace-class property of $\mathcal{U}_e$. Similarly, we have $(\mathcal{U}^\phi)_o = (\mathcal{U}_o)^\phi$ and so

$$\begin{aligned}\sum_{n\geq 1}\|(\mathcal{U}^\phi)_o\Phi^{-1}e_n\|^2_{L^2(F,\phi^*\lambda)} &= \sum_{n\geq 1}\|(\mathcal{U}_o)^\phi\Phi^{-1}e_n\|^2_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\|\Phi^{-1}\mathcal{U}_o e_n\|^2_{L^2(F,\phi^*\lambda)}\\ &= \sum_{n\geq 1}\|\mathcal{U}_o e_n\|^2_{L^2(E,\lambda)} < \infty,\end{aligned}$$

where the latter inequality follows by the Hilbert-Schmidt property of $\mathcal{U}_o$. The proof is completed. *Proof of* (v). Since $\mathcal{U}$ is J-Hermitian

$$\begin{aligned}&\lambda^{\otimes 2}(\{(\phi^{-1}(x),\phi^{-1}(y))\in E_i\times E_i:\ U(\phi^{-1}(x),\phi^{-1}(y))\neq\overline{U(\phi^{-1}(y),\phi^{-1}(x))}\})\\ &= (\phi^*\lambda)^{\otimes 2}(\{(x,y)\in F_i\times F_i:\ U^\phi(x,y)\neq\overline{U^\phi(y,x)}\}) = 0,\quad \text{for any } i\in\{1,2\}\end{aligned}$$

and

$$\begin{aligned}&\lambda^{\otimes 2}(\{(\phi^{-1}(x),\phi^{-1}(y))\in E_1\times E_2:\ U(\phi^{-1}(x),\phi^{-1}(y))\neq-\overline{U(\phi^{-1}(y),\phi^{-1}(x))}\})\\ &= (\phi^*\lambda)^{\otimes 2}(\{(x,y)\in F_1\times F_2:\ U^\phi(x,y)\neq-\overline{U^\phi(y,x)}\}) = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}U^\phi(x,y) = U(\phi^{-1}(x),\phi^{-1}(y)) &= \overline{U(\phi^{-1}(y),\phi^{-1}(x))}\\ &= \overline{U^\phi(y,x)},\quad \text{for } (\phi^*\lambda)^{\otimes 2}\text{-a.e. } (x,y)\in F_i^2,\ i=1,2\end{aligned}$$

and

$$\begin{aligned}U^\phi(x,y) = U(\phi^{-1}(x),\phi^{-1}(y)) &= -\overline{U(\phi^{-1}(y),\phi^{-1}(x))}\\ &= -\overline{U^\phi(y,x)},\quad \text{for } (\phi^*\lambda)^{\otimes 2}\text{-a.e. } (x,y)\in F_1\times F_2.\end{aligned}$$

The proof is completed. *Proof of* (vi). For any $\Lambda'\in\mathcal{B}(F)$,

$$(U^\phi)_{\Lambda'}(x,y) = \mathbb{1}_{\Lambda'}(x)U^\phi(x,y)\mathbb{1}_{\Lambda'}(y) = \mathbb{1}_{\Lambda'}(x)U(\phi^{-1}(x),\phi^{-1}(y))\mathbb{1}_{\Lambda'}(y)$$

and

$$
\begin{aligned}
(U_{\phi^{-1}(\Lambda')})^{\phi}(x,y) &= U_{\phi^{-1}(\Lambda')}(\phi^{-1}(x),\phi^{-1}(y)) \\
&= \mathbb{1}_{\phi^{-1}(\Lambda')}(\phi^{-1}(x))U(\phi^{-1}(x),\phi^{-1}(y))\mathbb{1}_{\phi^{-1}(\Lambda')}(\phi^{-1}(y)),
\end{aligned}
$$

so $(U^{\phi})_{\Lambda'} = (U_{\phi^{-1}(\Lambda')})^{\phi}$, and the proof is completed. *Proof of* (vii). Since $\mathcal{U}_{ii}$ is positive and locally of trace-class, for any $\widetilde{\Lambda} \in \mathcal{B}_0(E)$, the operator $(\mathcal{U}_{ii})_{\widetilde{\Lambda}}$ is positive and of trace-class. Since ϕ^{-1} is continuous, for any $\Lambda' \in \mathcal{B}_0(F)$, we have $\phi^{-1}(\Lambda') \in \mathcal{B}_0(E)$ and, arguing as in the proof of the previous point (vi), the following relations between kernels hold:

$$
[(U_{ii})_{\phi^{-1}(\Lambda')}]^{\phi}(x,y) = [(U^{\phi})_{ii}]_{\Lambda'}(x,y) \tag{3.3}
$$

and $(U_{ii})^{\phi} = (U^{\phi})_{ii}$. So by (ii) and the positivity of $\mathcal{U}_{ii}$, we have $(\mathcal{U}^{\phi})_{ii} = (\mathcal{U}_{ii})^{\phi} \geq 0$. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $L^2(E,\lambda)$, then $\{\Phi^{-1}e_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(F,\phi^*\lambda)$ and

$$
\begin{aligned}
\sum_{n\geq 1}\langle e_n, (\mathcal{U}_{ii})_{\phi^{-1}(\Lambda')}e_n\rangle_{L^2(E,\lambda)} &= \sum_{n\geq 1}\langle \Phi^{-1}e_n, \Phi^{-1}(\mathcal{U}_{ii})_{\phi^{-1}(\Lambda')}\Phi\,\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)} \\
&= \sum_{n\geq}\langle \Phi^{-1}e_n, [(\mathcal{U}_{ii})_{\phi^{-1}(\Lambda')}]^{\phi}\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)}, \\
&= \sum_{n\geq}\langle \Phi^{-1}e_n, [(\mathcal{U}^{\phi})_{ii}]_{\Lambda'}\Phi^{-1}e_n\rangle_{L^2(F,\phi^*\lambda)},
\end{aligned} \tag{3.4}
$$

where in (3.4) we used (3.3). Hence

$$
\mathrm{Tr}([(\mathcal{U}^{\phi})_{ii}]_{\Lambda'}) = \mathrm{Tr}((\mathcal{U}_{ii})_{\phi^{-1}(\Lambda')}) < \infty,
$$

where the latter inequality follows by the trace-class property of $(\mathcal{U}_{ii})_{\phi^{-1}(\Lambda')}$. The claim follows by the arbitrary of Λ'. □

4. Quasi-invariance

From now on, we assume that E is a connected, oriented, $\mathcal{C}^{\infty}$, non-compact and finite-dimensional Riemannian manifold and we denote by m the volume element on E and by $\mathrm{Diff}(E)$ the family of diffeomorphisms from E into itself. The reader is directed to the book by do Carmo [15] as a standard reference on Riemannian geometry. Assuming that λ is of the form $\lambda(\mathrm{d}x) = \rho(x)m(\mathrm{d}x)$, where $\rho : E \to (0,\infty)$ is a measurable and positive function, by the classical formula for the change of measure, for any $E' \in \mathcal{B}_0(E)$, measurable function $f : E' \to [0,\infty)$ and $\phi \in \mathrm{Diff}(E)$, it holds

$$
\begin{aligned}
\int_{E'} f(x)\,\phi^*\lambda(\mathrm{d}x) &= \int_{\phi^{-1}(E')} f\circ\phi(x)\rho(x)\,m(\mathrm{d}x) \\
&= \int_{E'} f(x)\rho(\phi^{-1}(x))|\mathrm{Jac}(\phi^{-1})(x)|\,m(\mathrm{d}x) \\
&= \int_{E'} f(x)\frac{\rho(\phi^{-1}(x))}{\rho(x)}|\mathrm{Jac}(\phi^{-1})(x)|\,\lambda(\mathrm{d}x),
\end{aligned} \tag{4.1}
$$

where $\mathrm{Jac}(\phi^{-1})$ is the Jacobian of ϕ^{-1}. The following theorem extends in various directions Theorem 8 in [10] with $\alpha = -1$.

Theorem 4.1. *Under Condition I, for any $\Lambda \in \mathcal{B}_0(E)$ and $\phi \in \mathrm{Diff}(E)$ such that $\max\{\|\mathcal{K}_\Lambda\|, \|\mathcal{K}_{\phi^{-1}(\Lambda)}\|\} < 1$, we have that $\phi^*\mu^{(K,\lambda)}$ is absolutely continuous with respect to $\mu^{(K,\lambda)}$ on Γ_Λ with density*

$$R_\Lambda^\phi(\xi) = \frac{\mathrm{Det}(I - \mathcal{K}_{\phi^{-1}(\Lambda)})}{\mathrm{Det}(I - \mathcal{K}_\Lambda)} \frac{\det(J[\phi^{-1}(\Lambda)]^\phi(x,y))_{x,y\in\xi}}{\det(J[\Lambda](x,y))_{x,y\in\xi}} \times \prod_{x\in\xi} \frac{\rho(\phi^{-1}(x))}{\rho(x)} |\mathrm{Jac}(\phi^{-1})(x)|.$$

The proof of this theorem is based on the following preliminary lemma, whose proof is given below. Hereafter, $\mathrm{Hom}(E,F)$ denotes the family of homeomorphisms from E into F.

Lemma 4.2. *Assume Condition II. Then, for any $\phi \in \mathrm{Hom}(E,F)$, we have:*
(i) $\mathcal{K}^\phi$ satisfies Condition I and $\|(\mathcal{K}^\phi)_{\phi(\Lambda)}\| < 1$.
(ii) $\mathrm{Det}(I - (\mathcal{K}^\phi)_{\phi(\Lambda)}) = \mathrm{Det}(I - \mathcal{K}_\Lambda)$.
(iii) $\mathcal{J}[\Lambda]^\phi = (I - (\mathcal{K}^\phi)_{\phi(\Lambda)})^{-1}(\mathcal{K}^\phi)_{\phi(\Lambda)}$, $\mathcal{J}[\Lambda]^\phi \in \mathcal{L}_{1|2}(L^2(F, \phi^*\lambda))$, $\mathcal{J}[\Lambda]^\phi$ is J-Hermitian, the operators $(\mathcal{J}[\Lambda]^\phi)_{F_i}$, $i = 1, 2$, are positive, $\det(J[\Lambda]^\phi(x,y))_{x,y\in\xi} \geq 0$*, for $L^{\phi^*\lambda}$-a.e. $\xi \in \Gamma_{F,0}$, and $J[\Lambda]^\phi(x,y) = J[\Lambda](\phi^{-1}(x), \phi^{-1}(y))$.*

Remark 4.3. Under the assumptions of Theorem 4.1 we have $J[\phi^{-1}(\Lambda)]^\phi = J^\phi[\Lambda]$ and therefore $\det(J[\phi^{-1}(\Lambda)]^\phi(x,y))_{x,y\in\xi} = \det(J^\phi[\Lambda](x,y))_{x,y\in\xi}$. Indeed, by Lemma 4.2(*iii*) and the definition of the operator $\mathcal{J}[\Lambda]$ we have $\mathcal{J}[\phi^{-1}(\Lambda)]^\phi = (I - (\mathcal{K}^\phi)_\Lambda)^{-1}(\mathcal{K}^\phi)_\Lambda = \mathcal{J}^\phi[\Lambda]$.

Proof. (Theorem 4.1). For ease of notation, throughout this proof, for $x \in E$ and $\Lambda \in \mathcal{B}_0(E)$, we set

$$\mathcal{R}_\rho^\phi(x) := \frac{\rho(\phi^{-1}(x))}{\rho(x)} |\mathrm{Jac}(\phi^{-1})(x)|, \ \mathcal{D}_\Lambda^\phi := \frac{\mathrm{Det}(I - \mathcal{K}_{\phi^{-1}(\Lambda)})}{\mathrm{Det}(I - \mathcal{K}_\Lambda)}.$$

By the mapping theorem, the definition of Janossy density, the definition of $\phi^*\lambda$-sample measure, relation (4.1), Theorem 2.6, Lemma 4.2(i), Lemma 3.2(vi) and Lemma 4.2(iii), for any measurable $\varphi : \Gamma_\Lambda \to [0,\infty)$ we have

$$\begin{aligned}
\int_{\Gamma_\Lambda} \varphi(\xi)\, \phi^*\mu^{(K,\lambda)}(\mathrm{d}\xi) &= \int_{\Gamma_\Lambda} \varphi(\xi)\, \mu^{(K^\phi, \phi^*\lambda)}(\mathrm{d}\xi) \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} \varphi(\{x_1,\dots,x_n\}) j^{(\mu_\Lambda^{(K^\phi,\phi^*\lambda)})}(\{x_1,\dots,x_n\}) \prod_{i=1}^n \mathcal{R}_\rho^\phi(x_i)\, \lambda(\mathrm{d}x_i) \\
&= \mathrm{Det}(I - (\mathcal{K}^\phi)_\Lambda) \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} \varphi(\{x_1,\dots,x_n\}) \det(J[\phi^{-1}(\Lambda)]^\phi(x_i,x_j))_{1\leq i,j\leq n} \\
&\qquad \times \prod_{i=1}^n \mathcal{R}_\rho^\phi(x_i)\, \lambda(\mathrm{d}x_i). \qquad (4.2)
\end{aligned}$$

Using Theorem 2.6, Remark 2.7, Lemma 4.2(ii) and (4.2), we deduce

$$\begin{aligned}
&\mathcal{D}^\phi_\Lambda \int_{\Gamma_\Lambda} \varphi(\xi) \frac{\det(J[\phi^{-1}(\Lambda)]^\phi(x,y))_{x,y\in\xi}}{\det(J[\Lambda](x,y))_{x,y\in\xi}} \prod_{x\in\xi} \mathcal{R}^\phi_\rho(x)\, \mu^{(K,\lambda)}(\mathrm{d}\xi) \\
&= \mathcal{D}^\phi_\Lambda \int_{\Gamma_\Lambda} \varphi(\xi) \frac{\det(J[\phi^{-1}(\Lambda)]^\phi(x,y))_{x,y\in\xi}}{\det(J[\Lambda](x,y))_{x,y\in\xi}} \prod_{x\in\xi} \mathcal{R}^\phi_\rho(x) j^{(\mu_\Lambda^{(K,\lambda)})}(\xi) L^\lambda_\Lambda(\mathrm{d}\xi) \\
&= \mathcal{D}^\phi_\Lambda \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} \varphi(\{x_1,\dots,x_n\}) \frac{\det(J[\phi^{-1}(\Lambda)]^\phi(x_i,x_j))_{1\leq i,j\leq n}}{\det(J[\Lambda](x_i,x_j))_{1\leq i,j\leq n}} \\
&\qquad\qquad \times \mathrm{Det}(I-\mathcal{K}_\Lambda) \det(J[\Lambda](x_i,x_j))_{1\leq i,j\leq n} \prod_{i=1}^n \mathcal{R}^\phi_\rho(x_i)\, \lambda(\mathrm{d}x_i) \\
&= \int_{\Gamma_\Lambda} \varphi(\xi)\, \phi^* \mu^{(K,\lambda)}(\mathrm{d}\xi).
\end{aligned}$$

The proof is completed. □

Proof. (Lemma 4.2). *Proof of* (i). By Lemma 3.2 $\mathcal{K}^\phi$ satisfies the corresponding Condition I. Since $\Lambda \in \mathcal{B}_0(E)$ and ϕ is continuous, we have $\phi(\Lambda) \in \mathcal{B}_0(F)$. By Lemma 3.2($vi$) we have $(\mathcal{K}^\phi)_{\phi(\Lambda)} = (\mathcal{K}_\Lambda)^\phi$ and so by Lemma 3.2(ii) $\|(\mathcal{K}^\phi)_{\phi(\Lambda)}\| = \|(\mathcal{K}_\Lambda)^\phi\| = \|\mathcal{K}_\Lambda\| < 1$. *Proof of* ($ii$). By Lemma 3.2 and Proposition 12 in [24] we have $\mathcal{K}_\Lambda \in \mathcal{L}_{1|2}(L^2(E,\lambda))$, $(\mathcal{K}^\phi)_{\phi(\Lambda)} \in \mathcal{L}_{1|2}(L^2(F,\phi^*\lambda))$, $(\mathcal{K}_\Lambda)_e$ and $[(\mathcal{K}^\phi)_{\phi(\Lambda)}]_e$ positive and Hermitian. So it is defined the Fredholm determinant of $I-\mathcal{K}_\Lambda$ and $I-(\mathcal{K}^\phi)_{\phi(\Lambda)}$. The claim follows noticing that

$$\begin{aligned}
&\mathrm{Det}(I-(\mathcal{K}^\phi)_{\phi(\Lambda)}) = \mathrm{Det}(I-(\mathcal{K}_\Lambda)^\phi) \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{F^n} \det((K_\Lambda)^\phi(x_i,x_j))_{1\leq i,j\leq n} \prod_{i=1}^n \phi^*\lambda(\mathrm{d}x_i) \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{F^n} \det(K_\Lambda(\phi^{-1}(x_i),\phi^{-1}(x_j)))_{1\leq i,j\leq n} \prod_{i=1}^n \phi^*\lambda(\mathrm{d}x_i) \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{E^n} \det(K_\Lambda(x_i,x_j))_{1\leq i,j\leq n} \prod_{i=1}^n \lambda(\mathrm{d}x_i) \\
&= \mathrm{Det}(I-\mathcal{K}_\Lambda).
\end{aligned}$$

Proof of (iii). We start checking that $[(\mathcal{K}^\phi)_{\phi(\Lambda)}]^n = \Phi^{-1}(\mathcal{K}_\Lambda)^n\Phi$, $n\geq 0$. We have

$$\begin{aligned}
\Phi^{-1}(\mathcal{K}_\Lambda)^n\Phi &= \underbrace{\Phi^{-1}\mathcal{K}_\Lambda\Phi\ldots\Phi^{-1}\mathcal{K}_\Lambda\Phi}_{n \text{ times}} \\
&= [(\mathcal{K}_\Lambda)^\phi]^n = [(\mathcal{K}^\phi)_{\phi(\Lambda)}]^n.
\end{aligned}$$

Since $\|\mathcal{K}_\Lambda\| = \|(\mathcal{K}^\phi)_{\phi(\Lambda)}\| < 1$, we have

$$\sum_{n\geq 0} (\mathcal{K}_\Lambda)^n = (I-\mathcal{K}_\Lambda)^{-1} \quad \text{and} \quad \sum_{n\geq 0} [(\mathcal{K}^\phi)_{\phi(\Lambda)}]^n = (I-(\mathcal{K}^\phi)_{\phi(\Lambda)})^{-1}.$$

So

$$\begin{aligned}\mathcal{J}[\Lambda]^{\phi} = \Phi^{-1}\mathcal{J}[\Lambda]\Phi &= \Phi^{-1}(I-\mathcal{K}_\Lambda)^{-1}\mathcal{K}_\Lambda\Phi\\ &= \Phi^{-1}(I-\mathcal{K}_\Lambda)^{-1}\Phi\Phi^{-1}\mathcal{K}_\Lambda\Phi\\ &= \sum_{n\geq 0}\Phi^{-1}(\mathcal{K}_\Lambda)^n\Phi\,(\mathcal{K}_\Lambda)^{\phi}\\ &= \sum_{n\geq 0}[(\mathcal{K}^{\phi})_{\phi(\Lambda)}]^n\,(\mathcal{K}^{\phi})_{\phi(\Lambda)}\\ &= (I-(\mathcal{K}^{\phi})_{\phi(\Lambda)})^{-1}\,(\mathcal{K}^{\phi})_{\phi(\Lambda)}.\end{aligned}$$

By Lemma 2.5(i) and (ii), we have $\mathcal{J}[\Lambda]^{\phi}\in\mathcal{L}_{1|2}(L^2(F,\phi^*\lambda))$, $\mathcal{J}[\Lambda]^{\phi}$ J-Hermitian, $(\mathcal{J}[\Lambda]^{\phi})_{F_i}\geq 0$, $i=1,2$, and $\det(J[\Lambda]^{\phi}(x,y))_{x,y\in\xi}\geq 0$, for $L^{\phi^*\lambda}$-a.e. $\xi\in\Gamma_{F,0}$. It remains to prove the claimed form of the kernel $J[\Lambda]^{\phi}$. Letting $J[\Lambda]$ denote the kernel of $\mathcal{J}[\Lambda]$, for any $f\in L^2(F,\phi^*\lambda)$, we have

$$\begin{aligned}\mathcal{J}[\Lambda]^{\phi}f(x) = \Phi^{-1}\mathcal{J}[\Lambda]\Phi f(x) &= \Phi^{-1}\int_E J[\Lambda](x,y)\Phi f(y)\,\lambda(\mathrm{d}y)\\ &= \Phi^{-1}\int_E J[\Lambda](x,y)f(\phi(y))\,\lambda(\mathrm{d}y)\\ &= \Phi^{-1}\int_F J[\Lambda](x,\phi^{-1}(y))f(y)\,\phi^*\lambda(\mathrm{d}y)\\ &= \int_F J[\Lambda](\phi^{-1}(x),\phi^{-1}(y))f(y)\,\phi^*\lambda(\mathrm{d}y).\end{aligned}$$

□

Remark 4.4. By Remark 2.7 and Theorem 4.1, we have $R_\Lambda^{\phi}(\xi)>0$ for $\mu^{(K,\lambda)}$-a.e. $\xi\in\Gamma_\Lambda$. Therefore, the measures $\mu^{(K,\lambda)}$ and $\phi^*\mu^{(K,\lambda)}$ are equivalent on Γ_Λ (see e.g. p. 8 in Bogachev [3]). Consequently, by the usual relations between equivalent measures we have that the density of $\mu^{(K,\lambda)}$ with respect to $\phi^*\mu^{(K,\lambda)}$ on Γ_Λ is

$$\begin{aligned}R_\Lambda^{\phi}(\xi)^{-1} = {}&\frac{\mathrm{Det}(I-\mathcal{K}_\Lambda)}{\mathrm{Det}(I-\mathcal{K}_{\phi^{-1}(\Lambda)})}\frac{\det(J[\Lambda](x,y))_{x,y\in\xi}}{\det(J[\phi^{-1}(\Lambda)]^{\phi}(x,y))_{x,y\in\xi}}\\ &\times\prod_{x\in\xi}\frac{\rho(x)}{\rho(\phi^{-1}(x))}|\mathrm{Jac}(\phi)(\phi^{-1}(x))|,\end{aligned}$$

where the equality follows by the classical relation

$$\mathrm{Jac}(\phi^{-1})(\phi(x)) = \mathrm{Jac}(\phi)(x)^{-1}.$$

Given $\phi\in\mathrm{Diff}(E)$ the support of ϕ, denoted by $\mathrm{supp}(\phi)$, is defined as the closure of the set $\{x\in E:\ \phi(x)\neq x\}$. In the following we denote by $\mathrm{Diff}_c(E)$ the subset of $\mathrm{Diff}(E)$ formed by the diffeomorphisms with compact support. We conclude this section with the following corollary of Theorem 4.1.

Corollary 4.5. *Assume Condition I and* $\|\mathcal{K}_\Lambda\|<1$*, for any* $\Lambda\in\mathcal{B}_0(E)$*. Then, for any* $\phi\in\mathrm{Diff}_c(E)$*, we have that* $\phi^*\mu^{(K,\lambda)}$ *is absolutely continuous with respect* *to* $\mu^{(K,\lambda)}$ *on* Γ_E *with density*

$$R^{\phi}(\xi) = \mathbb{1}_{\Gamma_{E\setminus \mathrm{supp}(\phi)}}(\xi) + \mathbb{1}_{\Gamma_{\mathrm{supp}(\phi)}}(\xi)\frac{\det(J[\mathrm{supp}(\phi)]^{\phi}(x,y))_{x,y\in\xi}}{\det(J[\mathrm{supp}(\phi)](x,y))_{x,y\in\xi}} \times \prod_{x\in\xi}\frac{\rho(\phi^{-1}(x))}{\rho(x)}|\mathrm{Jac}(\phi^{-1})(x)|.$$

Proof. Let $\phi \in \mathrm{Diff}_c(E)$ be fixed. By Theorem 4.1 the measure $\phi^*\mu^{(K,\lambda)}$ is absolutely continuous with respect to $\mu^{(K,\lambda)}$ on Γ_Λ, for any $\Lambda \in \mathcal{B}_0(E)$, and therefore $\phi^*\mu^{(K,\lambda)}$ is absolutely continuous with respect to $\mu^{(K,\lambda)}$ on Γ_E. Outside $\mathrm{supp}(\phi)$ we have $\phi(x) = x$ and so $\phi^*\mu^{(K,\lambda)} \equiv \mu^{(K,\lambda)}$ on $\Gamma_{E\setminus\mathrm{supp}(\phi)}$. The claim follows by applying Theorem 4.1 with $\Lambda = \mathrm{supp}(\phi)$ and noticing that $\phi^{-1}(\mathrm{supp}(\phi)) = \mathrm{supp}(\phi)$. □

5. Independent Thinning

In this short section we check that the class of fermion processes with J-Hermitian kernel is closed under independent thinning. More precisely, the following proposition holds.

Proposition 5.1. *Assume Condition I. The point process* μ *is obtained as an independent thinning of* $\mu^{(K,\lambda)}$ *with retention probability* $\varepsilon \in [0,1]$ *if and only if* $\mu \equiv \mu^{(K[\varepsilon],\lambda)}$*, i.e.* μ *is a fermion process with kernel* $K[\varepsilon] = \varepsilon K$ *and reference measure* λ*.*

The proof of this proposition is based on the following lemma.

Lemma 5.2. *Let* $\varepsilon \in [0,1]$ *be a fixed constant. If Condition I holds, then* $\mathcal{K}[\varepsilon] = \varepsilon\mathcal{K}$ *is a J-Hermitian bounded integral operator on* $L^2(E,\lambda)$*,* $\mathcal{K}[\varepsilon]_{11}$ *and* $\mathcal{K}[\varepsilon]_{22}$ *are locally of trace-class and* $0 \le \widehat{\mathcal{K}[\varepsilon]} \le I$ *(i.e.* $\mathcal{K}[\varepsilon]$ *satisfies Condition I.)*

Proof. (Proposition 5.1). We recall that the law of a point process μ on Γ_E is uniquely determined by its Laplace functional

$$f \mapsto \int_{\Gamma_E} \mathrm{e}^{-\sum_{x\in\xi} f(x)}\,\mu(\mathrm{d}\xi),$$

where $f : E \to [0,\infty)$ is a measurable, non-negative function with compact support. By Theorem 2 in [24] one has that, under Condition I,

$$\int_{\Gamma_E} \mathrm{e}^{-\sum_{x\in\xi} f(x)}\,\mu^{(K,\lambda)}(\mathrm{d}\xi) = \mathrm{Det}(I - \mathcal{K}[1-\mathrm{e}^{-f}]). \tag{5.1}$$

Let $\{U(x)\}_{x\in E}$ be a random field of independent Bernoulli random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\mathbb{P}(U(x)=1) = \varepsilon$ and $\{U(x)\}_{x\in E}$ is independent of $\mu^{(K,\lambda)}$. The claim follows combining the next identity with (5.1) and Lemma 5.2:

$$\begin{aligned}\mathbb{E}\left[\int_{\Gamma_E} \mathrm{e}^{-\sum_{x\in\xi} f(x)1\{U(x)=1\}}\,\mu^{(K,\lambda)}(\mathrm{d}\xi)\right] &= \mathbb{E}[\mathrm{Det}(I - \mathcal{K}[1-\mathrm{e}^{-f(\cdot)1\{U(\cdot)=1\}}])]\\ &= \mathbb{E}[\mathrm{Det}(I - \mathcal{K}[\mathbb{1}\{U(\cdot)=1\}(1-\mathrm{e}^{-f(\cdot)})])]\\ &= \mathrm{Det}(I - \mathcal{K}[\varepsilon][1-\mathrm{e}^{-f}])\end{aligned}$$

where in the latter equality we exchange the mean with the series by Fubini's Theorem. □

Proof. (Lemma 5.2). Since ε is a constant, we have that $\mathcal{K}[\varepsilon] = \varepsilon\mathcal{K}$ is a bounded integral operator on $L^2(E, \lambda)$ with kernel $K[\varepsilon](x, y) = \varepsilon K(x, y)$ and the J-Hermitianity of $\mathcal{K}[\varepsilon]$ is a straightforward consequence of the J-Hermitianity of $\mathcal{K}$. Let $\Lambda \in \mathcal{B}_0(E)$ and $i \in \{1, 2\}$ be fixed. By assumption $(\mathcal{K}_{ii})_\Lambda$ is of trace-class and so (arguing e.g. as in the proof of Proposition 2.3) we have that $(\mathcal{K}_{ii})_\Lambda[\varepsilon] = (\mathcal{K}[\varepsilon]_{ii})_\Lambda$ is of trace-class. Finally,

$$\begin{aligned}\widehat{\mathcal{K}[\varepsilon]} &= \mathcal{K}[\varepsilon]P_{E_1} + (I - \mathcal{K}[\varepsilon])P_{E_2} \\ &= \varepsilon\widehat{\mathcal{K}} + (1 - \varepsilon)P_{E_2}.\end{aligned}$$

Since $\widehat{\mathcal{K}} \geq 0$ and $P_{E_2} \geq 0$ we get $\widehat{\mathcal{K}[\varepsilon]} \geq 0$ and since $\widehat{\mathcal{K}} \leq I$ and $P_{E_2} \leq I$ we get $\widehat{\mathcal{K}[\varepsilon]} \leq I$. The proof is completed. □

6. Illustrating Examples

Let $\mathcal{K}$ be the integral operator defining a fermion process with the Whittaker kernel [6] or a fermion process with the matrix tail kernel [26] or a fermion process with the continuous hypergeometric kernel [7]. By Corollary 1 in [24], we have that $\mathcal{K}$ satisfies Condition I, and so Theorem 3.1 applies to such point processes. Let $\varepsilon \in (0, 1)$, $\Lambda \in \mathcal{B}_0(E)$ and $\phi \in \mathrm{Diff}(E)$ be fixed. By Proposition 7 in [24] we have $\|\mathcal{K}\| \leq 1$, therefore $\|\mathcal{K}[\varepsilon]\| = \varepsilon\|\mathcal{K}\| < 1$ and so $\max\{\|\mathcal{K}[\varepsilon]_\Lambda\|, \|\mathcal{K}[\varepsilon]_{\phi^{-1}(\Lambda)}\|\} < 1$. Consequently, due to Lemma 5.2 and Proposition 5.1, we have that Theorem 4.1 and Corollary 4.5 apply e.g. to thinned versions of the afore mentioned point processes.

Clearly, Theorems 3.1, 4.1 and Corollary 4.5 may be applied also to fermion processes with Hermitian kernel. We conclude this paper providing an application of Theorem 4.1, with some explicit computations, to fermion processes with the Ginibre kernel and fermion processes with the Bessel kernel. In the Ginibre case, we exploit the spectral properties of the projection on the complex balls of the operator $\mathcal{K}$ [similar computations may be implemented e.g. for fermion processes with the Bergman kernel (see [23])]; in the Bessel case, we use a result due to Borodin and Soshnikov [9].

Ginibre kernel Consider the Ginibre kernel

$$K(x, y) = \mathrm{e}^{x\overline{y}}, \quad x, y \in E = \mathbb{C}$$

and define

$$\lambda(\mathrm{d}x) = \frac{1}{\pi}\mathrm{e}^{-|x|^2}\, m(\mathrm{d}x),$$

where m is the Lebesgue measure on $\mathbb{C}$. Note that λ is nothing but that the standard complex Gaussian measure on $\mathbb{C}$. It is known that the integral operator $\mathcal{K}$ on $L^2(\mathbb{C}, \lambda)$ with kernel K satisfies Condition I (see [21]; see also [23]). Therefore there exists a unique fermion point process $\mu^{(K,\lambda)}$ on $\Gamma_{\mathbb{C}}$ with kernel K and reference measure λ. Note that

$$\rho(x) = \frac{1}{\pi}\mathrm{e}^{-|x|^2}.$$

Consider the map $\phi(x) = ax$, where $a \in \mathbb{C} \setminus \{0\}$. Clearly $\phi \in \mathrm{Diff}(\mathbb{C})$ and $\phi^{-1}(x) = x/a$. Set $\Lambda = b(O,r) = \{x : |x| \leq r\}$, $r > 0$, and $\mathcal{K}_r = \mathcal{K}_{b(O,r)}$. We have $\phi^{-1}(\Lambda) = b(O, r/|a|)$ and, by Lemma 3.2 in [27], the non-zero eigenvalues of $\mathcal{K}_r$ are

$$\kappa_{n,r} = \frac{1}{n!}\int_0^{r^2} t^n \mathrm{e}^{-t}\,\mathrm{d}t = \sum_{k\geq n+1} \frac{r^{2k}\mathrm{e}^{-r^2}}{k!}, \quad n = 0, 1, \dots$$

and the corresponding eigenfunctions are

$$\varphi_{n,r}(x) = \mathbb{1}_{b(O,r)}(x)\frac{x^n}{\sqrt{(n!)\kappa_{n,r}}}$$

(note that they form an orthonormal system of $L^2(b(O,r),\lambda)$). In particular, all the eigenvalues of $\mathcal{K}_r$ and $\mathcal{K}_{r/|a|}$ are non-negative and strictly less than 1 and so $\max\{\|\mathcal{K}_r\|, \|\mathcal{K}_{r/a}\|\} < 1$. Consequently, the assumptions of Theorem 4.1 are satisfied and therefore $\phi^*\mu^{(K,\lambda)}$ is absolutely continuous with respect to $\mu^{(K,\lambda)}$ on Γ_Λ. Now, we are going to compute the corresponding density. By Proposition 2.2 in [27] and Theorem 2.6, we deduce $\mathrm{Det}(I - \mathcal{K}_r) = \prod_{n\geq 0}(1 - \kappa_{n,r})$; indeed this infinite product equals the void probability of the Ginibre process $\mu^{(K,\lambda)}$ restricted to $\Gamma_{b(O,r)}$. Consequently,

$$\frac{\mathrm{Det}(I - \mathcal{K}_{\phi^{-1}(\Lambda)})}{\mathrm{Det}(I - \mathcal{K}_\Lambda)} = \frac{\mathrm{Det}(I - \mathcal{K}_{r/|a|})}{\mathrm{Det}(I - \mathcal{K}_r)} = \frac{\prod_{n\geq 0}(1 - \kappa_{n,r/|a|})}{\prod_{n\geq 0}(1 - \kappa_{n,r})}. \tag{6.1}$$

Since $|\mathrm{Jac}(\phi^{-1})(x)| = |a|^{-2}$, we have

$$\begin{aligned}\prod_{x\in\xi} \frac{\rho(\phi^{-1}(x))}{\rho(x)}|\mathrm{Jac}(\phi^{-1})(x)| &= \prod_{x\in\xi} |a|^{-2}\mathrm{e}^{-|x|^2(|a|^{-2}-1)} \\ &= |a|^{-2|\xi|}\mathrm{e}^{-(|a|^{-2}-1)\sum_{x\in\xi}|x|^2}. \end{aligned} \tag{6.2}$$

Plainly, hereafter the symbol $|\cdot|$ denotes both the cardinality of a finite configuration and the modulus of a complex number. Writing $\xi = \{x_1, \dots, x_{|\xi|}\}$, by e.g. Lemma 7 p. 13 in [22], for $1 \leq i, j \leq |\xi|$, we have

$$\begin{aligned} J[\Lambda](x_i, x_j) = J[b(O,r)](x_i, x_j) &= \sum_{n\geq 0} \frac{\kappa_{n,r}}{1 - \kappa_{n,r}}\varphi_{n,r}(x_i)\overline{\varphi_{n,r}(x_j)} \\ &= \sum_{n\geq 0}\frac{1}{1-\kappa_{n,r}}\mathbb{1}_{b(O,r)}(x_i)\frac{x_i^n\overline{x_j}^n}{n!}\mathbb{1}_{b(O,r)}(x_j) \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} J[\phi^{-1}(\Lambda)]^\phi(x_i, x_j) &= J[b(O, r/|a|)](x_i/|a|, x_j/|a|) \\ &= \sum_{n\geq 0} \frac{|a|^{-2n}}{1 - \kappa_{n,r/|a|}}\mathbb{1}_{b(O,r)}(x_i)\frac{x_i^n\overline{x_j}^n}{n!}\mathbb{1}_{b(O,r)}(x_j). \end{aligned} \tag{6.4}$$

Setting

$$c(n,r) = \frac{1}{1-\kappa_{n,r}},\ \mathbf{X}_n(\xi) = \left(\frac{x_i^n\overline{x_j}^n}{n!}\right)_{1\leq i,j\leq|\xi|},\ c(n,r,a) = \frac{|a|^{-2n}}{1-\kappa_{n,r/|a|}},$$

combining (6.1), (6.2), (6.3) and (6.4), we find the density

$$R^{\phi}_{b(O,r)}(\xi) = \frac{\prod_{n\geq 0}(1-\kappa_{n,r/|a|})}{\prod_{n\geq 0}(1-\kappa_{n,r})} \frac{\det(\sum_{n\geq 0} c(n,r,a)\mathbf{X}_n(\xi))}{\det(\sum_{n\geq 0} c(n,r)\mathbf{X}_n(\xi))} \times |a|^{-2|\xi|} \mathrm{e}^{-(|a|^{-2}-1)\sum_{x\in\xi}|x|^2}, \quad \xi \in \Gamma_{b(O,r)}.$$

Bessel kernel Consider the Bessel kernel

$$K(x,y) = \sum_{k,l\geq 0} \frac{(-1)^k x^k}{k!\Gamma(k+1)} \frac{(-1)^l y^l}{l!\Gamma(l+1)} \frac{1}{k+l+1}, \quad x,y \in \mathbb{R}$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is known that the integral operator $\mathcal{K}$ on $L^2(E, \mathrm{d}x)$, $E = (0,\infty)$, with kernel K satisfies Condition I and $\|\mathcal{K}_{(0,s)}\| < 1$ for any $s > 0$ (see e.g. [9]). In particular, there exists a unique fermion point process $\mu^{(K,\lambda)}$ with kernel K and reference measure $\lambda(\mathrm{d}x) = \mathrm{d}x$. Consider the map $\phi(x) = ax$, where $a \in (0,\infty)$. Clearly $\phi \in \mathrm{Diff}((0,\infty))$, $\phi^{-1}(x) = x/a$ and $\phi^{-1}((0,s)) = (0,s/a)$, for any $s > 0$. Consequently, by Theorem 4.1 we have that $\phi^*\mu^{(K,\lambda)}$ is absolutely continuous with respect to $\mu^{(K,\lambda)}$ on $\Gamma_{(0,s)}$, for any $s > 0$. Now, we are going to compute the corresponding density. By Corollary 1 in [9] and the comment after its proof, we have that the probability that the first point of $\mu^{(K,\lambda)}$ is bigger than or equal to $s > 0$ is equal to e^{-s}. Therefore, by Theorem 2.6 we have $\mathrm{Det}(I - \mathcal{K}_{(0,s)}) = \mathrm{e}^{-s}$ for any $s > 0$, and so

$$\frac{\mathrm{Det}(I-\mathcal{K}_{(0,s/a)})}{\mathrm{Det}(I-\mathcal{K}_{(0,s)})} = \mathrm{e}^{-(a^{-1}-1)s}. \tag{6.5}$$

Since $\rho \equiv 1$ and $|\mathrm{Jac}(\phi^{-1})(x)| = a^{-2}$, we have

$$\prod_{x\in\xi} \frac{\rho(\phi^{-1}(x))}{\rho(x)} |\mathrm{Jac}(\phi^{-1})(x)| = a^{-2|\xi|}. \tag{6.6}$$

By Proposition 1 in [9] we have

$$J[(0,s)](x,y) = K(x-s, y-s), \quad x,y \in (0,s) \tag{6.7}$$

and

$$J[(0,s/a)]^{\phi}(x,y) = K((x-s)/a, (y-s)/a), \quad x,y \in (0,s). \tag{6.8}$$

Finally, combining (6.5), (6.6), (6.7) and (6.8), we find the density

$$R^{\phi}_{(0,s)}(\xi) = \mathrm{e}^{-(a^{-1}-1)s} \frac{\det((K((x_i-s)/a,(x_j-s)/a))_{1\leq i,j\leq|\xi|})}{\det((K(x_i-s,x_j-s))_{1\leq i,j\leq|\xi|})} \times a^{-2|\xi|}, \quad s>0,\, \xi \in \Gamma_{(0,s)} \tag{6.9}$$

where $\xi = \{x_1, \ldots, x_{|\xi|}\}$.

Acknowledgment. The author thanks an anonymous referee for pointing out a mistake in a previous version of the paper.

References

1. Alastuey, A. and Forrester, P. J.: Correlations in two-components log-gas systems, *J. Statist. Phys.* **81** (1995), 579–627.
2. Blank, J., Exner, P., and Havlicek, M.: *Hilbert Space Operators in Quantum Physics*, AIP Press, New York, 1994.
3. Bogachev, V. I: *Differentiable measures and the Malliavin calculus*, American Mathematical Society, Providence, 2010.
4. Borodin, A., Okounkov, A., and Olshanski, G.: Asymptotics of Plancherel measures for symmetric groups, *J. Amer. Math. Soc.* **13** (2000), 481–515.
5. Borodin, A. and Olshanski, G.: Point processes and the infinite symmetric group, *Math. Res. Let.* **5** (1998), 799–816.
6. Borodin, A. and Olshanski, G.: Distributions on partitions, point processes, and the hypergeometric kernel, *Comm. Math. Phys.* **211** (2000), 335–358.
7. Borodin, A. and Olshanski, G.: Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, *Ann. of Math.* **161** (2005), 1319–1422.
8. Borodin, A. and Olshanski, G.: Random partitions and the gamma kernel, *Adv. Math.* **194** (2005), 141–202.
9. Borodin, A. and Soshnikov, A.: Janossy densities. I. Determinantal ensembles, *J. Statist. Phys.* **113** (2003), 595–610.
10. Camilier, I. and Decreusefond. L.: Quasi-invariance and integration by parts for determinantal and permanental processes, *J. Funct. Anal.* **259** (2010), 268–300.
11. Cornu, F. and Jancovici, B.: On the two-dimensional Coulomb gas, *J. Statist. Phys.* **49** (1987), 33–56.
12. Cornu, F. and Jancovici, B.: The electrical double layer: A solvable model, *J. Chem. Phys.* **90** (1989), 24–44.
13. Daley, D.J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. I*, Springer, New York, 2003.
14. Daley, D.J. and Vere-Jones, D.: *An Introduction to the Theory of Point Processes. Vol. II*, Springer, New York, 2008.
15. do Carmo, M. P.: *Riemannian geometry*, Birkhäuser, Boston, 1992.
16. Forrester, P. J.: Positive and negative charged lods alternating along the line: Exact results, *J. Statist. Phys.* **45** (1986), 153–169.
17. Forrester, P. J.: Solvable isotherms for a two-component system of charged rods on a line, *J. Statist. Phys.* **51** (1988), 457–479.
18. Forrester, P. J.: Exact results for correlations in a two-component log-gas, *J. Statist. Phys.* **54** (1989), 57–79.
19. Gaudin, M.: L'isotherme critique d'un plasma sur réseau ($\beta = 2$, $d = 2$, $n = 2$), *J. Physique* **46** (1985), 1027–1042.
20. Georgii, H. O. and Yoo, H. J.: Conditional intensity and Gibbsianness of determinantal point processes, *J. Stat. Phys.* **118** (2005), 55–84.
21. Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices, *J. Mathematical Phys.* **6** (1965), 440–449.
22. Goldman, A.: The Palm measure and the Voronoi tessellation for the Ginibre process, *The Annals of Applied Probability* **20** (2010), 90–128.
23. Hough, J. B., Krishnapur, M., Peres, Y. and Virág, B.: *Zeros of Gaussian analytic functions and determinantal point processes*, American Mathematical Society, Providence, 2009.
24. Lytvynov, E.: Determinantal point processes with J-Hermitian correlation kernels, *Ann. Probab.* **41** (2013), 2513–2543.
25. Macchi, O.: The coincidence approach to stochastic point processes, *Advances in Appl. Probability* **7** (1975), 83–122.
26. Olshanski, G.: Point processes and the infinite symmetric group. Part V: Analysis of the matrix Whittaker kernel, *ArXiv Preprint* (1998).
27. Shirai, T.: Large deviations for the Fermion point process associated with the exponential kernel, *J. Stat. Phys.* **123** (2006), 615–629.

28. Shirai, T. and Takahashi, Y.: Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes, *J. Funct. Anal.* **205** (2003), 414–463.
29. Soshnikov, A.: Determinantal random point fields, *Russian Mat. Surveys* **55** (2000), 923–975.

Giovanni Luca Torrisi: Istituto per le Applicazioni del Calcolo “Mauro Picone”, Consiglio Nazionale delle Ricerche (CNR), Via dei Taurini 19, I-00185, Rome, Italy.
E-mail address: g.torrisi@iac.cnr.it
URL: http://www.iac.rm.cnr.it/~torrisi/