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On an Effective Submodeling Procedure for Stresses Determined with Finite Element Analysis

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ON AN EFFECTIVE SUBMODELING PROCEDURE FOR STRESSES DETERMINED WITH FINITE ELEMENT ANALYSIS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctorate of Philosophy

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Abstract

Submodeling can enable stress analysts using finite elements to focus analysis on a subregion containing the stress concentration of interest, with consequent computational savings. Such benefits are only truly realized if the boundary conditions on the edges of the subregion that were originally contained within the global region are sufficiently accurate. These boundary conditions are drawn from initial global finite element analysis (FEA), and consequently themselves have errors that in turn lead to errors in the stresses sought. When these last boundary-condition errors are controlled, and the discretization errors incurred by the FEA of ensuing submodels are also controlled, submodeling is effective.

Here we furnish improved estimations of boundary-condition and discretization errors. These estimates are used in conjunction with precautions against underestimating errors in the presence of nonmonotonic convergence. To access the efficacy of our procedure, we apply it to four 2D and nine 3D test problems. These test problems have a range of stress concentration factors that exceed those normally encountered in practice. These test problems have exact solutions so that there is no ambiguity whatsoever as to the actual errors occasioned by their FEA. The performance of our approach is assessed with free and structured meshes, for elements of different orders, and for shape functions and cubic splines or bicubic surfaces for interpolating displacements in boundary conditions. For all these problems, whenever estimates of the boundary-condition errors indicate that there is a need to enlarge the subregion, actual errors due to cut boundary conditions confirm this, in fact, to be the case. Thereafter, whenever subregions are enlarged and estimates indicate that errors due to cut boundary conditions are then low enough to proceed with the FEA of submodels, actual errors also confirm this to be the case. Ultimately
for all thirteen test problems, accurate error estimates are made which are confirmed by actual error values, with significantly fewer degrees of freedom being used in submodel meshes.

Finally, we implement our submodeling procedure on two practical problems. The error estimates indicate that excellent results are obtained for both the applications with significant computational savings.
Chapter 1. Introduction

Accurately determining stress concentrations in engineering is an important activity. Peterson [1] provides accurate stress concentration factors for quite a wide array of configurations. Even so, in practice configurations are often encountered that are not present in [1]. Currently, finite element analysis (FEA) has become a principal method for determining stress raisers in such configurations.

Discretization error is intrinsic in the determination of stresses with FEA. If these stresses are to be determined with sufficient accuracy for design considerations, then such errors have to be controlled within reasonable limits. Controlling these errors can be challenging because of the presence of high stress gradients at stress raisers. One way to control discretization error is by systematically refining the FEA mesh used until further refinement results in no significant change in the peak stresses determined. One means of realizing such refinement is by halving the element sides. In two-dimensions (2D), this leads to quadrupling the number of elements with each successive refinement. In three-dimensions (3D), this leads to an increase of element numbers by a factor of eight. Moreover, 3D elements entail to more degrees of freedom per element than 2D elements. Although with today’s advanced computers this refinement is often possible, one can still exhaust the computational facility that is at their disposal without reaching the desired accuracy level. Then submodeling is one approach that can be used to refine meshes further to control these errors yet remain within computational capabilities.

Submodeling entails refining the mesh only inside a subregion or submodel around a stress concentrator as analysis proceeds. An initial global analysis is run to locate the region of interest. The so identified region is broken out as an FEA submodel and analyzed separately with finer meshes, with consequent computational savings. Such benefits are only truly realized if the
boundary conditions on the edges of the subregion that were originally contained within the global region – the cut boundaries – are sufficiently accurate. These boundary conditions are drawn from initial global FEA, and consequently themselves have errors that in turn lead to errors in the stresses sought. When these last boundary-condition errors are controlled, and the discretization errors incurred by the FEA of ensuing submodels are also controlled, submodeling is effective. To this end, Cormier et al. [2] provides a submodeling procedure along with a method to estimate discretization and boundary-condition errors. Here we intend to improve these error estimates and the submodeling procedure by a combination of mesh refinement and enlarging submodel regions.

In what follows, we describe the improved error estimates and submodeling procedure for 2D FEA in Chapter 2. Results of applying our submodeling procedure to 2D test problems and an application are also discussed in Chapter 2. Similar improved error estimates with precautions against underestimating errors in the presence of nonmonotonic convergence for 3D FEA are described in Chapter 3. Results of applying our submodeling procedure to 3D test problems and an application are also discussed in Chapter 3. We close, in Chapter 4, with some concluding remarks in the light of the results found and future outlook.
Chapter 2. Improved Submodeling of Two-Dimensional Stress Concentrations

2.1. Introduction

A practical 2D problem to illustrate submodeling is shown in Figure 2.1. Figure 2.1 shows an FEA of a keyway in a shaft. The initial global mesh of the shaft cross section with the keyway is shown in Figure 2.1(a). Close-ups of the initial and the finest global mesh of keyway region are shown in Figures 2.1(b) and (c), respectively. For this configuration the peak stress occurs at the center of the keyway corner highlighted by a red dot (Figures 2.1(b), (c)). The neighboring region, highlighted by a red line, then forms our submodel region. This submodel region is broken out and analyzed separately with finer mesh. Figure 2.1(d) shows an intermediate submodel mesh. The boundary conditions applied to these submodel meshes are taken from the finest mesh of the global configuration (Figure 2.1(c)). The finest submodel mesh has 66 thousand elements whereas the corresponding global mesh would have about 4.5 million elements.

Submodeling has been used to resolve local stresses in complex structures and has been successfully employed in several industries. Applications of the technique in some industries are: gas turbine [2, 3], ship building [4], and biomedical [5]. Submodeling has also been used to analyze some contact problems [6, 7]. While all of the foregoing references take advantage of submodeling to obtain desired FEA stresses, only two report developments of the method: Cormier et al. [2] and Kitamura et al. [4].

Cormier et al. [2] provides a set of procedures for implementing submodeling of 2D stress concentrations. In [2], displacement boundary conditions are used on the cut boundaries (e.g., the red line in Figure 2.1(b)); [2] also contributes a means of estimating the error due to these boundary conditions, hence furnishing a way of controlling this error. This lets their submodeling procedure be more aggressive than the earlier submodeling procedures cited in [2]. Here by being “aggressive” means using a smaller submodel area when compared to the original global area with
consequent computational savings. With the exception of Kitamura et al. [4], all of the preceding references use the approach for submodel boundary conditions of [2].

![Figure 2.1](image)

**Figure 2.1** Finite element meshes for stresses in the corner of a keyway: (a) initial global mesh \((m = 1)\); (b) close-up of initial mesh; (c) submodel region in finest global mesh \((m = 6)\); (d) intermediate submodel mesh \((m = 8)\).

Instead, Kitamura et al. [4] use stresses from the global meshes and apply corresponding tractions on the cut boundaries. To obtain like accuracy to displacements, the stresses are patch recovered, as in Zienkiewicz and Zhu [8], before applying their corresponding tractions to submodel cut boundaries. While the approach in [4] thus shows promise of achieving like accuracy to that in [2], here we continue to use displacements as in [2] because we think practicing finite element engineers are likely to find them easier to apply.

As analysis proceeds with finer submodel meshes, boundary conditions are required on all the intervening nodes on the cut boundaries. Cormier et al. [2] use cubic splines to fit nodal
displacements because cubic splines are continuous and continuously differentiable like the
displacements they are trying to replicate. An alternative is simply to use shape functions. While
they are continuous they are not continuously differentiable. As a result, they induce spurious
logarithmic stress singularities at nodes on submodel cut boundaries as shown in Sinclair and Epps
[9]. However they are simple to use, especially because they are incorporated in ANSYS [10] and
ABAQUS [11]. It may be that these stress singularities do not significantly effect the key stress
that is interior to the submodel region. Here, therefore, we use both cubic-spline fitted
displacements and displacement shape functions on cut boundaries.

Cormier et al. [2] use traditional discretization error estimates for a mesh sequence. Instead
here we use improved discretization error estimates that reflect the actual rate of convergence.
These improved estimates are the result of developments in the computational fluid dynamics
community in de Vahl Davis [12], and Roache [13, 14]. An excellent account of the entire
approach is given in Roache [15]. This is one of the approaches recommended by ASME’s guide
for verification and validation in computational solid mechanics, [16]. Like improvements are also
adopted for estimating displacement boundary-condition errors.

In what follows, we describe the improved convergence checks to determine discretization
error in Section 2.2. These checks are applicable to both global and submodel meshes. In Section
2.3, similar improved checks are developed for estimating boundary-condition error with
submodel meshes. Section 2.3 also describes the implementation of the procedure for 2D
configurations. In Section 2.4, the procedure is evaluated on a set of 2D test problems that have
known exact solutions for the peak stresses of interest. To demonstrate the implementation of our
procedure on a practical problem, in Section 2.5 it is used on the keyway in a shaft shown in Figure
2.1. We close, in Section 2.6, with some remarks in the light of the results found.
2.2. Improved Discretization Error Control

In this section we begin with a description of the mesh refinement scheme employed, then review the discretization error estimate used by Cormier et al. [2]. Thereafter we develop an improved error estimate that reflects the effective convergence rate being experienced.

An important step in analyzing any problem with FEA is to choose an effective mesh refinement scheme. Here we proceed as follows. We let $h_m$ denote the representative length of elements in mesh $m$. Then, as in [2], we reduce $h_m$ by a constant scale factor from the preceding mesh size throughout the mesh sequence. Thus, if $\lambda$ is the constant scale factor adopted,

$$h_m = h_{m-1}/\lambda$$

for $m \geq 2$ and $\lambda > 1$. For uniform meshes as is often true or nearly so with submodel meshes, $h_m$ for (2.1) is obvious. For nonuniform meshes, $h_m$ can be taken as the size of the elements in the critical region which contains the stress of interest (e.g., Figure 2.1(b) and (c)). An alternate means to estimate $h_m$ for nonuniform 2D meshes is $h_m = (A/N_m)^{1/2}$, where $A$ is the area of the region being meshed and $N_m$ is the number of elements in mesh $m$. Here, in our global meshes, (2.1) is adhered to with both definitions. However, if only one of these definitions is to apply, we prefer the first.

We typically form successive meshes simply by halving element sides and hence have $\lambda = 2$. This leads to four-fold increases in element numbers in two dimensions. In 2D elasticity, some justification is provided in Sinclair et al. [17] for the elements numbers to be increasing in this fashion rather than simply linearly (i.e., in accord with $N_m = 4^{m-1}N_1$ rather than $N_m = mN_1$). Here we primarily adopt this choice because it is easy to implement and is incorporated in standard codes (e.g., ANSYS [10] and ABAQUS [11]). In 2D elasticity, initially if a suitably coarse mesh is used, then mesh refinement in accord with (2.1) is not usually computationally problematic.
Alternatively, too, the approach can be implemented by mesh coarsening if one has a baseline mesh which is deemed to have sufficient accuracy to capture key stresses from FEA. This second approach is obviously computationally feasible.

We define the true discretization error in the stress of interest on mesh $m$, $e_m^d$, to be

$$e_m^d = \sigma_a - \sigma_m$$  \hspace{1cm} (2.2)

where $\sigma_a$ is the actual value of the stress sought and $\sigma_m$ is the FEA determination of $\sigma_a$ on mesh $m$. With a converging FEA, $e_m^d$ reduces with mesh refinement. As in [2], the stress increment attending mesh refinement from $m-1$ to $m$ is defined by

$$\Delta\sigma_m^d = \sigma_m - \sigma_{m-1}$$ \hspace{1cm} (2.3)

In [2] these stress increments need to be reducing in magnitude with mesh refinement for the FEA to be judged to be converging, and the ultimate mesh increment leading to $\sigma_m$ needs to be within the error level sought for the FEA to be judged to have converged. Then the estimate of discretization error in the stress of interest on mesh $m$, $\tilde{e}_m^d$, is defined as

$$\tilde{e}_m^d = \Delta\sigma_m^d$$ \hspace{1cm} (2.4)

Finally in [2], the estimate of absolute relative discretization error in $\sigma_m$, $\tilde{e}_m^d$, is taken to be

$$\tilde{e}_m^d = \frac{|\Delta\sigma_m^d|}{|\sigma_m|}$$ \hspace{1cm} (2.5)

usually expressed as a percentage. This error estimate is insensitive to the rate of convergence of the FEA employed.

In what follows we improve the estimate of (2.5) by taking into account an effective rate of discretization error convergence, $c_m^d$. This effective convergence rate is defined such that

$$e_m^d = e_{m-1}^d \left( \frac{h_m}{h_{m-1}} \right)^{c_m^d}$$ \hspace{1cm} (2.6)
With mesh refinement, if the errors have the same sign and reduce in magnitude, $c_m^d$ in (2.6) is real and positive. Then introducing (2.1) into (2.6), the errors reduce in accordance with $e_m^d = e_{m-1}^d/\lambda c_m^d$.

As $m \to \infty$, $c_m^d \sim c$ where $c$ is the asymptotic rate of convergence. A good review of the values of $c$ is provided in Cook et al. [18]. For example in 2D, for four-node quadrilateral elements (4Q) $c = 1$, while for eight-node quadrilaterals (8Q) $c = 2$. This is so provided stress fields are sufficiently continuous, otherwise $c$ can drop below these values. On the other hand, with the patch recovery technique of Zienkiewicz and Zhu [8], $c = 2$ for 4Q elements.

Irrespective of the value of $c$, $c_m^d$ is not necessarily equal to $c$ or even that close to $c$ in practice. This is because $c_m^d$ reflects the entire error going from mesh $m - 1$ to mesh $m$, not just the dominant terms. If the additional higher-order error contributions have the same sign as the dominant contribution, $c_m^d > c$. Conversely if they have an opposite sign, $c_m^d < c$. However, the approach adopted here can be effective without $c_m^d = c$, so that such deviations from asymptotic values are not of major concern.

An expression for $e_m^d$ can now be obtained in terms of $\Delta \sigma_m^d$ and $c_m^d$. Recognizing that from (2.2), (2.3), $\Delta \sigma_m^d = e_m^d - e_{m-1}^d$, then expressing $e_{m-1}^d$ in terms of $e_m^d$ using (2.6) with (2.1), we have

$$e_m^d = \frac{\Delta \sigma_m^d}{\lambda c_m^d - 1}$$

(2.7)

If the expression in (2.7) is to serve as an estimate of $e_m^d$, we need an estimate of $\lambda c_m^d$ from the quantities known from FEA. Introducing (2.1) into (2.6) and using (2.7), we get

$$\lambda c_m^d = \frac{e_{m-1}^d}{e_m^d} = \frac{\Delta \sigma_{m-1}^d (\lambda c_m^d - 1)}{\Delta \sigma_m^d (\lambda c_{m-1}^d - 1)}$$

(2.8)
If $c_m^d = c_{m-1}^d$ then $\lambda_m^d = \Delta\sigma_{m-1}^d / \Delta\sigma_m^d$. If this condition holds or nearly so, we can estimate $c_m^d$ with $\hat{c}_m^d$ such that

$$\lambda_m^d = \Delta\sigma_{m-1}^d / \Delta\sigma_m^d$$  \hspace{1cm} (2.9)

That is

$$\hat{c}_m^d = \frac{\ln(\Delta\sigma_{m-1}^d / \Delta\sigma_m^d)}{\ln \lambda}$$  \hspace{1cm} (2.10)

In de Vahl Davis [12], it is assumed that $c_m^d = c_{m-1}^d$ thereby resulting in the estimate of $c_m^d$ of (2.10). Observe that this estimate is valid under the given condition irrespective of whether or not $c_m^d = c$ or is even close to $c$.

The implementation of the improved convergence checks on a sequence of meshes that comply with (2.1) is as follows. First, the stress increments as in (2.3) are calculated. If the stress increments of (2.3) are of same sign and decreasing we judge the FEA to be converging and estimate the effective rate of convergence, $\hat{c}_m^d$, using (2.10). Then we estimate the absolute relative discretization error in $\sigma_m$, $\hat{\epsilon}_m^d$, from (2.7) and (2.10) as being

$$\hat{\epsilon}_m^d = \frac{|\Delta\sigma_m^d|}{|\sigma_m| (\lambda_m^d - 1)}$$  \hspace{1cm} (2.11)

The expression in (2.11) is usually expressed as a percentage. The sensitivity to convergence rates of the estimate of (2.11) is essentially what is introduced in Roache [15]. Alternatively, combining (2.9) with (2.11), we have

$$\hat{\epsilon}_m^d = \frac{|\Delta\sigma_m^d|}{|\sigma_m| (\Delta\sigma_{m-1}^d / \Delta\sigma_m^d - 1)}$$  \hspace{1cm} (2.12)

Equation (2.12) rather than (2.11) is what is used subsequently to estimate the absolute relative discretization error in our problems.
Finally in assigning merit to the absolute relative discretization error so obtained, we check for

$$
\hat{e}_M^d \leq \epsilon_s
$$

(2.13)

where $\hat{e}_M^d$ is given as percentage, $M$ is the last mesh in the global mesh sequence, and $\epsilon_s$ is the percentage error level sought in the FEA determination of $\sigma_a$. We classify $\epsilon_s$ in accordance with

$$
\begin{align*}
1 < \epsilon_s & \leq 5 \Rightarrow \text{satisfactory accuracy} \\
1/5 < \epsilon_s & \leq 1 \Rightarrow \text{good accuracy} \\
\epsilon_s & \leq 1/5 \Rightarrow \text{excellent accuracy}
\end{align*}
$$

(2.14)

We have found the ranges in (2.14) to be reasonable for stress concentration problems in practice, but certainly other ranges could be assigned to these three levels of accuracy. Here we seek excellent results, hence $\hat{e}_M^d \leq 1/5$. With global FEA, if $\hat{e}_M^d \leq 1/5$ is achieved, we accept $\sigma_M$ as the FEA determination of $\sigma_a$ on mesh $M$. Otherwise we continue mesh refinement, thereby increasing the value of $M$. If $\hat{e}_M^d$ remains greater than $1/5$ and further mesh refinement is not computationally possible, we proceed to submodel as described next.

### 2.3. Improved Boundary-Condition Error Control and Submodeling Procedure

Here we first review the boundary-condition error estimate used by Cormier et al. [2], then we develop an improved error estimate that reflects the effective convergence rate being experienced by the boundary conditions in submodeling. Thereafter we describe our submodeling procedure on a mock two-dimensional configuration.

As for the global meshes, here we index successively refined submodel meshes with $m \geq M + 1$, and $h_m$ denotes the representative length of elements in submodel mesh $m$. As previously we reduce $h_m$ by the same constant scale factor, $\lambda$, from the preceding mesh size throughout the mesh sequence. We typically form uniform successive submodel meshes by halving element sides.
and thus continue to have $\lambda = 2$. For 2D submodel meshes this refinement scheme is seldom if ever problematic computationally.

With submodel meshes, there are two sources of error. The first source is the discretization error. This is the error that is inherent with any FEA and consequently occurs in both global and submodel meshes. The second source is the *boundary-condition error*. This is the error in the stresses sought in the submodel that is incurred by taking values from the global analysis and using them as boundary conditions on the cut boundaries of the subregion. To estimate this error, Cormier et al. [2] use the displacements from the last mesh in the global sequence ($m = M$) and its predecessor ($m = M − 1$). In [2], the stress increment, on a submodel mesh $m$, attending boundary condition refinement from $(M − 1)$ to $M$ with global meshes is defined by

$$\Delta \sigma_m^{Mb} = \sigma_m^{Mb} - \sigma_m^{(M-1)b}$$  \hspace{1cm} (2.15)$$

for $m \geq M + 1$, where $\sigma_m^{Mb}$ and $\sigma_m^{(M-1)b}$ are the stresses found using boundary conditions from the $M$ and $(M − 1)$ global meshes, respectively. Then [2] simply estimates the boundary-condition error in the stress of interest on a submodel mesh $m$ for boundary conditions from mesh $M$, $\tilde{e}_m^b$, by

$$\tilde{e}_m^b = \Delta \sigma_m^{Mb}$$  \hspace{1cm} (2.16)$$

for $m \geq M + 1$. In [2] this estimate is compared with the actual boundary-condition error given by

$$e_m^{Mb} = \sigma_m^{ab} - \sigma_m^{Mb}$$  \hspace{1cm} (2.17)$$

where $\sigma_m^{ab}$ is the stress found using actual boundary conditions. These boundary conditions are from test problems with known exact solutions in [2]: then the boundary-condition error estimate of (2.17) is found to be uniformly conservative in [2].
To incorporate the effects of convergence rates in the boundary-condition error estimate, we begin by introducing an effective convergence rate for boundary-condition errors, $c^M_{mb}$. Analogously to (2.6), we define this rate such that

$$
e^M_{mb} = e^{(M-1)b}_m \cdot \left( \frac{h_M}{h_{M-1}} \right)^{c^M_{mb}}$$

That is, with (2.1),

$$c^M_{mb} = \frac{\ln \left( \frac{e^{(M-1)b}_m}{e^M_{mb}} \right)}{\ln \lambda}$$

This is the convergence rate of boundary-condition error with global mesh refinement. Then proceeding as previously for discretization error, we have our estimate of the effective rate of convergence for our boundary conditions, $\hat{c}^M_{mb}$, as

$$\hat{c}^M_{mb} = \frac{\ln \left( \frac{\Delta \sigma^{(M-1)b}_m}{\Delta \sigma^M_{mb}} \right)}{\ln \lambda}$$

Then our estimate of the absolute relative boundary-condition error in $\sigma^M_{mb}$, $\hat{e}^b_m$, is

$$\hat{e}^b_m = \frac{\left| \Delta \sigma^M_{mb} \right|}{\left| \sigma^M_{mb} \right| \left( \lambda^{c^M_{mb}} - 1 \right)}$$

for $m \geq M + 1$. The expression in (2.21) is usually expressed as a percentage. Alternatively, analogously to (2.12), replacing $\lambda^{c^M_{mb}}$ with the stress increment quotient of (2.20) in (2.21), we have

$$\hat{e}^b_m = \frac{\left| \Delta \sigma^M_{mb} \right|}{\left| \sigma^M_{mb} \right| \left( \frac{\Delta \sigma^{(M-1)b}_m}{\Delta \sigma^M_{mb}} - 1 \right)}$$
The estimate of (2.22) is used to estimate the absolute relative boundary-condition error in our problems. It is understood in (2.22) that we are only interested in the boundary-condition error resulting in using the boundary conditions from the most refined global mesh \((m = M)\). Since we use displacements in boundary conditions on cut boundaries, the estimate in (2.22) may be able to take advantage of the superior convergence rates enjoyed by displacements.

Taken together the two error estimates of (2.12) and (2.22) combine to give our absolute relative total error estimate for \(\sigma_m M^b\), \(\hat{\epsilon}_m^t\). That is

\[
\hat{\epsilon}_m^t = \hat{\epsilon}_m^d + \hat{\epsilon}_m^b \tag{2.23}
\]

For \(\hat{\epsilon}_m^t < \epsilon_s\), clearly neither \(\hat{\epsilon}_m^d\) nor \(\hat{\epsilon}_m^b\) can exceed \(\hat{\epsilon}_m^t\). From the experience in [2], the boundary-condition error is smaller than the discretization error on a given submodel mesh and so here we want its share to be less than half of the error level sought \((\epsilon_s)\). In line with this expectation, here we seek

\[
\hat{\epsilon}_m^b \leq \frac{1}{3} \epsilon_s \tag{2.24}
\]

The choice of 1/3 in (2.24) as a fraction somewhat less than 1/2 is simply convenient and is not critical in what follows. Since overall we seek excellent results, (2.14) in conjunction with (2.24) has \(\hat{\epsilon}_m^b \leq 0.067\%\).

To explain our submodeling procedure, we consider a mock two-dimensional FEA of a rectangular plate. The rectangular plate is loaded such that it features peak stress, \(\sigma_{\text{max}}\), at its bottom left-hand corner that is the stress of interest (Figure 2.2). First, we run a global analysis of this configuration with successively refined meshes. The first global mesh \((m = 1)\) is taken to have four elements. Refining this mesh by halving the element sides yields three further meshes \((m = 2 \rightarrow 4)\). Suppose this last mesh is the finest possible, hence \(M = 4\) in this mock FEA. Suppose further
that $\sigma_{\text{max}}$ has not converged to within the excellent level sought with the last mesh. Then we proceed to submodel the region of greatest interest (bottom left-hand corner containing $\sigma_{\text{max}}$).

Figure 2.2 Mock mesh sequence for demonstrating the submodeling procedure.
Second, we locate the boundaries of the submodel region, and as a result establish the size of the submodel region. We choose the submodel region such that the number of elements present in the first submodel mesh would be exactly the same as the number of elements in the finest global mesh in the same region. We do this to introduce a check on the implementation of the boundary conditions on the cut boundary. With this choice with structured meshes, the FEA estimate of $\sigma_{\text{max}}$ on $m = M$ and $m = M + 1$ should match exactly. Then for this submodel region, we successively refine the mesh by halving the element sides. In doing so we have four elements in our first submodel mesh (indicated for mesh $m = M + 1$, Figure 2.2), and sixteen in the second mesh and so on. This is the approach we adopt in general in terms of sizing the submodel region. Hence here the area contained in the subregion is 1/64 of that in the global region: in practice, with this sizing approach, it is typically a much smaller fraction of the area of the global region.

Third, we estimate the boundary-condition errors present in the submodel meshes using (2.22). If the so found boundary-condition error complies with (2.24), we deem the boundary conditions to have converged. If (2.24) is not complied with, we move the boundary of the submodel region further away from the stress of interest. We do this by doubling the length of extents of the sides of the submodel region. For this increased submodel region we use exactly the same number of elements as it has in the finest global mesh. Hence again we check on the correctness of submodel boundary conditions.

2.4. Test Problems

In this section for test problems, we consider an infinite elastic plate with an elliptical hole subjected to transverse uniform tension at infinity (Figure 2.3). The length of the semi-major axis of the elliptical hole is held constant while that of semi-minor axis is decreased to increase the notch acuity at the ends of ellipse. As notch acuity is increased, the stress concentration at the root
of the notch is also increased and hence challenges FEA. The exact solution for this configuration is given in Kolosoff [19] and Inglis [20]. Within the infinite plate, this solution is evaluated on an arc and the exact quantities so found are applied as boundary conditions there. This enables us to formulate a finite plate configuration with an exact solution for FEA. With exact solutions for the peak stresses acting, there is no ambiguity whatsoever in the errors present in the FEA of these test problems. We begin with a formal statement of our test problems. Then we describe the application of our submodeling procedure to these test problems. Thereafter we report the results found.

The chosen geometry has the following specifics. Transverse to the applied load $\sigma_0$, the lengths of semi-major and semi-minor axis of the elliptical hole are $a$ and $b$, respectively (Figure 2.3).

![Figure 2.3](image-url) Geometry and coordinates for test problems.
The configuration is most readily framed in elliptic cylindrical coordinates \((\xi, \eta)\). These coordinates share a common origin \(O\) with rectangular Cartesian coordinates \((x, y)\) at the center of the elliptical hole (Figure 2.3). Then these coordinates are related by

\[
x = c \ ch \xi \cos \eta, \quad y = c \ sh \xi \sin \eta
\]

for \(0 < \xi < \infty, 0 \leq \eta \leq 2\pi\), where \(c\) is the focus of the ellipse and is given by \(c = \sqrt{a^2 - b^2}\).

The boundary of the elliptical hole is

\[
(x/a)^2 + (y/b)^2 = 1
\]

(2.26)

The inner and outer boundaries of the region for FEA, in terms of the elliptical cylindrical coordinates \(\xi_i\) and \(\xi_o\) respectively, are taken to be

\[
\xi_i = \frac{1}{2} \ln \left( \frac{a + b}{a - b} \right), \quad \xi_o = \text{sh}^{-1} \left( \frac{10a}{c} \right)
\]

(2.27)

We use the symmetry of the geometry and loading to restrict our analysis to the upper rightmost quadrant of the elastic plate in Figure 2.3. Thus the finite region of interest, \(\mathcal{R}\), is defined by

\[
\mathcal{R} = \{(\xi, \eta) \mid \xi_i < \xi < \xi_o, 0 < \eta < \frac{\pi}{2}\}
\]

(2.28)

With these geometric preliminaries in place, we can formulate our test problems as next.

In general, we seek the plane strain stresses \(\sigma_\xi, \sigma_\eta, \tau_\xi\eta\), and their associated displacements \(u_\xi, u_\eta\), as functions of \(\xi\) and \(\eta\) throughout \(\mathcal{R}\) satisfying the 2D field equations of elasticity as well as the boundary conditions for our elliptical-hole test problems. The field equations are: the stress equations of equilibrium and the stress-displacement relations for a homogeneous and isotropic, linear elastic solid (these equations in \((\xi, \eta)\) coordinates can be assembled from equations given in [20]). The key boundary conditions that, in essence, apply the tension \(\sigma_0\) are actually displacement conditions on \(\xi = \xi_o\). From [19] and [2] these have
on $\xi = \xi_o$ for $0 < \eta < \pi/2$, wherein $\mu$ is the shear modulus and $\nu$ is Poisson’s ratio of the plate, and $h_\eta$ is the metric coefficient for the elliptical coordinates, $h_\eta = c \sqrt{\sin^2 \xi_o + \sin^2 \eta}$. The other boundary conditions are the stress-free conditions,

$$\sigma_\xi = 0, \tau_{\xi\eta} = 0$$

(2.30)
on $\xi = \xi_i$ for $0 < \eta < \pi/2$, and the symmetry conditions,

$$u_\eta = 0, \tau_{\xi\eta} = 0$$

(2.31)
on $\eta = 0, \pi/2$ for $\xi_i < \xi < \xi_o$.

In particular, we are interested in the normalized peak stress value, $\bar{\sigma}_{\text{max}} = \sigma_{\text{max}}/\sigma_0$ where $\sigma_{\text{max}} = \sigma_\eta$ at $\xi = \xi_i, \eta = 0$. From [19], the exact solution for this stress, $\sigma_e$, is given by

$$\bar{\sigma}_{\text{max}} = \sigma_e = 1 + \frac{2a}{b}$$

(2.32)

From our FEA we want an excellent value for $\sigma_e$. That is, we want to capture $\sigma_e$ to within 0.2 % for the entire range of ellipse heights considered.

Four values of $b$ are chosen which lead to aspect ratios $a/b$ of $2, 8^{1/2}, 18^{7/16}$, and $26^{1/2}$. These aspect ratios in turn give rise to normalized peak stresses, or stress concentration factors, of $5, 18, 37^{7/8},$ and $54$, respectively. This range of stress concentration factors exceeds that normally encountered in practice (cf., Peterson [1]). As long as the elastic moduli used to run the FEA are consistent with the moduli in (2.29), their precise values are of no consequence: this is a result of
the problem from which the test problems originated being a plane strain problem with only tractions applied.

All the global meshes need the displacement boundary conditions of (2.29) to be applied on the outer boundary (i.e., on $\xi = \xi_o$ for $0 < \eta < \pi/2$). For the ease in implementation of (2.29), we use their rectangular Cartesian counterparts, $u_x$ and $u_y$. These are given by

$$u_x = u_\xi \sin \phi - u_\eta \cos \phi$$
$$u_y = u_\xi \cos \phi + u_\eta \sin \phi$$  

where $\phi$ is the angle between the normal displacement ($u_\xi$) on outer boundary and the $y$ axis of the rectangular Cartesian coordinate system (Figure 2.3), and is given by

$$\phi = \tan^{-1}[\text{th}\xi_o \cot \eta]$$  

For these global meshes, initially we use 4Q elements (PLANE182, ANSYS [22]). We start our discretization with a structured initial mesh ($m = 1$) with uniform increments in elliptic cylindrical coordinates. By having 16 equal increments in $\xi$ and $\eta$, this initial mesh has 256 elements (Figure 2.4(a)). Uniform increments in these coordinates naturally produce element size reductions in the vicinity of the stress raiser. This initial mesh is systematically refined by halving the increments and hence $\lambda = 2$. Six further meshes ($m = 2$ - 7) are produced with such refinement. Our finest mesh in this sequence consequently has about 1 million elements.

In addition we run 4Q elements with free meshes for the first configuration with a stress concentration factor of 5. These free meshes are generated using an automatic mesh generator (AMESH, [10]). We adopt this approach because it is easier to implement and so likely to be used in practice. With some care in implementation, these meshes can be generated such that they are geometrically similar in element arrangements and have the same number of elements as their structured counterparts.
To access the performance of our procedure with higher-order elements, we analyze the last configuration with a stress concentration factor of 54 with 8Q elements (PLANE183, [22]). Similar to the meshes generated using 4Q elements, here we generate structured meshes with uniform increments in elliptic cylindrical coordinates. Our initial mesh ($m = 1$) has 8 equal increments in $\xi$ and $\eta$. Thus is comprised of 64 elements. This mesh is one mesh coarser than its corresponding mesh of 4Q elements. We do this here so that degrees of freedom are more comparable. All the subsequent meshes are generated by systematically halving the increments (i.e., with $\lambda = 2$).
Since $\sigma_e$ for the test problems is available from (2.32), we have the true absolute relative discretization error in the FEA determination of $\sigma_e$ on a global mesh $m$, $\varepsilon^d_m$, as

$$\varepsilon^d_m = \frac{|\sigma_e - \sigma_m|}{\sigma_e}$$

(2.35)

for $m \leq M$. For all the global meshes, the FEA values of $\bar{\sigma}_{\text{max}}$, the estimated discretization errors, $\hat{\varepsilon}^d_m$, and their corresponding true discretization errors, $\varepsilon^d_m$, are given in Tables A.1 – A.3 of Appendix A. For the global analysis for these test problems, the error estimates found using (2.12) are uniformly conservative. As is evident in these tables the estimated value of $\hat{\varepsilon}^d_m$ on the finest global mesh, for all the configurations, does not comply with the excellent criterion of (2.14), namely less than 0.2%. Hence our error estimates determine that our global mesh sequence is not sufficiently accurate. The corresponding true discretization errors of (2.35) confirm that, in fact, discretization errors do not comply with the excellent criterion. Because our finest global mesh is taxing our computational capabilities, we therefore look to submodeling to improve results.

Since the exact boundary conditions for these test problems are available, we run all the submodel meshes with the same. Then the true absolute relative boundary-condition error, $\varepsilon^b_m$, on a given submodel mesh $m$, is

$$\varepsilon^b_m = \frac{|\sigma_{e}^{eb} - \sigma_{m}^{Mb}|}{\sigma_e}$$

(2.36)

for $m \geq M + 1$, where $\sigma_{m}^{eb}$ is the stress found using exact boundary conditions. Further, the true absolute relative discretization error in the stress of interest on submodel mesh $m$, $\varepsilon^d_m$, is

$$\varepsilon^d_m = \frac{|\sigma_e - \sigma_{m}^{eb}|}{\sigma_e}$$

(2.37)
for \( m \geq M + 1 \). Then proceeding as previously for total error, we have our true absolute relative total error in the stress of interest on submodel mesh \( m \), 
\[
\varepsilon_m^t = \varepsilon_m^d + \varepsilon_m^b, \text{ for } m \geq M + 1. 
\]

For all the configurations analyzed with structured meshes of 4Q elements, the submodel region is chosen such that the number of elements present in the first submodel mesh is exactly the same as the number of elements in the finest global mesh in the same region. We do this here to check the correctness of the submodel boundary conditions. For these submodel regions, we successively refine the mesh by halving the element sides. For the mid-range stress concentration with \( \bar{\sigma}_{\text{max}} = 18 \), this is illustrated in Figure 2.4. Figure 2.4(a) shows the initial global mesh \((m = 1)\), while the close-ups in Figures 2.4(b) and (c) show the submodel region in the initial and the finest global mesh. The finest submodel mesh for this configuration is shown in Figure 2.4(d). These structured submodel meshes are run with both cubic-spline fitted displacements and displacement shape functions on their cut boundaries. For \( \bar{\sigma}_{\text{max}} = 18 \) our final submodel mesh has about 4 thousand elements. The global mesh with the same resolution would have about 16 million elements, hence a reduction in number of elements of 4000 to 1. The same reduction in number of elements occurs for all other configurations having structured meshes of 4Q elements.

For the first configuration with the lowest stress concentration factor of \( \bar{\sigma}_{\text{max}} = 5 \) run with free meshes of 4Q elements, the submodel region is chosen such that the area is exactly the same as that for structured meshes. On this submodel region we run three free meshes \((m = 8 \text{ to } 10)\) with \( \lambda = 2 \). To be consistent in comparing with structured meshes, the first of these submodel meshes has the same number of elements as the finest global mesh in the same region. The last two meshes are intended to improve the FEA determination of \( \bar{\sigma}_{\text{max}} \). We run these submodel meshes with displacement shape functions on their cut boundaries because shape functions are considerably easier than cubic splines to implement when using free meshes. The final submodel mesh has about
4 thousand elements whereas a global mesh with the same resolution would have about 16 million elements. Thus, again a reduction in number of elements of 4000 to 1.

To check how well our submodeling procedure works with higher-order elements, we analyze the configuration with the highest stress concentration factor of $\bar{\sigma}_{\text{max}} = 54$ with structured meshes of 8Q elements. The subregion is chosen as previously and meshes are refined by halving the element sides. Two further meshes ($m = 9$ and 10) are generated with such refinement. We run these submodel meshes with both cubic-spline fitted displacements and displacement shape functions on their cut boundaries. The final submodel mesh has about 1 thousand elements in contrast to a global mesh with the same resolution that would have about 4 million elements. Again, a reduction in number of elements of 4000 to 1.

We next present some representative results from applying our submodeling procedure. They are for $\bar{\sigma}_{\text{max}} = 5$, the lowest stress concentration factor; and for $\bar{\sigma}_{\text{max}} = 54$, the highest stress concentration factor. The lowest stress concentration factor demonstrates the use of free meshes, while the highest stress concentration factor demonstrates the use of higher order elements in conjunction with our submodeling procedure.

We begin with structured submodel mesh results with 4Q elements for $\bar{\sigma}_{\text{max}} = 5$ with shape function boundary conditions. These results serve as a benchmark for comparison of the results for free meshes. For these 4Q structured meshes, the FEA stresses and the accompanying boundary-condition errors are given in Table 2.1(a) (here and in Table 2.1(b), six decimal places are included to avoid round-off error when calculating error estimates and actual errors). The FEA stress value from our last global mesh (Table A.1 of Appendix A, for $m = 7$) and that for first submodel mesh (Table 2.1(a) for $m = 8$) are the same, which confirms that submodel boundary conditions have been correctly implemented. The estimated boundary-condition error from (2.22)
on meshes \( m = 9, 10 \) complies with the excellent criterion of (2.14) for (2.24), namely less than 0.067% (Table 2.1(a)). The true boundary-condition error from (2.36) confirms the same on both the meshes. The boundary-condition error estimate on the last submodel mesh is only 0.0026% which is close to its actual value of 0.0025% (Table 2.1(a) for \( m = 10 \)). The FEA stress values from submodel meshes run with \( M^{th} \) global mesh boundary conditions in Table 2.1(b) are the same as in Table 2.1(a), but now are accompanied by estimated and actual discretization and total errors. We estimate the discretization error with (2.12) on mesh \( m = 10 \) as 0.081% (Table 2.1(b)). This is the same as the true discretization error from (2.37). The estimated total error from (2.23) on mesh \( m = 10 \) is 0.084% which is in compliance with excellent criterion of (2.14), namely less than 0.2% (Table 2.1(b)). The true total error is also 0.084% and confirms that the true total error does comply with the excellent criterion.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{M}^{M_{b}} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \hat{\epsilon}<em>{M}^{M</em>{b}} )</th>
<th>( \epsilon_{m}^{b} ) (%)</th>
<th>( \epsilon_{m}^{b} ) (%)</th>
<th>( \epsilon_{m}^{t} ) (%)</th>
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<td>8</td>
<td>4.983859</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4.992016</td>
<td>4.992422</td>
<td>4.994130</td>
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<td>0.0025</td>
<td>4.991918</td>
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<td>0.0026</td>
<td>4.995954</td>
<td>2.1</td>
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</table>

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{M}^{M_{b}} )</th>
<th>( \hat{\epsilon}_{M}^{d} )</th>
<th>( \epsilon_{m}^{d} ) (%)</th>
<th>( \epsilon_{m}^{d} ) (%)</th>
<th>( \epsilon_{m}^{t} ) (%)</th>
<th>( \epsilon_{m}^{t} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.983859</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4.992016</td>
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<td>0.16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4.996081</td>
<td>1.0</td>
<td>0.081</td>
<td>1.0</td>
<td>0.081</td>
<td>0.084</td>
</tr>
</tbody>
</table>
The estimated boundary-condition errors from (2.22) are close to one another in Table 2.1(a). This is because they represent the error values introduced in submodel analysis due to the use of displacement boundary conditions from the $M^{th}$ global mesh. The estimated convergence rate of the boundary-condition error, $\hat{\epsilon}_{m}^{Mb}$ from (2.20), is due to the global mesh refinement (i.e., for $m \leq M$). The estimated convergence rate of the boundary-condition error on mesh $m = 10$ is 2.1. In fact, this value is same as the actual convergence rate from (2.19) for the same submodel mesh. The estimated convergence rate of the discretization error, $\hat{\epsilon}_{m}^{dl}$ from (2.10), on mesh $m = 10$ is 1.0. In fact, this is same as the actual convergence rate from (2.6) for the same submodel mesh. This is a confirmation of our expectation that boundary-condition error converges more rapidly than discretization error, at least with 4Q elements.

The FEA stresses and the accompanying errors for $\bar{\sigma}_{\text{max}} = 5$ when analyzed with free submodel meshes of 4Q elements are given in Tables 2.2(a), (b) and (c). Here because we are using free meshes, the stress from the last global free mesh does not completely match that from the first submodel mesh (Table A.1 of Appendix A cf., Table 2.2(a)), nor is it expected to match.

Table 2.2(a). Finite element stresses, estimated and actual boundary-condition errors from free submodel meshes with 4Q elements using shape function boundary conditions on the initial subregion for $\bar{\sigma}_{\text{max}} = 5$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{\epsilon}_{m}^{Mb}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$\epsilon_{m}^{eb}$ (%)</th>
</tr>
</thead>
<tbody>
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<td>8</td>
<td>4.9891</td>
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<td></td>
</tr>
<tr>
<td>9</td>
<td>4.9976</td>
<td>5.0231</td>
<td>4.9760</td>
<td>NA</td>
<td>NA</td>
<td>4.9918</td>
<td>2.4</td>
</tr>
<tr>
<td>10</td>
<td>5.0018</td>
<td>5.0278</td>
<td>4.9781</td>
<td>NA</td>
<td>NA</td>
<td>4.9958</td>
<td>2.4</td>
</tr>
</tbody>
</table>
Table 2.2(b). Finite element stresses, estimated and actual boundary-condition errors from free submodel meshes with 4Q elements using shape function boundary conditions on an enlarged subregion for $\sigma_{\text{max}} = 5$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{Mb}^{M}$</th>
<th>$\sigma_{Mb}^{(M-1)b}$</th>
<th>$\sigma_{Mb}^{(M-2)b}$</th>
<th>$\epsilon_{M}^{Mb}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
<th>$\sigma_{mb}$</th>
<th>$\epsilon_{mb}^{b}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>4.9846</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>4.9939</td>
<td>5.0040</td>
<td>5.0466</td>
<td>2.1</td>
<td>0.063</td>
<td>4.9907</td>
<td>2.1</td>
<td>0.064</td>
</tr>
<tr>
<td>16</td>
<td>4.9986</td>
<td>5.0087</td>
<td>5.0515</td>
<td>2.1</td>
<td>0.062</td>
<td>4.9953</td>
<td>2.0</td>
<td>0.066</td>
</tr>
</tbody>
</table>

Table 2.2(c). Finite element stresses, estimated and actual discretization and total errors from free submodel meshes with 4Q elements using shape function boundary conditions on the enlarged subregion for $\sigma_{\text{max}} = 5$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{Mb}^{M}$</th>
<th>$\epsilon_{M}^{d}$</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{mt}^{d}$ (%)</th>
<th>$\epsilon_{mt}^{d}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>4.9846</td>
<td></td>
<td></td>
<td></td>
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<td>4.9939</td>
<td>0.19</td>
<td>0.25</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>16</td>
<td>4.9986</td>
<td>0.98</td>
<td>0.096</td>
<td>0.98</td>
<td>0.094</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Since the stress increments due to boundary condition refinement have opposite signs, estimates of the convergence rate of the boundary-condition error, $\epsilon_{mb}^{Mb}$ from (2.20), are not available (NA) throughout the submodel mesh sequence (Table 2.2(a)). This ultimately leads to the boundary-condition error estimate of (2.22) being NA throughout the submodel mesh sequence (Table 2.2(a)). Also the true boundary-condition error values from (2.36) are also not less than 0.067%. Thus we increase the subregion and apply our submodeling procedure (meshes $m = 11$, 12, 13). We find that the estimated values of boundary-condition error from (2.22) is still not less than 0.067%, and further the true boundary-condition error values from (2.36) are also not less than 0.067%. Hence we enlarge the area of the subregion a second time and apply our submodeling procedure (meshes $m = 14$, 15, 16). The FEA values of stresses and accompanying boundary-condition errors for this enlarged subregion are given in Table 2.2(b). The estimate of boundary-condition error from (2.22) on the final submodel mesh, $m = 10$, is 0.062% (Table 2.2(b)), so now
less than 0.067%. The true value for this error from (2.36) on mesh $m = 10$ is 0.066%, thus is just less than 0.067%. The FEA stress values from submodel meshes run with $M^{th}$ global mesh boundary conditions in Table 2.2(c) are the same as in Table 2.2(b), but now are accompanied by estimated and actual discretization and total errors. We estimate the discretization error with (2.12) on mesh $m = 10$ as 0.096% and the true discretization error from (2.37) on the same mesh is 0.094% (Table 2.2(c)). The estimated total error from (2.23) on mesh $m = 10$ is 0.16% which is in compliance with excellent criterion of less than 0.2% (Table 2.2(c)). The true total error is also 0.16% and confirms compliance with the excellent criterion. Here the last submodel mesh has about 66 thousand elements. A global mesh with the same resolution would have about 17 million elements. Although the subregion is enlarged twice, the reduction in number of elements is still 250 to 1.

Total error estimate values are the same or nearly so as their corresponding true values irrespective of whether structured or free meshes of 4Q elements are used with shape function boundary conditions. However, for free meshes, three times as many submodel meshes are run compared to with structured meshes (9 cf., 3). This is so when both the global and the submodel meshes are free. If instead largely free global meshes that have structured local meshes are used in conjunction with structured submodel meshes, we would expect performance to be closer to that for the use of structured meshes throughout. Unfortunately this is difficult to check here because local structured meshes in elliptic cylindrical coordinates are not supported in ANSYS [10]. We do, though, consider the effects of such an approach in our next section for an application.

Next, we present the results from structured submodel meshes with 4Q elements for $\bar{\sigma}_{\text{max}} = 54$ with cubic-spline fitted boundary conditions because these results serve as a benchmark for comparison of results for structured meshes with 8Q elements. We find that the estimated
boundary-condition errors throughout the submodel mesh sequence on our first subregion are not less than 0.067% (meshes \( m = 8, 9, 10 \)). Further the true boundary-condition error values are also not less than 0.067%. Thus we increase the subregion and apply our submodeling procedure (meshes \( m = 11 - 16 \)). The FEA stresses along with estimated and actual boundary-condition errors for this enlarged subregion are given in Table 2.3(a). The FEA stress value from our last global mesh (Table A.3 of Appendix A, for \( m = 7 \)) and that for initial submodel mesh (Table 2.3(a) for \( m = 11 \)) are exactly the same, which confirms that submodel boundary conditions have been correctly implemented. The estimated and true boundary-condition error values throughout the submodel mesh sequence are nearly the same and both are distinctly less than 0.067% (Table 2.3(a)).

The FEA stress values from submodel meshes run with \( M^{th} \) global mesh boundary conditions along with estimated and actual discretization and total errors are given in Table 2.3(b). We estimate the discretization error on the final submodel mesh \( m = 16 \) as 0.14%, which is the same as the true discretization error (Table 2.3(b)). The estimated and true total error values on mesh \( m = 16 \) are the same, and are 0.18% which is in compliance with excellent criterion of less than 0.2% (Table 2.3(b)). Here the last submodel mesh has about 1 million elements. A global mesh with the same resolution would have about 1 billion elements. Although the subregion is enlarged once, the reduction in number of elements is still 1000 to 1.

Table 2.3(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using cubic-spline fit boundary conditions on an enlarged subregion for \( \bar{\sigma}_{\text{max}} = 54 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{m}^{Mb} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \dot{\epsilon}_{m}^{Mb} )</th>
<th>( \dot{\epsilon}_{m}^{b} ) (%)</th>
<th>( \sigma_{m}^{eb} )</th>
<th>( c_{m}^{Mb} )</th>
<th>( c_{m}^{b} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>51.7166</td>
<td>( \sigma_{m}^{(M-1)b} )</td>
<td>( \sigma_{m}^{(M-2)b} )</td>
<td>( \dot{\epsilon}_{m}^{Mb} )</td>
<td>( \dot{\epsilon}_{m}^{b} ) (%)</td>
<td>( \sigma_{m}^{eb} )</td>
<td>( c_{m}^{Mb} )</td>
<td>( c_{m}^{b} ) (%)</td>
</tr>
<tr>
<td>12</td>
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<td>52.7574</td>
<td>52.5288</td>
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<td>0.037</td>
<td>52.8345</td>
<td>2.0</td>
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</tr>
<tr>
<td>13</td>
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<td>53.3304</td>
<td>53.0994</td>
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<td>0.037</td>
<td>53.4080</td>
<td>2.0</td>
<td>0.036</td>
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</tbody>
</table>
Table 2.3(a continued)

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{\varepsilon}_{m}^{Mb}$</th>
<th>$\hat{\varepsilon}_{m}^{b}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$\epsilon_{m}^{Mb}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
</tr>
</thead>
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<td>53.7018</td>
<td>2.0</td>
<td>0.036</td>
</tr>
<tr>
<td>15</td>
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<td>53.7718</td>
<td>53.5390</td>
<td>2.0</td>
<td>0.037</td>
<td>53.8503</td>
<td>2.0</td>
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<td>16</td>
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<td>53.6132</td>
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<td>0.037</td>
<td>53.9250</td>
<td>2.0</td>
<td>0.036</td>
</tr>
</tbody>
</table>

Table 2.3(b). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 4Q elements using cubic-spline fit boundary conditions on the enlarged subregion for $\bar{\sigma}_{\text{max}} = 54$.

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\hat{\varepsilon}_{m}^{d}$</th>
<th>$\hat{\varepsilon}_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{d}$</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>0.57</td>
<td>0.99</td>
<td>0.55</td>
<td>0.61</td>
<td>0.59</td>
</tr>
<tr>
<td>13</td>
<td>53.3888</td>
<td>0.98</td>
<td>0.28</td>
<td>0.99</td>
<td>0.28</td>
<td>0.32</td>
<td>0.32</td>
</tr>
<tr>
<td>14</td>
<td>53.6822</td>
<td>0.97</td>
<td>0.57</td>
<td>0.99</td>
<td>0.55</td>
<td>0.61</td>
<td>0.59</td>
</tr>
<tr>
<td>15</td>
<td>53.8307</td>
<td>0.98</td>
<td>0.28</td>
<td>0.99</td>
<td>0.28</td>
<td>0.32</td>
<td>0.32</td>
</tr>
<tr>
<td>16</td>
<td>53.9053</td>
<td>0.99</td>
<td>0.14</td>
<td>1.0</td>
<td>0.14</td>
<td>0.18</td>
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</tr>
</tbody>
</table>

The FEA stresses and the accompanying estimated and actual boundary-condition errors for $\bar{\sigma}_{\text{max}} = 54$ when analyzed with structured submodel meshes of 8Q elements with cubic-spline fitted boundary conditions are given in Table 2.4(a) (again six decimal places are used to avoid round-off errors). The estimated boundary-condition errors on the submodel mesh sequence are only 0.00041%, far less than 0.067% (Table 2.4(a)). The true values for this error on the submodel mesh sequence are similar, and of the order of 0.0003%, thus also far less than 0.067% (Table 2.4(a)). The FEA stress values from submodel meshes run with $M^{th}$ global mesh boundary conditions along with estimated and actual discretization and total errors are given in Table 2.4(b).
Table 2.4(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8Q elements using cubic-spline fit boundary conditions for $\bar{\sigma}_{\text{max}} = 54$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{\sigma}_{mb}^M$</th>
<th>$\hat{\epsilon}_{mb}^b$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$\epsilon_{mb}^M$</th>
<th>$\epsilon_{mb}^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>53.535355</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
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<td>53.799644</td>
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<td>53.865955</td>
<td>4.5</td>
<td>0.00031</td>
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<tr>
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<td>53.963604</td>
<td>53.959995</td>
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<td>4.1</td>
<td>0.00041</td>
<td>53.963768</td>
<td>4.5</td>
<td>0.00030</td>
</tr>
</tbody>
</table>

Table 2.4(b). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8Q elements using cubic-spline fit boundary conditions for $\bar{\sigma}_{\text{max}} = 54$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\hat{\epsilon}_{mb}$ (%)</th>
<th>$\epsilon_{mb}^d$ (%)</th>
<th>$\epsilon_{mb}^e$ (%)</th>
<th>$\epsilon_{mb}$ (%)</th>
<th>$\epsilon_{mb}^t$ (%)</th>
<th>$\epsilon_{mb}$ (%)</th>
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</thead>
<tbody>
<tr>
<td>8</td>
<td>53.535355</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>53.865788</td>
<td>0.25</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>53.963604</td>
<td>1.8</td>
<td>0.076</td>
<td>1.9</td>
<td>0.067</td>
<td>0.067</td>
<td>0.067</td>
</tr>
</tbody>
</table>

We estimate the discretization error on the final submodel mesh $m = 10$ as 0.076% and the true discretization error on the same mesh is 0.067% (Table 2.4(b)). The estimated total error on the same submodel mesh is 0.076% which is in compliance with excellent criterion of less than 0.2% (Table 2.4(b)). The true total error on the same submodel mesh is 0.067% and therefore also complies with the excellent criterion. Although excellent results are achieved with both 4Q and 8Q elements, three times as many submodel meshes are run with 4Q elements compared to with 8Q elements (9 cf., 3). Hence, our submodeling procedure in conjunction with 8Q elements performs better than 4Q elements with far fewer meshes being used.

For comparison, we present results from structured submodel meshes of 8Q elements run with shape function boundary conditions for $\bar{\sigma}_{\text{max}} = 54$. We find that the estimated boundary-condition errors throughout the submodel mesh sequence are not less than 0.067% (meshes $m = 8, 9, 10$). Furthermore the true boundary-condition error values are also not less than 0.067%. Thus
we increase the subregion and apply our submodeling procedure (meshes \( m = 11, 12, 13 \)). The FEA stresses and the accompanying estimated and actual boundary-condition errors for this enlarged subregion are given in Table 2.5(a). The estimated boundary-condition errors on the submodel mesh sequence are 0.022\%, thus is less than 0.067\% (Table 2.5(a)). The true values for this error on the submodel mesh sequence are 0.021\%, thus is also less than 0.067\%. The FEA stress values from submodel meshes run with \( M^{th} \) global mesh boundary conditions along with estimated and actual discretization and total errors are given in Table 2.5(b). We estimate the discretization error on the final submodel mesh \( m = 13 \) as 0.073\% and the true discretization error on the same mesh is 0.067\% (Table 2.5(b)). The estimated total error on the same mesh is 0.095\% which is in compliance with excellent criterion of less than 0.2\% (Table 2.5(b)). The true total error on the same mesh is 0.088\% and confirms that the true total error does comply with the excellent criterion. Here the last submodel mesh has about 4 thousand elements. A global mesh with the same resolution would have about 4 million elements. Although the subregion is enlarged once, the reduction in number of elements is still 1000 to 1.

In sum, therefore, essentially for this test problem with the maximum stress concentration, the hierarchy in terms of computational performance of the different options considered for submodel meshes is as follows: first, 8Q elements with cubic spline boundary conditions; second, 8Q elements with shape function boundary conditions; third, 4Q elements with either cubic spline or shape function boundary conditions. This is because the first option has only three submodel meshes with an ultimate mesh with 64 hundred degrees of freedom, whereas the second option has six meshes ending with 25 thousand degrees of freedom, and the third option has nine meshes ending with 2.1 million degrees of freedom.
Table 2.5(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8Q elements using shape function boundary conditions on an enlarged subregion for $\bar{\sigma}_{\text{max}} = 54$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_m$</th>
<th>$\sigma^{(M-1)b}_m$</th>
<th>$\sigma^{(M-2)b}_m$</th>
<th>$\hat{\varepsilon}^M_m$</th>
<th>$\hat{\varepsilon}^b_m$ (%)</th>
<th>$\tilde{\sigma}^M_m$</th>
<th>$\varepsilon^b_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>53.5354</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>53.8775</td>
<td>53.9124</td>
<td>54.0521</td>
<td>2.0</td>
<td>0.022</td>
<td>53.8659</td>
<td>2.0</td>
</tr>
<tr>
<td>13</td>
<td>53.9754</td>
<td>54.0103</td>
<td>54.1501</td>
<td>2.0</td>
<td>0.022</td>
<td>53.9638</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Table 2.5(b). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8Q elements using shape function boundary conditions on the enlarged subregion for $\bar{\sigma}_{\text{max}} = 54$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_m$</th>
<th>$\hat{\varepsilon}^d_m$</th>
<th>$\hat{\varepsilon}^d_m$ (%)</th>
<th>$c^d_m$</th>
<th>$\varepsilon^d_m$ (%)</th>
<th>$\hat{\varepsilon}^t_m$</th>
<th>$\varepsilon^t_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>53.5354</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>53.8775</td>
<td></td>
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<td>0.25</td>
<td></td>
<td></td>
<td>0.27</td>
</tr>
<tr>
<td>13</td>
<td>53.9754</td>
<td>1.8</td>
<td>0.073</td>
<td>1.9</td>
<td>0.067</td>
<td>0.095</td>
<td>0.088</td>
</tr>
</tbody>
</table>

The FEA stresses, estimates of the boundary-condition, discretization and total errors along with their true counterparts from submodel meshes of alternate FEA of the configurations in Tables 2.1 – 2.5 and other test problems, namely $\bar{\sigma}_{\text{max}} = 18$ and $377/8$, are given in Tables B.1 – B.8 of Appendix B. Throughout these tables estimated boundary-condition and discretization errors agree well with corresponding true errors and our submodeling procedure ultimately always results in excellent FEA stress estimates (total error < 0.2%) that are confirmed by the true stresses. This is so with the 4Q elements used irrespective of whether cubic splines or shape functions are used to interpolate boundary conditions.

We close this section by comparing error estimates with those of Cormier et al. [2] on our set of test problems. For 4Q elements, the estimated discretization error of (2.5) from [2] is nonconservative for 18 instances whereas the present method is never nonconservative. The
boundary-condition error estimation of (2.16) from [2] uniformly overestimates its true value by a factor of 3.0, while the present method estimates these errors to within a factor of 1.1. For 8Q elements, the estimated discretization error from (2.5) is initially nonconservative on two occasions and then overestimates its true value. The boundary-condition error estimation of (2.16) from [2] grossly overestimates its true value by more than an order of magnitude. In contrast, the present method is conservative and estimates these errors to within a factor of 1.3. This is because we take advantage of the effective convergence rate these errors are experiencing. All told, therefore, markedly improved error estimation with the present discretization and boundary-condition error estimates of (2.12) and (2.22).

2.5. Application

Here, for our application, we consider a keyway in a circular shaft (Figure 2.5(a)). In fact this is the same configuration as introduced in Figure 2.1. For this configuration we wish to determine the peak stress in the keyway corner. We begin with a description of this application, then describe the implementation of our submodeling procedure. Thereafter we report the results found.

For a 2D plane strain analysis, we consider a circular shaft of radius \( r_s \) penetrated by a keyway of depth \( d \) and width \( w \) (Figure 2.5(b)). The root radius of the keyway is \( r_0 \) as shown in the close-up (Figure 2.5(b)). We take rectangular Cartesian coordinates \((x, y)\) with origin \( O \) as our basic coordinate system to formulate our application (Figure 2.5(b)). A high peak stress for the configuration occurs when the key abuts the keyway flanks without making contact at the root radius. To simulate such a situation we apply a normal pressure \( p \) on the keyway flanks on either side of the root radius (Figure 2.5(b)). In actuality the resultant of the force due to these pressures in the vertical \( y \) direction is balanced by out-of-plane shear stresses that are not active in the present 2D analysis. To balance the vertical forces, therefore, we constrain the cross section in the vertical
direction along the $x$ axis. In addition we use local, rotated, rectangular Cartesian coordinates ($x’, y’$) with origin $O’$ to facilitate in representing the applied normal tractions (close-up in Figure 2.5(b)). The $y$ axis of the original Cartesian coordinates passes through $O’$. The value of $y$ at the center of the root radius above $O’$ is $y_0$. With these geometric preliminaries in place, we next describe our application problem.

In general, we seek the plane strain stresses $\sigma_x, \sigma_y, \tau_{xy}$, and their associated displacements $u_x, u_y$, as functions of $x$ and $y$ throughout the shaft cross section satisfying the field equations of plane strain elasticity and the following boundary conditions. The applied normal pressure conditions that have

$$
\sigma_{y’} = -p, \tau_{x’y’} = 0
\sigma_{x’} = -p, \tau_{x’y’} = 0
$$

(2.38)

The first of (2.38) holds on $y’ = 0$ for $r_0 \leq x’ \leq d$, and the second holds on $x’ = 0$ for $r_0 \leq y’ \leq d$. The constraint conditions along the $x$ axis that take

$$
u_y = 0, \tau_{xy} = 0
$$

(2.39)

on $y = 0$ for $-\sqrt{2}r_s \leq x \leq 0$. All other surfaces are stress free. To stop translation in the $x$ direction, we further impose

$$
u_x = 0
$$

(2.40)

at the origin $O$. In particular, we seek the peak stress occurring in the keyway corner. Because of the local symmetry of the configuration (close-up in Figure 2.5(b)), we take the peak stress to occur at $x = 0$, and seek the value for $\bar{\sigma}_x = \sigma_x/p$ at $x = 0, y = y_0$. 


The specific geometry considered here is for a crank shaft in a two stroke engine. It has the following measured dimensions: $r_s = 6.35$ mm ($1/4"$), $w = 3.175$ mm ($1/8"$), $d = 1.5875$ mm ($1/16"$), and an average root radius of the keyway $r_0 = 0.096774$ mm ($0.00381"$).

To begin the FEA for this application, we use global meshes of $4Q$ elements, [22]. We start our discretization ($m = 1$) with a uniformly structured coarse mesh around the root radius to
facilitate our submodeling procedure. This region is picked such that it forms our submodel region. Outside of this region, free meshes are generated using an automatic mesh generator, [10]. The initial largely free global mesh used is shown in Figure 2.1(a) and has 269 elements. This mesh is systematically refined by halving the element sides in the vicinity of the root radius, and outside of this region is refined such that $\lambda \sim 2$. Five further meshes ($m = 2 - 6$) are produced with such refinement. Our finest mesh in this sequence has about 283 thousand elements.

We implement the submodeling procedure of Section 2.3. Again, we check for correctness of our submodel boundary conditions with our first structured submodel mesh ($m = 7$) that has about 4 thousand elements. Thereafter we refine this mesh by successively halving the element sides and produce two further meshes ($m = 8$ and $9$). We run these submodel meshes with both displacement shape functions and cubic-spline fitted displacements on their cut boundaries. Our last submodel mesh has about 66 thousand elements, when a global mesh for the same resolution would have about 4.5 million elements.

The FEA values for the normalized peak stress from our global analysis along with their estimated discretization error, using (2.12), are given in Table 2.6(a). The decreasing trend of the estimated discretization error values in Table 2.6(a) is consistent with a numerically converging analysis. The estimated discretization error on the finest possible global mesh is 0.31%. If a good accuracy level of (2.14) is sought, we would accept the FEA value of the normalized peak stress from our last global mesh. Here, however, we seek excellent results and the estimated discretization error from our last global mesh does not comply with the excellent criterion of (2.14), namely less than 0.2%. Hence we proceed to submodeling to improve results.

In following our submodeling procedure, we check for correctness of the boundary conditions. In doing so we found a discrepancy after 5 decimal places in the stress values from the
first submodel mesh and the last global mesh. Upon further examination, we found that displacement boundary conditions used are for nodes just outside the submodel region. These displacement boundary conditions are close, but nonetheless they are not correct. We subsequently implemented correct boundary conditions on the cut boundaries and the results that follow are for these boundary conditions. For us, this demonstrates the value of the check on the implementation of the boundary conditions on the cut boundary.

We illustrate the convergence of the peak stress resulting from our analysis in Figure 2.6. The distribution of $\sigma_x$ normalized by $\sigma_0$ from our global analysis is shown in Figure 2.6(a). These distributions are shown as functions of normalized distance $y/y_0$. The convergence of the normalized peak stress away from the center of the root radius is evident from Figure 2.6(a). The close-up in Figure 2.6(a) shows the convergence of the normalized peak stress near the center of the root radius. Then Figure 2.6(b) shows the distribution of the normalized peak stress from our submodel analysis using displacement shape function boundary conditions on an expanded vertical scale. The close-up in Figure 2.6(b) now shows the stresses near the center of the root radius are converging further.

The FEA stress values from the submodel analysis, using shape function boundary conditions, with their corresponding discretization and total error estimates are given in Table 2.6(a), below the dotted line. The FEA stress value from our last global mesh (Table 2.6(a) for $m = 6$) and the first submodel mesh (Table 2.6(a) for $m = 7$), are exactly the same. This confirms that eventually correct boundary conditions are chosen to run our submodel meshes. The estimated boundary-condition errors from (2.22) on our submodel mesh sequence are 0.026% (Table 2.6(b)), and so less than 0.067%. The estimated discretization error for the last submodel mesh from (2.12)
is 0.059% (Table 2.6(a)). The estimated total error for this last submodel mesh from (2.23) is 0.085%, thus now less than 0.2%.

Figure 2.6 Convergence of peak stress: (a) global meshes; (b) submodel meshes.

As expected for the low-order elements used, the corresponding results with cubic-spline fitted boundary conditions for this problem are essentially the same. The final total error estimate with cubic-spline fitted boundary conditions is 0.086% and so also less than 0.2%. For this and
like applications, therefore, when analyzed with 4Q elements, shape functions can be expected to be the preferred choice for finite element engineers because they are easier to implement.

Table 2.6(a). Finite element stresses, estimates of discretization error from global and submodel meshes, and total error estimates for the keyway application.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m, \sigma_{Mb}^M$</th>
<th>$\Delta\sigma_m$</th>
<th>$\hat{\epsilon}^{d}_m$</th>
<th>$\hat{\epsilon}^{d}_m$ (%)</th>
<th>$\hat{\epsilon}^{t}_m$ (%)</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.8551</td>
<td>0.0737</td>
<td></td>
<td></td>
<td></td>
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<td>7.9081</td>
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</tr>
<tr>
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<td>7.9456</td>
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</tr>
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<td>7.9702</td>
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<tr>
<td>6</td>
<td>7.9854</td>
<td>0.0152</td>
<td>0.69</td>
<td>0.31</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.6(b). Finite element stresses and estimates of boundary-condition error from structured submodel meshes using shape function boundary conditions for the keyway application.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{Mb}^M$</th>
<th>$\sigma_{m-1}^{(M-1)b}$</th>
<th>$\sigma_{m-2}^{(M-2)b}$</th>
<th>$\hat{\epsilon}^{Mb}_m$</th>
<th>$\hat{\epsilon}^{b}_m$ (%)</th>
</tr>
</thead>
<tbody>
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<td>7</td>
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<td></td>
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<td>8.0154</td>
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<td>0.026</td>
</tr>
<tr>
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<td>8.0050</td>
<td>8.0204</td>
<td>1.7</td>
<td>0.026</td>
</tr>
</tbody>
</table>

2.6. Concluding Remarks

The computational savings of submodeling can only be truly realized if errors due to boundary conditions applied to the submodel are controlled. When these errors as well as the discretization errors inherent in FEA are controlled, submodeling is effective.
Cormier et al., [2] apply displacement boundary conditions on submodel cut boundaries, with cubic splines used for intervening values, and furnish a means of estimating both boundary-condition error and discretization error. Here we offer improvements to the two error estimates introduced in [2] by taking into account effective rates of convergence. We also use both displacement shape functions and cubic-splines to fit displacement boundary conditions on the cut boundaries of the submodels.

These improvements are demonstrated to be effective on a set of four test problems with known exact solutions for peak stresses. These test problems have a range of stress concentration factors that exceeds that normally found in practice (cf., Peterson [1]). The improved discretization error estimates are conservative throughout global mesh sequences when compared to true error values. Although they are conservative, these error estimates suggests the use of submodeling when the true errors also indicate the same. For comparison, the discretization error estimate of [2] is overall less accurate and is also nonconservative on a number of occasions.

The improved boundary-condition error estimates correctly indicate when it is necessary to enlarge the subregion to gain control of these errors. Thereafter, subregions with increased areas so that control is achieved are always found. In comparison, the boundary-condition error estimates of [2] consistently overestimate their true values by a factor of three and thus are significantly less accurate than those from the present approach.

Results from submodel analysis of structured meshes with 4Q elements are essentially the same irrespective of whether shape functions or cubic splines are used to fit boundary conditions. Hence for these low-order elements, finite element engineers can be expected to employ shape functions for submodel boundary conditions because these are easier to implement. Although free meshes for both global and submodel meshes together with shape functions are even easier to
implement, what is demonstrated here is that one can expect to need significantly more submodel meshes when they are free instead of structured.

For higher stress concentrations, structured global and submodel meshes with 8Q elements are found to perform better than 4Q elements with degrees of freedom that are fewer by two orders of magnitude. This advantage is further increased if cubic splines instead of shape functions are used with 8Q elements.

The implementation of our submodeling procedure with 4Q elements is further demonstrated on an application problem. Our error estimates indicate that excellent results are achieved with both displacement shape functions and cubic splines as fits for displacement boundary conditions for this application. Again, therefore, one can expect shape functions to be preferred because of ease of use. Here partially structured global meshes in conjunction with structured submodel meshes appear to work as well as entirely structured meshes and hence are also likely to be the preferred approach for FEA engineers.

In sum, the approach to submodeling described here demonstrates the control of FEA errors in submodeling by a combination of mesh refinement and increasing submodel areas. This performance for the FEA of 2D stress concentration problems augurs well for attempting to extend the present approach to 3D stress concentrations.
Chapter 3. Improved Submodeling of Three-Dimensional Stress Concentrations

3.1. Introduction

An example to illustrate submodeling in 3D is shown in Figure 3.1 for a test problem. This test problem concerns a solid ellipsoid weakened by a hyperbolic notch encircling its equator under tension. The intent here is to use submodeling to accurately compute local notch stresses.

The initial global mesh of an octant with a hyperbolic notch is shown in Figure 3.1(a). Close-ups of the initial and the finest global mesh of the notch region are shown in Figure 3.1(b) and (c), respectively. For this configuration the peak stress occurs at the center of the notch corner highlighted by a red dot (Figures 3.1(b), (c) and (d)). The neighboring region, highlighted by a red line, then forms our submodel region (Figure 3.1(c)). This submodel region is broken out and analyzed separately with finer mesh. Figure 3.1(d) shows the final submodel mesh. The boundary conditions applied to these submodel meshes are taken from the finest mesh of the global configuration (Figure 3.1(c)). The finest submodel mesh has 32 thousand elements whereas the corresponding global mesh would have 134 million elements. Hence a reduction in number of elements is 4,000 to 1. This shows that computational savings in 3D can be significant, provided errors are controlled.

Submodeling has been used in industry to resolve local stresses in complex 3D structures. Applications of the technique in various industries include: automotive [23], biomedical [24 - 27], electronics packaging [28 - 32], gas turbine [33, 34], and ship building [35]. Submodeling has also been used to analyze different aspects of structural analysis such as: suspension bridges [36, 37], fretting fatigue problems [38 - 41], composite and bolted joints [42 - 45], and some contact problems [46]. All of the foregoing references take advantage of submodeling procedure of Cormier et al., [2] to obtain desired FEA stresses.
The development of improved error estimates and submodeling procedure in 2D is provided in Chapter 2. Here we intend to extend the approach of Chapter 2 to three dimensions. In doing so, we continue to use displacement shape functions as boundary conditions. Additionally, analogously to cubic splines here we use bicubic surface to interpolate displacement boundary conditions. We also employ some modifications to the error estimates to account for the presence of...
of nonmonotonic convergence in the FEA of stresses. Nonmonotonic convergence occurs quite frequently in 3D FEA.

To demonstrate the presence of nonmonotonic convergence in the FEA of 3D stresses, we consider the previous example of solid ellipsoid weakened by a hyperbolic notch (Figure 3.1). The radius of the solid at the notch root is $a = 0.9$. This ellipsoid is analyzed with eight-node hexahedral (8H) elements [22]. We start our discretization with uniform structured initial mesh ($m = 1$) with 4 equal increments in all the three directions. Subsequent meshes ($m = 2 - 6$) are formed by successively halving element extents throughout the mesh sequence. Finite element results ($\sigma_m$) for peak, normalized, tensile stress ($\bar{\sigma}_{\max}$) from all six meshes are given in Table 3.1.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m$</th>
<th>$\Delta\sigma_m^d$</th>
<th>$\epsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.641967</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.048764</td>
<td>0.406797</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>2.242300</td>
<td>0.193536</td>
<td>2.8</td>
</tr>
<tr>
<td>4</td>
<td>2.300289</td>
<td>0.057989</td>
<td>0.29</td>
</tr>
<tr>
<td>5</td>
<td>2.311374</td>
<td>0.011085</td>
<td>-0.19</td>
</tr>
<tr>
<td>6</td>
<td>2.311229</td>
<td>-0.000145</td>
<td>-0.18</td>
</tr>
</tbody>
</table>

In Table 3.1 $\Delta\sigma_m^d$ continues to be the stress increment with mesh refinement. Since the exact value ($\sigma_e$) of this stress (2.307001) is also known, the true discretization errors ($\epsilon_m^d$) of (2.35) in the FEA stress are also included in Table 3.1. With monotonic convergence, the actual errors will have a constant sign: whereas when convergence is nonmonotonic, they change sign. Here they do between mesh $m = 4$ and 5.

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In practical applications we do not have access to $\sigma_e$ and hence, actual errors are not available. In such situations we resort to using stress increments to estimate errors in FEA stress. Now treating the preceding example as application, the nonmonotonic convergence is also reflected by the sign change in stress increments between mesh $m = 5$ and 6. Given a sign change between mesh $m$ and $m - 1$, one way to guard against nonmonotonic convergence is to have error estimates from stress increments of either mesh to be not applicable (NA). Such an approach only works when a sign change is evident. If instead, results in Table 3.1 were only available for meshes $m = 1$ to 5, no such sign change is evident. Then estimating the discretization error for $m = 5$ with (2.12) of Chapter 2 leads to an error value of 0.11% whereas the true value of this error is 0.19%. Hence we underestimate the error by a factor of 1.7. Here, therefore, we modify the discretization error estimation of Chapter 2 by using the precautions as in Sinclair et al., [47] to guard against underestimating errors in the presence of nonmonotonic convergence but is yet to reveal itself with a sign change in stress increments. Like modifications are also adopted for estimating boundary-condition errors.

In what follows, we describe the basic convergence checks in Section 3.2. The precautions used to guard against underestimating errors in the presence of nonmonotonic convergence are given in Section 3.3. These precautions are applicable to both global and submodel meshes. In Section 3.4, similar improved checks are developed for estimating boundary-condition error with submodel meshes. In Section 3.5, the submodeling procedure is evaluated on a set of 3D test problems that have known exact solutions for the peak stresses of interest. To demonstrate the implementation of our procedure on a practical problem, in Section 3.6 it is used on a pin hole in a shaft. We close, in Section 3.7, with concluding remarks in the light of results found.
3.2. Basic Convergence Checks

The basic convergence checks are developed in Section 2.2 of Chapter 2. We state them here for completeness. Initially we begin with a description of the mesh refinement scheme employed, then we state the improved error estimate that reflects the effective convergence rate they are experiencing.

An important step in analyzing any problem with FEA is to choose an effective mesh refinement scheme. Here we proceed as follows. We let $h_m$ denote the representative length of elements in mesh $m$. We reduce $h_m$ by a constant scale factor from the preceding mesh size throughout the mesh sequence. Thus, if $\lambda$ is the constant scale factor adopted,

$$h_m = h_{m-1}/\lambda$$

for $m \geq 2$ and $\lambda > 1$. Here we continue to form successive meshes simply by halving element sides and hence have $\lambda = 2$. This leads to eight-fold increases in element numbers in three dimensions. In 3D elasticity, initially if a suitably coarse mesh is used, then mesh refinement in accord with (3.1) is not usually computationally problematic. Alternatively, too, the approach can be implemented by mesh coarsening if one has a baseline mesh which is deemed to have sufficient accuracy to capture key stresses from FEA. This second approach is obviously computationally feasible.

The stress increment attending mesh refinement from $m - 1$ to $m$ is defined by

$$\Delta \sigma_m^d = \sigma_m - \sigma_{m-1}$$

These stress increments need to be reducing in magnitude with mesh refinement for the FEA to be judged to be converging, and the ultimate mesh increment leading to $\sigma_m$ needs to be within the error level sought for the FEA to be judged to have converged.
In what follows we account for an effective rate of discretization error convergence, \( c_m^d \). Then following the development in Section 2.2, we estimate the rate of convergence, \( \hat{c}_m^d \), for discretization error on mesh \( m \) as

\[
\hat{c}_m^d = \frac{\ln(\Delta \sigma_m^d/\Delta \sigma_{m-1}^d)}{\ln \lambda}
\]  \hspace{1cm} (3.3)

Finally, we estimate the absolute relative discretization error in \( \sigma_m \), \( \hat{\epsilon}_m^d \), on mesh \( m \) as being

\[
\epsilon_m^d = \frac{|\Delta \sigma_m^d|}{|\sigma_m| (\lambda^{\hat{c}_m^d} - 1)}
\]  \hspace{1cm} (3.4)

The expression in (3.4) is usually expressed as a percentage. The sensitivity to convergence rates of the estimate of (3.4) is essentially what is introduced in Roache [15]. Alternatively, combining (3.3) with (3.4), we have

\[
\epsilon_m^d = \frac{|\Delta \sigma_m^d|}{|\sigma_m| (\Delta \sigma_m^{d-1} - 1)}
\]  \hspace{1cm} (3.5)

Equation (3.5) rather than (3.4) is what is used subsequently to estimate the absolute relative discretization error in our problems.

Now applying (3.3) and (3.5) to the results in Table 3.1 for \( m = 1 - 5 \), we estimate the convergence rate and discretization error as

\[
\hat{c}_5^d = 2.38, \quad \hat{\epsilon}_5^d = 0.11\%
\]  \hspace{1cm} (3.6)

While the true effective convergence rate and discretization error from (2.6) and (2.5), respectively are

\[
c_5^d = \text{NA}, \quad \epsilon_5^d = 0.19\%
\]  \hspace{1cm} (3.7)
Hence in this instance, underestimating errors by a factor of 1.7 has the potential to lead to misleading estimates. Next, we look to alleviating such a situation with precautions against underestimating errors in the presence of nonmonotonic convergence.

3.3. Precautions Against Underestimating Errors in the Presence of Nonmonotonic Convergence

Here we adopt a series of precautions aimed to prevent underestimating errors in the presence of nonmonotonic convergence. These precautions or procedures are developed in Sinclair et al., [47]; here we simply state them for completeness.

Procedure 1: This procedure takes advantage of sign change in stress increment which is an obvious signature of the presence of nonmonotonic convergence. This procedure has

$$\Delta \sigma^d_{m-1} \cdot \Delta \sigma^d_m < 0 \rightarrow \hat{\epsilon}^d_{m-1} \text{ and } \hat{\epsilon}^d_m = NA$$

(3.8)

Now applying (3.8) to the results in Table 3.1, the error estimate for $m = 6$ is removed since the inequality of (3.8) holds. Also the nonconservative error estimate of (3.6) for $m = 5$ is now removed (Table 3.1).

Alternative procedures are needed when sign change in stress increments are not evident yet. The underlying reason for errant estimates in the absence of sign change is that the stress increments approach zero when FEA starts to reverse direction. Under such situation $\hat{\epsilon}^d_m$ of (3.4) tends to zero because of two reasons: first, $\Delta \sigma^d_m$ in its numerator tends to zero; second, the increase in $\hat{\epsilon}^d_m$ of (3.3) when $\Delta \sigma^d_m \rightarrow 0$. Hence we adopt the following procedures to avoid such ill effects.

Here we focus on choosing the value of estimated convergence rate that leads to conservative error estimates instead of simply taking the value of $\hat{\epsilon}^d_m$ form (3.3). Since we continue to use $\lambda = 2$, we therefore take the discretization error estimate as

$$\hat{\epsilon}^d_m = \frac{|\Delta \sigma^d_m|}{|\sigma_m|(2\hat{\epsilon}^d_m - 1)}$$

(3.9)
where $\tilde{c}_m^d$ is the adjusted convergence rate for mesh $m$.

**Procedure 2:** This procedure takes advantage of slow convergence i.e., when $\hat{c}_m^d \leq 1$. Under these circumstances error estimates are increased over simply taking the quotient $|\Delta \sigma_m^d / \sigma_m|$. Hence the error estimate is recognizing to a degree the increased errors attending slow convergence.

Accordingly for any mesh $m$ we take

$$\tilde{c}_m^d = \hat{c}_m^d \text{ if } \hat{c}_m^d \leq 1$$

(3.10)

Conversely for $\hat{c}_m^d > 1$, we have three slightly different procedures for obtaining $\tilde{c}_m^d$ depending on mesh number. All of these use the variation in $\hat{c}_m^d$, $\delta \hat{c}_m^d$, defined as the change in $\hat{c}_m^d$ normalized by its average value, thus

$$\delta \hat{c}_m^d = \frac{2(\hat{c}_m^d - \hat{c}_{m-1}^d)}{(\hat{c}_m^d + \hat{c}_{m-1}^d)}$$

(3.11)

Then the procedures for $\hat{c}_m^d > 1$ depending on mesh number are as follows:

**Procedure 3:** For initial mesh ($m = m_i$)

$$\tilde{c}_{m_i}^d = 1$$

(3.12)

**Procedure 4:** For intermediate meshes ($m_i < m < M$)

$$\tilde{c}_m^d = \begin{cases} \hat{c}_m^d \text{ provided } & 0.01 \leq \delta \hat{c}_m^d \leq 0.5 \\ 1 \text{ otherwise} & \end{cases}$$

(3.13)

where $M$ is the last mesh in the sequence.

**Procedure 5:** For final mesh ($m = M$)

$$\tilde{c}_M^d = \begin{cases} \hat{c}_M^d \text{ provided } & 0.01 \leq \delta \hat{c}_M^d \leq 0.2 \\ 1 \text{ otherwise} & \end{cases}$$

(3.14)

and

$$\hat{c}_M^d \text{ is NA if } \delta \hat{c}_M^d > 0.5$$

(3.15)
Some justification for the preceding procedures is provided in [47]. Equations (3.8) – (3.15) then realize $\hat{e}_m^d$ for nonmonotonically converging FEA.

Now suppose the results in Table 3.1 were only available for meshes $m = 1$ to 5. We estimate the convergence rate for meshes $m = 4$ and 5 with (3.3) as

$$\hat{e}_4^d = 1.7 \text{ and } \hat{e}_5^d = 2.4$$

(3.16)

respectively. Then applying our modified procedures for $\hat{e}_m^d > 1$, we have $\delta \hat{e}_m^d$ from (3.11) on mesh $m = 5$ as

$$\delta \hat{e}_5^d = 0.34$$

(3.17)

Applying the modified procedure of (3.14) we take the convergence rate as one and estimate the discretization error on mesh $m = 5$ as

$$\hat{e}_5^d = 0.48\%$$

(3.18)

Previously in (3.6), we estimated the discretization error on mesh $m = 5$ to be 0.11\% whereas its true value as in (3.7) is 0.19\%. Since the discretization error estimate on mesh $m = 5$ of (3.18) is greater than its true value of (3.7), the nonconservative error estimate of (3.6) is hence removed.

The implementation of the improved convergence checks on a sequence of meshes that comply with (3.1) is as follows. First, the stress increments as in (3.2) are calculated. If the stress increments of (3.2) are of same sign and decreasing we judge the FEA to be converging. Second, we estimate the effective rate of convergence using (3.3). Third, we apply our modified procedures of (3.8) - (3.15) to guard against underestimating errors in the presence of nonmonotonic convergence and estimate the discretization error in our problems using (3.5).

Finally in assigning merit to the absolute relative discretization error so obtained, we check for

$$\frac{\hat{e}_m^d}{\epsilon_s} \leq \epsilon_s$$

(3.19)
where $\hat{\varepsilon}_M^d$ is given as percentage, and $\varepsilon_s$ is the percentage error level sought in the FEA determination of $\sigma_a$. We classify $\varepsilon_s$ in accordance with

\begin{align*}
1 < \varepsilon_s &\leq 5 \implies \text{satisfactory accuracy} \\
1/5 < \varepsilon_s &\leq 1 \implies \text{good accuracy} \\
\varepsilon_s &\leq 1/5 \implies \text{excellent accuracy}
\end{align*}

(3.20)

We have found the ranges in (3.20) to be reasonable for stress concentration problems in practice, but certainly other ranges could be assigned to these three levels of accuracy. Here we seek excellent results, hence $\hat{\varepsilon}_M^d \leq 1/5$. With global FEA, if $\hat{\varepsilon}_M^d \leq 1/5$ is achieved, we accept $\sigma_M$ as the FEA determination of $\sigma_a$ on mesh $M$. Otherwise we continue mesh refinement, thereby increasing the value of $M$. If $\hat{\varepsilon}_M^d$ remains greater than $1/5$ and further mesh refinement is not computationally possible, we proceed to submodel as described in Section 2.3.

### 3.4. Boundary-Condition Error Control

The improved boundary-condition error estimate is developed in Section 2.3 of Chapter 2. We state it here for completeness. Initially we begin with a description of the mesh refinement scheme employed in submodeling, then we give the improved boundary-condition error estimate that reflects the effective convergence being experienced by these errors in submodeling. Thereafter we describe the procedure employed to account for nonmonotonic convergence in FEA of stresses due to boundary condition refinement.

As for the global meshes, here we index successively refined submodel meshes with $m \geq M + 1$, and $h_m$ denotes the representative length of elements in submodel mesh $m$. As previously we reduce $h_m$ by the same constant scale factor, $\lambda$, from the preceding mesh size throughout the mesh sequence. Here, we continue to form uniform successive submodel meshes by halving element sides and thus continue to have $\lambda = 2$. 

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With submodel meshes, there are two sources of error. The first source is the discretization error. This is the error that is inherent with any FEA and consequently occurs in both global and submodel meshes. The second source is the boundary-condition error. This is the error in the stresses sought in the submodel that is incurred by taking values from the global analysis and using them as boundary conditions on the cut boundaries of the subregion.

As previously we estimate the boundary-condition error using stress increments. Then the stress increment, on a submodel mesh \( m \), attending boundary condition refinement from \((M-1)\) to \( M \) with global meshes is defined by

\[
\Delta \sigma_{mb}^M = \sigma_{mb}^M - \sigma_{mb}^{(M-1)b}
\]

for \( m \geq M + 1 \), where \( \sigma_{mb}^M \) and \( \sigma_{mb}^{(M-1)b} \) are the stresses found using boundary conditions from the \( M \) and \((M-1)\) global meshes, respectively. Then following the development in Section 2.3, we estimate the rate of convergence, \( \hat{\epsilon}_m^{mb} \), for discretization error on mesh \( m \) as

\[
\hat{\epsilon}_m^{mb} = \frac{\ln\left(\frac{\Delta \sigma_{mb}^{(M-1)b}}{\Delta \sigma_{mb}^M}\right)}{\ln \lambda}
\]

Finally, we estimate the absolute relative boundary-condition error in \( \sigma_{mb}^M \), \( \hat{\epsilon}_m^b \), as

\[
\hat{\epsilon}_m^b = \frac{|\Delta \sigma_{mb}^M|}{|\sigma_{mb}^M|\left(\lambda^{\hat{\epsilon}_m^{mb}} - 1\right)}
\]

for \( m \geq M + 1 \). The expression in (3.23) is usually expressed as a percentage. Alternatively, analogously to (3.5), replacing \( \lambda^{\hat{\epsilon}_m^{mb}} \) with the stress increment quotient of (3.22) in (3.23), we have

\[
\hat{\epsilon}_m^b = \frac{|\Delta \sigma_{mb}^M|}{|\sigma_{mb}^M|\left(\frac{\Delta \sigma_{mb}^{(M-1)b}}{\Delta \sigma_{mb}^M} - 1\right)}
\]
The estimate of (3.24) is used to estimate the absolute relative boundary-condition error in our problems. It is understood in (3.24) that we are only interested in the boundary-condition error resulting in using the boundary conditions from the most refined global mesh \( m = M \).

Taken together the two error estimates of (3.5) and (3.24) combine to give our absolute relative total error estimate for \( \sigma_m^{Mb}, \hat{\epsilon}_m^t \). That is

\[
\hat{\epsilon}_m^t = \hat{\epsilon}_m^d + \hat{\epsilon}_m^b
\]  

(3.25)

For \( \hat{\epsilon}_m^t < \epsilon_s \), clearly neither \( \hat{\epsilon}_m^d \) nor \( \hat{\epsilon}_m^b \) can exceed \( \hat{\epsilon}_m^t \). From the experience in Chapter 2, the boundary-condition error is smaller than the discretization error on a given submodel mesh and so here we want its share to be less than half of the error level sought \( (\epsilon_s) \). In line with this expectation, here we continue to seek

\[
\hat{\epsilon}_m^b \leq \frac{1}{3} \epsilon_s
\]  

(3.26)

The choice of \( 1/3 \) in (3.26) as a fraction somewhat less than \( 1/2 \) is simply convenient and is not critical in what follows. Since overall we seek excellent results, (3.20) in conjunction with (3.26) has \( \hat{\epsilon}_m^b \leq 0.067\% \). Additionally when multiple submodels are used, since the errors in the boundary conditions are being accumulated, we add the boundary-condition error estimate for the additional submodel mesh sequence with those from the previous submodel mesh sequence. These increased boundary-condition error estimates must now be less than 0.067%.

Based on our experience in Section 3.3, sign change in stress increment is the indication of the presence of nonmonotonic convergence. Taking advantage of this obvious signature, parallel to procedure 1 in Section 3.3 here we do not accept the boundary-condition error estimate of (3.24) on a submodel mesh \( \hat{\epsilon}_m \) when the stress increments of (3.21) change sign with boundary condition refinement and have
When the inequality of (3.27) holds on a sequence of 3 submodel meshes, we move the cut-boundaries of the submodel region further away from the stress of interest. We do this by doubling the length of extents of the sides of the submodel region. We follow the same steps as described in Section 2.3 to estimate the errors for this enlarged submodel region.

Also in lieu of sign change, when the stress increments of (3.21) on a submodel mesh \( m \) are increasing with boundary condition refinement, that is when

\[
\Delta \sigma^{(M-1)b}_m \cdot \Delta \sigma^{Mb}_m < 0 \rightarrow \hat{\epsilon}^b_m = \text{NA}
\]

\[(3.27)\]

\( \hat{\epsilon}^b_m \) of (3.22) is not available (NA), this ultimately leads to \( \hat{\epsilon}^b_m \) of (3.24) being NA. In such situations we enlarge the subregion and follow the preceding procedure. The application of procedures of (3.27) and (3.28) is demonstrated next in Section 3.5.

3.5. Test Problems

In this section for test problems, we consider a solid ellipsoid weakened by a hyperbolic notch encircling its equator and subjected to a tensile force \( F \) (Figure 3.2). The radius of the solid at the notch root is \( a \), while the notch root radius is \( r_0 \). Of greatest interest are the stresses induced by \( F \) at the notch root. In particular, here, we seek peak, normalized, tensile and hoop stress at the notch root. By reducing \( r_0 \), these stresses become increasingly concentrated and so challenge FEA.

Although this is an axisymmetric problem, when subject to FEA in rectangular Cartesian coordinates there are no coordinates where stresses do not vary with. In effect it is a 3D problem for FEA. Accordingly we formulate this problem in three dimensions. Exact analytical solutions for these stresses are derived in Neuber [47]. These solutions are for uniform tractions being applied at infinity. These solutions are evaluated on an ellipsoidal arc and the exact quantities so found are applied as displacement boundary conditions there. We use displacement here because
they are typically easy to apply in rectangular Cartesian coordinates and for that reason others may find it easy to apply. This enables us to formulate a finite solid ellipsoid with an exact solution for FEA. The exact solution of the peak stresses are used to assess the accuracy of FEA stresses, as well as the performance of convergence checks on these FEA stresses. Here we begin with a formal statement of our test problems. Then we describe the application of our submodeling procedure to these test problems. Thereafter we report the results found.

The configuration is most readily framed in oblate spheroidal coordinates \((\xi, \eta, \phi)\). These coordinates are related to their rectangular Cartesian counterparts, \((x, y, z)\) of Figure 3.2, by

\[
\begin{align*}
    x &= \text{sh} \xi \cos \eta, \\
    y &= \text{ch} \xi \sin \eta \cos \phi, \\
    z &= \text{ch} \xi \sin \eta \sin \phi
\end{align*}
\] (3.29)

With (3.29), lines of constant \(\xi\) are ellipsoids of revolution or spheroids, and lines of constant \(\eta\) are hyperboloids of one sheet.\(^{1}\) Since a 3D FEA solves this problem in \(x, y\) and \(z\) coordinates, it fails to recognize the axisymmetry. Hence in exact solutions of these problems the normal displacement \((u_\phi)\) and the shear stresses acting on a constant \(\phi\) plane \((\tau_{\phi\eta} \text{ and } \tau_{\phi\xi})\) are equal to zero (i.e., \(u_\phi = 0, \tau_{\phi\eta} = \tau_{\phi\xi} = 0\)). In what follows we use symmetry of the geometry and loading to restrict our analysis to an octant of the ellipsoid. Hence our finite region for the test problems, \(\mathcal{R}\), is given by

\[
\mathcal{R} = \{(\xi, \eta, \phi) \mid 0 < \xi < \xi_0, \ 0 < \eta < \eta_0, \ 0 < \phi < \pi/2\}
\] (3.30)

In (3.30), we take

\[
\xi_0 = \text{ch}^{-1} 4/a, \ \eta_0 = \sin^{-1} a
\] (3.31)

\(^{1}\) Usually there is a length scale on the right-hand sides of the equations in (3.1). Following [47], we set the value of this scale to be unity. While so simplifying expressions a little, this does have the effect of making some look questionable in terms of dimensions.
for $a < 1$. This results in lateral extents in the $y$ and $z$ directions of 4 as $a$ is varied, and thus nominal far-field stresses that are a factor of at least 16 times smaller than nominal net-section stresses. With these geometric preliminaries in place, we can formally state the class of notched-ellipsoid test problems for FEA as follows.

Figure 3.2  Geometry and coordinates for test problems: (a) Front view; (b) Side view.
In general, we seek the three-dimensional normal stress components \( \sigma_{\xi}, \sigma_{\eta}, \sigma_{\phi} \) and shear stress components \( \tau_{\xi\eta}, \tau_{\eta\phi}, \tau_{\phi\xi} \), together with their companion displacements \( u_{\xi}, u_{\eta}, u_{\phi} \), as functions of \( \xi, \eta, \) and \( \phi \) throughout \( \mathcal{R} \) satisfying the three-dimensional field equations of elasticity,\(^2\) and the following boundary conditions: the applied displacement conditions from Neuber [47],

\[
\begin{align*}
u_{\xi} &= \frac{C}{2Gh} \left\{ \left[ \sin^2 \eta_0 \cos \eta - \alpha_0 (1 - \cos \eta) \right] \tanh \xi + \beta_0 \cos \eta \cosh \xi \cot^{-1} (\sinh \xi) \right\} \\
u_{\eta} &= \frac{C}{2Gh} \left\{ \left[ \cos^2 \eta_0 - \alpha_0 \frac{\cos \eta}{1 + \cos \eta} \right] \sin \eta - \beta_0 \sin \eta \sinh \xi \cot^{-1} (\sinh \xi) \right\}
\end{align*}
\] (3.32)

on \( \xi = \xi_0 \) for \( 0 < \eta < \eta_0, 0 < \phi < \pi/2 \), where

\[
C = -\frac{\sigma_0}{2} \frac{1 + \cos \eta_0}{1 + 2\nu \cos \eta_0 + \cos^2 \eta_0} \\
\alpha_0 = (1 - 2\nu)(1 + \cos \eta_0) \\
\beta_0 = 1 + \cos^2 \eta_0 + (1 - 2\nu)(1 - \cos \eta_0) \\
h^2 = \sinh^2 \xi + \cos^2 \eta, \sigma_0 = \frac{F}{\pi a^2}
\] (3.33)

; the stress-free notch conditions,

\[
\sigma_{\eta} = \tau_{\xi\eta} = \tau_{\eta\phi} = 0
\] (3.34)

on \( \eta = \eta_0 \) for \( 0 < \xi < \xi_0, 0 < \phi < \pi/2 \); and the symmetry conditions,

\[
\begin{align*}
u_{\xi} &= 0, \tau_{\xi\eta} = \tau_{\xi\phi} = 0 \\
u_{\phi} &= 0, \tau_{\phi\eta} = \tau_{\phi\xi} = 0
\end{align*}
\] (3.35)

on \( \xi = 0 \) for \( 0 < \eta < \eta_0, 0 < \phi < \pi/2 \).

\[
\begin{align*}
u_{\phi} &= 0, \tau_{\phi\eta} = \tau_{\phi\xi} = 0
\end{align*}
\] (3.36)

on \( \phi = 0, \pi/2 \) for \( 0 < \xi < \xi_0, 0 < \eta < \eta_0 \).

---

\(^2\) See, e.g., Hughes and Gaylord [21], pp. 56, 62, 69, 150 for these equations.
In particular, we seek the peak, normalized, tensile stress at the notch apex \( (\xi = 0, \eta = \eta_0) \),

\[
\bar{\sigma}_{\text{max}} = \frac{\sigma_\xi}{\sigma_0}
\]  

(3.37)

and the peak, normalized, hoop stress at the notch apex \( (\xi = 0, \eta = \eta_0) \),

\[
\bar{\sigma}_h = \frac{\sigma_\phi}{\sigma_0}
\]  

(3.38)

From [47], the exact solution for these peak stresses are

\[
\bar{\sigma}_{\text{max}} = \gamma \left[ 2\sqrt{\kappa + 1} + 1 + 2\nu + 2(1 + \nu)\kappa^{-1} \right] 
\]

\[
\bar{\sigma}_h = \gamma \left[ 2\nu\sqrt{\kappa + 1} + 1 \right] 
\]  

(3.39)

where \( \kappa \) is the dimensionless, notch, root curvature given by

\[
\kappa = a/r_0 = \frac{a^2}{(1 - a^2)}
\]  

(3.40)

and

\[
\gamma = \left[ 2 + 4(1 + \nu\sqrt{\kappa + 1})\kappa^{-1} \right]^{-1}
\]  

(3.41)

More specifically for FEA of our test problems, we take four values of the radius of the solid at the notch root, \( a \), to have 0.800, 0.900, 0.990, and 0.999. As we show later these values lead to a wide range of peak stresses at the notch root. Resulting exact values of normalized peak stresses follow on substituting values of \( a \) and material properties into (3.39) to (3.41). Here we set the value of Poisson’s ratio and shear modulus to be 0.25 and \( 12 \times 10^6 \) psi, respectively. Viewing the FEA determination of one of the two peak stress in (3.37) and (3.38) for one of the neck radii as constituting one problem, we thus have 8 test problems for FEA. From our FEA we want excellent values for \( \bar{\sigma}_{\text{max}} \) and \( \bar{\sigma}_h \), respectively. That is, we want to capture \( \bar{\sigma}_{\text{max}} \) and \( \bar{\sigma}_h \) to within 0.2% for the entire range of the radius of the solid at the notch root.
For \( a = 0.8, 0.9 \), normalized, peak, hoop stresses have values that are less than unity, while other normalized peak stresses are only a little greater than unity. This reflects the fact that, for these neck radii, stress gradients in the \( yz \) plane of Figure 3.2 are modest. However, even for these \( a \), stress gradients down the \( x \) axis of Figure 3.2 are significant. To indicate that this is so, we consider gross stress concentration factors, \( K_{tg} \), based on peak stresses normalized by average far-field (\( \xi = \xi_0 \)) stresses. Accordingly we define

\[
K_{tg} \text{ (tensile stress)} = 16\bar{\sigma}_{max}/a^2
\]

and adopt a parallel definition for \( K_{tg} \) for the hoop stress. Table 3.2 then sets out values of these factors for all two stresses. Apparent in Table 3.2 is the significant stress intensification occurring even for \( a = 0.8, 0.9 \), as well as the increases produced by increasing notch root curvature (\( a = 0.990, 0.999 \)). Most of the values in Table 3.2 exceed those normally encountered in practice (see Pilkey and Pilkey [1]). Subsequently we find that the \( K_{tg} \) numbers for all the configurations in Table 3.2 would require submodeling to reach the desired level of accuracy with structured meshes. However, we would like to analyze a test problem with lower notch acuity with free submodel meshes. Hence, additionally, we seek normalized, peak, tensile stress of a solid ellipsoid with \( a = 0.400 \). Thus we have 9 test problems for FEA.

<table>
<thead>
<tr>
<th>Notch neck radius, ( a )</th>
<th>Dimensionless notch root curvature, ( \kappa )</th>
<th>Tensile stress</th>
<th>Hoop stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.800</td>
<td>1.8</td>
<td>41.4</td>
<td>8.8</td>
</tr>
<tr>
<td>0.900</td>
<td>4.3</td>
<td>45.6</td>
<td>12.2</td>
</tr>
<tr>
<td>0.990</td>
<td>49.3</td>
<td>118.0</td>
<td>33.3</td>
</tr>
<tr>
<td>0.999</td>
<td>499.3</td>
<td>362.0</td>
<td>95.1</td>
</tr>
</tbody>
</table>
All the global meshes need the displacement boundary conditions of (3.32) to be applied on the outer boundary (i.e., on $\xi = \xi_0$ for $0 < \eta < \eta_0$, $0 < \phi < \pi/2$). For the ease in implementation of (3.32), we use their rectangular Cartesian counterparts, $u_x$, $u_y$ and $u_z$. These are given by

$$
\begin{align*}
    u_x &= u_\xi \sin \theta - u_\eta \cos \theta \\
    u_y &= [u_\xi \cos \theta + u_\eta \sin \theta] \cos \phi \\
    u_z &= [u_\xi \cos \theta + u_\eta \sin \theta] \sin \phi 
\end{align*}
$$

(3.43)

where $\theta$ is the angle between the normal displacement ($u_\xi$) on the outer boundary and the $y$ axis of the rectangular Cartesian coordinate system, and is given by

$$
\theta = \tan^{-1} \left[ \frac{x}{r \sin^2 \xi_0} \right]
$$

(3.44)

Here $r$ in (3.42) is given by $r = \sqrt{y^2 + z^2}$.

For these global meshes, initially we use 8H elements (SOLID185, ANSYS [22]). We start our discretization with structured initial mesh ($m = 1$) with uniform increments in oblate spheroidal coordinates: 4 equal increments in $\xi$, $\eta$ and $\phi$. This initial mesh has 64 elements (Figure 3.1(a)). Uniform increments in these coordinates naturally produce element size reductions in the vicinity of the notch root radius. This initial mesh is systematically refined by halving the increments and hence have $\lambda = 2$. Five further meshes ($m = 2 - 6$) are produced with such refinement. Our finest mesh in this sequence consequently has about 2 million elements.

In addition we run 8H elements with free meshes for the first configuration with the solid notch root radius, $a = 0.4$. These free meshes are generated using an automatic mesh generator (AMESH, [10]). We adopt this approach because it is easier to implement and so likely to be used in practice. With some care in implementation, these meshes can be generated such that they are
geometrically similar in element arrangements and have the same number of elements as their structured counterparts.

To access the performance of our procedure with higher-order elements, we analyze the last configuration with the solid notch root radius $a = 0.999$ with twenty-node hexahedral elements (20H, SOLID186, [22]). Similar to the meshes generated using 8H elements, here we generate structured meshes with uniform increments in oblate spheroidal coordinates. Our initial mesh ($m = 1$) has 2 equal increments in $\xi$, $\eta$ and $\phi$. Thus is comprised of 8 elements. This mesh is one mesh coarser than its corresponding mesh of 8H elements. We do this here so that degrees of freedom are more comparable. All the subsequent five meshes are generated by systematically halving the increments (i.e., with $\lambda = 2$).

Since the exact value of the normalized peak stresses, $\sigma_e$, for the test problems is available from (3.39), we have the true absolute relative discretization error in the FEA determination of $\sigma_e$ on a global mesh $m$, $\epsilon_{md}^d$, as

$$
\epsilon_{md}^d = \frac{|\sigma_e - \sigma_m|}{\sigma_e} \quad (3.45)
$$

for $m \leq M$.

The FEA values of $\bar{\sigma}_{max}$, the estimated discretization errors, $\hat{\epsilon}_{md}^d$, from (3.5) and their corresponding true discretization errors, $\epsilon_{md}^d$ using (3.45) from global meshes for the first configuration with the solid notch root radius, $a = 0.4$ are given in Table 3.3. In applying our procedure to safe guard against nonmonotonic convergence we replace the value of $\hat{\epsilon}_{md}^d$ with one in accord with (3.12) on mesh, $m = 3$ (Whenever our procedures to safe guard against underestimating errors in the presence of nonmonotonic convergence are active, an asterisk is placed on top of $\hat{\epsilon}_{md}^d$ value in Table 3.3). The discretization error estimates are uniformly
conservative and the decreasing trend of these error estimates is consistent with a numerically converging analysis (Table 3.3). The estimated discretization error on the last global mesh ($M = 5$) is 0.049% which is in compliance with the excellent criterion of (3.20), namely less than 0.2% (Table 3.3). The true discretization error is 0.030% and confirms that the true discretization error does comply with the excellent criterion. Hence this configuration does not require submodeling when analyzed with structured meshes of 8H elements.

Table 3.3. Finite element stresses, estimated and actual discretization errors from structured global meshes with 8H elements for $a = 0.4$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.083524).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m$</th>
<th>$\bar{\epsilon}_m^d$</th>
<th>$\epsilon_m^d$ (%)</th>
<th>$\epsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9254</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.0420</td>
<td>1.9</td>
<td>3.8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0735</td>
<td>1.9*</td>
<td>2.9</td>
<td>2.1</td>
</tr>
<tr>
<td>4</td>
<td>1.0814</td>
<td>2.0</td>
<td>0.24</td>
<td>2.2</td>
</tr>
<tr>
<td>5</td>
<td>1.0832</td>
<td>2.1</td>
<td>0.049</td>
<td>2.7</td>
</tr>
</tbody>
</table>

For all other global meshes, the FEA values of $\bar{\sigma}_{\text{max}}$ and $\bar{\sigma}_h$, the estimated discretization errors, $\bar{\epsilon}_m^d$, and their corresponding true discretization errors, $\epsilon_m^d$, are given in Tables C.1 – C.6 of Appendix C. For the global analysis for these test problems, the error estimates found using (3.5) are uniformly conservative. As is evident in these tables the estimated value of $\bar{\epsilon}_m^d$ on the finest global mesh, for all the configurations, does not comply with the excellent criterion of (3.20), namely less than 0.2%. Hence our error estimates determine that our global mesh sequence is not sufficiently accurate. The corresponding true discretization errors of (3.45) confirm that, in fact, discretization errors do not comply with the excellent criterion. Because our finest global mesh is taxing our computational capabilities, we therefore look to submodeling to improve results.
Since the exact boundary conditions for these test problems are available, we run all the submodel meshes with the same. Then the true absolute relative boundary-condition error, $\epsilon^b_m$, on a given submodel mesh $m$, is

$$
\epsilon^b_m = \frac{\left| \sigma^e_m - \sigma^b_m \right|}{\sigma_e} \quad (3.46)
$$

for $m \geq M + 1$, where $\sigma^e_m$ is the stress found using exact boundary conditions. Further, the true absolute relative discretization error in the stress of interest on submodel mesh $m$, $\epsilon^d_m$, is

$$
\epsilon^d_m = \frac{\left| \sigma_e - \sigma^e_m \right|}{\sigma_e} \quad (3.47)
$$

for $m \geq M + 1$. Then proceeding as previously for total error, we have our true absolute relative total error in the stress of interest on submodel mesh $m$, $\epsilon^t_m = \epsilon^d_m + \epsilon^b_m$, for $m \geq M + 1$.

For all the configurations analyzed with structured meshes of 8H elements, the submodel region is chosen such that the number of elements present in the first submodel mesh is exactly the same as the number of elements in the finest global mesh in the same region. We do this here to check the correctness of the submodel boundary conditions. For these submodel regions, we successively refine the mesh by halving the element sides. For the third configuration with $a = 0.9$, this is illustrated in Figure 3.1. Figure 3.1(a) shows the initial global mesh ($m = 1$), while the close-up in Figure 3.1(c) shows the submodel region in the finest global mesh. The finest submodel mesh for this configuration is shown in Figure 3.1 (d). These structured submodel meshes are run with both bicubic surface fitted displacements and displacement shape functions on their cut boundaries. For $a = 0.9$ our final submodel mesh has about 4 thousand elements. The global mesh with the same resolution would have about 134 million elements, hence a reduction in number of
elements of 32,000 to 1. The same order of reduction in number of elements occurs for all other configurations having structured meshes of 8H elements.

For the first configuration with $a = 0.4$ run with free meshes of 8H elements, the submodel region is chosen such that the area is exactly the same as that for structured meshes. On this submodel region we run three free meshes ($m = 7$ to 9) with $\lambda = 2$. To be consistent in comparing with structured meshes, the first of these submodel meshes has the same number of elements as the finest global mesh in the same region. The last two meshes are intended to improve the FEA determination of $\bar{\sigma}_{\text{max}}$. We run these submodel meshes with displacement shape functions on their cut boundaries because shape functions are considerably easier than bicubic surfaces to implement when using free meshes. The final submodel mesh has about 4 thousand elements whereas a global mesh with the same resolution would have about 134 million elements. Thus, again a reduction in number of elements of 32,000 to 1.

To check how well our submodeling procedure works with higher-order elements, we analyze the configuration with the solid notch root radius $a = 0.999$ with structured meshes of 20H elements. The subregion is chosen as previously and meshes are refined by halving the element sides. Two further meshes ($m = 8$ and 9) are generated with such refinement. We run these submodel meshes with both bicubic surfaces fitted displacements and displacement shape functions on their cut boundaries. The final submodel mesh has about 5 hundred elements in contrast to a global mesh with the same resolution that would have about 16 million elements. Again, a reduction in number of elements of 32,000 to 1.

We next present some illustrative results from applying our submodeling procedure. These results are for peak, normalized, tensile stress ($\bar{\sigma}_{\text{max}}$) for $a = 0.9$, the third configuration; $a = 0.4$, the first configuration; and for $a = 0.999$, the last configuration. The third configuration
demonstrates the use of structured submodel meshes with 8H elements, the first configuration demonstrates the use of free submodel meshes with 8H elements, while the last configuration demonstrates the use of higher order elements (20H) in conjunction with our submodeling procedure.

We begin with free submodel mesh results with 8H elements for $a = 0.9$ with shape function boundary conditions. The FEA stresses and the accompanying boundary-condition errors for this configuration are given in Table 3.4(a) (here and in Table 3.4(b) and (c), six decimal places are included to avoid round-off error when calculating error estimates and actual errors). The FEA stress value from our last global free mesh (Table C.3 of Appendix C, for $m = 6$) and that for first submodel mesh (Table 3.4(a) for $m = 7$) are the same, which confirms that submodel boundary conditions have been correctly implemented. Since the stress increments due to boundary condition refinement have opposite signs, estimates of the convergence rate of the boundary-condition error, $\hat{e}^{Mb}_m$ from (3.22), are not available (NA) throughout the submodel mesh sequence (Table 3.4(a)). This ultimately leads to the boundary-condition error estimate of (3.24) being NA throughout the submodel mesh sequence (Table 3.4(a)). Thus we increase the subregion and apply our submodeling procedure (meshes $m = 10, 11, 12$). The FEA values of stresses and accompanying boundary-condition errors for this enlarged subregion are given in Table 3.4(b). The estimates of boundary-condition error from (3.24) are now available and their values on meshes $m = 11, 12$ comply with the excellent criterion of (3.20) with (3.26), namely less than 0.067% (Table 3.4(b)). The true boundary-condition error from (3.46) confirms the same on both the meshes. The boundary-condition error estimate on the last submodel mesh is 0.017% and its true value is 0.0037% (Table 3.4(b) for $m = 12$).
Table 3.4(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on the initial subregion for $a = 0.9$ for $\bar{\sigma}_{\text{max}}$ (exact value = 2.307001).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{M_{b}}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$c_{m}^{M_{b}}$</th>
<th>$\bar{\epsilon}^{b}_{m}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$c_{m}^{M_{b}}$</th>
<th>$\epsilon^{b}_{m}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2.311290</td>
<td>2.311650</td>
<td>2.309617</td>
<td></td>
<td>2.311511</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.309682</td>
<td>2.309725</td>
<td>2.305933</td>
<td>NA</td>
<td>NA</td>
<td>2.309749</td>
<td>NA</td>
<td>0.0029</td>
</tr>
<tr>
<td>9</td>
<td>2.308462</td>
<td>2.308546</td>
<td>2.304400</td>
<td>NA</td>
<td>NA</td>
<td>2.308500</td>
<td>NA</td>
<td>0.0016</td>
</tr>
</tbody>
</table>

Table 3.4(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.9$ for $\bar{\sigma}_{\text{max}}$ (exact value = 2.307001).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{M_{b}}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$c_{m}^{M_{b}}$</th>
<th>$\bar{\epsilon}^{b}_{m}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$c_{m}^{M_{b}}$</th>
<th>$\epsilon^{b}_{m}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.311290</td>
<td>2.311083</td>
<td>2.310531</td>
<td></td>
<td>2.311481</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2.309636</td>
<td>2.309418</td>
<td>2.309020</td>
<td>0.87</td>
<td>0.011</td>
<td>2.309747</td>
<td>1.6</td>
<td>0.0048</td>
</tr>
<tr>
<td>12</td>
<td>2.308414</td>
<td>2.308193</td>
<td>2.307852</td>
<td>0.63</td>
<td>0.017</td>
<td>2.308500</td>
<td>1.8</td>
<td>0.0037</td>
</tr>
</tbody>
</table>

Table 3.4(c). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.9$ for $\bar{\sigma}_{\text{max}}$ (exact value = 2.307001).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{M_{b}}$</th>
<th>$\hat{\epsilon}^{d}_{m}$ (%)</th>
<th>$\bar{\epsilon}^{t}_{m}$ (%)</th>
<th>$\epsilon^{d}_{m}$ (%)</th>
<th>$\epsilon^{t}_{m}$ (%)</th>
<th>$\epsilon^{t}_{m}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.311290</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2.309636</td>
<td>0.71</td>
<td>0.12</td>
<td>0.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>2.308414</td>
<td>0.44</td>
<td>0.15</td>
<td>0.87</td>
<td>0.065</td>
<td>0.18</td>
</tr>
</tbody>
</table>

The FEA stress values from submodel meshes run with $M_{th}$ global mesh boundary conditions in Table 3.4(c) are the same as in Table 3.4(b), but now are accompanied by estimated and actual discretization and total errors. We estimate the discretization error with (3.5) on mesh $m = 12$ as 0.15% and its true value from (3.47) is 0.065% (Table 3.3(c)). The estimated total error from (3.25) on mesh $m = 12$ is 0.18% which is in compliance with excellent criterion of (3.20), namely less than 0.2% (Table 3.4(c)). The true total error is 0.069% and confirms that the true total error does comply with the excellent criterion. Here the last submodel mesh has about 32 thousand
elements. A global mesh with the same resolution would have about 134 million elements. Although the subregion is enlarged once, the reduction in number of elements is still 4,000 to 1.

Next, we present results from free submodel meshes with 8H elements for $a = 0.4$ with shape function boundary conditions. The FEA stresses and the accompanying boundary-condition errors for this configuration are given in Table 3.5(a). Here because we are using free meshes, the stress from the last global free mesh does not completely match that from the first submodel mesh (Table C.1 of Appendix C cf., Table 3.5(a)), nor is it expected to match. We find that the estimated boundary-condition errors throughout the submodel mesh sequence (meshes $m = 7 - 9$) on our first subregion are not less than 0.067% (Table 3.5(a)). Further the true boundary-condition error values are also not less than 0.067%. Thus we increase the subregion and apply our submodeling procedure (meshes $m = 10 - 12$). We find that the estimated values of boundary-condition error are still not less than 0.067%, and further the true boundary-condition error values are also not less than 0.067%. Hence we enlarge the area of the subregion a second time and apply our submodeling procedure (meshes $m = 13 - 15$). Again, we find that the estimated values of boundary-condition error are still not less than 0.067%, and further the true boundary-condition error values are also not less than 0.067%. Hence we enlarge the area of the subregion a third time and apply our submodeling procedure (meshes $m = 16 - 18$). The FEA values of stresses and accompanying boundary-condition errors for this enlarged subregion are given in Table 3.5(b). The estimated boundary-condition error values on meshes $m = 17, 18$ complies with the good criterion of (3.20) with (3.26), namely less than 0.33% (Table 3.5(b)). Since further enlargement of subregion is not possible, we accept the good level of accuracy. The true boundary-condition error from (3.46) does not confirm the same on both the meshes. The boundary-condition error estimate on the last submodel mesh is 0.16% while the true value of this error is 0.5% (Table 3.5(b) for $m = 18$). The
FEA stress values from submodel meshes run with $M^{th}$ global mesh boundary conditions in Table 3.5(c) are the same as in Table 3.5(b), but now are accompanied by estimated and actual discretization and total errors. We estimate the discretization error on mesh $m = 18$ as 0.1% (Table 3.5(c)). The true discretization error on the same mesh is 0.36%. The estimated total error on the same mesh is 0.25% which is in compliance with good criterion of (3.20), namely less than 1% (Table 3.5(c)). The true total error is 0.86% and confirms that the true total error does comply with the good criterion. Here the last submodel mesh has about 2 million elements. A global mesh with the same resolution would have about 134 million elements. Although the subregion is enlarged thrice, the reduction in number of elements is still 67 to 1.

Table 3.5(a). Finite element stresses, estimated and actual boundary-condition errors from free submodel meshes with 8H elements using shape function boundary conditions on the initial subregion for $a = 0.4$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.083524).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{M_b}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{\epsilon}_{m}^{M_b}$</th>
<th>$\hat{\epsilon}_{m}^{b}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$c_{m}^{M_b}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.1121</td>
<td>1.0801</td>
<td>1.0239</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0921</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.1139</td>
<td>1.0796</td>
<td>1.0196</td>
<td>0.81</td>
<td>4.1</td>
<td>1.1026</td>
<td>NA</td>
<td>1.0</td>
</tr>
<tr>
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</tr>
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<td>0.80</td>
<td>4.3</td>
<td>1.1049</td>
<td>NA</td>
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</tr>
</tbody>
</table>

Table 3.5(b). Finite element stresses, estimated and actual boundary-condition errors from free submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.4$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.083524).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{M_b}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{\epsilon}_{m}^{M_b}$</th>
<th>$\hat{\epsilon}_{m}^{b}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$c_{m}^{M_b}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1.08074</td>
<td>1.07475</td>
<td>1.04732</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>1.08149</td>
<td>1.07545</td>
<td>1.04751</td>
<td>2.2</td>
<td>0.15</td>
<td>1.08916</td>
<td>0.83</td>
<td>0.71</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>1.08200</td>
<td>1.07595</td>
<td>1.04788</td>
<td>2.2</td>
<td>0.16</td>
<td>1.08745</td>
<td>1.1</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Table 3.5(c). Finite element stresses, estimated and actual discretization and total errors from free submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.4$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.083524).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\hat{\varepsilon}_{m}^d$</th>
<th>$\hat{\varepsilon}_{m}^d$ (%)</th>
<th>$\varepsilon_{m}^d$</th>
<th>$\varepsilon_{m}^d$ (%)</th>
<th>$\hat{\varepsilon}_{m}^t$ (%)</th>
<th>$\varepsilon_{m}^t$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1.08074</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>1.08149</td>
<td>0.58</td>
<td>0.52</td>
<td>0.52</td>
<td>0.36</td>
<td>0.25</td>
<td>0.86</td>
</tr>
<tr>
<td>18</td>
<td>1.08200</td>
<td>0.55</td>
<td>0.10</td>
<td>0.52</td>
<td>0.36</td>
<td>0.25</td>
<td>0.86</td>
</tr>
</tbody>
</table>

When the first configuration with solid notch root radius, $a = 0.4$ is analyzed with structure meshes of 8H elements excellent level of accuracy is achieved with global meshes. Hence submodeling is not required. However, when the same configuration is analyzed with free meshes, only good level of accuracy is achieved in spite of running 12 submodel free meshes. This suggests structured meshes perform better than free meshes, at least with 8H elements. This is so when both the global and the submodel meshes are free. If instead largely free global meshes that have structured local meshes are used in conjunction with structured submodel meshes, we would expect performance to be closer to that for the use of structured meshes throughout. Unfortunately this is difficult to check here because local structured meshes in oblate spheroidal coordinates are not supported in ANSYS [10]. We do, though, consider the effects of such an approach in our next section for an application.

Finally, we present the results from structured submodel meshes with 8H elements for the last configuration with solid notch root radius $a = 0.999$ with displacement boundary conditions fitted with bicubic surface because these results serve as a benchmark for comparison of results for structured meshes with 20H elements. For these 8H structured submodel meshes, the FEA stresses and the accompanying boundary-condition errors are given in Table 3.6(a). Since the stress increments due to boundary condition refinement are increasing, estimates of the convergence rate
of the boundary-condition error are not available (NA) throughout the submodel mesh sequence (Table 3.6(a)). This ultimately leads to the boundary-condition error estimate being NA throughout the submodel mesh sequence (Table 3.6(a)). Also the true boundary-condition error values are also not less than 0.067%. Thus we increase the subregion and apply our submodeling procedure (meshes \( m = 10, 11, 12 \)). We find that the estimated values of boundary-condition error is still not less than 0.067%, and further the true boundary-condition error values are also not less than 0.067%. Hence we enlarge the area of the subregion a second time and apply our submodeling procedure (meshes \( m = 13 - 16 \)). The FEA values of stresses and accompanying boundary-condition errors for this enlarged subregion are given in Table 3.6(b). The FEA stress value from our last global mesh (Table C.5 of Appendix C, for \( m = 6 \)) and that for first submodel mesh (Table 3.6(b) for \( m = 13 \)) are the same, which confirms that submodel boundary conditions have been correctly implemented. The estimated boundary-condition errors on meshes \( m = 14 - 16 \) complies with the good criterion of (3.20) with (3.26), namely less than 0.33% (Table 3.6(b)). Further enlargement of subregion leads to only one estimate of discretization and total error. Also considering the loss of computational efficiency we accept the good level of accuracy. The boundary-condition error estimate on the submodel mesh sequence is 0.14% which is close to its actual value of 0.12% (Table 3.6(b)). The FEA stress values from submodel meshes run with \( M^{th} \) global mesh boundary conditions in Table 3.6(c) are the same as in Table 3.6(b), but now are accompanied by estimated and actual discretization and total errors. In applying our procedure to safe guard against underestimating errors in the presence of non-monotonic convergence we replace the value of \( \hat{c}_m^d \) with one in accord with (3.12) on mesh, \( m = 15 \) and estimate a discretization error of 0.94% (Whenever our procedures to safe guard against underestimating errors in the presence of non-monotonic convergence are active, an asterisk is placed on top of \( \hat{c}_m^d \))
value in Table 3.6(c)). In accord with (3.15) the estimate of discretization error is not available on mesh, \( m = 16 \) (Table 3.6(c)). The estimated total error on mesh \( m = 15 \) is 1.1% which is in compliance with satisfactory criterion of (3.20), namely less than 5% (Table 3.6(c)). The true total error on the same mesh is 0.46% which satisfies the good criterion (Table 3.6(c)). Since the total error estimate does not satisfy the good criterion, we look to a fourth submodel to improve results by applying our submodeling procedure.

Table 3.6(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on the initial subregion for \( \alpha = 0.999 \) for \( \sigma_{\text{max}} \) (exact value = 22.578569).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{m}^{Mb} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \Delta_{m}^{Mb} )</th>
<th>( e_{m}^{b} ) (%)</th>
<th>( \sigma_{e}^{eb} )</th>
<th>( c_{m}^{Mb} )</th>
<th>( e_{m}^{b} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>21.4137</td>
<td>20.8188</td>
<td>20.3176</td>
<td></td>
<td></td>
<td>21.6044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>22.2906</td>
<td>21.6539</td>
<td>21.3026</td>
<td>NA</td>
<td>NA</td>
<td>22.4852</td>
<td>2.1</td>
<td>0.86</td>
</tr>
<tr>
<td>9</td>
<td>22.4709</td>
<td>21.8253</td>
<td>21.5316</td>
<td>NA</td>
<td>NA</td>
<td>22.6667</td>
<td>2.1</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Table 3.6(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for \( \alpha = 0.999 \) for \( \sigma_{\text{max}} \) (exact value = 22.578569).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{m}^{Mb} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \Delta_{m}^{Mb} )</th>
<th>( e_{m}^{b} ) (%)</th>
<th>( \sigma_{e}^{eb} )</th>
<th>( c_{m}^{Mb} )</th>
<th>( e_{m}^{b} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>22.4151</td>
<td>22.3336</td>
<td>22.0360</td>
<td>1.9</td>
<td>0.14</td>
<td>22.4428</td>
<td>2.0</td>
<td>0.12</td>
</tr>
<tr>
<td>15</td>
<td>22.6280</td>
<td>22.5457</td>
<td>22.2452</td>
<td>1.9</td>
<td>0.14</td>
<td>22.6560</td>
<td>2.0</td>
<td>0.12</td>
</tr>
<tr>
<td>16</td>
<td>22.6307</td>
<td>22.5484</td>
<td>22.2478</td>
<td>1.9</td>
<td>0.14</td>
<td>22.6587</td>
<td>2.0</td>
<td>0.12</td>
</tr>
<tr>
<td>17</td>
<td>22.6307</td>
<td>22.6146</td>
<td>22.5507</td>
<td></td>
<td></td>
<td>22.6641</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>22.5977</td>
<td>22.5815</td>
<td>22.5170</td>
<td>2.0</td>
<td>0.16</td>
<td>22.6310</td>
<td>0.57</td>
<td>0.15</td>
</tr>
<tr>
<td>19</td>
<td>22.5740</td>
<td>22.5578</td>
<td>22.4942</td>
<td>2.0</td>
<td>0.16</td>
<td>22.6073</td>
<td>0.57</td>
<td>0.15</td>
</tr>
</tbody>
</table>
Table 3.6(c). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $\alpha = 0.999$ for $\bar{\sigma}_{\text{max}}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_{mB}$</th>
<th>$\varepsilon^d_m$</th>
<th>$\varepsilon^d_m$ (%)</th>
<th>$\varepsilon^d_m$</th>
<th>$\varepsilon^d_m$ (%)</th>
<th>$\varepsilon^d_m$ (%)</th>
<th>$\varepsilon^t_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>21.4137</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>14</td>
<td>22.4151</td>
<td>3.1</td>
<td>0.60</td>
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<td></td>
<td></td>
<td>0.72</td>
</tr>
<tr>
<td>15</td>
<td>22.6280</td>
<td>2.2*</td>
<td>0.94</td>
<td>NA</td>
<td>0.34</td>
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</tr>
<tr>
<td>16</td>
<td>22.6307</td>
<td>6.3*</td>
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<td>NA</td>
<td>0.36</td>
<td>NA</td>
<td>0.48</td>
</tr>
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<td>22.6307</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>22.5977</td>
<td>0.71</td>
<td>0.23</td>
<td></td>
<td></td>
<td></td>
<td>0.38</td>
</tr>
<tr>
<td>19</td>
<td>22.5740</td>
<td>0.48</td>
<td>0.27</td>
<td>0.87</td>
<td>0.13</td>
<td>0.43</td>
<td>0.28</td>
</tr>
</tbody>
</table>

The FEA values of stresses and accompanying boundary-condition errors from our fourth submodel (meshes, $m = 17 - 19$) are given in Table 3.6(b) below the dotted line. The FEA stress value from our last mesh of third submodel (Table 3.6(b) for $m = 16$) and that for first mesh of fourth submodel (Table 3.6(b) for $m = 17$) are the same, which confirms that submodel boundary conditions have been correctly implemented. The estimated boundary-condition errors on the fourth submodel mesh sequence is 0.024% which is increased, by adding it with the boundary-condition error estimate from the previous submodel, to 0.16% (Table 3.6(b)), and is also less than 0.33%. The FEA stress values from fourth submodel meshes run with $M^{th}$ global mesh boundary conditions in Table 3.6(c) (below the dotted line) are the same as in Table 3.6(b) (below the dotted line), but now are accompanied by estimated and actual discretization and total errors. We estimate the discretization error on mesh $m = 19$ as 0.27% (Table 3.6(c)). The true discretization error on the same mesh is 0.13% (Table 3.6(c)). The estimated total error on the same mesh is 0.43% which
is in compliance with good criterion of (3.20), namely less than 1% (Table 3.6(c)). The true total error is 0.28% and confirms that the true total error does comply with the good criterion. Here the last submodel mesh has about 250 thousand elements. A global mesh with the same resolution would have about 69 billion elements. Although the subregion is enlarged twice, the reduction in number of elements is now 250,000 to 1.

For the last configuration with solid notch root radius $a = 0.999$, when analyzed with structured submodel meshes of 20H elements with bicubic surface fitted boundary conditions we find that the estimated boundary-condition errors throughout the submodel mesh sequence are not less than 0.067% (meshes $m = 7, 8, 9$). Furthermore the true boundary-condition error values are also not less than 0.067%. Thus we increase the subregion and apply our submodeling procedure (meshes $m = 10 - 14$). The FEA stresses and the accompanying estimated and actual boundary-condition errors for this enlarged subregion are given in Table 3.7(a). The FEA stress value from our last global mesh (Table C.6 of Appendix C, for $m = 6$) and that for the second submodel mesh (Table 3.7(a) for $m = 10$) are close but do not completely match. This is so because we do not use the displacement values from the mid-nodes of 20H elements, as spurious numerical noise (wobbles) persists on mid-side nodes of these elements. The estimated boundary-condition errors on the submodel mesh sequence are 0.062% (Table 3.7(a)), so now less than 0.067%. The true values for this error on the submodel mesh sequence are 0.060% (Table 3.7(a)), are also less than 0.067%. The FEA stress values from second submodel meshes run with $M^{th}$ global mesh boundary conditions in Table 3.7(b) are the same as in Table 3.7(a), but now are accompanied by estimated and actual discretization and total errors. Since the stress increments due to mesh refinement are increasing initially, estimate of the convergence rate of the discretization error, $e_m^d$ from (3.3), is not available (NA) on mesh, $m = 12$ (Table 3.7(b)). This ultimately leads to the discretization error
estimate being NA from the same submodel mesh (Table 3.7(b)). In applying our procedure to safe
guard against non-monotonic convergence we replace the value of $\hat{e}^d_m$ with one in accord with
(3.12) on mesh, $m = 13$ and estimate a discretization error of 0.33% (Table 3.7(b)). We estimate
the discretization error on the last submodel mesh $m = 14$ as 0.032% and the true discretization
error on the same mesh is 0.028% (Table 3.7(b)). The estimated total error on the same mesh is
0.094% which is in compliance with the excellent criterion of less than 0.067% (Table 3.7(b)). The
true total error on the same mesh is 0.088% and also confirms compliance with excellent criterion.
Here the last submodel mesh has about 260 thousand elements. A global mesh with the same
resolution would have about 1 billion elements. Although the subregion is enlarged once, the
reduction in number of elements is still 4000 to 1.

Excellent results are achieved with structured meshes of 20H elements, whereas structured
meshes of 8H elements produce only good results. Also 5 extra meshes are run with 8H elements
compared to with 20H elements (19 cf., 14). Hence our submodeling procedure in conjunction
with 20H elements performs better than 8H elements with fewer meshes being used.

Table 3.7(a). Finite element stresses, estimated and actual boundary-condition errors from
structured submodel meshes with 20H elements using bicubic surface fitted boundary conditions
on an enlarged subregion for $a = 0.999$ for $\bar{\sigma}_{max}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_{mb}$</th>
<th>$\sigma^{(M-1)b}_{m}$</th>
<th>$\sigma^{(M-2)b}_{m}$</th>
<th>$\Delta^M_{mb}$</th>
<th>$\hat{e}^b_m$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$c^M_{mb}$</th>
<th>$\hat{e}_m^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>23.0641</td>
<td>22.9636</td>
<td>22.1605</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>22.9507</td>
<td>22.8514</td>
<td>22.0630</td>
<td>3.0</td>
<td>0.062</td>
<td>22.9642</td>
<td>3.1</td>
<td>0.060</td>
</tr>
<tr>
<td>12</td>
<td>22.6679</td>
<td>22.5699</td>
<td>21.7923</td>
<td>3.0</td>
<td>0.062</td>
<td>22.6814</td>
<td>3.0</td>
<td>0.060</td>
</tr>
<tr>
<td>13</td>
<td>22.5923</td>
<td>22.4949</td>
<td>21.7217</td>
<td>3.0</td>
<td>0.062</td>
<td>22.6058</td>
<td>3.0</td>
<td>0.060</td>
</tr>
<tr>
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<td>22.5723</td>
<td>22.4749</td>
<td>21.7029</td>
<td>3.0</td>
<td>0.062</td>
<td>22.5858</td>
<td>3.0</td>
<td>0.060</td>
</tr>
</tbody>
</table>
Table 3.7(b). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 20H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $\alpha = 0.999$ for $\bar{\sigma}_{\text{max}}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\hat{\epsilon}_{m}^{d}$</th>
<th>$\hat{\epsilon}_{m}^{d}$ (%)</th>
<th>$c_{m}^{d}$</th>
<th>$\hat{\epsilon}_{m}^{d}$ (%)</th>
<th>$\hat{\epsilon}_{m}^{t}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
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<tbody>
<tr>
<td>10</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>22.9507</td>
<td>0.38</td>
<td>1.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>22.6679</td>
<td>NA</td>
<td>NA</td>
<td>2.1</td>
<td>0.40</td>
<td></td>
<td>0.46</td>
</tr>
<tr>
<td>13</td>
<td>22.5923</td>
<td>1.9*</td>
<td>0.33</td>
<td>2.7</td>
<td>0.061</td>
<td>0.39</td>
<td>0.12</td>
</tr>
<tr>
<td>14</td>
<td>22.5723</td>
<td>1.9</td>
<td>0.032</td>
<td>NA</td>
<td>0.028</td>
<td>0.094</td>
<td>0.088</td>
</tr>
</tbody>
</table>

For comparison, we present results from structured submodel meshes of 20H elements run with shape function boundary conditions for the last configuration with solid notch root radius $\alpha = 0.999$. The FEA stresses and the accompanying boundary-condition errors are given in Table 3.8(a) (here and in Table 3.8(b), six decimal places are included to avoid round-off error when calculating error estimates and actual errors). Again, the FEA stress value from our last global mesh (Table C.6 of Appendix C, for $m = 6$) and that for first submodel mesh (Table 3.7(a) for $m = 7$) are close but do not completely match. Since the stress increments due to boundary condition refinement change sign, estimate of the convergence rate of the boundary-condition error, $\hat{\epsilon}_{m}^{Mb}$, is not available (NA) for the submodel mesh $m = 8$ (Table 3.8(a)). This ultimately leads to the boundary-condition error estimate being NA for the same submodel mesh (Table 3.8(a)). We estimate the boundary-condition error on our last submodel mesh, $m = 11$, to be 0.00090%, far less than 0.067% (Table 3.8(a)). Whereas the true value of this error on the same submodel mesh is 0.43%, thus is not less than 0.067%. The FEA stress values from submodel meshes run with $M^{th}$ global mesh boundary conditions in Table 3.8(b) are the same as in Table 3.8(a), but now are accompanied by estimated and actual discretization and total errors.
Table 3.8(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 20H elements using shape function boundary conditions for $a = 0.999$ for $\bar{\sigma}_{\text{max}}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_{mb}$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\hat{c}^M_{mb}$</th>
<th>$\hat{e}^b_{mb}$ (%)</th>
<th>$\sigma^e_{mb}$</th>
<th>$c^M_{mb}$</th>
<th>$\epsilon^b_{mb}$ (%)</th>
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<td>23.256234</td>
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<td></td>
<td>23.978197</td>
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</tr>
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<td>23.002161</td>
<td>24.302681</td>
<td>NA</td>
<td>NA</td>
<td>23.103388</td>
<td>0.064</td>
<td>0.43</td>
</tr>
<tr>
<td>9</td>
<td>22.655308</td>
<td>22.673041</td>
<td>23.965865</td>
<td>6.2</td>
<td>0.0011</td>
<td>22.752034</td>
<td>NA</td>
<td>0.43</td>
</tr>
<tr>
<td>10</td>
<td>22.529528</td>
<td>22.546354</td>
<td>23.836805</td>
<td>6.3</td>
<td>0.00099</td>
<td>22.629514</td>
<td>NA</td>
<td>0.44</td>
</tr>
<tr>
<td>11</td>
<td>22.494674</td>
<td>22.510741</td>
<td>23.800625</td>
<td>6.3</td>
<td>0.00090</td>
<td>22.591976</td>
<td>NA</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Table 3.8(b). Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 20H elements using shape function boundary conditions for $a = 0.999$ for $\bar{\sigma}_{\text{max}}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma^M_{mb}$</th>
<th>$\hat{c}^d_{mb}$</th>
<th>$\hat{e}^d_{mb}$ (%)</th>
<th>$\sigma^d_{mb}$</th>
<th>$\hat{e}^d_{mb}$ (%)</th>
<th>$\sigma^t_{mb}$</th>
<th>$\epsilon^t_{mb}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>22.252278</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>23.006536</td>
<td>NA</td>
<td>1.9</td>
<td>NA</td>
<td>2.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>22.655308</td>
<td>NA</td>
<td>2.5</td>
<td>NA</td>
<td>0.34</td>
<td>NA</td>
<td>0.77</td>
</tr>
<tr>
<td>10</td>
<td>22.529528</td>
<td>1.5*</td>
<td>0.56</td>
<td>NA</td>
<td>0.22</td>
<td>0.56</td>
<td>0.66</td>
</tr>
<tr>
<td>11</td>
<td>22.494674</td>
<td>1.9*</td>
<td>0.15</td>
<td>NA</td>
<td>0.37</td>
<td>0.15</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Since the stress increments due to mesh refinement are changing sign on meshes $m = 8$ and 9, estimate of the convergence rate of the discretization error, $\hat{c}^d_{mb}$, is not available (NA) on mesh, $m = 9$ (Table 3.8(b)). This ultimately leads to the discretization error estimate being NA for the same submodel mesh (Table 3.8(b)). In applying our procedure to safeguard against underestimating errors in the presence of non-monotonic convergence we replace the value of $\hat{c}^d_{mb}$ with one in accord with (3.14) on the last submodel mesh, $m = 11$, and estimate a discretization error of 0.15% (Table 3.7(b)). Whereas the true value of this error on the same mesh is 0.37%. The
total error estimate on the final submodel mesh is 0.15% whereas it true value is 0.80%. Hence we under estimate the total error on the final mesh by a factor of 5.3.

In sum, therefore, essentially for this test problem with maximum stress concentration, the hierarchy in terms of accuracy of the different options considered for submodel meshes is as follows: first, 20H elements with bicubic surface fitted boundary conditions; second, 8H elements with either shape function or bicubic surface fitted boundary conditions; third, 20H elements with shape function boundary conditions. This is because with the first option we estimate the final total error at an excellent level of accuracy which is at the same level as its true value, whereas with second option we estimate the final total error at a good level of accuracy which is at the same level as its true value, and with third option we underestimate the final total error as at an excellent level when it really is at good level of accuracy.

The FEA stresses for $\bar{\sigma}_{\text{max}}$, estimates of the boundary-condition, discretization and total errors along with their true counterparts from submodel meshes of alternate FEA of the configurations in Tables 3.4(a) – 3.8(b) and other test problem configurations, namely $a = 0.8$ and 0.99, are given in Tables D.1(a) – D.8 of Appendix D. The FEA stresses for $\bar{\sigma}_h$, estimates of the boundary-condition, discretization and total errors along with their true counterparts from submodel meshes for all the test problem configurations are given in Tables D.9(a) – D.18 of Appendix D. Throughout these tables estimated boundary-condition and discretization errors agree well with corresponding true errors and our submodeling procedure ultimately results in excellent FEA stress estimates (total error < 0.2%) that are confirmed by the true stresses except for the case where $a = 0.999$ is analyzed with 8H elements. This is so with the 8H elements are used irrespective of whether bicubic surfaces or shape functions are used to interpolate boundary conditions.
We close this section by comparing our error estimates with those of Cormier et al. [2] on our set of test problems. For 8H elements, the estimated discretization error of (2.5) from [2] is nonconservative for 34 instances whereas the present method results in one nonconservative error estimate. The boundary-condition error estimation of (2.16) from [2] overestimates its true value by a factor of 3.0 for 56 instances and on one instance it overestimates by more than an order of magnitude. The present method is more accurate and estimates these errors to within a factor of 1.2. For 20H elements, the estimated discretization error from (2.5) is initially nonconservative on one occasion and then overestimates its true value by a factor of 3.0 whereas the present method is nonconservative twice. The boundary-condition error from (2.16) overestimates its true value by more than an order of magnitude on five occasions and under estimates its true value on five further occasions. In contrast, the present method is nonconservative only twice but otherwise more accurate and conservative. This is because of two reasons: first, we take advantage of the effective convergence rate these errors are experiencing; second, adopting a series of precautions to avoid nonconservative error estimates in the presence of nonmonotonic convergence. All told, therefore, markedly improved error estimation with present discretization and boundary-condition error estimates of (3.5) and (3.24).

3.6. Application

Here, for our application, we consider a pin hole in a circular shaft (Figure 3.3). For this configuration we wish to determine the peak stress in the pin hole corner. We begin with a description of this application, then describe the implementation of our submodeling procedure. Thereafter we report the results found.

The shaft is a crank shaft in a two stroke engine that is penetrated by a keyway and a pin hole (Figure 3.3(a)). This shaft is prone to failure across the pin hole region under pure torsion (Figure 3.3(a)). This failure occurs around the corners of the pin hole due to high stress
concentrations. Then the region of our interest, the pin hole, is shown in the close-up (Figure 3.3(b)).

![Image of shaft cross section showing pin hole and keyway](image)

**Figure 3.3** Photograph of shaft cross section: (a) Full cross section of the shaft with keyway and pin hole; (b) Close-up of pin hole.

We take rectangular Cartesian coordinates \((x, y, z)\) with its \(x\) and \(y\) axis passing through the center of the shaft and the pin hole, respectively, as our basic coordinate system to formulate our application (Figure 3.4(a) and (b)). The shaft has a length of \(2l\) and a radius \(r_s\) (Figure 3.4). The pin hole is of depth \(h_0\) and has diameters of \(d_1\) and \(d_2\) (close-up of Figure 3.4(b)). The pin hole also has three root radii at its corners namely, \(r_1\), \(r_2\) and \(r_3\) as shown in close-up of Figure 3.4(b). The keyway is of depth \(d\), width \(w\) and has a root radius of \(r_4\) (Figure 3.4(c)). To facilitate in applying shear tractions we use local cylindrical coordinates \((r, \theta)\) with origin \(O\) at the center of the shaft cross section at \(x = l\) (Figure 3.4(c)). The failure of the shaft under pure torsion occurs on a vertical plane along the line \(AA'\) as indicated in the top view of Figure 3.4(a). The vertical plane is located at an angle of \(45^\circ\) from the \(x\) axis where the maximum tensile stress occurs.
Figure 3.4  Geometry and coordinates for application: (a) top view; (b) front view; (c) end view.

We additionally use local cylindrical coordinates $(\tilde{r}, \tilde{\theta})$ with origin $\tilde{O}$ near the top corner of the pin hole, primarily to present results (Figure 3.5). This local coordinate is located on the
vertical plane along the line $AA'$ (Figure 3.5). The origin $O$ is located at the center of the curvature of the curved surface with radius $r_1$ (close-up of Figure 3.5). The $\tilde{x}$ and $\tilde{y}$ axes of the local coordinates are parallel to the $x$ and $y$ axes, respectively, of the shaft. Here $\tilde{\theta}$ varies between $0^\circ$ and $\phi$. The specific value of $\phi$ is given later in this section. With these geometric preliminaries in place, we next describe our application problem.

![Local cylindrical coordinates near the pin hole.](image)

In general, we seek the three-dimensional normal stress components $\sigma_x$, $\sigma_y$, $\sigma_z$ and shear stress components $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$, together with their companion displacements $u_x$, $u_y$, $u_z$, as functions of $x$, $y$, and $z$ throughout the entire shaft satisfying the three-dimensional field equations of elasticity and the following boundary conditions: the applied shear traction

$$\tau_{r\theta} = \frac{\tau_{\text{max}} r}{r_s}$$

(3.48)

on $x = l$ for $-r_s \leq y \leq r_s$, $-r_s \leq z \leq r_s$, where
\[
\tau_{\text{max}} = \frac{T r_s}{J}
\]  

(3.49)

where \( T \) is the torque applied to the shaft and \( J \) is the polar moment of inertia of the shaft cross section about its center; the clamped conditions

\[
u_x = u_y = u_z = 0
\]

(3.50)
on \( x = -l \) for \( -r_s \leq y \leq r_s, -r_s \leq z \leq r_s \); and stress-free conditions on other surfaces.

In particular, we seek the peak first principal stress occurring at the top corner of the pin hole having a radius of curvature \( r_1 \). Using the local cylindrical coordinates, we take the peak stress to occur at \( \tilde{r} = r_1 \) for \( 0^0 \leq \tilde{\theta} \leq \phi \), and seek the value for

\[
\bar{\sigma}_{\tilde{\theta}} = \frac{\sigma_{\tilde{\theta}}}{\tau_0}
\]

(3.51)

where

\[
\tau_0 = \frac{2T}{\pi r_s^3}
\]

(3.52)

A priori knowledge of the precise value of \( \tilde{\theta} \) where the peak stress would occur is not available because linearly varying shear tractions are being applied.

The specific measured dimensions of the geometry are as follows: \( r_s = 7.9375 \text{ mm (0.3125")}, l = 31.75 \text{ mm (1.25")}, w = 5.9436 \text{ mm (0.234")}, d = 2.9718 \text{ mm (0.117")}, h_0 = 8.0086 \text{ mm (0.3153")}, h_1 = 3.5941 \text{ mm (0.1415")}, h_2 = 2.2835 \text{ mm (0.0899")}, d_1 = 6.35 \text{ mm (1/4")}, d_2 = 3.175 \text{ mm (1/8")}, r_1 = 1.778 \text{ mm (0.07")}, r_2 = 0.762 \text{ mm (0.03")}, r_3 = 1.27 \text{ mm (0.05")}, r_4 = 0.2032 \text{ mm (0.008")}, \phi = 26^0 \).

To begin the FEA of this application, we use global meshes of 8H elements, (SOLID185, ANSYS [22]). We start our discretization \((m = 1)\) with a uniformly structured coarse mesh around.
the pin hole to facilitate our submodeling procedure (close-up of Figure 3.6(b)). Outside of this region, free meshes are generated using an automatic mesh generator, [10] (Figure 3.6(a)).

Figure 3.6 Finite element meshes for stresses in the corner of a pin hole: (a) initial global mesh \((m = 1)\); (b) close-up of initial mesh; (c) further close-up of the first subregion in the finest global mesh \((m = 4)\).

To apply shear tractions we use multipoint constraint elements (MPC184, [22]) as rigid beams. We create these elements after meshing the shaft with spatial elements (8H elements). A master node is created along \(x\) axis at some \(x > l\). Then the master node is connected to all the
nodes of 8H elements on the cross sectional surface at $x = l$ for $-r_s \leq y \leq r_s$, $-r_s \leq z \leq r_s$ using multipoint constraint elements. Then the torsion $T$ is applied to the master node which is then transferred to the spatial elements (8H elements) by multipoint constraint elements. The initial largely free global mesh ($m = 1$) used has 3966 spatial and 65 multipoint constraint elements. Hence a total of 4031 elements are present in our initial global mesh. This mesh is systematically refined by halving the element sides in the vicinity of the pin hole, and outside of this region is refined such that ($\lambda \sim 2$). Three further meshes ($m = 2 - 4$) are produced with such refinement. The actual numbers of elements for each global mesh are given in Table 3.9. Our finest global mesh has about 1.9 million elements.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Number of SOLID185 elements</th>
<th>Number of MPC184 elements</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3,966</td>
<td>65</td>
<td>4,031</td>
</tr>
<tr>
<td>2</td>
<td>31,656</td>
<td>201</td>
<td>31,857</td>
</tr>
<tr>
<td>3</td>
<td>237,118</td>
<td>732</td>
<td>237,850</td>
</tr>
<tr>
<td>4</td>
<td>1,984,297</td>
<td>2,793</td>
<td>1,987,090</td>
</tr>
</tbody>
</table>

We implement the submodeling procedure of Section 2.3. The region highlighted by red line in the close-ups of Figures 3.6(b) and (c) forms our first submodel region. We check for the correctness of our submodel boundary conditions with our first structured submodel mesh ($m = 5$) that has about 4 thousand spatial elements. Thereafter we refine this mesh by successively halving the sides and produce three further ($m = 6 - 8$). The finest mesh ($m = 8$) from our first submodel sequence shown in Figure 3.7(a) has about 2 million elements. The application at hand requires 2 submodels to achieve excellent results. The region highlighted by red line in the close-up of Figure 3.7(b) then forms our second subregion. Again, we run the second submodel following our procedure of Section 2.3. We check for the correctness of our submodel boundary conditions with our first structured mesh of second subregion ($m = 9$), which has about 4 thousand spatial elements.
Thereafter successively refined meshes are produced by halving the element sides. We further run two meshes \((m = 10 \text{ and } 11)\). All our submodel meshes are run using bicubic surface fitted displacements on their cut boundaries. Our last submodel mesh has about 262 thousand elements, when a global mesh for the same resolution would have about 64 million elements.

Figure 3.7  Submodel meshes for application: (a) finest submodel mesh of the first subregion \((m = 8)\); (b) close-up of the second subregion in \(m = 8\); (c) finest submodel mesh of the second subregion \((m = 11)\).

The FEA values for the normalized peak stresses on the vertical plane along the line \(AA'\) from our global analysis along with their estimated discretization error, using (3.5), at the pin hole corners with radius of curvature \(r_1, r_2\) and \(r_3\) are given in Tables 3.10(a), (b) and (c), respectively. We also report the FEA values for the normalized peak stress on the vertical plane along the line
from our global mesh sequence with their estimated discretization error at the keyway corner with radius of curvature $r_4$ in Table 3.10(d). In applying our procedure to safeguard against underestimating errors in the presence of nonmonotonic convergence we replace the value of $\hat{\varepsilon}_m^d$ with one in accord with (3.12) – (3.14) and estimate the discretization error in the preceding tables. The estimated discretization error on the finest possible global mesh is at satisfactory level in accord with (3.20) at all the corners (Tables 3.10(a) - (d)). The decreasing trend of the estimated discretization error values in Tables 3.10 is consistent with a numerically converging analysis. For this application one would expect the stress concentrations at the pin hole corners and the keyway corner would interact. As shown in Tables 3.10(a) - (d) the stress concentration at the pin hole corner with radius of curvature $r_1$ is more than that at the keyway corner by a factor of 2. Hence we focus our analysis at the pin hole corner with radius of curvature $r_1$. If a satisfactory accuracy level of (3.20) is sought, we would accept the FEA value of the normalized peak stress from our last global mesh (Table 3.10(a)). Here, however, we seek excellent results and the estimated discretization error from our last global mesh does not comply with the excellent criterion of (3.20), namely less than 0.2%. Hence we proceed to submodeling to improve results at the pin hole corner with radius of curvature $r_1$.

Table 3.10(a). Finite element stresses, estimates of discretization error from global meshes at the pin hole corner with radius of curvature $r_1$.  

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m$</th>
<th>$\Delta\sigma_m$</th>
<th>$\hat{\varepsilon}_m^d$</th>
<th>$\hat{\varepsilon}_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.98522</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.64344</td>
<td>0.65822</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3.93365</td>
<td>0.29021</td>
<td>1.2*</td>
<td>7.4</td>
</tr>
<tr>
<td>4</td>
<td>4.05829</td>
<td>0.12464</td>
<td>1.2</td>
<td>2.3</td>
</tr>
</tbody>
</table>
Table 3.10(b). Finite element stresses, estimates of discretization error from global meshes at the pin hole corner with radius of curvature \( r_2 \).  

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_m )</th>
<th>( \Delta \sigma_m )</th>
<th>( \dot{\varepsilon}_m )</th>
<th>( \dot{\varepsilon}_m^d )</th>
<th>( \dot{\varepsilon}_m^d ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.70912</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.86351</td>
<td>0.15439</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.93469</td>
<td>0.07118</td>
<td>1.2*</td>
<td>7.6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.96557</td>
<td>0.03088</td>
<td>1.2</td>
<td>2.4</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.10(c). Finite element stresses, estimates of discretization error from global meshes at the pin hole corner with radius of curvature \( r_3 \).  

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_m )</th>
<th>( \Delta \sigma_m )</th>
<th>( \dot{\varepsilon}_m )</th>
<th>( \dot{\varepsilon}_m^d )</th>
<th>( \dot{\varepsilon}_m^d ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.40674</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.50008</td>
<td>0.09334</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.53438</td>
<td>0.03430</td>
<td>1.5*</td>
<td>6.4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.54417</td>
<td>0.00979</td>
<td>1.8</td>
<td>0.70</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.10(d). Finite element stresses, estimates of discretization error from global meshes at the keyway corner with radius of curvature \( r_4 \).  

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_m )</th>
<th>( \Delta \sigma_m )</th>
<th>( \dot{\varepsilon}_m )</th>
<th>( \dot{\varepsilon}_m^d )</th>
<th>( \dot{\varepsilon}_m^d ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.72961</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.81185</td>
<td>0.08224</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.84611</td>
<td>0.03426</td>
<td>1.3*</td>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.85724</td>
<td>0.01113</td>
<td>1.6*</td>
<td>0.60</td>
<td></td>
</tr>
</tbody>
</table>

We illustrate the convergence of the peak stress resulting from our analysis in Figure 3.8. The distribution of \( \sigma_\theta \) normalized by \( \tau_0 \) from our global analysis is shown in Figure 3.8(a). These distributions are shown as functions of normalized distance \( y/h_0 \). The peak stress appears to be converging away from the pin hole corner having a radius of curvature \( r_1 \). Figure 3.8(b) shows the distribution of the normalized peak stress near the pin hole corner of interest from our first
submodel analysis, on an expanded vertical scale. The vertical scale is expanded by a factor of 10 (Figure 3.8(b)). The peak stress appear to be converging further near the pin hole corner. Figure 3.8(c) shows the distribution of the normalized peak stress near the pin hole corner of interest from our second submodel analysis. The vertical scale of Figure 3.8(c) is further expanded by a factor of 4. The peak stress appear to have converged away from the pin hole corner. The close-up in Figure 3.8(c) now shows the stresses near the pin hole corner of interest are converging further on a scale which is expanded by a factor of 40.

The FEA values from first submodel analysis are given above the dotted line in Table 3.11(a). These are also accompanied by their corresponding discretization and total error estimates. The FEA stress values from our last global mesh (Table 3.10(a) for $m = 4$) and the first mesh of first submodel (Table 3.11(a) for $m = 5$), are exactly the same, which confirms that submodel boundary conditions are correctly being implemented.

The estimated boundary condition error from (3.24) on our first submodel mesh sequence are 0.058% (Table 3.11(b)), and so less than 0.067%. The estimated discretization error from (3.5) for that last mesh in the first submodel mesh sequence ($m = 8$) is 0.28%. The estimated total error for the same mesh from (3.25) is 0.34%, still not less than 0.2%. Hence a further submodel was run (meshes $m = 9 - 11$). The FEA stress values from the second submodel analysis with their corresponding discretization and total error estimates are given in Table 3.11(a), below the dotted line. The FEA stress values from our last mesh of the first submodel (Table 3.11(a) for $m = 8$) and the first mesh of second submodel (Table 3.11(a) for $m = 9$), are exactly the same, which confirms that submodel boundary conditions are correctly being implemented.
Figure 3.8  Convergence of peak normalized stress: (a) global meshes; (b) first submodel mesh sequence; (c) second submodel mesh sequence.
The estimated boundary condition error on our last submodel mesh \((m = 11)\) is 0.0029\% which is increased, by adding it with the boundary-condition error estimate of the corresponding mesh from the previous submodel, to 0.061\% (Table 3.11(b)), and just less than 0.067\%. While using multiple submodels control over boundary-condition error is achieved only when the values of this estimate from the first submodel mesh sequence are significantly below the acceptable criterion of (3.26), i.e., for our analysis these values should be significantly less than 0.067\%. The estimated discretization error on the same mesh is 0.069\% (Table 3.11(a)). The estimated total error on the same mesh is 0.13\% (Table 3.11(a)), thus now less than 0.2\%. Hence for this and like applications, when analyzed with 8H elements in conjunction with bicubic surface fitted boundary conditions excellent results can be obtained.

Table 3.11(a). Finite element stresses, estimates of discretization and total error from structured submodel meshes at the pin hole corner with radius of curvature \(r_1\).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\sigma^M_{mb})</th>
<th>(\Delta\sigma_m)</th>
<th>(\hat{\epsilon}_m)</th>
<th>(\hat{\epsilon}_m^d) (%)</th>
<th>(\hat{\epsilon}_m^t) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.05829</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4.10220</td>
<td>0.04391</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4.12426</td>
<td>0.02206</td>
<td>0.99</td>
<td>0.54</td>
<td>0.60</td>
</tr>
<tr>
<td>8</td>
<td>4.13550</td>
<td>0.01124</td>
<td>0.97</td>
<td>0.28</td>
<td>0.34</td>
</tr>
<tr>
<td>9</td>
<td>4.13550</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4.14093</td>
<td>0.00543</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>4.14369</td>
<td>0.00276</td>
<td>0.98</td>
<td>0.069</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 3.11(b). Finite element stresses and estimates of boundary-condition error from structured submodel meshes for the pin hole corner with radius of curvature \(r_1\).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\sigma^M_{mb})</th>
<th>(\sigma^{(M-1)b}_m)</th>
<th>(\sigma^{(M-2)b}_m)</th>
<th>(\hat{\epsilon}_m^M)</th>
<th>(\hat{\epsilon}_m^b) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.05829</td>
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<td></td>
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<td></td>
</tr>
<tr>
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<td>4.09545</td>
<td>4.06940</td>
<td>1.9</td>
<td>0.058</td>
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</tbody>
</table>
(Table 3.11(b) continued)

<table>
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<tr>
<th>$m$</th>
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<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\varepsilon_{mb}^{M}$</th>
<th>$\varepsilon_{m}^{b}$ (%)</th>
</tr>
</thead>
<tbody>
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<td>0.058</td>
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<td>4.12868</td>
<td>4.10237</td>
<td>1.9</td>
<td>0.058</td>
</tr>
<tr>
<td>9</td>
<td>4.13550</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>4.14093</td>
<td>4.14065</td>
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<td>0.060</td>
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<tr>
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<td>4.14334</td>
<td>4.14198</td>
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<td>0.061</td>
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</tbody>
</table>

3.7. Concluding Remarks

Submodeling is effective when the discretization errors and boundary-condition errors are controlled. In this Chapter we show that computational savings in 3D are significant with our submodeling procedure.

Here we use improved estimates of both the errors. We further use the procedures to safeguard against underestimating errors in the presence of nonmonotonic convergence for both the errors. We also use both displacement shape functions and bicubic surface to fit displacement boundary conditions on the cut boundaries.

These improvements are demonstrated to be effective on a set of nine 3D test problems with known exact solutions for peak stresses. These test problems have a range of stress concentration factors that exceeds those normally found in practice (cf., Peterson [1]). The improved discretization error estimate along with the procedures to safeguard against underestimating errors in the presence of nonmonotonic convergence always result in conservative estimates throughout global mesh sequences when compared to true error values. Although they are conservative, these error estimates suggest the use of submodeling when true errors also indicate the same. With all the modifications including those of Sinclair et al., [47] we are able to promote conservative error estimates. Nonetheless with the present improved method we do
underestimate the discretization error one time by one level whereas without any modifications we underestimate the discretization error on 35 instances by one or more levels of accuracy.

The improved boundary-condition error estimate along with the procedures to safe guard against underestimating errors in the presence of nonmonotonic convergence correctly indicate when it is necessary to enlarge the subregion to gain control of these errors. Thereafter, these enlarged subregions so that control is achieved are always found. In comparison, the boundary-condition error estimate without any modifications consistently overestimates their true values by a factor of three and thus are significantly less accurate whereas the present improved method is more accurate and conservative.

Results from 3D submodel analysis of structured meshes with 8H elements also indicate that they are the same irrespective of whether bicubic surface or shape functions are used to fit boundary conditions. Hence for these low-order elements, finite element engineers can be expected to use shape functions for submodel boundary conditions because these are easier to implement. Although free meshes for both global and submodel meshes together with shape functions are easier to implement, then what is demonstrated here is that one can expect the need of significantly more submodel meshes to compute accurate results.

For higher stress concentrations, excellent results are achieved from submodel analysis of structured meshes with 20H elements whereas only good results are obtained using 8H elements. The excellent results with 20H elements are obtained only when bicubic surface fit boundary conditions are used. Hence structured submodel meshes with 20H elements using bicubic surface fit boundary conditions perform better than 8H elements with half the degrees of freedom being used.
The implementation of our submodeling procedure with 8H elements is further demonstrated on an application problem. Here we use bicubic surface to fit boundary conditions because the results from our test problems indicate that they are more numerically accurate. The error estimates indicate that excellent results are obtained for this application. Here partially structured global meshes in conjunction with structured submodel meshes appear to work as well as entirely structured meshes and hence can be a preferred approach for FEA engineers.

In toto, the approach to 3D submodeling described here demonstrates the control of FEA errors in submodeling by a combination of mesh refinement and increasing submodel regions.
Chapter 4. Concluding Remarks

We have successfully verified our improved submodeling procedure on a series of 2D and 3D test problems with known analytical closed-form solutions for their stress concentrations. Results demonstrate that accurate stress concentrations can be determined, even for high concentrations. These determinations are made in return for quite modest levels of computational effort. Finally we apply our submodeling procedure on applications in two and three dimensions. Excellent results are apparently obtained for both the application problems with similar computational efficiency. Thus continued computational savings with our submodeling procedure.

Using our verified submodeling approach other problems can be analyzed without significant modifications to the approach. We give some examples in what follows.

First, in 2D, we can apply our approach to compute acute stress concentrations at reentrant corners. For this problem high stresses are developed at the corner as the radius there approaches zero. The insertion of cohesive stress-separation laws is required to avoid stress singularities. A problem of such genre is an elliptical crack problem with cohesive stress-separation laws accurately solved by Sinclair et al., [49] using the submodeling technique given in [2]. Using our improved submodeling approach and error estimates, the stress for the corresponding corner should be able to be analyzed accurately.

Second, in 3D, we can apply our approach to compute stress intensity factors for the 3D crack problem when a crack intersects a free surface. Sinclair [50] and references therein give a review of analytical and numerical treatments of this problem. In [50] it is shown that stress singularities still persist at the interface of transverse crack and free surface in three dimensions. Then our submodeling approach should enable the accurate computation of corresponding stress intensity factors.
There are other potential areas of applications of our submodeling approach. Examples involving both 2D and 3D are fracture mechanics, contact problems, and elasto-plastic simulations. While these areas will require further development of our method, with suitable such adaptations the basic approach of using mesh refinement and convergence checks with varying submodel region sizes can be expected to yield accurate results.
References


38. Kim HS, Mall S. Investigation into three-dimensional effects of finite contact width on fretting fatigue. Finite Elements in Analysis and Design 2005; 41:1140-1159.


Appendix A: Stresses and Errors from Global Meshes for Two-Dimensional Test Problems

Finite element stresses from global meshes for all the two-dimensional test problems are given here in Tables A.1 – A.3, along with estimated and actual discretization errors.

Table A.1. Finite element stresses, estimated and actual errors from structured and free global meshes with 4Q elements for $\bar{\sigma}_{\text{max}} = 5$.

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma_m$</th>
<th>$\epsilon_m^d$ (%)</th>
<th>$\epsilon_m^d$ (%)</th>
<th>$\sigma_m$</th>
<th>$\epsilon_m^d$ (%)</th>
<th>$\epsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
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<td>2.8682</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.5310</td>
<td>9.4</td>
<td>3.4651</td>
<td>31</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.7544</td>
<td>6.0</td>
<td>4.9</td>
<td>79</td>
<td>20</td>
<td></td>
</tr>
<tr>
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<td>4.8740</td>
<td>2.8</td>
<td>2.5</td>
<td>28</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>5</td>
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<td>1.4</td>
<td>1.3</td>
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<td>0.64</td>
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<td>4.0</td>
</tr>
<tr>
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<td>4.9839</td>
<td>0.33</td>
<td>0.32</td>
<td>4.8937</td>
<td>2.6</td>
<td>2.1</td>
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</table>

Table A.2. Finite element stresses, estimated and actual errors from structured global meshes with 4Q elements for $\bar{\sigma}_{\text{max}} = 18$ and $377/8$.

<table>
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<tr>
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<td>$\sigma_m$</td>
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</tr>
<tr>
<td>2</td>
<td>12.3415</td>
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</tr>
<tr>
<td>3</td>
<td>14.6840</td>
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</tr>
<tr>
<td>4</td>
<td>16.1808</td>
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(Table A.2 continued)

<table>
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<th>$\bar{\sigma}_{\text{max}} = 37^{7/8}$</th>
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</thead>
<tbody>
<tr>
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<td>$\sigma_m$</td>
<td>$\dot{\epsilon}_m^d$ (%)</td>
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<tr>
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</table>

Table A.3. Finite element stresses, estimated and actual errors from structured global meshes with 4Q and 8Q elements for $\bar{\sigma}_{\text{max}} = 54$.  

<table>
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<th>$m$</th>
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<td>7</td>
<td>51.7166</td>
<td>5.1</td>
</tr>
</tbody>
</table>
Appendix B: Stresses and Errors from Submodel Meshes for Two-Dimensional Test Problems

Finite element stresses together with estimated and actual boundary-condition errors from submodel meshes for various test problems are given here in Tables B.1 – B.4. Finite element stresses and estimated and actual discretization and total errors are reported in Tables B.5 – B.8.

Table B.1. Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using cubic-spline fitted boundary conditions for $\sigma_{\text{max}} = 5$.

<table>
<thead>
<tr>
<th>$m$</th>
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<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
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</tr>
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<td>4.995954</td>
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</tr>
</tbody>
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Table B.2(a). Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using cubic-spline fitted boundary conditions for $\sigma_{\text{max}} = 18$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
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<td></td>
</tr>
<tr>
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<td>17.96231</td>
<td>17.94580</td>
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<td>17.96812</td>
<td>0.0081</td>
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</table>

Table B.2(b). Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using shape function boundary conditions for $\sigma_{\text{max}} = 18$.

<table>
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<tr>
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<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
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<tr>
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<tr>
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Table B.2(b) continued.

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<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\varepsilon_m^b$ (%)</th>
<th>$\varepsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
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</table>

Table B.3(a). Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using cubic-spline fitted boundary conditions for $\sigma_{\text{max}} = 377/8$.

<table>
<thead>
<tr>
<th>$m$</th>
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<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\varepsilon_m^b$ (%)</th>
<th>$\varepsilon_m^b$ (%)</th>
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<tr>
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<td>0.041</td>
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<td>0.041</td>
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</table>

Table B.3(b). Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using shape function boundary conditions for $\sigma_{\text{max}} = 377/8$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{M_b}^b$</th>
<th>$\sigma_{m}^{(M-1)b}$</th>
<th>$\sigma_{m}^{(M-2)b}$</th>
<th>$\varepsilon_m^b$ (%)</th>
<th>$\varepsilon_m^b$ (%)</th>
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</thead>
<tbody>
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<td>8</td>
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</tr>
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</table>
Table B.4. Finite element stresses as well as estimated and actual boundary-condition errors from structured submodel meshes with 4Q elements using shape function boundary conditions for $\bar{\sigma}_{\text{max}} = 54.$

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma^M_m$</th>
<th>$\sigma^{(M-1)b}_m$</th>
<th>$\sigma^{(M-2)b}_m$</th>
<th>$\varepsilon^b_m$ (%)</th>
<th>$\sigma^e_m$</th>
<th>$\varepsilon^e_m$ (%)</th>
</tr>
</thead>
<tbody>
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<td>11</td>
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</tr>
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<td>52.7154</td>
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<td>0.020</td>
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</tr>
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Table B.5. Finite element stresses, estimated and actual discretization, and total errors from structured submodel meshes with 4Q elements using cubic-spline fitted boundary conditions for $\bar{\sigma}_{\text{max}} = 5.$

<table>
<thead>
<tr>
<th>m</th>
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<th>$\varepsilon^d_m$ (%)</th>
<th>$\varepsilon^d^e_m$ (%)</th>
<th>$\varepsilon^t_m$ (%)</th>
<th>$\varepsilon^t_m$ (%)</th>
</tr>
</thead>
<tbody>
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<td></td>
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</tbody>
</table>

Table B.8. Finite element stresses, estimated and actual discretization, and total errors from structured submodel meshes with 4Q elements using shape function boundary conditions for $\bar{\sigma}_{\text{max}} = 54.$

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma^M_m$</th>
<th>$\varepsilon^d_m$ (%)</th>
<th>$\varepsilon^d^e_m$ (%)</th>
<th>$\varepsilon^t_m$ (%)</th>
<th>$\varepsilon^t_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>51.7166</td>
<td>2.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>52.8238</td>
<td>1.2</td>
<td>1.1</td>
<td>1.2</td>
<td>1.1</td>
</tr>
<tr>
<td>13</td>
<td>53.3996</td>
<td>0.57</td>
<td>0.55</td>
<td>0.59</td>
<td>0.57</td>
</tr>
<tr>
<td>14</td>
<td>53.6936</td>
<td>0.28</td>
<td>0.28</td>
<td>0.30</td>
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<tr>
<td>15</td>
<td>53.8422</td>
<td>0.14</td>
<td>0.14</td>
<td>0.16</td>
<td>0.15</td>
</tr>
<tr>
<td>16</td>
<td>53.9169</td>
<td>0.14</td>
<td>0.14</td>
<td>0.16</td>
<td>0.15</td>
</tr>
</tbody>
</table>
Table B.6. Finite element stresses, estimated and actual discretization, and total errors from structured submodel meshes with 4Q elements using both shape functions and cubic-spline fitted boundary conditions for $\sigma_{max}^\text{c} = 18$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{M_b}$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\varepsilon_m^l$ (%)</th>
<th>$\sigma_m^{M_b}$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\varepsilon_m^l$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>17.74855</td>
<td>17.74855</td>
<td></td>
<td></td>
<td>17.74855</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>17.87210</td>
<td>0.70</td>
<td>0.71</td>
<td></td>
<td>17.87210</td>
<td>0.70</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>17.93496</td>
<td>0.36</td>
<td>0.35</td>
<td>0.36</td>
<td>17.93496</td>
<td>0.36</td>
<td>0.35</td>
<td>0.36</td>
</tr>
<tr>
<td>11</td>
<td>17.96666</td>
<td>0.18</td>
<td>0.18</td>
<td>0.19</td>
<td>17.96666</td>
<td>0.18</td>
<td>0.18</td>
<td>0.19</td>
</tr>
</tbody>
</table>

Table B.7. Finite element stresses, estimated and actual discretization, and total errors from structured submodel meshes with 4Q elements using both shape functions and cubic-spline fitted boundary conditions for $\sigma_{max}^\text{c} = 37/8$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{M_b}$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\varepsilon_m^l$ (%)</th>
<th>$\sigma_m^{M_b}$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\varepsilon_m^l$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>36.74499</td>
<td></td>
<td></td>
<td></td>
<td>36.74499</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>37.28824</td>
<td>1.5</td>
<td>1.5</td>
<td></td>
<td>37.28824</td>
<td>1.5</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>37.57021</td>
<td>0.81</td>
<td>0.76</td>
<td>0.80</td>
<td>37.57021</td>
<td>0.80</td>
<td>0.76</td>
<td>0.81</td>
</tr>
<tr>
<td>11</td>
<td>37.71389</td>
<td>0.40</td>
<td>0.38</td>
<td>0.45</td>
<td>37.71389</td>
<td>0.39</td>
<td>0.38</td>
<td>0.40</td>
</tr>
<tr>
<td>12</td>
<td>37.78643</td>
<td>0.20</td>
<td>0.19</td>
<td>0.25</td>
<td>37.78643</td>
<td>0.20</td>
<td>0.19</td>
<td>0.21</td>
</tr>
<tr>
<td>13</td>
<td>37.82283</td>
<td>0.097</td>
<td>0.097</td>
<td>0.14</td>
<td>37.82283</td>
<td>0.097</td>
<td>0.097</td>
<td>0.11</td>
</tr>
</tbody>
</table>

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Appendix C: Stresses and Errors from Global Meshes for Three-Dimensional Test Problems

Finite element stresses from global meshes for all the three-dimensional test problems are given here in Tables C.1 – C.6, along with estimated and actual discretization errors. Whenever an asterisk is placed atop a $\hat{\epsilon}_m^d$ value, this value has been calculated by replacing $\frac{\Delta \sigma_{m-1}^d}{\Delta m}$ with 2 in (3.5) in accordance with (3.12) – (3.14).

Table C.1. Finite element stresses, estimated and actual discretization errors from free global meshes with 8H elements for $a = 0.4$ for $\bar{\sigma}_{max}$ (exact value = 1.083524).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m$</th>
<th>$\hat{\epsilon}_m^d$ (%)</th>
<th>$\epsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4265</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.5343</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.9167</td>
<td>NA</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>1.0332</td>
<td>11*</td>
<td>4.6</td>
</tr>
<tr>
<td>5</td>
<td>1.0869</td>
<td>4.9*</td>
<td>0.31</td>
</tr>
<tr>
<td>6</td>
<td>1.1235</td>
<td>7.0</td>
<td>3.7</td>
</tr>
</tbody>
</table>

Table C.2. Finite element stresses, estimated and actual discretization errors from structured global meshes with 8H elements for $a = 0.8$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{max} = 1.654618$</th>
<th>$\bar{\sigma}_h = 0.353414$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_m$</td>
<td>$\hat{\epsilon}_m^d$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>1.3386</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.5549</td>
<td>6.0</td>
</tr>
<tr>
<td>3</td>
<td>1.6334</td>
<td>4.8*</td>
</tr>
<tr>
<td>4</td>
<td>1.6533</td>
<td>1.2*</td>
</tr>
<tr>
<td>5</td>
<td>1.6565</td>
<td>NA</td>
</tr>
<tr>
<td>6</td>
<td>1.6563</td>
<td>NA</td>
</tr>
</tbody>
</table>
Table C.3. Finite element stresses, estimated and actual discretization errors from structured global meshes with 8H elements for $a = 0.9$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{\text{max}} = 2.307001$</th>
<th>$\bar{\sigma}_{\text{h}} = 0.617614$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>1.641967</td>
<td>0.51799</td>
</tr>
<tr>
<td>2</td>
<td>2.048764 11</td>
<td>0.60275 2.4</td>
</tr>
<tr>
<td>3</td>
<td>2.242300 8.6*</td>
<td>0.63271 4.7* 2.4</td>
</tr>
<tr>
<td>4</td>
<td>2.300289 2.5*</td>
<td>0.63343 NA 2.6</td>
</tr>
<tr>
<td>5</td>
<td>2.311374 NA 0.19</td>
<td>0.62794 NA 1.7</td>
</tr>
<tr>
<td>6</td>
<td>2.311290 NA 0.19</td>
<td>0.62343 3.3 0.94</td>
</tr>
</tbody>
</table>

Table C.4. Finite element stresses, estimated and actual discretization errors from structured global meshes with 8H elements for $a = 0.99$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{\text{max}} = 7.230177$</th>
<th>$\bar{\sigma}_{\text{h}} = 2.042294$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>2.3288</td>
<td>1.0383</td>
</tr>
<tr>
<td>2</td>
<td>3.8019 47</td>
<td>1.4901 27</td>
</tr>
<tr>
<td>3</td>
<td>5.4790 NA 24</td>
<td>1.8962 190 7.2</td>
</tr>
<tr>
<td>4</td>
<td>6.6414 40 8.1</td>
<td>2.0980 9.5 2.7</td>
</tr>
<tr>
<td>5</td>
<td>7.1191 6.7* 1.5</td>
<td>2.1340 NA 4.5</td>
</tr>
<tr>
<td>6</td>
<td>7.2414 1.7* 0.16</td>
<td>2.1098 NA 3.3</td>
</tr>
</tbody>
</table>

Table C.5. Finite element stresses, estimated and actual discretization errors from structured global meshes with 8H elements for $a = 0.999$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{\text{max}} = 22.578569$</th>
<th>$\bar{\sigma}_{\text{h}} = 5.934853$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
<td>$\sigma_m$ $\varepsilon_d^m$ (%) $\varepsilon_d^m$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>2.6105</td>
<td>1.1463</td>
</tr>
<tr>
<td>2</td>
<td>4.6972 79</td>
<td>1.8776 68</td>
</tr>
</tbody>
</table>
(Table C.5. continued)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{\text{max}}$</th>
<th>$\bar{\sigma}_h$</th>
<th>$\sigma_m$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\sigma_m$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8.2988</td>
<td>NA</td>
<td>63</td>
<td>3.1500</td>
<td>NA</td>
<td>47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>13.4845</td>
<td>NA</td>
<td>40</td>
<td>4.6885</td>
<td>NA</td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>18.5012</td>
<td>805</td>
<td>18</td>
<td>5.7932</td>
<td>49</td>
<td>2.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>21.4137</td>
<td>19</td>
<td>5.2</td>
<td>6.2167</td>
<td>6.8*</td>
<td>4.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table C.6. Finite element stresses, estimated and actual discretization errors from structured global meshes with 20H elements for $a = 0.999$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{\sigma}_{\text{max}} = 22.578569$</th>
<th>$\bar{\sigma}_h = 5.934853$</th>
<th>$\sigma_m$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\sigma_m$</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.1013</td>
<td>NA</td>
<td>1.6426</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.0763</td>
<td>69</td>
<td>2.7028</td>
<td>54</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>11.5939</td>
<td>NA</td>
<td>49</td>
<td>4.2264</td>
<td>NA</td>
<td>29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>NA</td>
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<td>5.6976</td>
<td>725</td>
<td>4.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>21.2531</td>
<td>85</td>
<td>5.9</td>
<td>6.4124</td>
<td>NA</td>
<td>8.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>23.0047</td>
<td>7.6*</td>
<td>1.9</td>
<td>6.3907</td>
<td>NA</td>
<td>7.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix D: Stresses and Errors from Submodel Meshes for Three-Dimensional Test Problems

Finite element stresses of $\bar{\sigma}_{\text{max}}$ together with estimated and actual boundary-condition, discretization and total errors from submodel meshes for various test problems are given here in Tables D.1(a) - D.8. Finite element stresses of $\bar{\sigma}_h$ together with estimated and actual boundary-condition, discretization and total errors are reported in Tables D.9(a) - D.18. Whenever an asterisk is placed atop a $\hat{\epsilon}_m^d$ value, this value has been calculated by replacing $\frac{\Delta \sigma_m^d}{\Delta \sigma_m}$ with 2 in (3.5) in accordance with (3.12) – (3.14).

Table D.1(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.8$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.654618).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{Mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\hat{\epsilon}_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.656258</td>
<td>1.656226</td>
<td>1.655986</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.655605</td>
<td>1.655755</td>
<td>1.655349</td>
<td>0.00028</td>
<td>1.655616</td>
<td>0.00066</td>
</tr>
<tr>
<td>12</td>
<td>1.655150</td>
<td>1.655121</td>
<td>1.654999</td>
<td>0.00026</td>
<td>1.655161</td>
<td>0.00066</td>
</tr>
</tbody>
</table>

Table D.1(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions for $a = 0.8$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.654618).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{Mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\hat{\epsilon}_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.656257</td>
<td>1.656472</td>
<td>1.657036</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.655629</td>
<td>1.655731</td>
<td>1.656810</td>
<td>0.00064</td>
<td>1.655615</td>
<td>0.00085</td>
</tr>
<tr>
<td>9</td>
<td>1.655178</td>
<td>1.655300</td>
<td>1.656425</td>
<td>0.00090</td>
<td>1.655161</td>
<td>0.0010</td>
</tr>
</tbody>
</table>

Table D.2. Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.9$ for $\bar{\sigma}_{\text{max}}$ (exact value = 2.307001).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{Mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\hat{\epsilon}_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.311290</td>
<td>2.310648</td>
<td>2.307916</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2.309532</td>
<td>2.308895</td>
<td>2.306236</td>
<td>0.0087</td>
<td>2.309747</td>
<td>0.0093</td>
</tr>
</tbody>
</table>

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Table D.3(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.99$ for $\sigma_{\text{max}}$ (exact value = 7.230177).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^{\text{M}}$</th>
<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_{mb}^b$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\dot{\varepsilon}_{mb}^e$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>2.308285</td>
<td>2.307651</td>
<td>2.305015</td>
<td>0.0087</td>
<td>2.308500</td>
<td>0.0093</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^{\text{M}}$</th>
<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_{mb}^b$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\dot{\varepsilon}_{mb}^e$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>7.2414</td>
<td>7.2294</td>
<td>7.1832</td>
<td>0.059</td>
<td>7.2454</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>7.2524</td>
<td>7.2404</td>
<td>7.1942</td>
<td>0.058</td>
<td>7.2564</td>
<td>0.055</td>
</tr>
<tr>
<td>15</td>
<td>7.2443</td>
<td>7.2324</td>
<td>7.1863</td>
<td>0.057</td>
<td>7.2483</td>
<td>0.055</td>
</tr>
<tr>
<td>16</td>
<td>7.2366</td>
<td>7.2246</td>
<td>7.1786</td>
<td>0.059</td>
<td>7.2406</td>
<td>0.055</td>
</tr>
</tbody>
</table>

Table D.3(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.99$ for $\sigma_{\text{max}}$ (exact value = 7.230177).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^{\text{M}}$</th>
<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_{mb}^b$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\dot{\varepsilon}_{mb}^e$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>7.241386</td>
<td>7.231451</td>
<td>7.193232</td>
<td>0.062</td>
<td>7.235750</td>
<td>0.047</td>
</tr>
<tr>
<td>14</td>
<td>7.252881</td>
<td>7.242885</td>
<td>7.204470</td>
<td>0.048</td>
<td>7.256411</td>
<td>0.049</td>
</tr>
<tr>
<td>15</td>
<td>7.244928</td>
<td>7.234930</td>
<td>7.196518</td>
<td>0.049</td>
<td>7.248338</td>
<td>0.047</td>
</tr>
<tr>
<td>16</td>
<td>7.237181</td>
<td>7.227191</td>
<td>7.188809</td>
<td>0.049</td>
<td>7.240560</td>
<td>0.047</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^{\text{M}}$</th>
<th>$\sigma_{mb}^{(M-1)b}$</th>
<th>$\sigma_{mb}^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_{mb}^b$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\dot{\varepsilon}_{mb}^e$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>7.237181</td>
<td>7.237145</td>
<td>7.237012</td>
<td>0.062</td>
<td>7.240742</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>7.232324</td>
<td>7.232274</td>
<td>7.232137</td>
<td>0.048</td>
<td>7.235750</td>
<td>0.047</td>
</tr>
<tr>
<td>19</td>
<td>7.229638</td>
<td>7.229588</td>
<td>7.229448</td>
<td>0.049</td>
<td>7.233033</td>
<td>0.047</td>
</tr>
</tbody>
</table>

115
Table D.4. Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.999$ for $\bar{\sigma}_{\text{max}}$ (exact value = 22.578569).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{\text{mb}}^M$</th>
<th>$\sigma_{\text{mb}}^{(M-1)}$</th>
<th>$\sigma_{\text{mb}}^{(M-2)}$</th>
<th>$\varepsilon_{\text{mb}}^B$ (%)</th>
<th>$\varepsilon_{\text{mb}}^B$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>22.4177</td>
<td>22.3470</td>
<td>22.0946</td>
<td>0.12</td>
<td>22.4428</td>
</tr>
<tr>
<td>15</td>
<td>22.6313</td>
<td>22.5598</td>
<td>22.3044</td>
<td>0.12</td>
<td>22.6560</td>
</tr>
<tr>
<td>16</td>
<td>22.6341</td>
<td>22.5626</td>
<td>22.3070</td>
<td>0.12</td>
<td>22.6587</td>
</tr>
<tr>
<td>17</td>
<td>22.6341</td>
<td>22.6237</td>
<td>22.5830</td>
<td>22.6641</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>22.6025</td>
<td>22.5920</td>
<td>22.5513</td>
<td>0.14</td>
<td>22.6310</td>
</tr>
<tr>
<td>19</td>
<td>22.5792</td>
<td>22.5687</td>
<td>22.5280</td>
<td>0.14</td>
<td>22.6073</td>
</tr>
</tbody>
</table>

Table D.5. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.8$ for $\bar{\sigma}_{\text{max}}$ (exact value = 1.654618).

<table>
<thead>
<tr>
<th>Interpolation method</th>
<th>$m$</th>
<th>$\sigma_{\text{mb}}^{M}$</th>
<th>$\varepsilon_{\text{mb}}^d$ (%)</th>
<th>$\varepsilon_{\text{mb}}^d$ (%)</th>
<th>$\varepsilon_{\text{mb}}^t$ (%)</th>
<th>$\varepsilon_{\text{mb}}^t$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bicubic surface</td>
<td>10</td>
<td>1.656258</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>1.655605</td>
<td>0.060</td>
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<td></td>
<td></td>
</tr>
<tr>
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<td>1.655150</td>
<td>0.063</td>
<td>0.033</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.656258</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shape function</td>
<td>8</td>
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<td>0.060</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>1.655178</td>
<td>0.069</td>
<td>0.033</td>
<td>0.070</td>
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</tr>
</tbody>
</table>

116
Table D.6. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions for \(a = 0.9\) for \(\bar{\sigma}_{\text{max}}\) (exact value = 2.307001).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\sigma_{M}^{Mb})</th>
<th>(\epsilon_{d}^{M}) (%)</th>
<th>(\epsilon_{d}^{d}) (%)</th>
<th>(\epsilon_{d}^{t}) (%)</th>
<th>(\epsilon_{m}^{t}) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.311290</td>
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</tr>
<tr>
<td>11</td>
<td>2.309532</td>
<td>0.12</td>
<td>0.13</td>
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<td>2.308285</td>
<td>0.13</td>
<td>0.065</td>
<td>0.14</td>
<td>0.074</td>
</tr>
</tbody>
</table>

Table D.7. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for \(a = 0.99\) for \(\bar{\sigma}_{\text{max}}\) (exact value = 7.230177).

<table>
<thead>
<tr>
<th>Interpolation method</th>
<th>(m)</th>
<th>(\sigma_{M}^{Mb})</th>
<th>(\epsilon_{d}^{M}) (%)</th>
<th>(\epsilon_{d}^{d}) (%)</th>
<th>(\epsilon_{d}^{t}) (%)</th>
<th>(\epsilon_{m}^{t}) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bicubic surface</td>
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<td></td>
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</tr>
<tr>
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<td>7.2524</td>
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<td>0.42</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>15</td>
<td>7.2443</td>
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<td>0.25</td>
<td>NA</td>
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<tr>
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<td>0.15</td>
<td>2.1</td>
<td>0.21</td>
</tr>
<tr>
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<td></td>
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<tr>
<td></td>
<td>18</td>
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<td>0.079</td>
<td>0.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>7.2289</td>
<td>0.044</td>
<td>0.040</td>
<td>0.11</td>
<td>0.097</td>
</tr>
<tr>
<td>Shape function</td>
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<td></td>
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<td>0.41</td>
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<td></td>
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<td>0.14</td>
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</tr>
<tr>
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<td>7.237181</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>7.232324</td>
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<td>0.12</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>7.229638</td>
<td>0.046</td>
<td>0.040</td>
<td>0.095</td>
<td>0.087</td>
</tr>
</tbody>
</table>
Table D.8. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using shape function boundary conditions for \( a = 0.999 \) for \( \bar{\sigma}_{\text{max}} \) (exact value = 22.578569).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{mb}^{M} )</th>
<th>( \bar{\varepsilon}_{m}^{d} (%) )</th>
<th>( \varepsilon_{m}^{d} (%) )</th>
<th>( \bar{\varepsilon}_{m}^{l} (%) )</th>
<th>( \varepsilon_{m}^{l} (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>21.4137</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>22.4177</td>
<td>0.60</td>
<td>0.71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>22.6313</td>
<td>0.94*</td>
<td>0.34</td>
<td>1.1</td>
<td>0.45</td>
</tr>
<tr>
<td>16</td>
<td>22.6341</td>
<td>NA</td>
<td>0.36</td>
<td>NA</td>
<td>0.47</td>
</tr>
<tr>
<td>17</td>
<td>22.6341</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>22.6025</td>
<td>0.23</td>
<td>0.36</td>
<td></td>
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</tr>
<tr>
<td>19</td>
<td>22.5792</td>
<td>0.29</td>
<td>0.13</td>
<td>0.43</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table D.9(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for \( a = 0.8 \) for \( \bar{\sigma}_{h} \) (exact value = 0.353414).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{mb}^{M} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \bar{\varepsilon}_{m}^{b} (%) )</th>
<th>( \varepsilon_{m}^{b} (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.355534</td>
<td>0.355510</td>
<td>0.355388</td>
<td></td>
<td>0.355542</td>
</tr>
<tr>
<td>11</td>
<td>0.354524</td>
<td>0.354501</td>
<td>0.354390</td>
<td>0.0017</td>
<td>0.354531</td>
</tr>
<tr>
<td>12</td>
<td>0.353978</td>
<td>0.353957</td>
<td>0.353848</td>
<td>0.0014</td>
<td>0.353986</td>
</tr>
</tbody>
</table>

Table D.9(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions for \( a = 0.8 \) for \( \bar{\sigma}_{h} \) (exact value = 0.353414).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \sigma_{mb}^{M} )</th>
<th>( \sigma_{m}^{(M-1)b} )</th>
<th>( \sigma_{m}^{(M-2)b} )</th>
<th>( \bar{\varepsilon}_{m}^{b} (%) )</th>
<th>( \varepsilon_{m}^{b} (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
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<td>0.355678</td>
<td>0.355508</td>
<td></td>
<td>0.355545</td>
</tr>
<tr>
<td>8</td>
<td>0.354536</td>
<td>0.354560</td>
<td>0.354671</td>
<td>0.0019</td>
<td>0.354532</td>
</tr>
<tr>
<td>9</td>
<td>0.353990</td>
<td>0.354012</td>
<td>0.354111</td>
<td>0.0018</td>
<td>0.353986</td>
</tr>
</tbody>
</table>
Table D.10(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.9$ for $\bar{\sigma}_h$ (exact value = 0.617614).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon^b_m$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\epsilon^b_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.623433</td>
<td>0.623245</td>
<td>0.622421</td>
<td>0.623527</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.620647</td>
<td>0.620464</td>
<td>0.619675</td>
<td>0.620710</td>
<td>0.0089</td>
<td>0.010</td>
</tr>
<tr>
<td>12</td>
<td>0.619139</td>
<td>0.618957</td>
<td>0.618181</td>
<td>0.619201</td>
<td>0.0090</td>
<td>0.010</td>
</tr>
<tr>
<td>13</td>
<td>0.618356</td>
<td>0.618175</td>
<td>0.617403</td>
<td>0.618418</td>
<td>0.0090</td>
<td>0.010</td>
</tr>
</tbody>
</table>

Table D.10(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.9$ for $\bar{\sigma}_h$ (exact value = 0.617614).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon^b_m$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\epsilon^b_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.623433</td>
<td>0.623342</td>
<td>0.623041</td>
<td>0.623427</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.620668</td>
<td>0.620565</td>
<td>0.620327</td>
<td>0.620710</td>
<td>0.013</td>
<td>0.0068</td>
</tr>
<tr>
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<td>0.619056</td>
<td>0.618837</td>
<td>0.619201</td>
<td>0.017</td>
<td>0.0062</td>
</tr>
<tr>
<td>13</td>
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<td>0.618272</td>
<td>0.618057</td>
<td>0.618418</td>
<td>0.018</td>
<td>0.0062</td>
</tr>
</tbody>
</table>

Table D.11(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.99$ for $\bar{\sigma}_h$ (exact value = 2.042294).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_{mb}^M$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon^b_m$ (%)</th>
<th>$\sigma_{mb}^e$</th>
<th>$\epsilon^b_m$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.0947</td>
<td>2.1109</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
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<td>2.0785</td>
<td>2.0666</td>
<td>2.0826</td>
<td>0.052</td>
<td>0.049</td>
</tr>
<tr>
<td>15</td>
<td>2.0630</td>
<td>2.0600</td>
<td>2.0482</td>
<td>2.0641</td>
<td>0.050</td>
<td>0.054</td>
</tr>
<tr>
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<td>2.0526</td>
<td>2.0495</td>
<td>2.0378</td>
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<td>0.054</td>
<td>0.049</td>
</tr>
<tr>
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<td>2.0524</td>
<td>2.0519</td>
<td>2.0536</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2.0469</td>
<td>2.0463</td>
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<td>0.054</td>
</tr>
<tr>
<td>19</td>
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<td>2.0440</td>
<td>2.0435</td>
<td>2.0452</td>
<td>0.051</td>
<td>0.054</td>
</tr>
</tbody>
</table>
Table D.11(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.99$ for $\bar{\sigma}_h$ (exact value = 2.042294).

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma_m^{Mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\varepsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.096352</td>
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<td>2.110890</td>
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<td>2.053588</td>
<td>0.045</td>
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Table D.12(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).

<table>
<thead>
<tr>
<th>m</th>
<th>$\sigma_m^{Mb}$</th>
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<th>$\sigma_m^{(M-2)b}$</th>
<th>$\dot{\varepsilon}_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\varepsilon_m^b$ (%)</th>
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<td>0.12</td>
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<td>5.9403</td>
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<td>0.12</td>
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120
Table D.12(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 8H elements using shape function boundary conditions on an enlarged subregion for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).

<table>
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<tr>
<th>$m$</th>
<th>$\sigma_m^{mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
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<td>6.0455</td>
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Table D.13(a). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 20H elements using bicubic surface fitted boundary conditions on an enlarged subregion for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
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</thead>
<tbody>
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<td>5.7213</td>
<td>0.063</td>
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<td>5.9098</td>
<td>5.7093</td>
<td>0.063</td>
<td>5.9391</td>
<td>0.062</td>
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</table>

Table D.13(b). Finite element stresses, estimated and actual boundary-condition errors from structured submodel meshes with 20H elements using shape function boundary conditions for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\sigma_m^{mb}$</th>
<th>$\sigma_m^{(M-1)b}$</th>
<th>$\sigma_m^{(M-2)b}$</th>
<th>$\epsilon_m^b$ (%)</th>
<th>$\sigma_m^{eb}$</th>
<th>$\epsilon_m^b$ (%)</th>
</tr>
</thead>
<tbody>
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<td>6.454356</td>
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<td>6.703814</td>
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</tr>
<tr>
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<td>6.177619</td>
<td>6.505083</td>
<td>0.00012</td>
<td>6.197157</td>
<td>0.36</td>
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<td>6.324086</td>
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(Table D.13(b) continued)

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<th>$\epsilon_{m}^{b}$ (%)</th>
<th>$\sigma_{m}^{eb}$</th>
<th>$\epsilon_{m}^{b}$ (%)</th>
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</table>

Table D.14. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.8$ for $\bar{\sigma}_{h}$ (exact value = 0.353414).

<table>
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<th>$m$</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
</tr>
</thead>
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<tr>
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<td>0.18</td>
<td>0.16</td>
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<td>0.16</td>
<td>0.19</td>
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</table>

Table D.15. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.9$ for $\bar{\sigma}_{h}$ (exact value = 0.617614).

<table>
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<tr>
<th>Interpolation method</th>
<th>$m$</th>
<th>$\sigma_{m}^{Mb}$</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{d}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
<th>$\epsilon_{m}^{t}$ (%)</th>
</tr>
</thead>
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</table>
Table D.16. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.99$ for $\bar{\sigma}_h$ (exact value = 2.042294).

<table>
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<th>$\sigma_m^{Mb}$</th>
<th>$\dot{\varepsilon}_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\dot{\varepsilon}_m^t$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\sigma_m^{Mb}$</th>
<th>$\dot{\varepsilon}_m^d$ (%)</th>
<th>$\varepsilon_m^d$ (%)</th>
<th>$\dot{\varepsilon}_m^t$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
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Table D.17. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 8H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).

<table>
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<tr>
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<th>$\sigma_m^{Mb}$</th>
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<th>$\varepsilon_m^d$ (%)</th>
<th>$\dot{\varepsilon}_m^t$ (%)</th>
<th>$\varepsilon_m^t$ (%)</th>
<th>$\sigma_m^{Mb}$</th>
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<th>$\varepsilon_m^d$ (%)</th>
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<td>$\dot{\epsilon}_m^t$ (%)</td>
<td>$\dot{\epsilon}_m^t$ (%)</td>
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</table>

Table D.18. Finite element stresses, estimated and actual discretization and total errors from structured submodel meshes with 20H elements using both shape function and bicubic surface fitted boundary conditions for $a = 0.999$ for $\bar{\sigma}_h$ (exact value = 5.934853).
Vita

Ajay Kardak was born in Gwalior and raised in Bangalore, India. After finishing higher secondary education he attended Visvesvaraya Technological University, Bangalore where he received his Bachelor of Engineering degree in Mechanical Engineering in 2003. Between 2003 and 2005 he worked at Indian Institute of Science, Bangalore, India. To further his knowledge in the field of science and technology, he enrolled in the Department of Mechanical and Industrial Engineering at Louisiana State University (LSU) in Fall 2005. He received his Master of Science degree in Fall of 2008. He started his doctoral program at LSU in the summer of 2009 under the guidance of Dr. Glenn B. Sinclair and expects to graduate in Dec of 2015.