Communications on Stochastic Analysis

Volume 7 | Number 4 | Article 6

12-1-2013

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Recommended Citation
Kuo, Hui-Hsiung; Peng, Yun; and Szozda, Benedykt (2013) "Generalization of the anticipative Girsanov theorem," Communications on Stochastic Analysis: Vol. 7 : No. 4 , Article 6.
DOI: 10.31390/cosa.7.4.06
Available at: https://digitalcommons.lsu.edu/cosa/vol7/iss4/6
GENERALIZATION OF THE ANTICIPATIVE GIRSANOV
THEOREM

HUI-HSIUNG KUO, YUN PENG, AND BENEDYKT SZOZDA*

Abstract. We study the Itô formula and Girsanov theorem in the anticipa-
tive setting using the stochastic integral of adapted and instantly independent
processes. The results of the present paper extend several of the previously
known theorems. The generalization presented here can be summarized as
a domain extension as we allow for a more general class of processes to be
treated by the Itô formula and more general shifts to be used in the change of
measure in the Girsanov theorem. Finally, we apply our results to present a
toy problem of the Black–Scholes formula for a market that knows the future
but not the past.

1. Introduction

In the present paper, we extend and generalize the results of [10] to obtain a
version of the Girsanov theorem for a Brownian motion translated by a mixture
of adapted and backward-adapted terms. Our setup and notation follow closely
those of [10]. We let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(B_t\) be a Brownian
motion defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and \(\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}\); be its natural filtration, that is
\(\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}\). Since we are only interested in a finite time horizon, we
fix it to be \(T\). For the sake of brevity, we will write \(\mathcal{F}_t\) for \(\{\mathcal{F}_s : 0 \leq s \leq T\}\) and
\(\{f_t\}\) for \(\{f_t : 0 \leq t \leq T\}\). If \(\{f_t\}\) is a square-integrable stochastic process adapted to
\(\mathcal{F}_t\), we denote by \(\mathcal{E}_t(f)\) the stochastic exponential associated to \(f\) defined by
\[
\mathcal{E}_t(f) = \exp\left\{ \int_0^t f_s \, dB_s - \frac{1}{2} \int_0^t f_s^2 \, ds \right\}.
\]

First, we recall several theorems that are building blocks in many areas of
application of stochastic analysis, e.g. financial mathematics. Namely, the Itô
formula and the Girsanov theorem. We are concerned with generalization of these
theorems to an anticipative setting that is based on a new stochastic integral
introduced by Ayed and Kuo in [1, 2] and later developed by Kuo, Sae-Tang and
Szozda in [11, 12, 13] and by Khalifa et al. in [8].

Received 2013-11-12; Communicated by the editors.
2010 Mathematics Subject Classification. Primary 60H05; Secondary 60H20.
Key words and phrases. Brownian motion, Itô integral, Itô formula, adapted stochastic pro-
cesses, instantly independent stochastic processes, anticipating stochastic processes, stochastic
integral, anticipating integral, Girsanov theorem, Black–Scholes formula.
* Benedykt Szozda acknowledges support from the T.N. Thiele Centre for Applied Mathematics in Natural Science and from CREATE (DNRF78), funded by the Danish National Research Foundation.
The first statement of the Girsanov theorem in the setting of the new stochastic integral appears in [10]. In the present paper we present a generalization of the results of [10] as well as generalization of some of the results on Itô formula from [12]. We review the relevant results on the new stochastic integral in Section 2 and recall the results that we intend to generalize in Section 3 where we also present the first extensions.

Section 4 contains an extension of the results of [12] while Section 5 contains the main results of the present paper, that is the generalization of the Girsanov theorem to the Brownian motion shifted by mixture of adapted and backward-adapted stochastic processes. Finally, in Section 6 we apply the results of Section 3 to obtain a Black–Scholes type formula for stock whose price is driven by backward-adapted processes.

To conclude the introduction, let us state two classic results that are generalized in the forthcoming sections. Namely the Itô formula and the Girsanov theorem.

**Theorem 1.1** (Itô Formula – adapted). Suppose that \( \{X_t^{(i)}: i = 1, 2, \ldots, n\} \) are continuous martingales with respect to \( \{\mathcal{F}_t: 0 \leq t \leq T\} \) and \( f(x_1, x_2, \ldots, x_n) \) is a twice continuously differentiable real function on \( \mathbb{R}^n \). Then

\[
\begin{align*}
&f(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)}) = f(X_0^{(1)}, X_0^{(2)}, \ldots, X_0^{(n)}) \\
&\quad + \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n)}) \, dX_i \\
&\quad + \frac{1}{2} \int_0^t \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n)}) \, d\langle X_i, X_j \rangle_s,
\end{align*}
\]

where \( \langle X, Y \rangle_t \) stands for predictable covariation of \( \{X_t\} \) and \( \{Y_t\} \).

**Theorem 1.2** (Girsanov, 1960, [6]). Let \( \{f_t\} \) be a square-integrable stochastic process adapted to \( \{\mathcal{F}_t\} \) such that \( \mathbb{E}_P[\mathcal{E}_t(f)] < \infty \) for all \( t \in [0, T] \). Then,

\[
\tilde{B}_t = B_t - \int_0^t f_s \, ds
\]

is a Brownian motion with respect to an equivalent probability measure \( Q \), given by

\[
dQ = \mathcal{E}_T(f) \, dP.
\]

2. The New Integral

As we have mentioned in the introduction, Ayed and Kuo [1, 2] introduced a new approach to stochastic integration of anticipating stochastic processes. Below we briefly recall their construction.

A stochastic process \( \{\varphi_t\} \) is said to be *instantly independent* of the filtration \( \{\mathcal{F}_t\} \) if for each \( t \in [0, T] \), the random variable \( \varphi_t \) and the \( \sigma \)-field \( \mathcal{F}_t \) are independent. For example \( \varphi(B_1 - B_t) \) is instantly independent of \( \{\mathcal{F}_t: t \in [0, 1]\} \) for any real measurable function \( \varphi(x) \). Observe that for \( t \geq 1 \), \( \varphi(B_1 - B_t) \) is adapted to \( \{\mathcal{F}_t\} \).
Definition 2.1. Suppose that \( \{ f_t \} \) is a stochastic process adapted to the filtration \( \{ \mathcal{F}_t \} \) and \( \{ \varphi_t \} \) is instantly independent of the same filtration. We define the stochastic integral of \( f_t \varphi_t \) as

\[
\int_0^T f_t \varphi_t \, dB_t = \lim_{\| \Delta_n \| \to 0} \sum_{i=1}^n f_{t_{i-1}} \varphi_{t_i} \Delta B_{t_i},
\]

where \( \Delta_n = \{ 0 = t_0 < t_1 < \ldots < t_n = T \} \) is a partition of the interval \([0, T]\) and \( \Delta B_t = B_{t_i} - B_{t_{i-1}} \) and \( \| \Delta_n \| = \max \{ t_i - t_{i-1} : i = 1, \ldots, n \} \), provided the limit exists in probability.

One of the central concepts in stochastic analysis is that of a martingale. Below we recall some basic facts about martingales and their instantly independent counterpart, near-martingales. The latter kind of processes were introduced and studied by Kuo, Sae-Tang and Szozda in [11]. They also appear in [3], where the authors call them increment martingales.

Definition 2.2. A stochastic process \( \{ X_t \} \) is said to be a martingale with respect to a filtration \( \{ \mathcal{F}_t \} \) if \( E|X_t| < \infty \) for all \( t \in [0, T] \) and

\[
E[X_t | \mathcal{F}_s] = X_s, \quad 0 \leq s < t \leq T.
\]

Taking into consideration the definition of the conditional expectation, we immediately see that \( \{ X_t \} \) as defined above is adapted to \( \{ \mathcal{F}_t \} \). It is, therefore, not feasible in the anticipating setting. However, in [11], authors define a near-martingale, which, as we will see, serves as an instantly independent counterpart to martingales.

Definition 2.3. We say that a process \( \{ X_t \} \) is a near-martingale with respect to a filtration \( \{ \mathcal{F}_t \} \) if \( E|X_t| < \infty \) for all \( 0 \leq t \leq T \) and \( E[X_t - X_s | \mathcal{F}_s] = 0 \) for all \( 0 \leq s < t \leq T \).

It is not hard to see that an adapted near-martingale is a martingale. For more properties of near-martingales we refer to [11].

It is a well-known fact that the Itô integral is a martingale, that is \( X_t = \int_0^t f_s \, dB_s \) is a martingale with respect to \( \{ \mathcal{F}_t \} \), for any adapted stochastic process \( \{ f_t \} \) that is integrable with respect to \( B_t \) on the interval \([0, T]\). Similarly, if \( f_t \) and \( \varphi_t \) are as in Definition 2.1, and \( Y^{(t)} = \int_0^t f_s \varphi_s \, dB_s \) exists for all \( t \in [0, T] \), then \( Y^{(t)} \) is a near-martingale with respect to \( \{ \mathcal{F}_t \} \) (see [11, Theorem 3.5]). Furthermore, \( Y^{(t)} \) is also a near-martingale with respect to a natural backward filtration \( \{ \mathcal{G}^{(t)} \} \) of \( B_t \) defined by

\[
\mathcal{G}^{(t)} = \sigma \{ B_T - B_s : t \leq s \leq T \}.
\]

For details see [11, Theorem 3.7]. In general, a backward filtration is any decreasing family of \( \sigma \)-fields, i.e. \( \{ \mathcal{G}^{(t)} \} \) satisfies \( \mathcal{G}^{(t)} \subseteq \mathcal{G}^{(s)} \) for any \( 0 \leq s \leq t \leq T \). A similar concept is also used in [14]. A process adapted to the natural backward Brownian filtration will be called backward-adapted.

Before we proceed, let us introduce a backward Brownian motion \( B^{(t)} \), that is a process given by

\[
B^{(t)} = B_T - B_{T-t}.
\]
It is in fact a Brownian motion in the filtration $\overline{G}^{(t)}$ (see [10, Proposition 3.2]) given by

$$\overline{G}^{(t)} = G^{(T-t)}.$$  

Notice that this is a forward filtration induced by the backward filtration $G^{(t)}$ of the underlying Brownian motion $B_t$.

3. From instantly independent to Backward-Adapted Processes

In this section we present a generalization of several results from [10]. Before we proceed with proofs of the new results, let us recall the versions of the theorems from [10].

**Theorem 3.1** (Itô formula, [10, Theorem 3.4]). Suppose that

$$Y_i^{(t)} = \int_t^T h_i(B_t - B_s) dB_s + \int_t^T g_i(B_t - B_s) ds, \quad i = 1, 2, \ldots, n,$$

where $h_i, g_i, \quad i = 1, 2, \ldots, n$ are continuous, square-integrable functions. Then for any $i$, $Y_i$ is instantly independent with respect to $\{F_t\}$. Furthermore, let $f(x_1, x_2, \ldots, x_n)$ be a function in $C^2(\mathbb{R}^n)$. Then

$$df(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}) dY_i^{(t)} - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)})(dY_i^{(t)})(dY_j^{(t)}).$$

**Theorem 3.2** ([10, Theorem 4.4]). Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and $\varphi(x)$ is a square-integrable function on $\mathbb{R}$. Let

$$\tilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s) ds.$$  

Then $\tilde{B}_t$ is a continuous near-martingale with respect to the probability measure $Q$ given by

$$dQ = \exp\left\{-\int_0^T \varphi(B_T - B_s) dB_s - \frac{1}{2} \int_0^T \varphi^2(B_T - B_s) ds\right\} dP.$$  

**Theorem 3.3** ([10, Theorem 4.5]). Suppose that the assumptions of Theorem 3.2 hold and

$$\tilde{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) ds.$$  

Then $\tilde{B}_t^2 - (T - t)$ is a continuous $Q$-near-martingale.

**Theorem 3.4** ([10, Theorem 4.6]). Suppose that the assumptions of Theorem 3.2 hold. Then the $Q$-quadratic variation of $\tilde{B}$ on the interval $[0, t]$ is equal to $t$.

In the present paper, we weaken the assumptions of Theorems 3.1–3.4. In general, the main improvement lies in the fact that we drop the explicit dependence on the tail of Brownian motion in favor of adaptedness to the backward filtration.
That is, instead of representing the underlying function as \( f(B_T - B_t) \), we assume that \( f_t \) is a backward-adapted stochastic process.

Notice that this is in fact a generalization for if \( f \) is any measurable real-valued function, it follows that \( f(B_T - B_t) \) is adapted to the natural backward Brownian filtration \( \mathcal{G}^t \). Moreover, it is a nontrivial generalization. A simple example that is not in the scope of the theory of [10] is the following. For a square-integrable real-valued function \( g \) define

\[
\theta_t = \int_t^T g(B_T - B_s) \, dB_s.
\]

Then \( \{\theta_t\} \) is backward-adapted and (in general) cannot be expressed as \( \theta_t = f(B_T - B_t) \). Of course, backward-adapted processes are instantly independent, but not all instantly independent stochastic processes are backward-adapted.

We begin with a generalized version of the Itô formula in Theorem 3.1. Namely, we replace the integrands \( h_i(B_T - B_t) \) and \( g_i(B_T - B_t) \) by any processes that are backward-adapted.

To prove the Itô Formula, we need the following technical lemma. It is a direct generalization of [10, Lemma 3.3].

**Lemma 3.5.** Suppose that \( B_t \) is a Brownian motion and \( \{B^i(t)\} \) is its backward Brownian motion, that is \( B^i(t) = B_T - B_{T-t} \) for all \( 0 \leq t \leq T \). Suppose also that \( g_t \) is a square-integrable process adapted to \( \mathcal{G}^t \). Then the following two identities hold

\[
\begin{align*}
\int_t^T g_s \, ds & = \int_0^{T-t} g_{t-s} \, ds \quad \text{(3.1)} \\
\int_t^T g_s \, dB_s & = \int_0^{T-t} g_{t-s} \, dB^i(s). \quad \text{(3.2)}
\end{align*}
\]

**Proof.** Let us first show that Equation (3.1) holds. Note that application of a change of variables \( \bar{s} = T - s \) in the right side of Equation (3.1) yields

\[
\int_0^{T-t} g_{t-s} \, ds = - \int_t^T g_{t-s} \, d\bar{s} = \int_t^T g_{t-s} \, d\bar{s}.
\]

Thus the validity of Equation (3.1) is proven.

Next, we show that Equation (3.2) holds. By the definition of the stochastic integral the right side of Equation (3.2) becomes

\[
\int_0^{T-t} g_{t-s} \, dB^i(s) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n g_{T-t_{i-1}} (B^i(t_i) - B^i(t_{i-1}))
\]

\[
= \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n g_{T-t_{i-1}} (B_{T-t_{i-1}} - B_{T-t_i}), \quad \text{(3.3)}
\]

where \( \Delta_n \) is a partition of the interval \([0, T-t]\) and the convergence is understood to be in probability on the space \( (\Omega, \mathcal{G}^{(T)}, P) \). A change of variables, \( t_i = T - t_i \),
$i = 1, 2, \ldots, n$ transforms Equation (3.3) into
\[
\int_0^{T-t} g_{T-s} dB(s) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} g_{T_{i-1}}(B_{T_{i-1}} - B_{T_i})
\]
(3.4)

Since $T = \tilde{t}_0 > \tilde{t}_1 > \tilde{t}_2 > \cdots > \tilde{t}_n = t$ can be chosen arbitrarily, and the probability space $(\Omega, \mathcal{G}(T), P)$ coincides with $(\Omega, \mathcal{F}_T, P)$, see [10, Theorem 3.1], by the definition of the new stochastic integral, the last term in Equation (3.4) converges in probability to the new stochastic integral
\[
\int_t^T g_s dB_s.
\]
Hence the Equation (3.2) holds. 

Now we are ready to prove the generalization of the Itô formula.

**Theorem 3.6.** Suppose that
\[
Y_i(t) = \int_t^T h_i(s) dB(s) + \int_t^T g_i(s) ds \quad i = 1, 2, \ldots, n,
\]
where $h_i(s), g_i(s)$ for $i = 1, 2, \ldots, n$ are continuous square-integrable stochastic processes that are adapted to $\mathcal{G}(t)$. Then for any $i = 1, 2, \ldots, n$, $Y_i$ is instantly independent with respect to $\mathcal{F}_t$. Let furthermore $f(x_1, x_2, \ldots, x_n)$ be a function in $C^2(\mathbb{R}^n)$, we have following Itô Formula,
\[
df(Y_1(t), Y_2(t), \ldots, Y_n(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Y_1(t), Y_2(t), \ldots, Y_n(t)) dY_i(t)
- \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1(t), Y_2(t), \ldots, Y_n(t)) \left(dY_i(t)\right) \left(dY_j(t)\right).
\]
(3.5)

**Proof.** Since the only difference between the arguments establishing the one- and multi-dimensional cases is the amount of bookkeeping, we will only show that Equation (3.5) holds with $n = 1$. For the sake of clarity of notation, we let
\[
Y(t) = \int_t^T h_s dB_s + \int_t^T g_s ds.
\]
Let us define
\[
X_t = \int_0^t h_{T-s} dB(s) + \int_0^t g_{T-s} ds.
\]
Since $h_s$ and $g_s$ are adapted to $\mathcal{G}(t)$, we can view $X_t$ as an Itô integral on the probability space $(\Omega, \mathcal{G}(0), P)$. Application of the classic Itô Formula and the Itô
table yield

\[
f(X_{T-t}) - f(X_0) = \int_0^{T-t} f'(X_s) \, dX_s + \frac{1}{2} \int_0^{T-t} f''(X_s) \, (dX_s)^2 \\
= \int_0^{T-t} f'(X_s) h_{T-s} \, dB(s) + \int_0^{T-t} f'(X_s) g_{T-s} \, ds \\
+ \frac{1}{2} \int_0^{T-t} f''(X_s) h_{T-s}^2 \, ds. \tag{3.6}
\]

By Lemma 3.5 we have the following identities

\[
X_{T-t} = Y(t), \\
\int_0^{T-t} f'(X_s) h_{T-s} \, dB(s) = \int_t^T f'(X_{T-s}) h_s \, dB_s, \\
\int_0^{T-t} f'(X_s) g_{T-s} \, ds = \int_t^T f'(X_{T-s}) g_s \, ds, \\
\int_0^{T-t} f''(X_s) h_{T-s}^2 \, ds = \int_t^T f''(X_{T-s}) h_s^2 \, ds. \tag{3.7}
\]

Putting Equations (3.6) and (3.7) together gives

\[
f(Y(t)) - f(Y(T)) = \int_t^T f'(Y(s)) h_s \, dB_s + \int_t^T f'(Y(s)) g_s \, ds \\
+ \frac{1}{2} \int_t^T f''(Y(s)) h_s^2 \, ds. \tag{3.8}
\]

\[
\frac{1}{2} \int_t^T f''(Y(s)) h_s^2 \, ds. \tag{3.9}
\]

Notice that \( dY(t) = -h_t \, dB_t - g_t \, dt \) and \( (dY(t))^2 = h_t^2 \, dt \). Using the above in Equation (3.9) and changing to the differential notation yields

\[
df(Y(t)) = f'(Y(t)) \, dY(t) - \frac{1}{2} f''(Y(t)) \, (dY(t))^2,
\]

which ends the proof. \(\square\)

Since it is not difficult to derive a corollary to Theorem 3.6 that covers the case when the function \( f \) depends explicitly on time, we state it without a proof.

**Corollary 3.7.** Suppose that

\[
Y^{(i)}_t = \int_t^T h^{(i)}_s \, dB(s) + \int_t^T g^{(i)}_s \, ds \quad i = 1, 2, \ldots, n,
\]

where \( h^{(i)}_s, g^{(i)}_s \) for \( i = 1, 2, \ldots, n \) are continuous square-integrable stochastic processes that are adapted to \( \mathcal{G}^{(i)} \). Suppose also that \( f(x_1, x_2, \ldots, x_n, t) \) is a function twice continuously differentiable in the first \( n \) variables and once continuously
differentiable in the last variable. Then,
\[
df(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}, t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}, t) \, dY_i^{(t)} - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}, t) \left( dY_i^{(t)} \right) \left( dY_j^{(t)} \right) + \frac{\partial f}{\partial t}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}, t).
\]

Using Theorem 3.6, we can easily find the counterpart to the exponential process for any process \( \theta_t \) adapted to the backward filtration \( \{G_t^{(t)}\} \).

**Example 3.8.** Suppose that \( \theta_t \) is a square-integrable stochastic process adapted to \( \{G_t^{(t)}\} \) and let
\[
E^{(t)}(\theta) = \exp \left\{ - \int_{t}^{T} \theta_s \, dB_s - \frac{1}{2} \int_{t}^{T} \theta_s^2 \, ds \right\}.
\]
Then
\[
dE^{(t)}(\theta) = \theta_t E^{(t)}(\theta) \, dB_t.
\]
The process \( E^{(t)}(\theta) \) is called an exponential process of the backward-adapted process \( \theta_t \).

**Proof.** Let \( f(x) = e^x \) and define
\[
Y_t = - \int_{t}^{T} \theta_s \, dB_s - \frac{1}{2} \int_{t}^{T} \theta_s^2 \, ds.
\]
Since \( f(x) = f'(x) = f''(x) \) and \( f(Y_t) = E^{(t)}(\theta) \), application of Theorem 3.6 to \( f(Y_t) \), yields
\[
dE^{(t)}(\theta) = df(Y_t)
\]
\[
= f'(Y_t) \, dY_t - \frac{1}{2} f''(Y_t) \, (dY_t)^2
\]
\[
= e^{Y_t} \left( \theta_t \, dB_t + \frac{1}{2} \theta_t^2 \, dt \right) - \frac{1}{2} e^{Y_t} \theta_t^2 \, dt
\]
\[
= \theta_t E^{(t)}(\theta) \, dB_t.
\]
Above we have used the fact that \( dY_t = \theta_t \, dB_t + \frac{1}{2} \theta_t^2 \, dt \). \( \square \)

Next, we generalize Theorems 3.2–3.4 in the same spirit as Theorem 3.6 generalizes Theorem 3.1. We begin with a theorem that is an extension of Theorem 3.2.

**Theorem 3.9.** Suppose that \( \{B_t\} \) is a Brownian motion on \((\Omega, \mathcal{F}_T, P)\) and \( \varphi_t \) is a square-integrable real-valued stochastic process adapted to \( \{G_t^{(t)}\} \). Let
\[
\tilde{B}_t = B_t + \int_{0}^{t} \varphi(B_T - B_s) \, ds.
\]
Then \( \bar{B}_t \) is a continuous near-martingale with respect to the probability measure \( Q \) given by

\[
dQ = \mathcal{E}^{(0)}(\varphi) \, dP = \exp \left\{ - \int_0^T \varphi_s \, dB_s - \frac{1}{2} \int_0^T \varphi_s^2 \, ds \right\} dP.
\] (3.11)

The following theorem generalizes Theorem 3.3.

**Theorem 3.10.** Suppose that \( \{B_t\} \) is a Brownian motion on \((\Omega, \mathcal{F}_T, P)\). Suppose also that \( \varphi_t \) is a square-integrable real-valued stochastic process adapted to \( \{G^{(1)}\} \) and \( Q \) is the probability measure given by Equation (3.11). Let

\[
\bar{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) \, ds.
\]

Then \( \bar{B}_t^2 - (T - t) \) is a continuous \( Q \)-near-martingale.

Finally, we state the generalization of Theorem 3.4.

**Theorem 3.11.** Suppose that \( \{B_t\} \) is a Brownian motion in the probability space \((\Omega, \mathcal{F}_T, P)\), \( Q \) is a measure given by Equation (3.11) and \( \tilde{B} \) be given by Equation (3.10). Then the \( Q \)-quadratic variation of \( \tilde{B} \) on the interval \([0, t]\) is equal to \( t \).

Proofs of Theorems 3.9–3.11 follow the lines of the proofs of Theorems 4.4–4.6 of [10] with processes of the form \( g(B_T - B_t) \) substituted for backward-adapted processes \( g_t \). For the sake of brevity we omit the details and refer an interested reader to [10].

### 4. Itô Formula for Mixed Terms

In Section 3, we proved the Itô formula for the backward-adapted Itô processes. The obvious limitation of the aforementioned Itô formula is the fact that it can treat functions that depend on the backward-adapted processes only. In the present section, we prove a more general result that is applicable to function depending on adapted and backward-adapted Itô processes.

The **adapted Itô process** is a stochastic process of the form

\[
X_t = \int_0^t h_s \, dB(s) + \int_0^t g_s \, ds,
\] (4.1)

where \( h_t, g_t \) are adapted square-integrable processes. The **backward-adapted Itô process** is a stochastic process of the form

\[
Y^{(t)} = \int_t^T \eta_s \, dB(s) + \int_t^T \zeta_s \, ds,
\] (4.2)

where \( \eta_t, \zeta_t \) are backward-adapted processes.

The classic Itô formula is applicable to functions of \( X_t \), while Theorem 3.6 is applicable to functions of \( Y^{(t)} \). The next theorem constitutes an Itô formula for functions that depend on both types of processes. It is a first step towards a general Itô formula and it only applies to functions of the form \( \theta(X_t, Y^{(t)}) \), where
\( \theta(x, y) = f(x) \varphi(y) \). The first Itô formula of this type was introduced in [12, Theorem 5.1], where authors treated only the case when \( \eta, \zeta \) are deterministic functions. Thus, while our arguments are similar to those of [12], our result extends the result of [12] substantially.

**Theorem 4.1.** Suppose that \( \theta(x, y) \) is a function of the form \( \theta(x, y) = f(x) \varphi(y) \), where \( f \) and \( \varphi \) are twice continuously differentiable real-valued functions. Suppose also that \( X_t \) and \( Y_t \) are defined as in Equations (4.1) and (4.2) respectively. Then

\[
\theta(X_T, Y_T) = \theta(X_0, Y_0) + \int_0^T \frac{\partial \theta}{\partial x}(x_t, y_t)\, dx_t + \frac{1}{2} \int_0^T \frac{\partial^2 \theta}{\partial x^2}(x_t, y_t)\, (dx_t)^2
\]

\[
+ \int_0^T \frac{\partial \theta}{\partial y}(x_t, y_t)\, dy_t - \frac{1}{2} \int_0^T \frac{\partial^2 \theta}{\partial y^2}(x_t, y_t)\, (dy_t)^2.
\]

**Proof.** We begin by writing out \( \theta(x_t, y_t) - \theta(x_0, y_0) \) as a telescoping sum. For any partition \( \delta_n = \{ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \} \), we have

\[
\theta(x_t, y_t) - \theta(x_0, y_0) = \sum_{i=1}^n [\theta(x_{t_i}, y_{t_i}) - \theta(x_{t_{i-1}}, y_{t_{i-1}})]
\]

\[
= \sum_{i=1}^n [f(x_{t_i}) \varphi(y_{t_i}) - f(x_{t_{i-1}}) \varphi(y_{t_{i-1}})]. \tag{4.3}
\]

Now, we apply the Taylor expansion to \( f \) and \( \varphi \), to obtain

\[
f(x_{t_i}) = \sum_{k=0}^\infty \frac{1}{k!} f^{(k)}(x_{t_{i-1}})(\delta x_i)^k
\]

\[
\varphi(y_{t_{i-1}}) = \sum_{k=0}^\infty \frac{1}{k!} \varphi^{(k)}(y_{t_{i-1}})(-\delta y_i)^k,
\]

where \( \Delta X_t = X_{t_i} - X_{t_{i-1}} \) and \( \Delta Y_t = Y_{t_i} - Y_{t_{i-1}} \). Using the standard approximation results for the Brownian motion and adapted Itô processes, we obtain the following approximations

\[
\Delta X_t \approx b_{t_{i-1}} \Delta B_t + g_{t_{i-1}} \Delta t_i.
\]

\[
(\Delta X_t)^2 \approx h_{t_{i-1}}^2 \Delta t_i
\]

\[
(\Delta X_t)^k = o(\Delta t_i) \quad \text{for } k \geq 3 \tag{4.4}
\]

To obtain a result for \( \Delta Y_t \) analogous to the first of Equations (4.4), we employ Lemma 3.5

\[
\Delta Y_t = \int_{t_i}^{t} \eta_s \, dB_s + \int_{t_i}^{T} \zeta_s \, ds - \int_{t_{i-1}}^{T} \eta_s \, dB_s - \int_{t_{i-1}}^{t} \zeta_s \, ds
\]

\[
= \int_{0}^{T-t_{i-1}} \eta_{t-s} \, dB^{(s)} - \int_{0}^{T-t_{i-1}} \eta_{t-s} \, dB^{(s)} - \int_{t_{i-1}}^{t} \zeta_s \, ds \tag{4.5}
\]

\[
= -\int_{T-t_i}^{T-t_{i-1}} \eta_{t-s} \, dB^{(s)} - \int_{t_{i-1}}^{t} \zeta_s \, ds.
\]
Now, the first of the integrals in Equation (4.5) can be viewed as a standard Itô integral of an adapted process with respect to a Brownian motion \( B^{(t)} \) in its natural filtration \( \mathcal{G}^{(t)} \). Notice that since \( T - t_{i-1} > T - t_i \), Equation (4.5) can be approximated as
\[
\Delta Y_i \approx -\eta_i \Delta B_i - \zeta_i \Delta t_i = -\eta_i \Delta B_i - \zeta_i \Delta t_i.
\]
Thus,
\[
(\Delta Y_i)^2 \approx \eta_i^2 \Delta t_i \quad \text{and} \quad (\Delta Y_i)^k \approx o(\Delta t_i) \text{ for } k \geq 3. \tag{4.6}
\]
Putting Equations (4.3) and (4.5)–(4.6) together yields
\[
\theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)})
= \sum_{i=1}^{n} \left\{ f'(X_{t_{i-1}})\varphi(Y^{(t_i)}) [h_{t_{i-1}} \Delta B_i + g_{t_{i-1}} \Delta t_i] + \frac{1}{2} f''(X_{t_{i-1}})\varphi(Y^{(t_i)})h_{t_{i-1}}^2 \Delta t_i + f(X_{t_{i-1}})\varphi'(Y^{(t_i)}) [-\eta_i \Delta B_i - \zeta_i \Delta t_i] - \frac{1}{2} f(X_{t_{i-1}})\varphi''(Y^{(t_i)}) \eta_i^2 \Delta t_i \right\}.
\]
Using Definition 2.1 of the new stochastic integral, the definition of the Itô integral and letting \( n \) go to \( \infty \), we obtain
\[
\theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)})
\]
\[
= \int_0^T f'(X_t)\varphi(Y^{(t)})h_t \, dB_t + \int_0^T f'(X_t)\varphi(Y^{(t)})g_t \, dt + \frac{1}{2} \int_0^T f''(X_t)\varphi(Y^{(t)})h_t^2 \, dt - \frac{1}{2} \int_0^T f(X_t)\varphi'(Y^{(t)})\eta_t \, dB_t + \int_0^T f(X_t)\varphi'(Y^{(t)})\zeta_t \, dt - \frac{1}{2} \int_0^T f(X_t)\varphi''(Y^{(t)})\eta_t^2 \, dt.
\]
This proves our claim. \( \square \)

5. Girsanov Theorem for Mixture of Adapted and Anticipative Shifts

The main result of the present paper follows from application of the classic Girsanov Theorem 1.2 as well as Theorems 3.9–3.11 that constitute an anticipative version of the Girsanov theorem. The improvement of Theorems 5.1–5.4 over Theorems 3.9–3.11 lays in the fact that we allow for the translations of Brownian motion that can be decomposed into a sum of processes that are either adapted to \( \{\mathcal{F}_t\} \) or adapted to \( \{\mathcal{G}^{(t)}\} \). In this setting, we find that the Girsanov type results have the exact same form as Theorem 1.2.

**Theorem 5.1.** Suppose that \( \{B_t\} \) is a Brownian Motion and \( \{\mathcal{F}_t\} \) is its natural filtration on probability space \((\Omega, \mathcal{F}, P)\). Let \( f_t \) and \( g_t \) be continuous square-integrable
stochastic processes such that $f_t$ is adapted to $\{F_t\}$ and $g_t$ is adapted to $\{G^{(t)}\}$, i.e. the backward Brownian filtration. Let

$$\tilde{B}_t = B_t + \int_0^t (f_s + g_s) \, ds.$$  

Then $\tilde{B}_t$ is a near-martingale with respect to $(\Omega, F, Q)$, where

$$dQ = \exp\left\{ - \int_0^T (f_t + g_t) \, dB_t - \frac{1}{2} \int_0^T (f_t + g_t)^2 \, dt \right\} \, dP. \tag{5.1}\label{5.1}$$

**Remark 5.2.** This theorem can be proved with methods similar to the ones used in [10], that is by defining the exponential process for a sum of processes $f$ and $g$ adapted to $\{F_t\}$ and $\{G^{(t)}\}$ respectively, and using the results of the preceding sections to repeat the calculations done in [10]. However, since the results that are applicable to translations of Brownian motion by $\int_0^t f(s) \, ds$ and $\int_0^t g(s) \, ds$ separately already exist, we can apply them to obtain a shorter proof.

**Proof.** First, let us rewrite $\tilde{B}_t$ as

$$\tilde{B}_t = B_t + \int_0^t f_s \, ds + \int_0^t g_s \, ds$$

and define $W_t = B_t + \int_0^t f_s \, ds$. Thus $\tilde{B}_t = W_t + \int_0^t g_s \, ds$. Since $f_t$ is adapted, application of the original Girsanov theorem yields that $W_t$ is a Brownian motion with respect to $(\Omega, F_T, Q_1)$ where

$$dQ_1 = \exp\left\{ - \int_0^T f_t \, dB_t - \frac{1}{2} \int_0^T f_t^2 \, dt \right\} \, dP. \tag{5.2}\label{5.2}$$

Now, since $W_t$ is a Brownian motion on $(\Omega, F_T, Q_1)$ and $g_t$ is adapted to the backward filtration $\{G^{(t)}\}$, we can apply Theorem 3.11. Therefore, $\tilde{B}(t)$ is a near-martingale with respect to $(\Omega, F_T, Q)$, where

$$dQ = \exp\left\{ - \int_0^T g_t \, dW_t - \frac{1}{2} \int_0^T g_t^2 \, dt \right\} \, dQ_1. \tag{5.3}\label{5.3}$$

with $dQ_1$ given by Equation (5.2).

It remains to show that the measure $Q$ in Equation (5.3) coincides with the measure $Q$ in Equation (5.1). To this end, we put together the identity $dW_t = dB_t + f_t \, dt$, Equation (5.2) and Equation (5.3) obtain

$$dQ = \exp\left\{ - \int_0^T g_t \, dW_t - \frac{1}{2} \int_0^T g_t^2 \, dt \right\} dQ_1$$

$$= \exp\left\{ - \int_0^T g_t \, dB_t - \frac{1}{2} \int_0^T g_t^2 \, dt - \int_0^T f_t \, dB_t - \frac{1}{2} \int_0^T f_t^2 \, dt \right\} dP$$

$$= \exp\left\{ - \int_0^T g_t \, dB_t - \int_0^T g_t f_t \, dt - \frac{1}{2} \int_0^T g_t^2 \, dt - \int_0^T f_t \, dB_t - \frac{1}{2} \int_0^T f_t^2 \, dt \right\} dP$$

$$= \exp\left\{ - \int_0^T (g_t + f_t) \, dB_t - \frac{1}{2} \int_0^T (g_t + f_t)^2 \, dt \right\} dP.$$

Thus the theorem holds. \qed
Next we state generalization of Theorem 3.10.

**Theorem 5.3.** Suppose that assumptions of Theorem 5.1 hold. Let
\[
\hat{B}_t = B_T - B_t + \int_t^T (f_s + g_s) \, ds.
\]
Then \(\hat{B}_t^2 - (T - t)\) is a continuous \(Q\)-near-martingale.

Finally, we give the generalization of Theorem 3.11.

**Theorem 5.4.** Suppose that the assumptions of Theorem 5.1 hold. Then the \(Q\)-quadratic variation of \(\hat{B}\) on the interval \([0, t]\) is equal to \(t\).

Note that the proofs of Theorems 5.3 and 5.4 follow the same reasoning as the proof of Theorem 5.1, that is one first applies the adapted version of the Girsanov theorem (see Theorem 1.2) and then applies one of Theorems 3.10 or 3.11. We omit these proofs for the sake of brevity.

**Remark 5.5.** Using the relationship between probability measures \(Q\) and \(Q_1\) given by Equation (5.3) from the proof of Theorem 5.1 we can deduce an interesting stochastic differential equation. To this end we will follow the lines of Example 3.8. From Equation (5.3) we have
\[
dQ = \exp\left\{ -\int_0^T g_t \, dW_t - \frac{1}{2} \int_0^T g_t^2 \, dt \right\} dQ_1.
\]
Let us define
\[
\theta^{(t)}(g) = \exp\left\{ -\int_t^T g_s \, dW_s - \frac{1}{2} \int_t^T g_s^2 \, ds \right\},
\]
Clearly, according to Example 3.8, \(\theta^{(t)}(g)\) is a backward exponential process for the backward-adapted stochastic process \(g_t\) in the space \((\Omega, \mathcal{F}_T, Q_1)\). Thus we have the following SDE
\[
d\theta^{(t)}(g) = g_t \theta^{(t)}(g) \, dW_t
\]
\[
= g_t \theta^{(t)}(g) (dB_t + f_t \, dt)
\]
\[
= g_t \theta^{(t)}(g) dB_t + f_t g_t \theta^{(t)}(g) \, dt.
\]
The above equation may give some insight into Itô formulas for processes that are adapted to neither \(\{\mathcal{F}_t\}\) nor \(\{\mathcal{G}^{(t)}\}\) as the last term in the above equation is a stochastic process of the form
\[
X_t = \int_0^t f_s \varphi_s \, ds,
\]
with \(f\) and \(\varphi\) being adapted to \(\{\mathcal{F}_t\}\) and \(\{\mathcal{G}^{(t)}\}\) respectively.

We conclude this section with an example.

**Example 5.6.** Let
\[
X_t = B_t + \int_0^t B_1 \, dB_s,
\]
where \( B_s \) is a Brownian motion on the probability space \((\Omega, \mathcal{F}_T, P)\). Define the equivalent probability measure \( Q \) by

\[
dQ = \exp\{-\int_0^T B_1 dB_t - \frac{1}{2} \int_0^T B_1^2 dt\}.
\]

Using Theorems 5.1–5.4, we conclude that \( X_t \) is a near-martingale in the probability space \((\Omega, \mathcal{F}_T, Q)\), its quadratic variation on the interval \([0, t]\) is equal to \( t \) and if

\[
\tilde{X}_t = X_T - X_t = B_T - B_t + \int_t^T B_1 dB_s,
\]

then \( \tilde{X}_t^2 - (T - t) \) is a near martingale on \((\Omega, \mathcal{F}_T, Q)\).

Note that the conclusions of this example cannot be obtained with the classic Girsanov Theorem 1.2 as the \( B_1 \) is not adapted to \( \{\mathcal{F}_t\} \). It is also not possible to approach this example with results of [10] or of Section 3 of the present paper because \( B_1 \) is not adapted to \( \{\mathcal{G}^{(i)}\} \). However, we can rewrite \( B_1 \) as

\[
B_1 = (B_1 - B_t) + B_t,
\]

where \( (B_1 - B_t) \) is adapted to \( \{\mathcal{G}^{(i)}\} \) and \( B_t \) is adapted to \( \{\mathcal{F}_t\} \). In the view of the above equation, Theorems 5.1–5.4 are applicable.

6. Black–Scholes Equation in the Backward Case

In this section we discuss a simple scenario of Black–Scholes model in the backward-adapted setting. The outline of this approach comes from [4, Chapter 7]. In our setting, the market is composed of two assets. The first asset is a risk-free bond whose price \( D_t \) is driven by a deterministic differential equation

\[
dD_t = rD_t dt,
\]

where \( r \) is the risk-free interest rate. The second asset is a stock (or some security) \( S_t \), whose price is dependent on the information right after time \( t \) and driven by a stochastic differential equation

\[
dS_t = S_t \alpha_t dt + S_t \sigma_t dB_t,
\]

where \( \alpha_t \) and \( \sigma_t \) are both adapted to \( \{\mathcal{G}^{(i)}\} \). This can be viewed as a special case of “insider information”, where the “insider” uses only the knowledge unavailable to the rest of the market as the processes \( \alpha_t \) and \( \sigma_t \) are completely out of the scope of the natural forward Brownian filtration, but instead are adapted to the natural backward Brownian filtration. The backward Brownian filtration describes exactly the future information generated by the driving Brownian process that is independent of the current or past state of the market.

We assume there is a contingent claim \( \Phi(S_T) \), which is tradable on the market and whose price process is given by

\[
\Pi_t = F(t, S_t)
\]
for some smooth function $F(x, y)$. Our goal is to find a function $F$ such that the market is arbitrage-free. Using Corollary 3.7, we have

$$d\Pi_t = dF(t, S_t)$$

$$= \frac{\partial F}{\partial y}(t, S_t) dS_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) (dS_t)^2 + \frac{\partial F}{\partial x}(t, S_t) dt$$

$$= \frac{\partial F}{\partial y}(t, S_t)(S_t \alpha_t dt + S_t \sigma_t dB_t) - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) (S_t^2 \sigma_t^2 dt) + \frac{\partial F}{\partial x}(t, S_t) dt$$

$$= \left( \frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t) \right) dt$$

$$+ \left( \frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t \right) dB_t$$

$$= \alpha^\Pi_t \Pi_t dt + \sigma^\Pi_t \Pi_t dB_t,$$

where

$$\alpha^\Pi_t = \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t)}{F(t, S_t)},$$

$$\sigma^\Pi_t = \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t}{F(t, S_t)}.$$
Equations (6.2) and (6.4) yield

$$u^S_t = -\frac{\sigma^I_t}{\sigma_t - \sigma^I_t}, \quad u^I_t = \frac{\sigma_t}{\sigma_t - \sigma^I_t}. \quad (6.5)$$

Putting together Equations (6.5) and (6.1), we obtain

$$u^S_t = \frac{\frac{\partial F}{\partial y}(t, S_t)S_t}{\frac{\partial F}{\partial y}(t, S_t)S_t - F(t, S_t)}, \quad u^I_t = \frac{F(t, S_t)}{F(t, S_t) - \frac{\partial F}{\partial y}(t, S_t)S_t} \quad (6.6)$$

Now, together with Equation (6.3) and the terminal condition that comes from the form of the contingent claim II, Equation (6.6) yields

$$\begin{cases} \frac{\partial F}{\partial x}(t, S_t) + \frac{\partial F}{\partial y}(t, S_t)rS_t - \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(t, S_t)S_t^2\sigma_t^2 - F(t, S_t)r = 0 \\ F(T, s) = \Phi(s) \end{cases}$$

Observe that unlike with the classic Black–Scholes formula, in the above PDE we have a minus in front of the term with \(\frac{\partial^2 F}{\partial y^2}\). This change of sign enters through the Itô formula for the backward-adapted processes. Intuitively this can be explained by the fact that the difference between the classic Black–Scholes model and our example is that of a different point of view. That is the former model looks forward with the information on the past and the latter looks backward with the information from the future. Thus the influence of the volatility (\(\sigma_t\)) will have opposite effects in the two models.

Of course, the above example is rather simple and not realistic on its own, however one might use it together with the classic Black–Scholes model to study the influence of the insider information on the market.

References


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