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PORTFOLIO OPTIMIZATION UNDER PARTIAL INFORMATION WITH EXPERT OPINIONS: A DYNAMIC PROGRAMMING APPROACH*

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Abstract. This paper investigates optimal portfolio strategies in a market where the drift is driven by an unobserved Markov chain. Information on the state of this chain is obtained from stock prices and expert opinions in the form of signals at random discrete time points. As in Frey et al. (2012), Int. J. Theor. Appl. Finance, 15, No. 1, we use stochastic filtering to transform the original problem into an optimization problem under full information where the state variable is the filter for the Markov chain. The dynamic programming equation for this problem is studied with viscosity-solution techniques and with regularization arguments.

1. Introduction

It is well-known that optimal investment strategies in dynamic portfolio optimization depend crucially on the drift of the underlying asset price process. On the other hand it is notoriously difficult to estimate drift parameters from historical asset price data. Hence it is natural to include expert opinions or investors' views as additional source of information in the computation of optimal portfolios. In the context of the classical one-period Markowitz model this leads to the well-known Black-Littermann approach, where Bayesian updating is used to improve return predictions (see Black & Litterman [1]).

Frey et al. [7] consider expert opinions in the context of a dynamic portfolio optimization problem in continuous time. In their paper the asset price process is modelled as diffusion whose drift is driven by a hidden finite-state Markov chain Y. Investors observe the stock prices and in addition a marked point process with jump-size distribution depending on the current state of Y that represents expert opinions. Frey et al. [7] derive a finite-dimensional filter p_t with jump-diffusion dynamics for the state of Y and they reduce the portfolio optimization problem to a problem under complete information with state variable given by the filter p_t. Moreover they write down the dynamic programming equation for the value function V of that problem and, assuming that the dynamic programming

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* This version of the paper grew out of an earlier, unpublished version that was authored only by Frey and Wunderlich; see [8].

equation admits a classical solution, they compute a candidate solution for the optimal strategy. The precise mathematical meaning of these preliminary results is however left open.

This issue is addressed in the present paper. A major challenge in the analysis of the dynamic programming equation is the fact that the equation is not strictly elliptic if the number of states of Y is larger than the number of assets. In fact, due to this non-ellipticity it is not possible to apply any of the known results on the existence of classical solutions to this equation. We study two ways to address this problem. First, following the analysis of Pham [13] we show that the value function is a viscosity solution of the associated dynamic programming equation. Since the comparison principle for viscosity solutions applies to our model, this yields an elegant characterization of the value function. However, the viscosity-solution methodology does not provide any information on the form of (nearly) optimal strategies.

For this reason we study a second approach based on regularization arguments. Here an additional noise term of the form $m^{-\frac{1}{2}} d\widetilde{B}_t$, $\widetilde{B}$ an independent Brownian motion of suitable dimension and $m \in \mathbb{N}$ large, is added to the dynamics of the state process p. The dynamic programming equation associated with the regularized optimization problem is strictly elliptical so that recent results of Davis & Lleo [4] imply the existence of a classical solution V^m. Moreover, the optimal strategy for the regularized problem can be characterized as solution of a quadratic optimization problem that involves V^m and its first derivatives. We show that for $m \to \infty$ reward- and value function for the regularized problem and the original problem converge uniformly for all admissible strategies. This uniform convergence implies that for m sufficiently large the optimal strategy for the regularized problem is a nearly-optimal strategy in the original problem, so that we have solved the problem of finding good strategies. In order to carry out this program we need an explicit representation of jump-diffusion processes as a solution of an SDE driven by Brownian motion and - this is the new part - some *exogenous* Poisson random measure; we refer the reader to Section 5 below for details.

The related literature on portfolio optimization under partial information is discussed in detail in the companion paper [7]. Here we just mention the papers Rieder & Bäeuerle [14] and Sass & Haussmann[16] that are concerned with portfolio optimization in models with Markov-modulated drift but without any extra information.

The paper is organized as follows. In Section 2 we introduce the model of the financial market and formulate the portfolio optimization problem. For this problem we derive in Section 3 the dynamic programming equation in the case of power utility. In Section 4 we reformulate the state equation in terms of an exogenous Poisson random measure. For this reformulated state equation we provide in Section 5 an explicit construction of the jump coefficient. The main results of this paper are presented in Sections 6 and 7. Here we show that the value function is a viscosity solution of the dynamic programming equation. Moreover, we study a regularized version of the dynamic programming equation and investigate nearly optimal strategies.

2. Model and Optimization Problem

The setting is based on [7]. For a fixed date $T > 0$ representing the investment horizon, we work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$, with filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ satisfying the usual conditions. All processes are assumed to be $\mathbb{G}$-adapted. For a generic $\mathbb{G}$-adapted process H we denote by $\mathbb{G}^H$ the filtration generated by H.

2.1. Price dynamics. We consider a market model for one risk-free bond with price $S_t^0 = 1$ and n risky securities with prices $S_t = (S_t^1, \dots, S_t^n)^\top$ given by

$$dS_t^i = S_t^i \left(\mu^i(Y_t)dt + \sum_{j=1}^n \sigma^{ij} dW_t^j\right), \quad S_0^i = s^i, \quad i = 1, \cdots, n. \tag{2.1}$$

Here $\mu = \mu(Y_t) \in \mathbb{R}^n$ denotes the mean stock return or drift which is driven by some factor process Y described below. The volatility $\sigma = (\sigma^{ij})_{1\le i,j\le n}$ is assumed to be a constant invertible matrix and $W_t = (W_t^1, \cdots, W_t^n)$ is an n-dimensional $\mathbb{G}$-adapted Brownian motion. The invertibility of σ always can be ensured by a suitable parametrization if the covariance matrix $\sigma\sigma^\top$ is positive definite. The factor process Y is a finite-state Markov chain independent of the Brownian motion W with state space $\{e_1, \dots, e_d\}$ where e_i is the ith unit vector in $\mathbb{R}^d$. The generator matrix is denoted by Q and the initial distribution by $\widetilde{p} = (\widetilde{p}^1, \dots, \widetilde{p}^d)^\top$. The states of the factor process Y are mapped onto the states $\mu_1, \dots, \mu_d$ of the drift by the function $\mu(Y_t) = MY_t$, where $M_{lk} = \mu_k^l = \mu^l(e_k)$, $1 \le l \le n$, $1 \le k \le d$.

Define the return process R associated with the price process S by $dR_t^i = dS_t^i/S_t^i$, $i = 1, \dots, n$. Note that R satisfies $dR_t = \mu(Y_t)dt + \sigma dW_t$, and it is easily seen that $\mathbb{G}^R = \mathbb{G}^{\log S} = \mathbb{G}^S$. This is useful, since it allows us to work with R instead of S in the filtering part. For details we refer to [7].

2.2. Investor Information. We assume that the investor does not observe the factor process Y directly; he does however know the model parameters, in particular the initial distribution $\widetilde{p}$, the generator matrix Q and the functions $\mu^i(\cdot)$. Moreover, he has noisy observations of the hidden process Y at his disposal. More precisely we assume that the investor observes the return process R and that he receives at discrete points in time T_n noisy signals about the current state of Y. These signals are to be interpreted as expert opinions; specific examples can be found in the companion paper [7].

We model expert opinions by a marked point process $I = (T_n, Z_n)$, so that at T_n the investor observes the realization of a random variables Z_n whose distribution depends on the current state Y_{T_n} of the factor process. The T_n are modeled as jump times of a standard Poisson process with intensity λ, independent of Y, so that the timing of the information arrival does not carry any useful information. The signal Z_n takes values in some set $\mathcal{Z} \subset \mathbb{R}^\kappa$, and we assume that given $Y_{T_n} = e_k$, the distribution of Z_n is absolutely continuous with Lebesgue-density $f_k(z)$. We identify the marked point process $I = (T_n, Z_n)$ with the associated counting measure denoted by $I(dt, dz)$. Note that the $\mathbb{G}$-compensator of I is $\lambda dt \sum_{k=1}^d 1_{\{Y_t = e_k\}} f_k(z)dz$.

Summarizing, the information available to the investor is given by the *investor filtration* $\mathbb{F}$ with

$$\mathcal{F}_t = \mathcal{G}_t^R \vee \mathcal{G}_t^I\,, \quad 0 \le t \le T. \tag{2.2}$$

2.3. Portfolio and optimization problem. We describe the selffinancing trading of an investor by the initial capital $x_0 > 0$ and the n-dimensional $\mathbb{F}$-adapted trading strategy h where h_t^i, $i = 1, \dots, n$, represents the proportion of wealth invested in stock i at time t. It is well-known that in this setting the wealth process $X^{(h)}$ has the dynamics

$$\frac{dX_t^{(h)}}{X_t^{(h)}} = \sum_{i=0}^{n} h_t^i \frac{dS_t^i}{S_t^i} \quad = \quad h_t^\top \mu(Y_t)dt + h_t^\top \sigma dW_t, \quad X_0^{(h)} = x_0. \tag{2.3}$$

We assume that for all $t \in [0,T]$ the strategy h_t takes values in some non-empty convex and compact subset K of $\mathbb{R}^n$ that can be described in terms of r linear constraints. In mathematical terms,

$$K = \{h \in \mathbb{R}^n \colon \Psi_l^\top h \le \nu_l,\ 1 \le l \le r,\ \text{ for given } (\Psi_1, \nu_1), \dots, (\Psi_r, \nu_r) \in \mathbb{R}^n \times \mathbb{R}\}\,. \tag{2.4}$$

We assume that there is some $h^0 \in \mathbb{R}^n$ such that $\Psi_l^\top h^0 < \nu_l$ for all $1 \le l \le r$ and that $0 \in K$. The set K models constraints on the portfolio. Moreover, the assumption that $h_t \in K$ for all t facilitates many technical estimates in the paper. For a specific example fix constants $c_1 < 0$, $c_2 > 1$, and let

$$K = \{h \in \mathbb{R}^n \colon h_i \ge c_1 \text{ for all } 1 \le i \le n \text{ and } \sum_{i=1}^{n} h_i \le c_2\}\,.$$

This choice of K hat would correspond to a limit $|c_1|$ on the amount of shortselling and a limit c_2 for leverage.

We denote the class of *admissible trading strategies* by

$$\mathcal{H} = \{h = (h_t)_{t\in[0,T]} \colon\ h \text{ is } \mathbb{F}\text{-adapted and } h_t \in K \text{ for all } t\}\,. \tag{2.5}$$

Since $\mu(Y_t)$ is bounded and since σ is constant, equation (2.3) is well defined for all $h \in \mathcal{H}$.

We assume that the investor wants to maximize the expected utility of terminal wealth for power utility $U(x) = \frac{x^\theta}{\theta}$, $\theta < 1$, $\theta \neq 0$.[1] The optimization problem thus reads as

$$\max\{E(U(X_T^{(h)})) \colon h \in \mathcal{H}\}. \tag{2.6}$$

This is a maximization problem under partial information since we have required that the strategy h is adapted to the investor filtration $\mathbb{F}$.

[1]The case $\theta = 0$ corresponds to logarithmic utility $U(x) = \ln x$ which is treated in [7].

2.4. Partial information and filtering. Next we explain how the control problem (2.6) can be reduced to a control problem with complete information via filtering arguments. We use the following notation: for a generic process H we denote by $\widehat{H}_t = E(H|\mathcal{F}_t)$ its optional projection on the filtration $\mathbb{F}$, and the filter for the Markov chain Y_t is denoted by $p_t = (p_t^1, \cdots, p_t^d)$ with $p_t^k = P(Y_t = e_k|\mathcal{F}_t),\ k = 1\ldots, d$. Note that for a process of the form $H_t = h(Y_t)$ the optional projection is given by $\widehat{h(Y_t)} = \sum_{k=1}^d h(e_k)p_t^k$. In particular, the projection of the drift equals

$$\widehat{\mu(Y_t)} = \sum_{k=1}^d \mu(e_k)p_t^k = Mp_t\,.$$

The following two processes will drive the dynamics of p_t. First, let

$$\widetilde{W}_t \quad := \quad \sigma^{-1}(R_t - \int_0^t Mp_s ds).$$

By standard results from filtering theory $\widetilde{W}$ is an $\mathbb{F}$-Brownian motion (the so-called innovations process). Second, define the predictable random measure

$$\nu_I(dt, dz) = \lambda dt \sum_{k=1}^d p_{t-}^k f_k(z)dz.$$

By standard results on point processes ν_I is the $\mathbb{F}$-compensator of I, see for instance Bremaud [2]. The compensated random measure will be denoted by $\widetilde{I}(dt, dz) := I(dt, dz) - \nu_I(dt, dz)$.

Using a combination of the HMM filter (see e.g. Wonham [17], Elliott et al. [5], Liptser & Shiryaev [11]) and Bayesian updating, in [7] the following d-dimensional SDE system for the dynamics of the filter p is derived

$$dp_t = Q^\top p_t dt + \beta^\top(p_t)d\widetilde{W}_t + \int_{\mathcal{Z}} \gamma_I(p_{t-}, z)\widetilde{I}(dt, dz) \tag{2.7}$$

with initial condition $p_0^k = \widetilde{p}^k$. Here, the matrix $\beta = \beta(p) = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^{n\times d}$ and the vector $\gamma_I = \gamma_I(p, z) = (\gamma_I^1, \ldots, \gamma_I^d)^\top \in \mathbb{R}^d$ are defined by

$$\beta_k(p) = p^k\Big(\sigma^{-1}\big(\mu_k - \sum_{j=1}^d p^j\mu_j\big)\Big) = p^k\sigma^{-1}M(e_k - p) \in \mathbb{R}^n$$

$$\text{and} \quad \gamma_I^k(p, z) = p^k\left(\frac{f_k(z)}{\overline{f}(z, p)} - 1\right),\ 1 \le k \le d, \quad \text{with} \quad \overline{f}(z, p) = \sum_{k=1}^d p^k f_k(z). \tag{2.8}$$

It is well-known (see e.g. Lakner [10], Sass & Haussmann [16]) that the $\mathbb{F}$-semimartingale decomposition of X is given by

$$\frac{dX_t^{(h)}}{X_t^{(h)}} \quad = \quad h_t^\top\, Mp_t\, dt + h_t^\top \sigma d\widetilde{W}_t. \tag{2.9}$$

Now note that for a constant strategy $h_t \equiv h \in K$ the $(d+1)$-dimensional process $(X^{(h)}, p)$ is an $\mathbb{F}$-Markov process as is immediate from the dynamics in (2.7) and (2.9). Hence the optimization problem (2.6) can be considered as a control problem

under complete information with the $(d+1)$-dimensional state variable process $(X^{(h)}, p)$. This control problem is studied in the remainder of the paper.

3. Dynamic Programming Equation for the Case of Power Utility

3.1. A simplified optimization problem. As a first step, we simplify the control problem by a change of measure. As shown in Nagai & Runggaldier [12] this measure change leads to a new problem where the set of state variables is reduced to p and where the dynamic programming equation takes on a simpler form. First we compute for an admissible strategy $h \in \mathcal{H}$ the utility of terminal wealth $U(X_T^{(h)}) = \frac{1}{\theta}(X_T^{(h)})^\theta$. From (2.9) it follows that

$$\frac{1}{\theta}(X_T^{(h)})^\theta = \frac{x_0^\theta}{\theta} \exp\Big\{\theta \int_0^T \Big(h_s^\top M p_s - \frac{1}{2}\left|\sigma^\top h_s\right|^2\Big)ds + \theta \int_0^T h_s^\top \sigma d\widetilde{W}_s\Big\}, \quad (3.1)$$

where $|.|$ denotes the Euclidean norm. Define now the random variable $L_T^{(h)} = \exp\big\{\int_0^T \theta h_s^\top \sigma d\widetilde{W}_s - \frac{1}{2}\int_0^T \left|\theta\sigma^\top h_s\right|^2 ds\big\}$ and the function

$$b(p,h;\theta) = -\theta\Big(h^\top M p - \frac{1-\theta}{2}\left|\sigma^\top h\right|^2\Big). \quad (3.2)$$

With this notation (3.1) can be written in the form

$$\frac{1}{\theta}(X_T^{(h)})^\theta = \frac{x_0^\theta}{\theta} L_T^{(h)} \exp\Big\{\int_0^T -b(p_s, h_s;\theta)ds\Big\}. \quad (3.3)$$

Since σ is deterministic and since h is bounded, the Novikov condition implies that $E(L_T^{(h)}) = 1$. Hence we can define an equivalent measure P^h on $\mathcal{F}_T$ by $dP^h/dP = L_T^{(h)}$, and Girsanov's theorem guarantees that $B_t := \widetilde{W}_t - \theta\int_0^t \sigma^\top h_s ds$ is a standard $\mathbb{F}$-Brownian motion. Substituting into (2.7) we find the following dynamics for the filter under P^h

$$dp_t = \alpha(p_t, h_t)dt + \beta^\top(p_t)dB_t + \int_{\mathcal{Z}} \gamma_I(p_{t-}, z)\widetilde{I}(dt, dz) \quad (3.4)$$

$$\text{where} \quad \alpha = \alpha(p,h) = Q^\top p + \theta\beta^\top(p)\sigma^\top h. \quad (3.5)$$

In view of these transformations, for $0 < \theta < 1$ the optimization problem (2.6) is equivalent to

$$\max\Big\{E\Big(\exp\Big\{\int_0^T -b(p_s^{(0,\widetilde{p},h)}, h_s;\theta)ds\Big\}\Big) : h \in \mathcal{H}\Big\} \quad (3.6)$$

where we denote by $p_s^{(t,p,h)}$ the solution of (3.4) for $s \in [t,T]$ starting at time $t \in [0,T]$ with initial value $p \in \mathcal{S}$ for strategy $h \in \mathcal{H}$. For $\theta < 0$ on the other hand (2.6) is equivalent to minimizing the expectation in (3.6). In the sequel we will concentrate on the case $0 < \theta < 1$; the necessary changes for $\theta < 0$ will be indicated where appropriate. Moreover, θ will be largely removed from the

notation. The reward and value function for this control problem are given by

$$\begin{aligned} v(t,p,h) &= E\Big(\exp\Big\{\int_t^T -b(p_s^{(t,p,h)},h_s)ds\Big\}\Big) \quad \text{for } h\in\mathcal{H}, \\ V(t,p) &= \sup\{v(t,p,h)\colon h\in\mathcal{H}\}. \end{aligned} \tag{3.7}$$

Note that $v(T,p,h)=V(T,p)=1$.

3.2. The dynamic programming equation. Next, we derive the form of the dynamic programming equation for $V(t,p)$. We begin with the generator of the *state process* p_t (the solution of the SDE (3.4)) for a constant strategy $h_t\equiv h$. Denote by $\mathcal{S}=\{p\in\mathbb{R}^d : \sum_{i=1}^d p^i=1, p^i\geq 0, i=1,\dots,d\}$ the unit simplex in $\mathbb{R}^d$. Standard arguments show that the solution of this SDE is a Markov process whose generator $\mathcal{L}^h$ operates on $g\in\mathcal{C}^2(\mathcal{S})$ as follows

$$\begin{aligned} \mathcal{L}^h g(p) = \frac{1}{2}\sum_{i,j=1}^d \beta_i^\top(p)\beta_j(p) g_{p^ip^j} &+ \sum_{i=1}^d \alpha^i(p,h) g_{p^i} \\ &+ \lambda\int_{\mathcal{Z}}\{g(p+\gamma_I(p,z))-g(p)\}\overline{f}(z,p)dz. \end{aligned} \tag{3.8}$$

By standard arguments the dynamic programming equation associated to this optimization problem is

$$V_t(t,p)+\sup_{h\in K}\Big\{\mathcal{L}^hV(t,p)-b(p,h;\theta)V(t,p)\Big\}=0, \quad (t,p)\in[0,T)\times\mathcal{S}, \tag{3.9}$$

with terminal condition $V(T,p)=1$. In case that $\theta<0$ the equation is similar, but the sup is replaced by an inf. Plugging in $\mathcal{L}^h$ as given in (3.8) and $b(p,h)$ as given in (3.2) into (3.9) the dynamic programming equation can be written more explicitly as

$$\begin{aligned} 0 = V_t(t,p) &+ \frac{1}{2}\sum_{k,l=1}^d \beta_k^\top(p_t)\beta_l(p_t)V_{p^kp^l}(t,p) + \sum_{k=1}^d\Big\{\sum_{l=1}^d Q^{lk}p^l\Big\}V_{p^k}(t,p) \\ &+ \lambda\int_{\mathcal{Z}}\{V(t,p+\gamma_I(p,z))-V(t,p)\}\overline{f}(z,p)dz \\ &+ \sup_{h\in K}\Big\{\sum_{k=1}^d \beta_k^\top(p_t)\sigma^\top\theta h V_{p^k}(t,p) + \theta V(t,p)\Big(h^\top Mp - \frac{1}{2}\left|\sigma^\top h\right|^2(1-\theta)\Big)\Big\}. \end{aligned} \tag{3.10}$$

Suppose for the moment that a classical solution to (3.10) exists. The argument of the supremum in the last line of (3.10) is quadratic in h and strictly concave (as $\sigma\sigma^\top$ is positive definite). Hence this function attains a unique maximum h^* on the convex set K. Moreover, as shown in Davis and Lleo [4], Proposition 3.6, h^* can be chosen as a measurable function of t and p. Hence there exists a solution p^* of the SDE (3.4) with $h_t=h^*(t,p_t^*)$; this can be verified by a similar application of Girsanov's theorem as in the derivation of the equation (3.4). Then standard verification arguments along the lines of Theorem 3.1 of Fleming & Soner [6] or Theorem 5.5 of Davis and Lleo[4] immediately give that V is the value function of the control problem (3.6) and that $h_t^*:=h^*(t,p_t^*)$ is the optimal strategy.

Remark 3.1. If for some (t,p) $h^*(t,p)$ is inner point of K, an explicit formula for $h^*(t,p)$ can be given. In that case $h^*(t,p)$ is given by the solution h^* of the following linear equation (the first-order condition for the unconstrained problem)

$$\sigma \sum_{k=1}^{d} \beta_k(p) V_{p^k}(t,p) + V(t,p)\Big(Mp - \sigma\sigma^\top h(1-\theta)\Big) = 0.$$

Since σ is an invertible matrix h^* equals

$$h^* = h^*(t,p) = \frac{1}{(1-\theta)}(\sigma\sigma^\top)^{-1}\Big\{Mp + \frac{1}{V(t,p)}\sigma \sum_{k=1}^{d} \beta_k(p) V_{p^k}(t,p)\Big\}.$$

However, the existence of a classical solution of equation (3.10) is an open issue. The main problem is the fact that one cannot guarantee that the equation is *uniformly elliptic*. To see this note that the coefficient matrix of the second derivatives in (3.10) is given by $C(p) = \beta^\top(p)\beta(p)$. By definition equation (3.10) is uniformly elliptic if the matrix $C(p)$ is strictly positive definite uniformly in p. A necessary condition for this is that there are no non-trivial solutions of the linear equation $\beta x = 0$ so that we need to have the inequality $n \geq d$ (at least as many assets as states of the Markov chain Y). Such an assumption is hard to justify economically; imposing it nonetheless out of mathematical necessity would severely limit the applicability of our approach.

In the present paper we therefore study two alternative routes to giving a precise mathematical meaning to the dynamic programming equation (3.10). First, following the analysis of Pham [13], in Section 6 we show that the value function is a viscosity solution of the associated dynamic programming equation. Since the comparison principle for viscosity solutions applies in our case, this provides an elegant characterization of the value function. However, the viscosity-solution methodology does not provide any information on the form of the optimal strategies. For this reason, in Section 7 we use regularization arguments to find approximately optimal strategies. More precisely, we add a term $\frac{1}{\sqrt{m}} d\widetilde{B}_t$, with $m \in \mathbb{N}$ and $\widetilde{B}$ a Brownian motion of suitable dimension and independent of B, to the dynamics of the state equation (3.4). The dynamic programming equation associated with these regularized dynamics has an additional term $\frac{1}{2m}\Delta V$, Δ the Laplace operator, and is therefore uniformly elliptic. Hence the results of Davis & Lleo [4] apply directly to the modified equation, yielding the existence of a classical solution V^m. Moreover, the optimal strategy ${}^m h^*$ of the regularized problem is given by the argument of the supremum in the last line of (3.10) with V^m instead of V. We then derive convergence results for the reward- and the value function of the regularized problem as $m \to \infty$. In particular, we show in Theorem 7.5 that for m sufficiently large ${}^m h^*$ is approximately optimal in the original problem.

4. Reformulation of the State Equation

To carry out the program described above we have to reformulate the state equation for a number of reasons. First, in our model the state variable process p (the solution of (3.4)) takes values in the simplex $\mathcal{S}$ which is a subset of a $d-1$-dimensional hyperplane of $\mathbb{R}^d$. If we introduce the announced regularization

to the diffusion part of the state equation then the state variable will leave this hyperplane and takes values in the whole $\mathbb{R}^d$ so that the normalization property of p is violated, which creates technical difficulties. Second, in our analysis we need to apply results from the literature on the theory of dynamic programming of controlled jump diffusions, such as Pham [13] and Davis & Lleo [4]. These papers consider models where the jump part of the state variable is driven by an *exogenous* Poisson random measure, and this structure is in fact essential for many arguments in these papers. In our model, on the other hand, the measure $\widetilde{I}$ is not an exogenous Poisson random measure since the law of the compensator ν_I depends on the solution π_t. Hence we need to reformulate the dynamics of the state variable process in terms of an exogenous Poisson random measure.

4.1. Restriction to a $d-1$-dimensional state. We rewrite the state equation in terms of the 'restricted' $(d-1)$-dimensional process $\pi = (\pi^1, \ldots, \pi^{d-1})^\top = (p^1, \ldots, p^{d-1})^\top$. Then the original state p can be recovered from π by using the normalization property for the last component p^d and we define $p = R\pi := \left(\pi_1, \ldots, \pi_{d-1}, 1 - \sum_{i=1}^{d-1} \pi^i\right)^\top$. Assuming $p \in \mathcal{S}$ implies that the restricted state process takes values in

$$\underline{\mathcal{S}} = \Big\{\pi \in \mathbb{R}^{d-1} : \sum_{i=1}^{d-1} \pi^i \leq 1, \pi^i \geq 0,\ i = 1, \ldots, d-1\Big\}.$$

Now the state equation for $\pi \in \underline{\mathcal{S}}$ associated to (3.4) reads as

$$d\pi_t \quad = \quad \underline{\alpha}(\pi_t, h_t)dt + \underline{\beta}^\top(\pi_t)dB_t + \int_{\mathcal{Z}} \underline{\gamma}_I(\pi_{t-}, z)\widetilde{I}(dt, dz) \tag{4.1}$$

where the coefficients are given by

$$\underline{\alpha}(\pi, h) = (\alpha^1(R\pi, h), \ldots, \alpha^{d-1}(R\pi, h))^\top \in \mathbb{R}^{d-1} \tag{4.2}$$

$$\underline{\beta}(\pi) = (\beta_1(R\pi), \ldots, \beta_{d-1}(R\pi)) \in \mathbb{R}^{n \times d-1} \tag{4.3}$$

$$\underline{\gamma}_I(\pi, z) = (\gamma_I^1(R\pi, z), \ldots, \gamma_I^{d-1}(R\pi, z))^\top \in \mathbb{R}^{d-1}. \tag{4.4}$$

It is straightforward to give an explicit expression for $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}_I$, but such an expression is not needed in the sequel. The original state can be recovered from π by setting $p = R\pi$.

4.2. Exogenous Poisson random measure. In the remainder of the paper we assume that the state process solves the following SDE

$$d\pi_t \quad = \quad \underline{\alpha}(\pi_t, h_t)dt + \underline{\beta}^\top(\pi_t)dB_t + \int_{\mathcal{U}} \underline{\gamma}(\pi_{t-}, u)\widetilde{N}(dt, du), \tag{4.5}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are defined above, $\underline{\gamma} : \underline{\mathcal{S}} \times \mathcal{U} \to \mathbb{R}^{d-1}$, and $\widetilde{N}$ is the compensated measure to some finite activity Poisson random measure N with jumps in a set $\mathcal{U} \subset \mathbb{R}^\kappa$. The compensator of N is denoted by $\nu(du)\lambda dt$, i.e. we have $\widetilde{N}(dt, du) = N(dt, du) - \nu(dz)\lambda dt$. In the next section we show that for a proper choice of $\underline{\gamma}(\cdot)$ and $\widetilde{N}(dt, du)$ the solution of (4.5) has the same law as the original state process from (4.1).

In order to ensure that SDE (4.5) has for each control $h \in \mathcal{H}$ a unique strong solution and for the proof of some of the estimates in Section 7 the coefficients $\underline{\alpha}, \underline{\beta}$ and $\underline{\gamma}$ have to satisfy certain Lipschitz and growth conditions (see [9] and [13]). These conditions are given below. For technical reasons we require that the conditions hold not only for $\pi \in \underline{\mathcal{S}}$ but also for a slightly larger set $\underline{\mathcal{S}}_\varepsilon \supset \underline{\mathcal{S}}$ defined for sufficiently small $\varepsilon \geq 0$ by

$$\underline{\mathcal{S}}_\varepsilon := \{\pi \in \mathbb{R}^{d-1} : \operatorname{dist}(\pi, \underline{\mathcal{S}}) \leq \varepsilon\},$$

where we denoted the distance of $\pi \in \mathbb{R}^{d-1}$ to $\underline{\mathcal{S}}$ by $\operatorname{dist}(\pi, \underline{\mathcal{S}}) := \inf\{|\pi - \pi_0|_\infty \colon \pi_0 \in \underline{\mathcal{S}}\}$, for $|\pi|_\infty$ the maximum norm on $\mathbb{R}^{d-1}$.

Assumption 4.1 (Lipschitz and growth conditions)**.** There exist constants $C_L, \bar{\varepsilon} > 0$ and a function $\rho : \mathcal{U} \to \mathbb{R}_+$ with $\int_{\mathcal{U}} \rho^2(u)\nu(du) < \infty$ such that for all $\pi_1, \pi_2 \in \mathcal{S}_\varepsilon$, $\varepsilon < \bar{\varepsilon}$ and $k = 1, \ldots, d$

$$\begin{aligned}
\sup_{h \in K} |\underline{\alpha}(\pi_1, h) - \underline{\alpha}(\pi_2, h)| + \left|\underline{\beta}_k(\pi_1) - \underline{\beta}_k(\pi_2)\right| &\leq C_L \left|\pi_1 - \pi_2\right|, && (4.6)\\
|\underline{\alpha}(\pi, h)| + \left|\underline{\beta}_k(\pi)\right| &\leq C_L(1 + |\pi|), && (4.7)\\
\left|\underline{\gamma}(\pi_1, u) - \underline{\gamma}(\pi_2, u)\right| &\leq \rho(u)\left|\pi_1 - \pi_2\right|, && (4.8)\\
\left|\underline{\gamma}(\pi, u)\right| &\leq \rho(u)(1 + |\pi|). && (4.9)
\end{aligned}$$

In our case the coefficients $\underline{\alpha}$ and $\underline{\beta}$ are continuously differentiable functions of π on the compact set $\underline{\mathcal{S}}_\varepsilon$ and $h \in K$ is bounded. Hence, the Lipschitz and growth condition (4.6) and (4.7) are fulfilled. Specific conditions on the densities $f_k(\cdot)$ that guarantee (4.8) and (4.9) are given in the next section.

For the optimization problem (3.6) we can give an equivalent formulation in terms of the restricted state variable π with dynamics given in (4.5), that is the equation driven by an exogenous Poisson random measure. For this it is convenient to denote for a given strategy $h \in \mathcal{H}$ the solution of the SDE (4.5) starting at time $t \leq T$ in the state $\pi \in \underline{\mathcal{S}}$ by $\pi^{(t,\pi,h)}$. This control problem reads as

$$\max\Big\{E\Big(\exp\Big\{\int_0^T -b(R\pi_s^{(0,\tilde{\pi},h)}, h_s; \theta)ds\Big\}\Big) : h \in \mathcal{H}\Big\}. \tag{4.10}$$

The associated reward and value function for $(t, \pi) \in [0, T] \times \underline{\mathcal{S}}$ are

$$\begin{aligned}
v(t, \pi, h) &= E\Big(\exp\Big\{\int_t^T -b(R\pi_s^{(t,\pi,h)}, h_s)\, ds\Big\}\Big) \quad \text{for } h \in \mathcal{H},\\
V(t, \pi) &= \sup\{v(t, \pi, h) \colon h \in \mathcal{H}\}.
\end{aligned} \tag{4.11}$$

The generator associated to the solution of the state equation (4.5) reads as

$$\begin{aligned}
\mathcal{L}^h g(\pi) = \frac{1}{2}\sum_{i,j=1}^{d-1} \underline{\beta}_i^\top(\pi)\underline{\beta}_j(\pi) g_{\pi^i\pi^j} + \sum_{i=1}^{d-1} \underline{\alpha}^i(\pi, h) g_{\pi^i} \qquad (4.12)\\
+ \int_{\mathcal{U}} \{g(\pi + \underline{\gamma}(\pi, u)) - g(\pi)\}\nu(du)
\end{aligned}$$

and the associated dynamic programming equation is

$$V_t(t,\pi) + \sup_{h\in K}\Big\{\mathcal{L}^h V(t,\pi) - b(R\pi,h;\theta)V(t,\pi)\Big\} = 0,\ (t,\pi)\in[0,T)\times\underline{\mathcal{S}}. \quad (4.13)$$

5. State Equation with Exogenous Poisson Random Measure

In this section we show how a solution of the state equation (4.1) can be constructed by means of an SDE of the form (4.5) that is driven by an exogenous Poisson random measure. The main tool for constructing $\underline{\gamma}$ will be the so-called inverse Rosenblatt or distributional transform, see Rüschendorf [15], which is an extension of the quantile transformation to the multivariate case.

We impose the following regularity conditions on the functions $f_k(\cdot)$ that represent the conditional densities of Z_n given $Y_{T_n} = e_k$.

Assumption 5.1. All densities $f_k(z)$, $1 \le k \le d$, are continuously differentiable and have the common support $\mathcal{Z}$. We assume that $\mathcal{Z}$ is a κ-dimensional rectangle $[a,b]\subset\mathbb{R}^\kappa$, i.e.

$$\mathcal{Z} = \{z\in\mathbb{R}^\kappa : -\infty < a_k \le z_k \le b_k < \infty,\ k=1,\dots,\kappa\}.$$

Moreover, there is some $0 < C_1$ such that $f_k(z) > C_1$ for all $z\in\mathcal{Z}, k = 1,\dots,d$.

Remark 5.2. Examples for densities that satisfy the above assumption are easily constructed. Start with C^1-densities $\widetilde{f}_k$ and choose some (large) rectangle $\mathcal{Z}$ then

$$f_k(z) = (1-\varepsilon)\frac{\widetilde{f}_k(z)}{\int_{\mathcal{Z}}\widetilde{f}_k(u)du}1_{\mathcal{Z}}(z) + \varepsilon\frac{1}{|\mathcal{Z}|}1_{\mathcal{Z}}(z) \quad \text{where} \quad |\mathcal{Z}| = \prod_{i=1}^{\kappa}(b_i-a_i) \quad (5.1)$$

$k = 1,\dots,\kappa,\ \varepsilon\in(0,1]$, satisfy the requirements of Assumption 5.1. Intuitively, (5.1) corresponds to a mixture of the original density $\widetilde{f}_k$ and the uniform distribution. The latter distribution carries no information so there is uniform a lower bound on the information carried by a single expert opinion.

5.1. Inverse Rosenblatt Transform. In order to write our state equation (4.1) in the form (4.5) with exogenous Poisson random measure we apply the inverse Rosenblatt transform, for instance see [15]. Denote by $\mathcal{U} = [0,1]^\kappa$ the unit cube in $\mathbb{R}^\kappa$. In our context the inverse Rosenblatt transform is a mapping $G:\mathcal{U}\to\mathcal{Z}$ such that for a uniform random variable U on $[0,1]^\kappa$ the random variable $Z = G(U)$ has density $\overline{f}(z,p) = \sum_{j=1}^d p^j f_j(z),\ p = R\pi$; the mapping G can thus be viewed as a generalization of the well-known quantile transform.

Now we explain the construction of the transformation G in detail. First, we define for $k = 1,\dots,\kappa-1$, $p = R\pi$, the marginal densities

$$f_{Z_1\dots Z_k}(z_1,\dots,z_k,p) = \int_{a_{k+1}}^{b_{k+1}}\cdots\int_{a_\kappa}^{b_\kappa}\overline{f}(z_1,\dots,z_k,s_{k+1},\dots,s_\kappa,p)ds_\kappa\dots ds_{k+1}. \quad (5.2)$$

For $k=\kappa$ we set $f_{Z_1\dots Z_\kappa} := \overline{f}$. Next we define for $k = 2,\dots,\kappa$ the conditional densities

$$f_{Z_k|Z_1\dots Z_{k-1}}(z_k|z_1,\dots z_{k-1},p) \quad := \quad \frac{f_{Z_1\dots Z_k}(z_1,\dots,z_k,p)}{f_{Z_1\dots Z_{k-1}}(z_1,\dots,z_{k-1},p)}$$

and the associated distribution functions

$$F_{Z_k|Z_1\dots Z_{k-1}}(z_k|z_1,\dots z_{k-1},p) \;=\; \int_{a_k}^{z_k} f_{Z_k|Z_1\dots Z_{k-1}}(s_k|z_1,\dots z_{k-1},p)ds_k;$$

for $k=1$ we denote by F_{Z_1} the distribution function of Z_1. Now we introduce the Rosenblatt transform $\widetilde{F}:\mathcal{Z}\to[0,1]^\kappa=\mathcal{U}$, $z\mapsto(\widetilde{F}_1(z,p),\dots,\widetilde{F}_\kappa(z,p))^\top$ by

$$\widetilde{F}_1(z,p)=F_{Z_1}(z_1,p) \text{ and } \widetilde{F}_k(z,p)=F_{Z_k|Z_1\dots Z_{k-1}}(z_k|z_1,\dots z_{k-1},p),\ k=2,\dots,\kappa. \tag{5.3}$$

Clearly, $\widetilde{F}_k(z,p)$ depends on the first k variables $z_1,\dots,z_k$, only. The desired transformation G will be the inverse of $\widetilde{F}$, and the explicit form of $\widetilde{F}$ is needed when we estimate the derivatives of G in the proof of Lemma 5.4 below.

Assumption 5.1 ensures that the joint density $\overline{f}(z,p)$ is finite and bounded away from zero. Hence, the conditional densities $f_{Z_k|Z_1\dots Z_{k-1}}(z|z_1,\dots z_{k-1},p)$ are strictly positive, and the mapping $z\mapsto F_{Z_k|Z_1\dots Z_{k-1}}(z|z_1,\dots z_{k-1},p)$ is strictly increasing and hence invertible. In the sequel we denote the corresponding inverse function by $F^{-1}_{Z_k|Z_1\dots Z_{k-1}}(\cdot|z_1,\dots z_{k-1},p)$.

Now the desired transformation $Z=G(U)=G(U,p)$ with transformation function $G:\mathcal{U}\to\mathcal{Z}$, $u\mapsto(G_1(u,p),\dots,G_\kappa(u,p))^\top$ can be defined recursively by

$$\begin{aligned} &G_1(u,p)=F^{-1}_{Z_1}(u_1,p)\,, \text{ and for } k=2,\dots,\kappa,\\ &G_k(u,p)=F^{-1}_{Z_k|Z_1\dots Z_{k-1}}\big(u_k\mid G_1(u,p),\dots,G_{k-1}(u,p)\big) \end{aligned} \tag{5.4}$$

Note that by construction it holds $G(\widetilde{F}(z,p),p)=z$. From [15] it is known that if U is uniformly distributed in $[0,1]^\kappa$, the random vector $Z=(Z_1,\dots,Z_\kappa)^\top=G(U,p)$ has the joint distribution density $\overline{f}(z,p)$.

With the transformation G at hand we define the jump coefficient $\underline{\gamma}(\pi,u)$ by

$$\underline{\gamma}^k(\pi,u)=\pi^k\Big(\frac{f_k(G(u,R\pi))}{\overline{f}(G(u,R\pi),R\pi)}-1\Big) \text{ for } u\in\mathcal{U},\ k=1,\dots,d-1. \tag{5.5}$$

Moreover, we choose the Poisson random measure $N(dt,du)$ in (4.5) such that the associated compound Poisson process has constant intensity λ and jump heights which are uniformly distributed on $\mathcal{U}=[0,1]^\kappa$. Then, the compensator of N is $\nu(du)dt=\lambda du\,dt$ and the compensated measure reads as $\widetilde{N}(dt,du)=N(dt,du)-\lambda du\,dt$. Note that with this definition the solution π_t of (4.5) satisfies for some Borel set $A\subset\mathbb{R}^{d-1}$

$$\begin{aligned} P(\Delta\pi_{T_n}\in A\mid\mathcal{F}_{T_n-}) &= \int_{\mathcal{U}} 1_A\big(\underline{\gamma}(\pi_{T_n-},u)\big)du\\ &= \int_{\mathcal{U}} 1_A\Big(\underline{\gamma}_I\big(\pi_{T_n-},G(u,R\pi_{T_n-})\big)\Big)du\\ &= \int_{\mathcal{Z}} 1_A\big(\underline{\gamma}_I(\pi_{T_n-},z)\big)\,\overline{f}(z,R\pi_{T_n-})\,dz\,. \end{aligned}$$

Hence with the above choice of γ and $N(dt,du)$, for constant h the process $R\pi$, π the solution of the SDE (4.5), solves the martingale problem associated to the generator $\mathcal{L}^h$ from (3.8). Below we show that under Assumption 5.1 the Lipschitz and growth conditions from Assumption 4.1 hold, so that the SDE (4.5) has

a unique solution. It is well-known that this implies that the martingale problem associated with $\mathcal{L}^h$ has a unique solution, see for instance Jacod & Shiriaev [9], Theorem III.2.26. Hence $R\pi$ has the same law as the state variable process p in (3.4), which shows that we have achieved the desired reformulation of the dynamics of the problem in terms of an exogenous Poisson random measure.

Remark 5.3. Admittedly, the construction of G and $\underline{\gamma}$ is quite involved. The main reason for this is the fact that we consider the case of multidimensional expert opinions with values in $\mathbb{R}^\kappa$ for some $\kappa > 1$. Note however, that such a multivariate situation arises naturally in a model with more than one risky asset.

5.2. Lipschitz and growth conditions. The next Lemma states that under Assumption 5.1 the functions $\underline{\gamma}^k(\pi, u)$ satisfy the Lipschitz and growth conditions (4.8) and (4.9). The proof is given in Appendix A.

Lemma 5.4. *Under Assumption 5.1 and for* $\varepsilon \leq \bar{\varepsilon} := \frac{C_1}{(d-1)C_2}$ *the coefficient* $\underline{\gamma}(\pi, u)$ *defined in* (5.5) *satisfies for* $\pi \in \underline{\mathcal{S}}_\varepsilon$ *the Lipschitz and growth condition* (4.8) *and* (4.9).

6. Viscosity Solution

In this section we show that the value function of the control problem (4.11) is a viscosity solution of the dynamic programming equation (4.13). Since it is known from the literature that the comparison principle holds for these equation (a precise reference is given below) we obtain an interesting characterization of the value function as viscosity solution of (4.13). This part of our analysis is based to a large extent on the work of Pham [13].

6.1. Preliminaries. The following estimates are crucial in proving that the value function $V(t,p)$ is a viscosity solution of (4.13).

Proposition 6.1. *For any* $r \in [0,2]$ *there exists a constant* $C > 0$ *such that for all* $\delta \geq 0, t \in [0,T]$, $\pi, \xi \in \underline{\mathcal{S}}$, $h \in \mathcal{H}$ *and all stopping times* τ *between* t *and* $T \wedge t + \delta$

$$E\big(\ |\pi_\tau^{(t,\pi,h)}|^r\ \big) \leq C(1 + |\pi|^r) \tag{6.1}$$

$$E\big(\ |\pi_\tau^{(t,\pi,h)} - \pi|^r\ \big) \leq C(1 + |\pi|^r)\delta^{\frac{r}{2}} \tag{6.2}$$

$$E\Big(\Big\{ \sup_{t \leq s \leq t+\delta} \Big|\pi_s^{(t,\pi,h)} - \pi\Big| \Big\}^r\Big) \leq C(1 + |\pi|^r)\delta^{\frac{r}{2}} \tag{6.3}$$

$$E\big(\ |\pi_\tau^{(t,\pi,h)} - \pi_\tau^{(t,\xi,h)}|^r\ \big) \leq C\,|\pi - \xi|^r\,. \tag{6.4}$$

The proof is given in Appendix B.

Next we state the dynamic programming principle associated to the control problem (4.10).

Proposition 6.2 (Dynamic Programming Principle)**.** *For* $t \in [0,T]$, $\pi \in \underline{\mathcal{S}}$ *and every stopping time* δ *such that* $0 \leq \delta \leq T - t$ *we have*

$$V(t,\pi) = \sup_{h \in \mathcal{H}} E\Big(\exp\Big\{ \int_t^{t+\delta} -b(R\pi_s^{(t,\pi,h)}, h_s)ds \Big\} V\big(t+\delta, \pi_{t+\delta}^{(t,\pi,h)}\big)\Big)$$

For the proof of dynamic programming principle we refer to Pham [13], Proposition 3.1.

Applying the dynamic programming principle yields the next proposition on the continuity of the value function. The proof is given in Appendix C.

Proposition 6.3. *There exists a constant $C > 0$ such that for all $t, s \in [0,T]$ and $\pi_1, \pi_2 \in \underline{\mathcal{S}}$*

$$|V(t,\pi_1) - V(s,\pi_2)| \leq C\big[(1+|\pi_1|)|t-s|^{\frac{1}{2}} + |\pi_1 - \pi_2|\big]. \tag{6.5}$$

6.2. Viscosity Solution. Following Pham [13] we adapt the notion of a viscosity solution introduced by Crandall and Lions [3] to the case of integro-differential equations. This concept consists in interpreting equation (4.13) in a weaker sense. To simplify notation we split the generator $\mathcal{L}^h$ given in (4.12) into

$$\mathcal{L}^h g(\pi) = \mathcal{A}^h g(\pi) + \mathcal{B}(\pi)$$

where for $g \in C^2(\underline{\mathcal{S}})$ the linear second-order differential operator $\mathcal{A}^h$ is defined by

$$\mathcal{A}^h g(\pi) = \frac{1}{2}\sum_{i,j=1}^{d-1} \underline{\beta}_i^\top(\pi)\underline{\beta}_j(\pi) g_{\pi^i\pi^j}(\pi) + \sum_{i=1}^{d-1} \underline{\alpha}^i(\pi,h) g_{\pi^i}(\pi)$$

and $\mathcal{B}$ is the integral operator

$$\mathcal{B}g(\pi) = \lambda \int_{\mathcal{U}} \{g(\pi + \underline{\gamma}(\pi,u)) - g(\pi)\}\nu(du).$$

Moreover $D_\pi g$ and $D_\pi^2 g$ denote the gradient and Hessian matrix of g w.r.t π.

Definition 6.4.
(1) A function $V \in \mathcal{C}^0([0,T]\times\mathcal{S})$ is a viscosity *supersolution* (*subsolution*) of equation (3.9) if

$$-\frac{\partial\psi}{\partial t}(\bar t,\bar\pi) - \sup_{h\in K}\Big(-b(R\bar\pi,h)V(\bar t,\bar\pi) + \mathcal{A}^h\psi(\bar t,\bar\pi)\Big) - \mathcal{B}\psi(\bar t,\bar\pi) \geq 0 \tag{6.6}$$

(resp. ≤ 0) for all $(\bar t,\bar\pi) \in [0,T]\times\mathcal{S}$ and for all $\psi \in \mathcal{C}^{1,2}([0,T]\times\mathcal{S})$ with Lipschitz continuous derivatives $\psi_t, D_\pi^2\psi$ such that $(\bar t,\bar\pi)$ is a global minimizer (maximizer) of the difference $V-\psi$ on $[0,T]\times\mathcal{S}$ with $V(\bar t,\bar\pi) = \psi(\bar t,\bar\pi)$.
(2) V is a viscosity solution of (3.9) if it is both super and subsolution of that equation.

Proposition 6.5 (Viscosity solution)**.** *The value function $V(t,\pi)$ associated to the optimization problem (3.6) is a viscosity solution of* (3.9)

Proof. **Supersolution inequality.** Let be ψ such that

$$0 = (V-\psi)(\bar t,\bar\pi) = \min_{[0,T]\times\mathcal{S}}(V-\psi). \tag{6.7}$$

We apply the dynamic programming principle for a fixed time $\delta \in [0, T-t]$ to get

$$V(\bar t,\bar\pi) = \psi(\bar t,\bar\pi) = \sup_{h\in\mathcal{H}} E\Big(\exp\Big\{\int_{\bar t}^{\bar t+\delta} -b(R\pi_s^{(\bar t,\bar\pi,h)},h_s)ds\Big\}V(\bar t+\delta,\pi_{\bar t+\delta}^{(\bar t,\bar\pi,h)})\Big).$$

From (6.7) we obtain

$$0 \geq \sup_{h\in\mathcal{H}} E\Big(\exp\Big\{ \int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)ds\Big\}\psi(\overline{t}+\delta, \pi_{\overline{t}+\delta}^{(\overline{t},\overline{\pi},h)}) - \psi(\overline{t},\overline{\pi})\Big). \quad (6.8)$$

We now define for $u \in [\overline{t}, T]$

$$\eta_u := \exp\Big\{ \int_{\overline{t}}^{u} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)ds\Big\} \quad \text{and} \quad Z_u := \eta_u\, \psi(u, \pi_u^{(\overline{t},\overline{\pi},h)}). \quad (6.9)$$

Then, we apply Itô's formula to $Z_{\overline{t}+\delta}$, where we use the shorthand notation π_s for $\pi_s^{(\overline{t},\overline{\pi},h)}$. Since $dZ_t = -b(R\pi_t, h_t)\eta_t\psi(t,\pi_t)dt + \eta_t d\psi(t,\pi_t)$, we have

$$\begin{aligned} Z_{\overline{t}+\delta} &= \psi(\overline{t},\overline{\pi}) + \int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s, h_s)\eta_s\psi(s,\pi_s)ds \\ &\quad + \int_{\overline{t}}^{\overline{t}+\delta} \eta_s\big\{\psi_t(s,\pi_s) + \mathcal{A}^h\psi(s,\pi_s) + \mathcal{B}\psi(s,\pi_s)\big\}ds \\ &\quad + \int_{\overline{t}}^{\overline{t}+\delta} \eta_s D_\pi\psi(s,\pi_s)\beta^\top(\pi_s)dB_s \\ &\quad + \int_{\overline{t}}^{\overline{t}+\delta} \eta_s \int_{\mathcal{U}} \big(\psi(s,\pi_s+\gamma(\pi_s,u)) - \psi(s,\pi_s)\big)\widetilde{N}(ds\times du). \end{aligned}$$

Due to our assumptions on b and ψ, the last two terms are martingales with zero expectations. From (6.8) we therefore obtain

$$\begin{aligned} 0 &\geq \sup_{h\in\mathcal{H}} E(Z_{\overline{t}+\delta} - \psi(\overline{t},\overline{\pi})) \\ &= \sup_{h\in\mathcal{H}} E\Big(\int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)\eta_s\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)})ds \quad (6.10)\\ &\qquad + \int_{\overline{t}}^{\overline{t}+\delta} \eta_s\{\psi_t(s,\pi_s^{(\overline{t},\overline{\pi},h)}) + \mathcal{A}^h\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)}) + \mathcal{B}\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)})\}ds\Big). \end{aligned}$$

We now show for the first integral that

$$\begin{aligned} E\Big(\int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)\eta_s\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)})ds\Big) \geq \\ E\Big(\int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi, h_s)\psi(\overline{t},\overline{\pi})ds\Big) - \delta\varepsilon(\delta). \quad (6.11) \end{aligned}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Using the Lipschitz continuity of ψ we obtain the inequality $|\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)}) - \psi(\overline{t},\overline{\pi})| \leq C(|s-\overline{t}| + |\pi_s^{(\overline{t},\overline{\pi},h)} - \overline{\pi}|)$, which leads to

$$\begin{aligned} &\sup_{h\in\mathcal{H}} E\Big(\int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)\eta_s\psi(s,\pi_s^{(\overline{t},\overline{\pi},h)})\Big)ds \geq \\ &\sup_{h\in\mathcal{H}} E\Big(\int_{\overline{t}}^{\overline{t}+\delta} -b(R\pi_s^{(\overline{t},\overline{\pi},h)}, h_s)\eta_s\psi(\overline{t},\overline{\pi})ds\Big) - C\delta\Big\{\delta + E\Big(\sup_{0\leq s\leq\delta} |\pi_s^{(\overline{t},\overline{\pi},h)} - \overline{\pi}|\Big)\Big\}\Big). \end{aligned}$$

By Proposition 6.1 we have $E\Big(\sup_{0\le s\le\delta}|\pi_s^{(\bar t,\overline\pi,h)}-\overline\pi|\Big)\le C(1+|\overline\pi|)\delta^{\frac12}$ and hence we obtain

$$E\Big(\int_{\bar t}^{\bar t+\delta}-b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)\eta_s\psi(s,\pi_s^{(\bar t,\overline\pi,h)})ds\Big)\ge E\Big(\int_{\bar t}^{\bar t+\delta}-b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)\eta_s\psi(\bar t,\overline\pi)ds\Big)-\delta\varepsilon(\delta). \quad (6.12)$$

Recall from (3.2) that $b(R\pi,h)=-\theta\Big(h^\top MR\pi-\frac{1-\theta}{2}\left|\sigma^\top h\right|^2\Big)$. Since this expression depends linearly on π and h_s and since h_s takes values in the compact set K we have

$$|b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)-b(R\pi,h_s)|\le C|\pi_s^{(\bar t,\overline\pi,h)}-\pi|.$$

Using $|\eta_s-\eta_{\bar t}|=|\eta_s-1|\le C|s-\bar t|$ and the same computations to get (6.12) it yields

$$E\Big(\int_{\bar t}^{\bar t+\delta}-b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)\eta_s\psi(s,\pi_{t+s}^{(\bar t,\overline\pi,h)})ds\Big)\ge E\Big(\int_{\bar t}^{\bar t+\delta}-b(R\overline\pi,h_s)\psi(\bar t,\overline\pi)ds\Big)-\delta\varepsilon(\delta).$$

Applying similar computations to the other terms in (6.10) by using the estimates for the state process π and the Lipschitz continuity of $D_\pi^2\psi$ we obtain

$$\varepsilon(\delta)\ \ge\ \frac1\delta\sup_{h\in\mathcal H}E\Big(\int_{\bar t}^{\bar t+\delta}\{-b(\overline\pi,h_s)\psi(\bar t,\overline\pi)+\psi_t(\bar t,\overline\pi)+\mathcal A^h\psi(\bar t,\overline\pi)+\mathcal B\psi(\bar t,\overline\pi)\}ds\Big).$$

Replacing $h\in\mathcal H$ by a constant strategy in the above sup we get

$$\varepsilon(\delta)\ge\frac1\delta\Big(\int_{\bar t}^{\bar t+\delta}\sup_{h\in K}\Big(-b(R\overline\pi,h_s)\psi(\bar t,\overline\pi)+\psi_t(\bar t,\overline\pi)+\mathcal A^h\psi(\bar t,\overline\pi)+\mathcal B\psi(\bar t,\overline\pi)\Big)ds.$$

Applying the Mean Value Theorem and sending δ to 0 we get the supersolution viscosity inequality:

$$-\frac{\partial\psi}{\partial t}(\bar t,\overline\pi)-\sup_{h\in K}\Big(-b(R\overline\pi,h)V(\bar t,\overline\pi)+A^h\psi(\bar t,\overline\pi)\Big)-\mathcal B\psi(\bar t,\overline\pi)\ge0.$$

Subsolution inequality. Let ψ be such that

$$0=(V-\psi)(\bar t,\overline\pi)=\max_{[0,T]\times\mathcal S}(V-\psi) \quad (6.13)$$

As a consequence of dynamic programming principle in Proposition 6.2 we have

$$V(\bar t,\overline\pi)\ =\ \sup_{h\in\mathcal H}E\Big(\exp\Big\{\int_{\bar t}^{\bar t+\delta}-b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)ds\Big\}V(t+\delta,\pi_{t+\delta}^{(\bar t,\overline\pi,h)})\Big)$$

Equation (6.13) implies that

$$0\ \le\ \sup_{h\in\mathcal H}E\Big(\exp\Big\{\int_{\bar t}^{\bar t+\delta}-b(R\pi_s^{(\bar t,\overline\pi,h)},h_s)ds\Big\}\psi(t+\delta,\pi_{t+\delta}^{(\bar t,\overline\pi,h)})-\psi(\bar t,\overline\pi)\Big).$$

Using similar computations by applying Itô's formula to the process Z_u given in (6.9) and using the estimates for the state process π we obtain

$$\varepsilon(\delta)\le\frac1\delta\sup_{h\in\mathcal H}E\Big(\int_{\bar t}^{\bar t+\delta}\{-b(R\overline\pi,h_s)\psi(\bar t,\overline\pi)+\psi_t(\bar t,\overline\pi)+\mathcal A^h\psi(\bar t,\overline\pi)+\mathcal B\psi(\bar t,\overline\pi)\}ds\Big).$$

Replacing $h \in \mathcal{H}$ by a constant strategy in the above sup, applying the Mean Value Theorem and sending δ to 0 we obtain the subsolution viscosity inequality

$$-\psi_t(\bar{t},\overline{\pi}) - \sup_{h\in K}\Big(- b(R\overline{\pi},h)V(\bar{t},\overline{\pi}) + \mathcal{A}^h\psi(\bar{t},\overline{\pi})\Big) - \mathcal{B}\psi(\bar{t},\overline{\pi}) \leq 0.$$

□

6.3. Comparison principle. Here we quote the following result, which is Theorem 4.1 of [13].

Theorem 6.6. *Suppose that Assumption 4.1 holds and that u_1 and u_2 are continuous functions on $[0,T]\times\underline{\mathcal{S}}$ such that u_1 is a subsolution and u_2 is a supersolution of the dynamic programming equation (3.9). If $u_1(T,\pi) \leq u_2(T,\pi)$ for all $\pi \in \underline{\mathcal{S}}$, then*

$$u_1(t,\pi) \leq u_2(t,\pi) \text{ for all } (t,\pi) \in [0,T]\times\underline{\mathcal{S}}\,.$$

Together with Proposition 6.5, this result implies immediately that the value function $V(t,\pi)$ associated to the optimization problem (3.6) is the unique continuous viscosity solution of (3.9).

7. Regularized Dynamic Programming Equation

In this section we introduce the regularized version of our dynamic programming problem and we discuss the convergence of reward and value functions as the regularization-terms converge to zero. In Corollary 7.5 we finally show that optimal strategies in the regularized problem are nearly optimal in the original problem.

7.1. Regularized state equation. Since regularization will drive the state process outside the set $\underline{\mathcal{S}}$ we need to extend the definition of the coefficients $\underline{\alpha},\underline{\beta}$ and $\underline{\gamma}$ from $\underline{\mathcal{S}}$ to the whole $\mathbb{R}^{d-1}$. For $\pi\in\mathbb{R}^{d-1}$, $h\in K$ and $\varepsilon>0$ we define

$$\widetilde{\underline{\alpha}}(\pi,h) := \begin{cases} \underline{\alpha}(\pi,h)(1-\operatorname{dist}(\pi,\underline{\mathcal{S}})/\varepsilon) & \text{for} \quad \pi\in\underline{\mathcal{S}}_\varepsilon \\ 0 & \text{otherwise.}\end{cases}$$

Note that $\underline{\mathcal{S}}\subset\underline{\mathcal{S}}_\varepsilon$ and there is a continuous transition to zero if $\operatorname{dist}(\pi,\underline{\mathcal{S}})$ reaches ε. Moreover, on $\underline{\mathcal{S}}$ it holds $\widetilde{\underline{\alpha}}(\pi,h)=\underline{\alpha}(\pi,h)$, i.e. the coefficients coincide. Analogously we define $\widetilde{\underline{\beta}}$ and $\widetilde{\underline{\gamma}}$ as extensions of $\underline{\beta}$ and $\underline{\gamma}$.

Lemma 7.1. *Under the assumptions of Lemma 4.1 the coefficients $\widetilde{\underline{\alpha}},\widetilde{\underline{\beta}}$ and $\widetilde{\underline{\gamma}}$ satisfy the Lipschitz and growth conditions* (4.6) *to* (4.9) *for $\pi\in\mathbb{R}^{d-1}$.*

Proof. The Lipschitz and growth conditions for the coefficients $\underline{\alpha},\underline{\beta}$ and $\underline{\gamma}$ given in Lemma 4.1 hold for $\pi\in\underline{\mathcal{S}}_\varepsilon$ for $\varepsilon\leq\overline{\varepsilon}$. Multiplication of these functions by the bounded and Lipschitz continuous function $1-\operatorname{dist}(\pi,\underline{\mathcal{S}})/\varepsilon$ preserves the Lipschitz and growth property. □

For the sake of simplicity of notation in the sequel we will suppress the tilde and simply write $\underline{\alpha},\underline{\beta}$ and $\underline{\gamma}$ instead of $\widetilde{\underline{\alpha}},\widetilde{\underline{\beta}}$ and $\widetilde{\underline{\gamma}}$.

Next we define the dynamics of the regularized state process ${}^m\pi_t$

$$d\,{}^m\pi_t = \underline{\alpha}({}^m\pi_t,h_t)dt + \underline{\beta}^\top({}^m\pi_t)dB_t + \int_{\mathcal{U}}\underline{\gamma}({}^m\pi_{t-},u)\widetilde{N}(dt,du) + \frac{1}{\sqrt{m}}d\widetilde{B}_t \quad (7.1)$$

where $\widetilde{B}_t$ denotes a $d-1$-dimensional Brownian motion independent of B_t. This state process is now driven by an $n+d-1$-dimensional Brownion motion. Note that the diffusion coefficient of the regularized equation

$$(\underline{\beta}^\top(\pi_t), \frac{1}{\sqrt{m}} I_{d-1})^\top$$

satisfies the Lipschitz and growth condition (4.6) and (4.7) given in Lemma 4.1 since $\underline{\beta}(\pi_t)$ satisfies these conditions and $\frac{1}{\sqrt{m}} I_{d-1}$ does not depend on p.

7.2. L_2-Convergence ${}^m\pi_t \to \pi_t$. We now compare the solution ${}^m\pi_t$ of the regularized state equation (7.1) with the solution π_t of the unregularized state equation (4.5) and study asymptotic properties for $m \to \infty$. This will be crucial for establishing convergence of the associated reward function of the regularized problem to the original optimization problem.

We assume that both processes start at time $t_0 \in [0,T]$ with the same initial value $q \in \underline{\mathcal{S}}$, i.e. ${}^m\pi_{t_0} = \pi_{t_0} = q$. The corresponding solutions are denoted by ${}^m\pi_t^{(t_0,q,h)}$ and $\pi_t^{(t_0,q,h)}$.

Lemma 7.2 (Uniform L_2-convergence w.r.t. $h \in \mathcal{H}$)**.**
For $m \to \infty$ the following holds

$$E\Big(\sup_{t_0 \le t \le T} \Big| {}^m\pi_t^{(t_0,q,h)} - \pi_t^{(t_0,q,h)} \Big|^2 \Big) \longrightarrow 0 \quad \textit{uniformly for } h \in \mathcal{H}.$$

Proof. To simplify the notation we suppress the superscript (t_0,q,h) and write π_t and ${}^m\pi_t$. Moreover, we denote by C a generic constant.

We give the proof for $t_0 = 0$, only. Using the corresponding representation as stochastic integrals for the solutions of the above SDEs we find

$$\begin{aligned}
{}^m\pi_t - \pi_t &= A_t^m + M_t^m \qquad \text{where} \\
A_t^m &:= \int_0^t (\underline{\alpha}({}^m\pi_s, h_s) - \underline{\alpha}(\pi_s, h_s)) ds \quad \text{and} \\
M_t^m &= \int_0^t (\underline{\beta}({}^m\pi_s) - \underline{\beta}(\pi_s))^\top dB_s + \int_0^t \int_{\mathcal{U}} (\underline{\gamma}({}^m\pi_s, u) - \underline{\gamma}(\pi_s, u)) \widetilde{N}(ds, du) \\
&\quad + \frac{1}{\sqrt{m}}\, d\widetilde{B}_t.
\end{aligned}$$

Note that here we have used the fact that the SDE for ${}^m\pi$ and for π is driven by an exogenous Poisson random measure, since this permits us to write the difference of the jump-terms as stochastic integral with respect to the same compensated random measure.

Denoting $G_t^m := E\Big(\sup_{s \le t} |{}^m\pi_s - \pi_s|^2 \Big)$ the following holds

$$G_t^m = E\Big(\sup_{s \le t} |A_s^m + M_s^m|^2 \Big) \le 2E\Big(\sup_{s \le t} |A_s^m|^2 \Big) + 2E\Big(\sup_{s \le t} |M_s^m|^2 \Big). \tag{7.2}$$

For the first term on the r.h.s. we find by applying Cauchy-Schwarz inequality and the Lipschitz condition (4.6) for α

$$\begin{aligned}\sup_{s\le t}|A_s^m|^2 &= \sup_{s\le t}\Big|\int_0^s \big(\underline{\alpha}({}^m\pi_u,h_u)-\underline{\alpha}(\pi_u,h_u)\big)du\Big|^2\\ &\le \sup_{s\le t}\, s\cdot\int_0^s \Big|\underline{\alpha}({}^m\pi_u,h_u)-\underline{\alpha}(\pi_u,h_u)\big)\Big|^2 du\\ &\le t\cdot\int_0^t C_L|{}^m\pi_u-\pi_u|^2du \le t\cdot\int_0^t C_L\sup_{v\le u}|{}^m\pi_v-\pi_v|^2du.\end{aligned}$$

Note that the constant C_L does not depend on h. Taking expectation it follows

$$E\Big(\sup_{s\le t}|A_s^m|^2\Big) \le t\cdot C_L\int_0^t E\Big(\sup_{v\le s}|{}^m\pi_v-\pi_v|^2ds\Big)\le C\int_0^t G_s^m ds. \quad (7.3)$$

For the second term on the r.h.s. of (7.2) Doob's inequality for martingales yields

$$\begin{aligned}E\Big(\sup_{s\le t}|M_s^m|^2\Big) &\le 4E(|M_t^m|^2)\\ &= 4\Big(\int_0^t E\big(\operatorname{tr}[(\underline{\beta}({}^m\pi_s)-\underline{\beta}(\pi_s))^\top(\underline{\beta}({}^m\pi_s)-\underline{\beta}(\pi_s))]\big)ds \quad (7.4)\\ &\quad +\int_0^t\int_{\mathcal{U}} E\big(\,|\underline{\gamma}({}^m\pi_s,u)-\underline{\gamma}(\pi_s,u))|^2\,\big)\nu(du)ds+\frac{(d-1)t}{m}\Big).\end{aligned}$$

Using the Lipschitz conditions (4.6) and (4.8) for the coefficients β and γ it follows

$$\begin{aligned}E\big(\operatorname{tr}[(\underline{\beta}({}^m\pi_s)-\underline{\beta}(\pi_s))^\top(\underline{\beta}({}^m\pi_s)-\underline{\beta}(\pi_s))]\big) &\le C_L^2E(|{}^m\pi_s-\pi_s|^2)\\ &\le C_L^2E\big(\sup_{v\le s}|{}^m\pi_v-\pi_v|^2\big)=C_L^2G_s^m\\ E\big(\,|\underline{\gamma}({}^m\pi_s,u)-\underline{\gamma}(\pi_s,u))|^2\,\big) &\le \rho^2(u)E(|{}^m\pi_s-\pi_s|^2)\\ &\le \rho^2(u)E\big(\sup_{v\le s}|{}^m\pi_v-\pi_v|^2\big)=\rho^2(u)G_s^m.\end{aligned}$$

Substituting the above estimates into (7.4) it follows that

$$\begin{aligned}E\Big(\sup_{s\le t}|M_s^m|^2\Big) &\le 4\Big(\int_0^t C_L^2G_s^mds+\int_0^t G_s^mds\int_{\mathcal{U}}\rho^2(u)\nu(du)+\frac{(d-1)t}{m}\Big)\\ &\le C\int_0^t G_s^mds+\frac{4(d-1)t}{m}. \quad (7.5)\end{aligned}$$

Substituting (7.3) and (7.5) into (7.2) we find

$$G_t^m\le\frac{4(d-1)T}{m}+C\int_0^t G_s^mds.$$

Finally we apply Gronwall Lemma to derive

$$G_T^m\le\frac{4(d-1)T}{m}\,e^{CT}\to 0 \quad \text{for } m\to\infty$$

which concludes the proof. □

Note that the L_2-convergence for the restricted state process ${}^m\pi_t$ established in Lemma 7.2 also holds for the associated d-dimensional process ${}^m p_t = R\,{}^m\pi_t$.

We now extend the notions of reward and value function given in (3.7) to the process ${}^m p_t = R\,{}^m\pi_t$ with ${}^m\pi_t$ satisfying the regularized state equation (7.1). Since ${}^m p_t$ takes values in $\mathbb{R}^d$ (and not only in $\mathcal{S}$) we extend the function b given in (3.2) to $p = R\pi \in \mathbb{R}^d$. With the notation $b_* = \min\{b(p,h),\ p \in \mathcal{S}, h \in K\}$ and $b^* = \max\{b(p,h),\ p \in \mathcal{S}, h \in K\}$ we define

$$\widetilde{b}(p,h) := (b(p,h) \vee b_*) \wedge b^*.$$

Then $\widetilde{b}$ is bounded on $\mathbb{R}^d \times K$ and for $p \in \mathcal{S}$ the function $\widetilde{b}$ coincides with b. In the sequel we simply write b instead of $\widetilde{b}$. We define the reward and value function associated to the regularized state equation (7.1) by

$$\begin{aligned} v^m(t,\pi,h) &= E\Big(\exp\Big\{\int_t^T -b(R({}^m\pi_s^{(t,\pi,h)}),h_s)ds\Big\}\Big) \quad \text{for } h \in \mathcal{H}, \\ V^m(t,\pi) &= \sup\{v^m(t,\pi,h)\colon h \in \mathcal{H}\}. \end{aligned}$$

Recall that $v(t,\pi,h)$ and $V(t,\pi)$ defined in (4.11) denote the reward and value function associated to the unregularized state equation (4.5). The generator associated to the solution of the regularized state equation (7.1) reads as

$$\begin{aligned} {}^m\mathcal{L}^h g(\pi) &= \frac{1}{2}\sum_{i,j=1}^{d-1} \underline{\beta}_i^\top(\pi)\underline{\beta}_j(\pi) g_{\pi^i\pi^j} + \frac{1}{2m}\sum_{i=1}^{d-1} g_{\pi^i\pi^i} + \sum_{i=1}^{d-1} \underline{\alpha}^i(\pi,h) g_{\pi^i} \\ &\qquad + \int_{\mathcal{U}} \{g(\pi + \underline{\gamma}(\pi,u)) - g(\pi)\}\nu(du) \end{aligned}$$

and the associated dynamic programming equation is

$$V_t^m(t,\pi) + \sup_{h\in K}\Big\{{}^m\mathcal{L}^h V^m(t,\pi) - b(R\pi,h;\theta)V^m(t,\pi)\Big\} = 0,\ (t,\pi) \in [0,T)\times\mathbb{R}^{d-1}. \tag{7.6}$$

Note that for the generator ${}^m\mathcal{L}^h$ the *ellipticity condition* for the coefficients of the second derivatives holds: we have for all $z \in \mathbb{R}^{d-1}\setminus\{0\}$

$$z^\top(\underline{\beta}^\top\underline{\beta} + \frac{1}{2m}I_{d-1})z = z^\top\underline{\beta}^\top\underline{\beta}z + \frac{1}{2m}z^\top z = |\underline{\beta}z|^2 + \frac{1}{2m}|z|^2 > 0.$$

Hence the results of Davis & Lleo [4] apply to this dynamic programming problem. equation. According to Theorem 3.8 of their paper, there is a classical solution V^m of (7.6). Moreover, for every (t,π) there is a unique maximizer ${}^m h^*$ of the problem

$$\sup_{h\in K}\Big\{{}^m\mathcal{L}^h V^m(t,\pi) - b(R\pi,h;\theta)V^m(t,\pi)\Big\}.$$

The maximizer ${}^m h^*$ can be chosen as a Borel-measurable function of t and π and the optimal strategy is given by ${}^m h_t^* = {}^m h^*(t, {}^m\pi_t)$; see also the discussion preceding Remark 3.1.

7.3. Convergence of reward and value functions. The next theorem on the uniform convergence of reward functions is our main result; convergence of the value function and ε-optimality of ${}^m h^*$ follow easily from this theorem.

Theorem 7.3 (Uniform Convergence of reward functions)**.**
For $m \to \infty$ the following holds

$$\sup_{h\in\mathcal{H}} |v^m(t,\pi,h) - v(t,\pi,h)| \to 0 \quad \textit{for } t \in [0,T],\ \pi \in \underline{\mathcal{S}}.$$

Proof. We introduce the notation

$$J := \int_t^T -b(R\pi_s^{(t,\pi,h)}, h_s)ds \quad \text{and} \quad J^m := \int_t^T -b(R\,{}^m\pi_s^{(t,\pi,h)}, h_s)ds.$$

Then the reward functions read as $v(t,\pi,h) = E(e^J)$ and $v^m(t,\pi,h) = E(e^{J^m})$ and it holds

$$\begin{aligned} |v^m(t,\pi,h) - v(t,\pi,h)| &= |E(e^{J^m} - e^J)| \le E(|e^{J^m} - e^J|) \\ &\le CE(|J^m - J|), \end{aligned} \tag{7.7}$$

where we used Lipschitz continuity of $f(x) = e^x$ on bounded intervals and the boundedness of J and J^m which follows from the the boundedness of b. Using Lipschitz continuity of b we derive

$$\begin{aligned} E(|J^m - J|) &= E\Big(\Big|\int_t^T \big[b(R\pi_s^{(t,\pi,h)}, h_s) - b(R\,{}^m\pi_s^{(t,\pi,h)}, h_s)\big]ds\Big|\Big) \\ &\le \int_t^T C\,E(|{}^m\pi_s^{(t,\pi,h)} - \pi_s^{(t,\pi,h)}|)\,ds \\ &\le C\int_t^T \Big(E(|{}^m\pi_s^{(t,\pi,h)} - \pi_s^{(t,\pi,h)}|^2\Big)^{1/2} ds \to 0 \end{aligned} \tag{7.8}$$

for $m \to \infty$ and uniformly w.r.t. $h \in \mathcal{H}$ which follows from Lemma 7.2. Plugging (7.8) into (7.7) we find $\sup_{h\in\mathcal{H}} |v^m(t,\pi,h) - v(t,\pi,h)| \to 0$ for $m \to \infty$. □

Corollary 7.4 (Convergence of value functions)**.**
For $m \to \infty$ the following holds

$$V^m(t,\pi) \to V(t,\pi) \quad \textit{for } t \in [0,T],\ \pi \in \underline{\mathcal{S}}.$$

Proof. For $\theta \in (0,1)$ the assertion follows from

$$\begin{aligned} |V^m(t,\pi) - V(t,\pi)| &= \Big|\sup_{h\in\mathcal{H}} v^m(t,\pi,h) - \sup_{h\in\mathcal{H}} v(t,\pi,h)\Big| \\ &\le \sup_{h\in\mathcal{H}} |v^m(t,\pi,h) - v(t,\pi,h)| \end{aligned}$$

and Lemma 7.3. Analogously, for $\theta < 0$ it follows

$$\begin{aligned} |V^m(t,\pi) - V(t,\pi)| &= \Big|\inf_{h\in\mathcal{H}} v^m(t,\pi,h) - \inf_{h\in\mathcal{H}} v(t,\pi,h)\Big| \\ &= \Big|\sup_{h\in\mathcal{H}}(-v^m(t,\pi,h)) - \sup_{h\in\mathcal{H}}(-v(t,\pi,h))\Big| \\ &\le \sup_{h\in\mathcal{H}} |v^m(t,\pi,h) - v(t,\pi,h)|. \end{aligned}$$

□

7.4. On ε-optimal stratgies. Finally we show that the optimal strategy ${}^m h^*$ for the regularized problem is ε-optimal in the original problem. This gives a method for computing (nearly) optimal strategies.

Corollary 7.5 (ε-optimality)**.** *For every $\varepsilon > 0$ there exists some $m_0 \in \mathbb{N}$ such that*

$$|V(t,\pi) - v(t,\pi,{}^m h^*)| \leq \varepsilon \quad \text{for } m \geq m_0,$$

i.e. ${}^m h^*$ is an ε-optimal strategy for the original control problem.

Proof. It holds

$$\begin{aligned} &|V(t,\pi) - v(t,\pi,{}^m h^*)| \\ &\qquad \leq \quad |V(t,\pi) - v^m(t,\pi,{}^m h^*)| + |v^m(t,\pi,{}^m h^*) - v(t,\pi,{}^m h^*)| \\ &\qquad = \quad |V(t,\pi) - V^m(t,\pi)| + |v^m(t,\pi,{}^m h^*) - v(t,\pi,{}^m h^*)| \end{aligned} \tag{7.9}$$

where for the first term on the r.h.s. we used $v^m(t,\pi,{}^m h^*) = V^m(t,\pi)$. Using the convergence properties for the reward function given in Lemma 7.3 and for the value function given in Corollary 7.4 we can find for every $\varepsilon > 0$ some $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ it holds

$$|V(t,\pi) - V^m(t,\pi)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |v^m(t,\pi,{}^m h^*) - v(t,\pi,{}^m h^*)| \leq \frac{\varepsilon}{2}.$$

Plugging the above estimates into (7.9) it follows for $m \geq m_0$

$$|V(t,\pi) - v(t,\pi,{}^m h^*)| \quad \leq \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Remark 7.6. Note that in the proof of the corollary we use that the sequence of reward functions v^m converges to v *uniformly* in h. This is a stronger property than convergence of the value functions V^m to V so that standard stability results for dynamic programming equations are not sufficient to proof the corollary.

Appendix A. Proof of Lemma 5.4

Proof. We give the proof for the maximum norm $|.|_\infty$ in $\mathbb{R}^d$. From this the assertion for the Euclidean norm can be deduced from the equivalence of norms.

Note that the fact that all densities are $\mathcal{C}^1$ with compact support $\mathcal{Z} = [a,b]$ implies the existence of constants $C_2, C_d < \infty$ such that for all $1 \leq k \leq d$, $z \in \mathcal{Z}$,

$$f_k(z) \leq C_2 \text{ and } \left|\frac{\partial}{\partial z_i} f_j(z)\right| \leq C_d, \quad i = 1,\dots,\kappa. \tag{A.1}$$

Boundedness of $\overline{f}(z,R\pi)$. First we show that for $\pi \in \underline{\mathcal{S}}_\varepsilon$, $z \in \mathcal{Z}$ and $\varepsilon < \bar{\varepsilon}$ there are constants $0 < C_* \leq C^* < \infty$ such that

$$C_* \leq \overline{f}(z,R\pi) \leq C^*. \tag{A.2}$$

For this, observe that for $p = R\pi$,

$$\overline{f}(z,p) = \sum_{j=1}^d p^j f_j(z) = \sum_{p^j<0} p^j f_j(z) + \sum_{p^j \geq 0} p^j f_j(z). \tag{A.3}$$

For the lower bound we deduce

$$\begin{aligned}\overline{f}(z,p) &\geq \sum_{p^j<0}(-\varepsilon)\max_j f_j(z) + \sum_{p^j\geq 0} p^j \min_j f_j(z)\\ &\geq -\varepsilon(d-1)C_2 \quad + \quad \Big(1-\sum_{p^j<0} p^j\Big)\cdot C_1 \geq -\varepsilon\frac{C_1}{\overline{\varepsilon}} + 1\cdot C_1 = C_*,\end{aligned}$$

where we used Assumption 5.1, (A.1), $p^j \geq -\varepsilon$ and $\sum_{j=1}^d p^j = 1$. For the upper bound from A.3 we find

$$\overline{f}(z,p) \leq 0 + \sum_{p^j\geq 0} p^j \max_j f_j(z) \leq \Big(1-\sum_{p^j<0} p^j\Big)C_2 \leq (1+\varepsilon(d-1))C_2 = C^*$$

Note that the lower bound in (A.2) implies that $\overline{f}(\cdot, R\pi)$ is strictly positive for $\pi \in \underline{\mathcal{S}}_\varepsilon$. Moreover, since the components of $p = R\pi$ sum up to one by definition $\overline{f}(\cdot, R\pi)$ is a strictly positive probability density for $\pi \in \underline{\mathcal{S}}_\varepsilon$. Hence, the inverse Rosenblatt transform $G(u, R\pi)$ and thus the function $\underline{\gamma}(\pi, u)$ defined in (5.5) are well defined for $\pi \in \underline{\mathcal{S}}_\varepsilon$ (and not just for $\pi \in \underline{\mathcal{S}}$).

Lipschitz condition (4.8)**.** Clearly, (4.8) holds for some constant function $\rho(u) = \overline{\rho}$ if we can show that the derivatives of $\underline{\gamma}(\pi, u)$ with respect to π^j are bounded for all $1 \leq j \leq d-1$. This is obviously equivalent to estimating the derivatives of

$$\gamma^k(p,u) = p^k\Big(\frac{f_k(G(u,p))}{\overline{f}(G(u,p),p)} - 1\Big)$$

with respect to the components p^j where $p = R\pi$. Let

$$c_j^k(p,u) := \frac{\partial}{\partial p^j}\Big(\frac{f_k(G(u,p))}{\overline{f}(G(u,p),p)} - 1\Big), \quad j,k = 1,\ldots,d.$$

Then it holds

$$\frac{\partial}{\partial p^j}\gamma^k(p,u) = \delta_{jk}\Big(\frac{f_k(G(u,p))}{\overline{f}(G(u,p),p)} - 1\Big) + p^k c_j^k(p,u).$$

The first term on the r.h.s. is bounded since it holds for $k = 1,\ldots,d$ and $\varepsilon < \overline{\varepsilon}$

$$\frac{f_k(G(u,p))}{\overline{f}(G(u,p),p)} \leq \frac{C_2}{C_*}, \tag{A.4}$$

where we have used (A.1) and the lower bound for $\overline{f}(z,p)$ given in (A.2).

It remains to show that $c_j^k(p,u)$ is bounded. Abbreviating $z = z(p) = G(u,p)$ we find

$$\begin{aligned}c_j^k(p,u) &= \frac{1}{(\overline{f}(z,p))^2}\Big(\sum_{l=1}^{\kappa}\frac{\partial}{\partial z_l}f_k(z)\frac{\partial}{\partial p^j}G_l(u,p)\cdot\overline{f}(z,p)\\ &\qquad -f_k(z)\cdot\Big(f_j(z) + \sum_{i=1}^d p^i\sum_{l=1}^{\kappa}\frac{\partial}{\partial z_l}f_i(z)\frac{\partial}{\partial p^j}G_l(u,p)\Big)\Big).\end{aligned} \tag{A.5}$$

Using (A.1), $\sum_{j=1}^d |p^j| \le 1+(d-1)\overline{\varepsilon}$ and estimate (A.2) for $\overline{f}$, we derive for $\varepsilon < \overline{\varepsilon}$

$$\begin{aligned} |c_j^k(p,u)| &\le \frac{1}{C_*^2}\Big(C_d \sum_{l=1}^{\kappa} \Big|\frac{\partial}{\partial p^j} G_l(u,p)\Big| C^* \\ &\quad + C_2 \cdot \Big(C_2 + (1+(d-1)\overline{\varepsilon}) C_d \sum_{l=1}^{\kappa} \Big|\frac{\partial}{\partial p^j} G_l(u,p)\Big|\Big)\Big). \quad \text{(A.6)} \end{aligned}$$

In Lemma A.1 below we show that the derivatives $\frac{\partial}{\partial p^j} G_l(u,p)$ are bounded, that is there is some $C>0$ such that for $j=1,\dots,d$ and $l=1,\dots,\kappa$ $\big|\frac{\partial}{\partial p^j} G_l(u,p)\big| \le C$. From this the boundedness of c_j^k follows immediately.

Growth condition (4.9)**.** Here we apply estimate (A.4) and find

$$|\gamma^j(p,u)| = \Big|p^j\Big(\frac{f_j(G(u,p))}{\overline{f}(G(u,p),p)} - 1\Big)\Big| \le |p^j|\Big(\frac{C_2}{C_*}+1\Big) \le (1+|p|_\infty)\Big(\frac{C_2}{C_*}+1\Big)$$

and hence $|\gamma(p,u)|_\infty \le \overline{\rho}(1+|p|_\infty)$ with some constant $\overline{\rho}$. □

Lemma A.1. *Under the assumptions of Lemma 5.4 there exists a constant $C>0$ such that for $j=1,\dots,d$ and $l=1,\dots,\kappa$*

$$\Big|\frac{\partial}{\partial p^j} G_l(u,p)\Big| \le C.$$

Proof. We derive from differentiating $G_l(\widetilde{F}(z,p),p) = z_l$ w.r.t. p_j using the chain rule

$$\sum_{i=1}^{\kappa} \frac{\partial}{\partial u_i} G_l(\widetilde{F}(z,p),p) \frac{\partial}{\partial p^j}\widetilde{F}_i(z,p) + \frac{\partial}{\partial p^j} G_l(\widetilde{F}(z,p),p) = 0.$$

Substituting $u = \widetilde{F}(z,p)$ we obtain the estimate

$$\Big|\frac{\partial}{\partial p^j} G_l(u,p)\Big| \le \sum_{i=1}^{\kappa} \Big|\frac{\partial}{\partial u_i} G_l(u,p)\Big| \Big|\frac{\partial}{\partial p^j}\widetilde{F}_i(z,p)\Big|. \quad \text{(A.7)}$$

(i) For the proof of the boundedness of the derivatives on the r.h.s. we need the following auxiliary estimates for the marginal densities $f_{Z_1\dots Z_k}, k=1,\dots,\kappa$ given in (5.2). From estimate (A.2) for $\overline{f}$ we derive the estimate

$$C_* \prod_{i=k+1}^{\kappa}(b_i-a_i) \le f_{Z_1\dots Z_k}(z_1,\dots,z_k,p) \le C^* \prod_{i=k+1}^{\kappa}(b_i-a_i). \quad \text{(A.8)}$$

For the derivatives of the marginal densities w.r.t. p^j the definition of $\overline{f}$ in (2.8) yields

$$\frac{\partial}{\partial p^j} f_{Z_1\dots Z_k}(z_1,\dots,z_k,p) = \int_{a_{k+1}}^{b_{k+1}} \dots \int_{a_\kappa}^{b_\kappa} f_j(z_1,\dots,z_k,s_{k+1},\dots,s_\kappa) ds_{k+1}\dots ds_\kappa.$$

From Assumption 5.1 and (A.1) it follows

$$0 < C_1 \prod_{i=k+1}^{\kappa}(b_i-a_i) \le \frac{\partial}{\partial p^j} f_{Z_1\dots Z_k}(z_1,\dots,z_k,p) \le C_2 \prod_{i=k+1}^{\kappa}(b_i-a_i). \quad \text{(A.9)}$$

For the derivatives of the marginal densities w.r.t. $z_j,\ j = 1\ldots,k$ we find

$$\begin{aligned}
&\Big|\frac{\partial}{\partial z_j} f_{Z_1\ldots Z_k}(z_1,\ldots,z_k,p)\Big| \\
&\quad\leq \int_{a_{k+1}}^{b_{k+1}}\cdots\int_{a_\kappa}^{b_\kappa}\sum_{l=1}^{d} p^l\Big|\frac{\partial}{\partial z_j} f_l(z_1,\ldots,z_k,s_{k+1},\ldots,s_\kappa)\Big| ds_{k+1}\ldots ds_\kappa \\
&\quad\leq C_d\prod_{i=k+1}^{\kappa}(b_i-a_i), \qquad\qquad (A.10)
\end{aligned}$$

where the upper bound from (A.1) on the derivatives of the densities f_j has been used.

(ii) Now we can prove the boundedness for the second term r.h.s. of (A.7). For $k = 2,\ldots,\kappa$ we obtain from the definition of $\widetilde{F}(z,p)$ in (5.3)

$$\begin{aligned}
\Big|\frac{\partial}{\partial p^j}\widetilde{F}_i(z,p)\Big| &= \Big|\int_{a_k}^{z_k}\frac{\partial}{\partial p^j} f_{Z_k|Z_1\ldots Z_{k-1}}(s_k|z_1,\ldots z_{k-1},p)ds_k\Big| \\
&= \Big|\int_{a_k}^{z_k}\frac{\partial}{\partial p^j}\frac{f_{Z_1\ldots Z_k}(z_1,\ldots,z_{k-1},s_k,p)}{f_{Z_1\ldots Z_{k-1}}(z_1,\ldots,z_{k-1},p)}ds_k\Big| \\
&\leq \int_{a_k}^{z_k}\frac{1}{f^2_{Z_1\ldots Z_{k-1}}(\cdot)}\Big(\Big|\frac{\partial}{\partial p^j} f_{Z_1\ldots Z_k}(\cdot)\Big| f_{Z_1\ldots Z_{k-1}}(\cdot) + \\
&\qquad\qquad f_{Z_1\ldots Z_k}(\cdot)\Big|\frac{\partial}{\partial p^j} f_{Z_1\ldots Z_{k-1}}(\cdot)\Big|\Big)ds_k,
\end{aligned}$$

which is bounded. Here, we have used estimate (A.8), which states that the marginal densities are bounded from above and bounded away from zero, and (A.9) for the boundedness of the derivatives of the marginal densities w.r.t. p^j.

For $k = 1$ we observe that

$$\frac{\partial}{\partial p^j}\widetilde{F}_1(z,p) = \frac{\partial}{\partial p^j}F_{Z_1}(z_1,p) = \int_{a_1}^{z_1}\frac{\partial}{\partial p^j} f_{Z_1}(s_1,p)ds_1.$$

The boundedness $\partial/\partial p^j\widetilde{F}_1(z,p)$ is a consequence of estimate (A.9).

(iii) For proving the boundedness of $\partial/\partial u_i G_l(u,p)$ in (A.7) we consider the Jacobian matrices for $G(u)$ and $\widetilde{F}(z)$ defined by

$$J^G(u) := \Big(\frac{\partial}{\partial u_j}G_i(u,p)\Big)_{i,j=1\ldots,\kappa} \quad\text{and}\quad J^{\widetilde{F}}(z) := \Big(\frac{\partial}{\partial z_j}\widetilde{F}_i(z,p)\Big)_{i,j=1\ldots,\kappa}.$$

Below we show that for $z = G(u,p)$ the matrix $J^{\widetilde{F}}(z)$ is regular, hence $J^G(u) = J^{-1}_{\widetilde{F}}(G(u,p))$, since $G(\widetilde{F}(z,p),p) = z$. From the definition of $\widetilde{F}$ in (5.3) it follows that $J^{\widetilde{F}}(z)$ is a lower triangular matrix since $\widetilde{F}_k$ depends on $z_1,\ldots,z_k$ only.

Next we consider the diagonal elements of $J^{\widetilde{F}}(z)$. Using (A.8) we find constants $\underline{C}$ and $\overline{C}$ such that $\underline{C} \leq f_{Z_1\ldots Z_k}(z_1,\ldots,z_k,p) \leq \overline{C}$ for all $k = 1\ldots,\kappa$. Then with

$\delta := \min\{\underline{C},\, \underline{C}\,/\,\overline{C}\}$ the following holds

$$\begin{aligned}
\frac{\partial}{\partial z_1}\widetilde{F}_1(z,p) &= f_{Z_1}(z_1,p) \geq \delta \quad \text{and} \\
\frac{\partial}{\partial z_k}\widetilde{F}_k(z,p) &= f_{Z_k|Z_1\dots Z_{k-1}}(z_k|z_1,\dots z_{k-1},p) = \frac{f_{Z_1\dots Z_k}(z_1,\dots,z_k,p)}{f_{Z_1\dots Z_{k-1}}(z_1,\dots,z_{k-1},p)} \geq \delta,
\end{aligned}$$

for $k = 2,\dots,\kappa$. Since $J^{\widetilde{F}}(z)$ is triangular, its determinant is

$$\det(J^{\widetilde{F}}(z)) = \prod_{k=1}^{\kappa} \frac{\partial}{\partial z_k}\widetilde{F}_k(z,p) \geq \delta^\kappa > 0,$$

hence $J^{\widetilde{F}}(z)$ is invertible.

Next we show that the the non-zero off-diagonal elements of $J^{\widetilde{F}}$ are bounded. It holds for $k = 2,\dots,\kappa,\ j = 1,\dots,k-1$

$$\begin{aligned}
\frac{\partial}{\partial z_j}\widetilde{F}_k(z,p) &= \int_{a_k}^{z_k} \frac{\partial}{\partial z_j} f_{Z_k|Z_1\dots Z_{k-1}}(s_k|z_1,\dots z_{k-1},p)ds_k \\
&= \int_{a_k}^{z_k} \frac{\partial}{\partial z_j} \frac{f_{Z_1\dots Z_k}(z_1,\dots,z_{k-1},s_k,p)}{f_{Z_1\dots Z_{k-1}}(z_1,\dots,z_{k-1},p)} ds_k \\
&\leq \int_{a_k}^{z_k} \frac{1}{f^2_{Z_1\dots Z_{k-1}}(\cdot)} \Big(\Big|\frac{\partial}{\partial z_j} f_{Z_1\dots Z_k}(\cdot)\Big| f_{Z_1\dots Z_{k-1}}(\cdot) \\
&\qquad + f_{Z_1\dots Z_k}(\cdot)\Big|\frac{\partial}{\partial z_j} f_{Z_1\dots Z_{k-1}}(\cdot)\Big| \Big) ds_k,
\end{aligned}$$

which is bounded. Here again we have used that the marginal densities are bounded from above and bounded away from zero, and (A.10) for the boundedness of the derivatives of the marginal densities w.r.t. z_j.

For proving the boundedness of $\partial/\partial u_i G_l(u,p)$ in (A.7) which are the entries of the Jacobian matrix $J^G(u)$ we use that J^G is the inverse of $J^{\widetilde{F}}$. Since $J^{\widetilde{F}}$ is a triangular matrix the entries of J^G can be computed recursively by Gaussian elimination starting with the first row. This gives that for $k,l = 1,\dots,\kappa$

$$J^G_{kl} = \frac{1}{J^{\widetilde{F}}_{kk}} \Big(\delta_{kl} - \sum_{j=1}^{k-1} J^{\widetilde{F}}_{kj}\, J^G_{jl} \Big),$$

i.e. the entry J^G_{kl} can be represented by an affine linear combination of the bounded off-diagonal entries in row k of $J^{\widetilde{F}}$ divided by $J^{\widetilde{F}}_{kk}$. The latter is strictly positive and bounded from below by $\delta > 0$. Hence, all entries of J^G are bounded. □

Appendix B. Proof of Proposition 6.1

Proof. We give the proof for $r = 2$. The assertions for $r \in [0,2]$ follow from Hölder's inequality. We denote by C a generic constant which may vary from line to line.

Inequality (6.1): $E(|\pi_\tau^{(t,\pi,h)}|^2) \le C(1+|\pi|^2)$**.**
We recall the state equation

$$d\pi_t = \underline{\alpha}(\pi_t, h_t)dt + \underline{\beta}^\top(\pi_t)dB_t + \int_{\mathcal{U}} \underline{\gamma}(\pi_t, u)\widetilde{N}(dt \times du) \tag{B.1}$$

and for the sake of shorter notation we denote by $\pi_\tau = \pi_\tau^{(t,\pi,h)}$ the solution of equation $(B.1)$ starting from π at time t using strategy h for $\tau \ge t$. Then it holds

$$\begin{aligned} |\pi_\tau|^2 \quad \le \quad & C\Big(|\pi|^2 + \Big| \int_t^\tau \underline{\alpha}(\pi_s, h_s)ds\Big|^2 + \Big| \int_t^\tau \underline{\beta}(\pi_s)dB_s\Big|^2 \\ & + \Big| \int_t^\tau \int_{\mathcal{U}} \underline{\gamma}(\pi_s, u)\widetilde{N}(ds \times du)\Big|^2\Big) \end{aligned}$$

Taking expectation and using Itô-Levy isometry implies

$$\begin{aligned} E(|\pi_\tau|^2) \quad \le \quad & C\Big(|\pi|^2 + E\Big(\int_t^\tau |\underline{\alpha}(\pi_s, h_s)|^2 ds\Big) + E\Big(\int_t^\tau tr(\underline{\beta}^\top(\pi_s)\underline{\beta}(\pi_s))ds\Big) \\ & + E\Big(\int_t^\tau \int_{\mathcal{U}} |\underline{\gamma}(\pi_s, u)|^2 \nu(du)ds)\Big)\Big). \end{aligned}$$

We now use the linear growth of $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ and the integrability property for ρ (see Assumption 4.1) to obtain

$$\begin{aligned} E(|\pi_\tau|^2) &\le C\Big\{ |\pi|^2 + E\Big(\int_t^\tau (1+|\pi_s|^2)ds\Big)\Big\} \le C\Big\{ |\pi|^2 + E(\tau) + E\Big(\int_t^\tau |\pi_s|^2 ds\Big)\Big\} \\ &\le C\Big\{ |\pi|^2 + 1 + E\Big(\int_t^\tau |\pi_s|^2 ds\Big)\Big\}. \end{aligned} \tag{B.2}$$

For any deterministic time $\tau = u$ Fubini's Theorem gives

$$E(|\pi_u|^2) \le C\Big\{ |\pi|^2 + 1 + \int_t^u E(|\pi_s|^2)ds\Big\}$$

and applying Gronwall's Lemma to $G_u := E(|\pi_u|^2)$ implies

$$E(|\pi_u|^2) \le C(|\pi|^2 + 1)e^{C(u-t)} \le C(|\pi|^2 + 1).$$

Finally, we note, that for any stopping time $\tau \in [t, T \wedge t + \delta]$ it holds

$$E\Big(\int_t^\tau |\pi_s|^2 ds\Big) \le \int_t^{t+\delta} E(|\pi_s|^2)ds \le C(1+|\pi|^2).$$

Substituting the upper estimate back into $(B.2)$ proves the assertion.

Inequality (6.2): $E(|\pi_\tau^{(t,\pi,h)} - \pi|^2) \le C(1+|\pi|^2)\delta$**.**
The process $(\pi_\tau - \pi)$ starts from 0 and hence the computations for π_τ in the above proof inequality (6.2) give for $\tau \in [t, T \wedge t + \delta]$

$$\begin{aligned} E(|\pi_\tau - \pi|^2) \quad &\le \quad C\int_t^\tau (1 + E(|\pi_s|^2))ds \le C\int_t^{t+\delta} (1 + E(|\pi_s|^2))ds \\ &\le \quad C\int_t^{t+\delta} (1 + C(1 + (|\pi|^2))ds \le C(1+|\pi|^2)\delta. \end{aligned}$$

Inequality (6.3): $E\Big(\Big\{\sup_{t\le s\le t+\delta}|\pi_s^{(t,\pi,h)}-\pi|\Big\}^2\Big)\le C(1+|\pi|^2)\delta$**.**
We give the proof for $t=0$ from which the claim for general t follows immediately. Using the corresponding representation as stochastic integrals for the solution of equation (B.1) we find

$$\begin{aligned}\pi_s-\pi &= A_s+M_s \qquad \text{where}\\ A_s &= \int_0^s \underline{\alpha}(\pi_r,h_r)dr \text{ and } M_s=\int_0^s \underline{\beta}^\top(\pi_r)dB_r+\int_0^s \underline{\gamma}(\pi_r,u)\widetilde{N}(dr\times du).\end{aligned}$$

Then it holds

$$\begin{aligned}E\big(\{\sup_{0\le s\le\delta}|\pi_s-\pi|\}^2\big) &= E\big(\{\sup_{0\le s\le\delta}|A_s+M_s|\}^2\big)\\ &\le 2E\big(\sup_{0\le s\le\delta}|A_s|^2\big)+2E\big(\sup_{0\le s\le\delta}|M_s|^2\big). \qquad \text{(B.3)}\end{aligned}$$

For the first term on the r.h.s. we find by applying Cauchy-Schwarz inequality and the growth condition (4.7) for $\underline{\alpha}$

$$\begin{aligned}\sup_{0\le s\le\delta}|A_s|^2 &= \sup_{0\le s\le\delta}\left|\int_0^s \underline{\alpha}(\pi_r,h_r)dr\right|^2 \le \sup_{0\le s\le\delta} s\int_0^s|\underline{\alpha}(\pi_r,h_r)|^2\,dr.\\ &\le \delta\int_0^\delta C(1+|\pi_r|^2)dr.\end{aligned}$$

Taking expectation and applying estimate (6.1) we find

$$E\big(\sup_{0\le s\le\delta}|A_s|^2\big)\le\delta\int_0^\delta C(1+|\pi|^2)dr\le\delta C(1+|\pi|^2). \qquad \text{(B.4)}$$

For the second term on the r.h.s. of (B.3) Doob's inequality for martingales and Itô-Levy isometry yields

$$\begin{aligned}E\Big(\sup_{0\le s\le\delta}|M_s|^2\Big)\le 4E(|M_\delta|^2) &= 4\Big(\int_0^\delta E\big(tr[\underline{\beta}^\top(\pi_r)\underline{\beta}(\pi_r)]\big)dr\\ &\quad+\int_0^\delta\int_{\mathcal{U}}E\big(\,|\underline{\gamma}(\pi_r,u)|^2\,\big)\nu(du)dr\Big).\end{aligned}$$

Applying the growth conditions (4.9), (4.7) and estimate (6.1) it yields

$$\begin{aligned}E\Big(\sup_{0\le s\le\delta}|M_s|^2\Big) &\le C\Big(\int_0^\delta E(1+|\pi_r|^2)dr+\int_0^\delta\int_{\mathcal{U}}\rho^2(u)E(1+|\pi_r|^2)\nu(du)dr\Big)\\ &\le C(1+|\pi|^2)\int_0^\delta\Big(1+\int_{\mathcal{U}}\rho^2(u)\nu(du)\Big)dr\le C\delta(1+|\pi|^2). \qquad \text{(B.5)}\end{aligned}$$

Substituting (B.4) and (B.5) into (B.3) yields the assertion.

Inequality (6.4): $E(|\pi_\tau^{(t,\pi,h)}-\pi_\tau^{(t,\xi,h)}|^2)\le C(\pi-\xi)^2$**.**
For the sake of shorter notation we write $\pi_s=\pi_s^{(t,\pi,h)}$ and $\xi_s=\pi_s^{(t,\xi,h)}$ and we set $\Delta\underline{\alpha}(\pi,\xi,h)=\underline{\alpha}(\pi,h)-\underline{\alpha}(\xi,h)$, $\Delta\underline{\beta}(\pi,\xi)=\underline{\beta}(\pi)-\underline{\beta}(\xi)$ and $\Delta\underline{\gamma}(\pi,\xi)=$

$\underline{\gamma}(\pi,u)-\underline{\gamma}(\xi,u)$. Then,

$$\begin{aligned} Y_\tau &:= \pi_\tau-\xi_\tau=\pi-\xi+\int_t^\tau \Delta\underline{\alpha}(\pi_s,\xi_s,h_s)ds+\int_t^\tau \Delta\underline{\beta}^\top(\pi_s,\xi_s)dB_s \\ &\quad +\int_t^\tau\int_{\mathcal{U}}\Delta\underline{\gamma}(\pi_s,\xi_s,u)\widetilde{N}(ds\times du). \end{aligned}$$

Applying Itô's lemma to Y_s^2 and using Itô-Levy isometry we obtain

$$\begin{aligned} E(|Y_\tau|^2) &= |\pi-\xi|^2+E\Big(\int_t^\tau\Big\{2Y_s^\top\Delta\underline{\alpha}(\pi_s,\xi_s,h_s)+tr\Big(\Delta\underline{\beta}(\pi_s,\xi_s)\Delta\underline{\beta}^\top(\pi_s,\xi_s)\Big) \\ &\quad +\int_{\mathcal{U}}|\Delta\underline{\gamma}(\pi_s,\xi_s,u)|^2\nu(du)\Big\}ds\Big). \end{aligned}$$

Hence we obtain from the Lipschitz continuity of $\underline{\alpha},\underline{\beta},\underline{\gamma}$ given in Assumption 4.1

$$E(|Y_\tau|^2)\le|\pi-\xi|^2+CE\Big(\int_t^\tau|Y_s|^2ds\Big).$$

For any deterministic time $\tau=u$ Fubini's Theorem gives

$$E(|Y_u|^2)\le|\pi-\xi|^2+CE\Big(\int_t^u|Y_s|^2ds\Big)$$

and applying Gronwall's Lemma to $G_u:=E(|Y_u|^2)$ implies

$$E(|Y_u|^2)\le|\pi-\xi|^2e^{C(u-t)}\le C|\pi-\xi|^2.$$

Finally, we note, that for any stopping time $\tau\in[t,T\wedge t+\delta]$ it holds

$$E(|Y_\tau|^2)\le|\pi-\xi|^2+CE\Big(\int_t^{t+\delta}|Y_s|^2ds\Big)\le C|\pi-\xi|^2.$$

□

Appendix C. Proof of Proposition 6.3

Proof. **Boundedness of V.** We recall that $V(t,\pi)=\sup_{h\in\mathcal{H}}v(t,\pi,h)$ where

$$\begin{aligned} v(t,\pi,h) &= E\Big(\exp\Big\{\int_t^T-b(R\pi_s^{(t,\pi,h)},h_s)ds\Big\}\Big) \\ \text{with}\quad b(p,h) &= -\theta\Big(h^\top Mp-\frac{1-\theta}{2}\left|\sigma^\top h\right|^2\Big), \end{aligned}$$

and $\pi_s^{(t,\pi,h)}$ is the solution of the SDE (4.5) with initial value $\pi_t=\pi$.

The function b is bounded, since it is continuous and $\pi\in\underline{\mathcal{S}}$ and $h\in K$ take values in compact sets, i.e. $|b(R\pi,h)|\le C_b$ with some constant $C_b>0$. Hence $0\le v(t,\pi,h)\le e^{C_b(T-t)}\le e^{C_bT}$ for all $h\in\mathcal{H}$ which implies that $0\le V(t,\pi)\le e^{C_bT}$.

Note that since the value function V is bounded, it also satisfies the *linear growth condition* $V(t,\pi)\le C(1+|\pi|)$ since $|\pi|_\infty\le1$.

Lipschitz continuity in π. The reward function can be written as

$v(t,\pi,h) = E(e^{J(\pi)})$ where $J(\pi) := \int_t^T -b(R\pi_s^{(t,\pi,h)}, h_s)ds$. For $\theta \in (0,1)$ the following holds

$$\begin{aligned} |V(t,\pi) - V(t,\xi)| &= \big|\sup_{h\in\mathcal{H}} E(e^{J(\pi)}) - \sup_{h\in\mathcal{H}} E(e^{J(\xi)})\big| \le \sup_{h\in\mathcal{H}} \big|E(e^{J(\pi)} - e^{J(\xi)})\big| \\ &\le \sup_{h\in\mathcal{H}} E(|e^{J(\pi)} - e^{J(\xi)}|) \le \sup_{h\in\mathcal{H}} CE(|J(\pi) - J(\xi)|), \quad \text{(C.1)} \end{aligned}$$

where we used Lipschitz continuitiy of $f(x) = e^x$ on bounded intervals and the boundedness of $J(\pi)$ which follows, since b is bounded. For $\theta < 0$ we use $V(t,\pi) = \inf_{h\in\mathcal{H}} E(e^{J(\pi)}) = \sup_{h\in\mathcal{H}} -E(e^{J(\pi)})$ and apply analogous estimates.

Using that b is linear in the first variable and that $h_t \in K$ is uniformly bounded we derive

$$\begin{aligned} E(|J(\pi) - J(\xi)|) &= E\Big(\Big|\int_t^T \big[b(R\pi_s^{(t,\xi,h)}, h_s) - b(R\pi_s^{(t,\pi,h)}, h_s)\big]ds\Big|\Big) \\ &\le \int_t^T C\, E(|\pi_s^{(t,\pi,h)} - \pi_s^{(t,\xi,h)}|)\, ds \\ &\le C\int_t^T |\pi - \xi|\, ds \le C\,|\pi - \xi|\,, \quad \text{(C.2)} \end{aligned}$$

for every $h \in \mathcal{H}$, where we used estimate (6.4). Note that as always C denotes a generic constant. Plugging the above estimate into (C.1) it follows $|V(t,\pi) - V(t,\xi)| \le C\,|\pi - \xi|$, which proves the Lipschitz continuity of $V(t,\pi)$ in π.

Continuity in t. Let $0 \le t < s \le T$, then the dynamic programming principle to $V(t,\pi)$ implies

$$\begin{aligned} 0 &\le |V(t,\pi) - V(s,\pi)| \\ &= \sup_{h\in\mathcal{H}} E\Big(\exp\Big\{-\int_t^s b(R\pi_u^{(t,\pi,h)}, h_u)du\Big\} V(s,\pi_s^{(t,\pi,h)}) - V(s,\pi)\Big) \\ &\le \sup_{h\in\mathcal{H}} E\Big(\exp\Big\{-\int_t^s b(R\pi_u^{(t,\pi,h)}, h_u)du\Big\} \Big|V(s,\pi_s^{(t,\pi,h)}) - V(s,\pi)\Big|\Big) \\ &+ \sup_{h\in\mathcal{H}} E\Big(\Big|\exp\Big\{-\int_t^s b(R\pi_u^{(t,\pi,h)}, h_u)du\Big\} V(s,\pi) - V(s,\pi)\Big|\Big). \end{aligned}$$

Using the Lipschitz continuity of V in π the first term can be estimated by

$$C \sup_{h\in\mathcal{H}} E\Big(|\pi_s^{(t,\pi,h)} - \pi|\Big) \le C|s-t|^{\frac{1}{2}}$$

where we have used (6.3). Again C denotes a generic constant. For the second term the boundedness of b and V yields the estimate

$$|e^{C_b(s-t)} - 1|V(s,\pi) \le C|s-t|$$

where we have used that $f(x) = e^x$ is Lipschitz continuous on bounded intervals. Finally, we obtain

$$|V(t,\pi) - V(s,\pi)| \le C(|s-t|^{\frac{1}{2}} + |s-t|) \le (C + T^{\frac{1}{2}})|s-t|^{\frac{1}{2}}.$$

□

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